On Progressive Waves. *By* Lord RAYLEIGH, M.A., F.R.S.

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It has often been remarked that, when a group of waves advances into still water, the velocity of the group is less than that of the individual waves of which it is composed; the waves appear to advance through the group, dying away as they approach its anterior limit. This phenomenon was, I believe, first explained by Stokes, who regarded the group as formed by the superposition of two infinite trains of waves, of equal amplitudes and of nearly equal wave-lengths, advancing in the same direction. My attention was called to the subject about two years since by Mr. Froude, and the same explanation then occurred to me independently.* In my book on the "Theory of Sound" (§ 191), I have considered the question more generally, and have shewn that, if V be the velocity of propagation of any kind of waves whose wave-length is λ, and $\kappa = 2\pi\lambda^{-1}$, then U, the velocity of a group composed of a great number of waves, and moving into an undisturbed part of the medium, is expressed by

$$U = \frac{d(\kappa V)}{d\kappa} \quad \text{.............................} \quad (1),$$

or, as we may also write it,

$$U : V = 1 + \frac{d \log V}{d \log \kappa} \quad \text{.........................} \quad (2).$$

Thus, if $V \propto \lambda^n$, $\qquad U = (1-n)\,V$ ....(3).

In fact, if the two infinite trains be represented by $\cos\kappa(Vt-x)$ and $\cos\kappa'(V't-x)$, their resultant is represented by

$$\cos\kappa(Vt-x) + \cos\kappa'(V't-x),$$

which is equal to

$$2\cos\left\{\frac{\kappa'V'-\kappa V}{2}t - \frac{\kappa'-\kappa}{2}x\right\} . \cos\left\{\frac{\kappa'V'+\kappa V}{2}t - \frac{\kappa'+\kappa}{2}x\right\}.$$

* Another phenomenon, also mentioned to me by Mr. Froude, admits of a similar explanation. A steam launch moving quickly through the water is accompanied by a peculiar system of diverging waves, of which the most striking feature is the obliquity of the line containing the greatest elevations of successive waves to the wave-fronts. This wave pattern may be explained by the superposition of two (or more) infinite trains of waves, of slightly differing wave-lengths, whose directions and velocities of propagation are so related in each case that there is no change of position relatively to the boat. The mode of composition will be best understood by drawing on paper two sets of parallel and equidistant lines, subject to the above condition, to represent the crests of the component trains. In the case of two trains of slightly different wave-lengths, it may be proved that the tangent of the angle between the line of maxima and the wave-fronts is half the tangent of the angle between the wave-fronts and the boat's course.

If $\kappa' - \kappa$, $V' - V$ be small, we have a train of waves whose amplitude varies slowly from one point to another between the limits 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. The position at time t of the middle of that group, which was initially at the origin, is given by

$$(\kappa' V' - \kappa V)\, t - (\kappa' - \kappa)\, x = 0,$$

which shews that the velocity of the group is $(\kappa' V' - \kappa V) \div (\kappa' - \kappa)$. In the limit, when the number of waves in each group is indefinitely great, this result coincides with (1).

The following particular cases are worth notice, and are here tabulated for convenience of comparison :—

$V \propto \lambda$,	$U = 0$,	Reynolds' disconnected pendulums.
$V \propto \lambda^{\frac{1}{2}}$,	$U = \frac{1}{2} V$,	Deep-water gravity waves.
$V \propto \lambda^{0}$,	$U = V$,	Aërial waves, &c.
$V \propto \lambda^{-\frac{1}{2}}$,	$U = \frac{3}{2} V$,	Capillary water waves.
$V \propto \lambda^{-1}$,	$U = 2V$,	Flexural waves.

The capillary water waves are those whose wave-length is so small that the force of restitution due to capillarity largely exceeds that due to gravity. Their theory has been given by Thomson (Phil. Mag., Nov. 1871). The flexural waves, for which $U = 2V$, are those corresponding to the bending of an elastic rod or plate ("Theory of Sound," § 191).

In a paper read at the Plymouth meeting of the British Association (afterwards printed in "Nature," Aug. 23, 1877), Prof. Osborne Reynolds gave a dynamical explanation of the fact that a group of deep-water waves advances with only half the rapidity of the individual waves. It appears that the energy propagated across any point, when a train of waves is passing, is only one-half of the energy necessary to supply the waves which pass in the same time, so that, if the train of waves be limited, it is impossible that its front can be propagated with the full velocity of the waves, because this would imply the acquisition of more energy than can in fact be supplied. Prof. Reynolds did not contemplate the cases where *more* energy is propagated than corresponds to the waves passing in the same time; but his argument, applied conversely to the results already given, shews that such cases must exist. The ratio of the energy propagated to that of the passing waves is $U : V$; thus the energy propagated in the unit time is $U : V$ of that existing in a length V, or U times that existing in the unit length. Accordingly

Energy propagated in unit time : Energy contained (on an average) in unit length $= d(\kappa V) : d\kappa$, by (1).

As an example, I will take the case of small irrotational waves in water of finite depth l.* If z be measured downwards from the surface, and the elevation (h) of the wave be denoted by

$$h = H\cos(nt-\kappa x) \quad\text{.........................(4)},$$

in which $n = \kappa V$, the corresponding velocity-potential (ϕ) is

$$\phi = -VH\frac{e^{\kappa(z-l)}+e^{-\kappa(z-l)}}{e^{\kappa l}-e^{-\kappa l}}\sin(nt-\kappa x) \quad\text{.............. (5)}.$$

This value of ϕ satisfies the general differential equation for irrotational motion ($\nabla^2\phi = 0$), makes the vertical velocity $\frac{d\phi}{dz}$ zero when $z = l$, and $-\frac{dh}{dt}$ when $z = 0$. The velocity of propagation is given by

$$V^2 = \frac{g}{\kappa}\,\frac{e^{\kappa l}-e^{-\kappa l}}{e^{\kappa l}+e^{-\kappa l}} \quad\text{.......................... (6)}.$$

We may now calculate the energy contained in a length x, which is supposed to include so great a number of waves that fractional parts may be left out of account.

For the potential energy we have

$$V_1 = g\rho\iint_0^h z\,dz\,dx = \tfrac{1}{2}g\rho\int h^2\,dx = \tfrac{1}{4}g\rho H^2.x \quad\text{...........(7)}.$$

For the kinetic energy,

$$T = \tfrac{1}{2}\rho\iint\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right\}dx\,dz$$

$$= \tfrac{1}{2}\rho\int\left(\phi\frac{d\phi}{dz}\right)_{z=0}dx = \tfrac{1}{4}g\rho H^2.x \quad\text{....................(8)},$$

by (1) and (6). If, in accordance with the argument advanced at the end of this paper, the equality of V_1 and T be assumed, the value of the velocity of propagation follows from the present expressions. The whole energy in the waves occupying a length x is therefore (for each unit of breadth) $V_1+T = \tfrac{1}{2}g\rho H^2.x$ .... (9), H denoting the maximum elevation.

We have next to calculate the energy propagated in time t across a plane for which x is constant, or, in other words, the work (W) that must be done in order to sustain the motion of the plane (considered as a flexible lamina) in the face of the fluid pressures acting upon the front of it. The variable part of the pressure (δp), at depth z, is given by

$$\delta p = -\rho\frac{d\phi}{dt} = -nVH\frac{e^{\kappa(z-l)}+e^{-\kappa(z-l)}}{e^{\kappa l}-e^{-\kappa l}}\cos(nt-\kappa x),$$

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* Prof. Reynolds considers the trochoidal wave of Rankine and Froude, which involves molecular rotation.

while for the horizontal velocity

$$\frac{d\phi}{dx} = \kappa V H \frac{e^{\kappa(z-l)} + e^{-\kappa(z-l)}}{e^{\kappa l} - e^{-\kappa l}} \cos(nt - \kappa x);$$

so that $$W = \iint \delta p \frac{d\phi}{dx} dz\, dt = \tfrac{1}{4} g\rho H^2 . Vt . \left[1 + \frac{4\kappa l}{e^{2\kappa l} - e^{-2\kappa l}}\right] \quad \ldots\ldots(10),$$

on integration. From the value of V in (6) it may be proved that

$$\frac{d(\kappa V)}{d\kappa} = \tfrac{1}{2} V \left\{1 + \frac{1}{V^2} \frac{d(\kappa V^2)}{d\kappa}\right\} = \tfrac{1}{2} V \left\{1 + \frac{4\kappa l}{e^{2\kappa l} - e^{-2\kappa l}}\right\};$$

and it is thus verified that the value of W for a unit time

$$= \frac{d(\kappa V)}{d\kappa} \times \text{energy in unit length.}$$

As an example of the direct calculation of U, we may take the case of waves moving under the joint influence of gravity and cohesion.

It is proved by Thomson that

$$V^2 = \frac{g}{\kappa} + T'\kappa \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where T' is the cohesive tension. Hence

$$U = \tfrac{1}{2} V \left\{1 + \frac{1}{V^2} \frac{d(\kappa V^2)}{d\kappa}\right\} = \tfrac{1}{2} V \frac{g + 3\kappa^2 T'}{g + \kappa^2 T'} \quad \ldots\ldots\ldots (12).$$

When κ is small, the surface tension is negligible, and then $U = \tfrac{1}{2} V$; but when, on the contrary, κ is large, $U = \tfrac{3}{2} V$, as has already been stated. When $T'\kappa^2 = g$, $U = V$. This corresponds to the minimum velocity of propagation investigated by Thomson.

Although the argument from interference groups seems satisfactory, an independent investigation is desirable of the relation between energy existing and energy propagated. For some time I was at a loss for a method applicable to all kinds of waves, not seeing in particular why the comparison of energies should introduce the consideration of a variation of wave-length. The following investigation, in which the increment of wave-length is *imaginary*, may perhaps be considered to meet the want:—

Let us suppose that the motion of every part of the medium is resisted by a force of very small magnitude proportional to the mass and to the velocity of the part, the effect of which will be that waves generated at the origin gradually die away as x increases. The motion, which in the absence of friction would be represented by $\cos(nt - \kappa x)$, under the influence of friction is represented by $e^{-\mu x} \cos(nt - \kappa x)$; where μ is a small positive coefficient. In strictness the value of κ is also altered by the friction; but the alteration is of the second order as regards the frictional forces, and may be omitted under the circumstances here supposed. The energy of the waves per unit length at any stage of degradation is proportional to the square of the amplitude, and thus the whole energy on the positive side of the origin is to the energy of so much of the waves at their greatest value, *i.e.*, at the origin, as would be contained in the unit of length, as $\int_0^\infty e^{-2\mu x}\,dx : 1$, or as $(2\mu)^{-1} : 1$. The energy transmitted through the origin in the unit time is the same as the energy dissipated; and, if the frictional force acting on the element of mass m be hmv, where v is the velocity of the element and h is constant, the energy dissipated in unit time is $h\Sigma mv^2$ or $2hT$, T being the kinetic energy. Thus, on the assumption that the kinetic energy is half the whole energy, we find that the energy transmitted in the unit time is to the greatest energy existing in the unit length as $h : 2\mu$. It remains to find the connection between h and μ.

For this purpose it will be convenient to regard $\cos(nt-\kappa x)$ as the real part of $e^{int}e^{-i\kappa x}$, and to inquire how κ is affected, when n is given, by the introduction of friction. Now the effect of friction is represented in the differential equations of motion by the substitution of $\frac{d^2}{dt^2}+h\frac{d}{dt}$ in place of $\frac{d^2}{dt^2}$, or, since the whole motion is proportional to e^{int}, by substituting $-n^2+ihn$ for $-n^2$. Hence the introduction of friction corresponds to an alteration of n from n to $n-\frac{1}{2}ih$ (the square of h being neglected); and accordingly κ is altered from κ to $\kappa-\frac{1}{2}ih\frac{d\kappa}{dn}$. The solution thus becomes $e^{-\frac{1}{2}h\frac{d\kappa}{dn}x}e^{i(nt-\kappa x)}$, or, when the imaginary part is rejected, $e^{-\frac{1}{2}h\frac{d\kappa}{dn}x}\cos(nt-\kappa x)$; so that $\mu=\frac{1}{2}h\frac{d\kappa}{dn}$, and $h : 2\mu=\frac{dn}{d\kappa}$. The ratio of the energy transmitted in the unit time to the energy existing in the unit length is therefore expressed by $\frac{dn}{d\kappa}$ or $\frac{d(\kappa V)}{d\kappa}$, as was to be proved.

It has often been noticed, in particular cases of progressive waves, that the potential and kinetic energies are equal; but I do not call to mind any general treatment of the question. The theorem is not usually true for the individual parts of the medium,* but must be understood to refer either to an integral number of wave-lengths, or to a space so considerable that the outstanding fractional parts of waves may be left out of account. As an example well adapted to give insight into the question, I will take the case of a uniform stretched circular membrane ("Theory of Sound," § 200) vibrating with a given

* Aërial waves are an important exception.

number of nodal circles and diameters. The fundamental modes are not quite determinate in consequence of the symmetry, for any diameter may be made nodal. In order to get rid of this indeterminateness, we may suppo e the membrane to carry a small load attached to it anywhere except on a nodal circle. There are then two definite fundamental modes, in one of which the load lies on a nodal diameter, thus producing no effect, and in the other midway between nodal diameters, where it produces a maximum effect ("Theory of Sound," § 208). If vibrations of both modes are going on simultaneously, the potential and kinetic energies of the whole motion may be calculated by *simple addition* of those of the components. Let us now, supposing the load to diminish without limit, imagine that the vibrations are of equal amplitude and differ in phase by a quarter of a period. The result is a *progressive* wave, whose potential and kinetic energies are the sums of those of the stationary waves of which it is composed. For the first component we have $V_1 = E\cos^2 nt$, $T_1 = E\sin^2 nt$; and for the second component, $V_2 = E\sin^2 nt$, $T_2 = E\cos^2 nt$; so that $V_1 + V_2 = T_1 + T_2 = E$, or the potential and kinetic energies of the progressive wave are equal, being the same as the whole energy of either of the components. The method of proof here employed appears to be sufficiently general, though it is rather difficult to express it in language which is appropriate to all kinds of waves.

Notes on Vortex-Motion, on the Triple Generation of Three-bar Curves, and on the Mass-Centre of an Octahedron. By Prof. CLIFFORD, F.R.S.

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On Vortex-Motion.

Let σ be the velocity, and ω the rotation, at any point of a moving substance. It is known that $2\omega = V\nabla\sigma$; viz., this is equivalent to the three equations ordinarily written thus:

$$2\xi = \delta_z v - \delta_y w,$$
$$2\eta = \delta_x w - \delta_z u,$$
$$2\zeta = \delta_y u - \delta_x v.$$

If, moreover, k be the *expansion*, or the logarithmic rate of change of the volume, we have $k = -S\nabla\sigma$; viz., this is $\delta_x u + \delta_y v + \delta_z w$. Hence the quaternion q, $= -k + 2\omega$, is simply $\nabla\sigma$. The problem solved by