On the Sum of the Products of r different Terms of a Series.

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Let $a_0, a_1, a_2, \ldots$ form a series, and let $C_r(a_0, a_1, \ldots a_{n-1})$ denote the sum of the products of r different terms, taken from the first n terms of this series.

Then

$$C_r(a_0, a_1, \ldots a_{n-1}, a_n) = C_r(a_0, a_1, \ldots a_{n-1}) + a_n C_{r-1}(a_0, a_1, \ldots a_{n-1}).$$

Or, writing $\phi(n, r)$ instead of $C_r(a_0, a_1, \ldots a_{n-1})$,

$$\phi(n+1, r) = \phi(n, r) + a_n \phi(n, r-1).$$

This gives

$$\begin{aligned} \phi(n, r) &= \Sigma a_n \phi(n, r-1) \\ &= \Sigma a_n \Sigma a_n \phi(n, r-2) \\ &= \ldots \quad \ldots \quad \ldots \end{aligned}$$

where Σ means summation with respect to n, and each Σ operates on everything that follows it.

Continuing this process, we obtain

$$\phi(n, r) = \Sigma a_n \Sigma a_n \ldots \Sigma a_n \phi(n, 1);$$

there being $r-1$ operations, each expressed by Σa_n, performed on $\phi(n, 1)$ in succession; Σa_n meaning that its subject of operation is to be multiplied by a_n and then summed with respect to n.

This result may be written

$$\phi(n, r) = (\Sigma a_n)^{r-1} \phi(n, 1).$$

But

$$\phi(n, 1) = a_0 + a_1 + a_2 + \ldots + a_{n-1} = \Sigma a_n.$$

Thus

$$\phi(n, r) = (\Sigma a_n)^r \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$\phi(n, r)$ or $(\Sigma a_n)^r$ is not the same as $C_r(a_0, a_1, \ldots a_{n-1})$ except in the case where each of the r arbitrary constants, implied in the r summations, is properly determined.

Let these constants be $c_1, c_2, \ldots c_r$.

Then

$$C_1(a_0, a_1, \ldots a_{n-1}) = \Sigma a_n + c_1.$$

Multiplying by a_n and summing with respect to n,

$$C_2(a_0, a_1, \ldots a_{n-1}) = (\Sigma a_n)^2 + c_1 \Sigma a_n + c_2.$$

Proceeding in this manner, the final result is

$$C_r(a_0, a_1, \ldots a_{n-1}) = (\Sigma a_n)^r + c_1(\Sigma a_n)^{r-1} + c_2(\Sigma a_n)^{r-2} + \ldots + c_r \ldots (2).$$

The constants may now be found from the conditions that

$$\begin{aligned} C_1(a_0, a_1, \ldots a_{n-1}) &= 0 \text{ when } n = 0, \\ C_2(a_0, a_1, \ldots a_{n-1}) &= 0 \text{ when } n = 0 \text{ or } 1, \\ \ldots \quad \ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \\ C_r(a_0, a_1, \ldots a_{n-1}) &= 0 \text{ when } n = 0, 1, 2, \ldots r-1. \end{aligned}$$

We have

$$\left.\begin{array}{l} C_1(a_0, a_1, \dots a_{n-1}) = \phi(n,1)+c_1 \\ C_2(a_0, a_1, \dots a_{n-1}) = \phi(n,2)+c_1\phi(n,1)+c_2 \\ C_3(a_0, a_1, \dots a_{n-1}) = \phi(n,3)+c_1\phi(n,2)+c_2\phi(n,1)+c_3 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ C_r(a_0, a_1, \dots a_{n-1}) = \phi(n,r)+c_1\phi(n,r-1)+c_2\phi(n,r-2)+\dots+c_r \end{array}\right\} \quad (3).$$

In each of the equations (3) put $n=0$ and solve for c_r.

Then

$$\frac{1}{\begin{vmatrix} 1 & 0 & 0 & \dots \\ \phi(0,1) & 1 & 0 & \dots \\ \phi(0,2) & \phi(0,1) & 1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{r \text{ rows}}} = \frac{(-)^r c_r}{\begin{vmatrix} \phi(0,1) & 1 & 0 & \dots \\ \phi(0,2) & \phi(0,1) & 1 & \dots \\ \phi(0,3) & \phi(0,2) & \phi(0,1) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{r \text{ rows}}}$$

or

$$c_r = (-)^r \begin{vmatrix} \phi(0,1) & 1 & 0 & \dots \\ \phi(0,2) & \phi(0,1) & 1 & \dots \\ \phi(0,3) & \phi(0,2) & \phi(0,1) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{r \text{ rows}} \quad \dots\dots\dots\dots\dots \ (4).$$

Again, in the first equation, put $n=0$; in the second, $n=1$; in the third, $n=2$, and so on; and solve for c_r.

Then, as before,

$$c_r = (-)^r \begin{vmatrix} \phi(0,1) & 1 & 0 & \dots \\ \phi(1,2) & \phi(1,1) & 1 & \dots \\ \phi(2,3) & \phi(2,2) & \phi(2,1) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{r \text{ rows}} \quad \dots\dots\dots\dots\dots \ (5).$$

From (4) and (5) we obtain the identity

$$\begin{vmatrix} \phi(0,1) & 1 & 0 & \dots \\ \phi(0,2) & \phi(0,1) & 1 & \dots \\ \phi(0,3) & \phi(0,2) & \phi(0,1) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \equiv \begin{vmatrix} \phi(0,1) & 1 & 0 & \dots \\ \phi(1,2) & \phi(1,1) & 1 & \dots \\ \phi(2,3) & \phi(2,2) & \phi(2,1) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

Since, in order that the left-hand members of (3) may vanish, in the first equation n must be put $=0$; in the second, 0 or 1; in the third, 0, 1, or 2, and so on; there are $\lfloor r$ equations to find the r quantities $c_1, c_2, \dots c_r$, and therefore $\lfloor r - r$ identical relations between the other quantities.

If $C_r(a_0, a_1, \dots a_{n-1})$ be expressed in the form $(\Sigma a_n)^r$, where each summation is performed between certain limits, these limits are, for the first summation, from 0 to n; for the second, from 0 to n or from 1 to n; for the third, from 0 to n, from 1 to n, or from 2 to n; and so on.

Thus, $C_r(a_0, a_1, \ldots a_{n-1})$ may be found in $\lfloor r$ different ways by taking different limits of summation; but the results will be in every case identical.

The most convenient limits are from 0 to n for each summation.

For example,

$$C_1(a_0, a_1, \ldots a_{n-1}) = \sum_0^n a_n,$$

$$C_2(a_0, a_1, \ldots a_{n-1}) = (\sum_0^n a_n)^2 \equiv \sum_1^n a_n \sum_0^n a_n,$$

$$C_3(a_0, a_1, \ldots a_{n-1}) = (\sum_0^n a_n)^3 \equiv \sum_1^n a_n (\sum_0^n a_n)^2 \equiv \sum_2^n a_n (\sum_0^n a_n)^2$$

$$\equiv \sum_0^n a_n \sum_1^n a_n \sum_0^n a_n \equiv (\sum_1^n a_n)^2 \sum_0^n a_n \equiv \sum_2^n a_n \sum_1^n a_n \sum_0^n a_n.$$

Corresponding to the determination of c_r in (4), we have

$$C_r(a_0, a_1, \ldots a_{n-1}) = (\sum_0^n a_n)^r \quad \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

this is the most convenient form.

And, corresponding to the determination of c_r in (5),

$$C_r(a_0, a_1, \ldots a_{n-1}) = \sum_{r-1}^n a_n \sum_{r-2}^n a_n \ldots \sum_0^n a_n \quad \ldots\ldots\ldots\ldots (7).$$

In the case where $\phi(\infty, r) = 0$ for all values of r, or, what is the same thing, where $C_r(a_0, a_1, \ldots a_\infty)$ tends to a finite limit for all values of r, we have, from (3),

$$\left.\begin{array}{l} c_1 = C_1(a_0, a_1, \ldots a_\infty) \\ c_2 = C_2(a_0, a_1, \ldots a_\infty) \\ c_3 = C_3(a_0, a_1, \ldots a_\infty) \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ c_r = C_r(a_0, a_1, \ldots a_\infty) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Hence c_r is a constant with respect to n only; *i. e.*, it is not independent of r or of the form of $a_0, a_1, a_2, \ldots$, as is also evident from either (4) or (5).

In the most simple case, when $a_n = a^n$,

$$\phi(n, 1) = \Sigma a_n = \frac{a^n}{a-1},$$

$$\phi(n, 2) = (\Sigma a^n)^2 = \Sigma \frac{a^{2n}}{a-1} = \frac{a^{2n}}{(a-1)(a^2-1)},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\phi(n, r) = (\Sigma a^n)^r = \frac{a^{rn}}{(a-1)(a^2-1)\ldots(a^r-1)};$$

and, from (3),

$$C_r(1, a, a^2, \dots a^{n-1}) = \phi(n, r) + c_1\phi(n, r-1) + c_2\phi(n, r-2) + \dots + c_r$$
$$= \frac{a^{nr} + A_1 a^{n(r-1)} + A_2 a^{n(r-2)} + \dots + A_r}{(a-1)(a^2-1)\dots(a^r-1)}.$$

The constants $A_1, A_2, \dots A_r$ are found from the condition that $C_r(1, a, a^2, \dots a^{n-1}) = 0$ when $n = 0, 1, \dots r-1$, and the final result is

$$C_r(1, a, a^2, \dots a^{n-1}) = \frac{(a^n-1)(a^n-a)\dots(a^n-a^{r-1})}{(a-1)(a^2-1)\dots(a^r-1)} \quad \dots\dots (9).$$

The term independent of n is c_r.

Thus,
$$c_r = (-)^r \frac{a^{1+2+\dots+(r-1)}}{(a-1)(a^2-1)\dots(a^r-1)}$$
$$= (-)^r \frac{a^{\frac{r(r-1)}{2}}}{(a-1)(a^2-1)\dots(a^r-1)}.$$

To verify (4) and (5), put $r = 2$.

Then
$$c_2 = \frac{a}{(a-1)(a^2-1)};$$

and

$$\begin{vmatrix} \phi(0,1) & 1 \\ \phi(0,2) & \phi(0,1) \end{vmatrix} = \begin{vmatrix} \frac{1}{a-1} & 1 \\ \frac{1}{(a-1)(a^2-1)} & \frac{1}{a-1} \end{vmatrix}$$
$$= \frac{1}{(a-1)^2} - \frac{1}{(a-1)(a^2-1)} = \frac{a}{(a-1)(a^2-1)},$$

$$\begin{vmatrix} \phi(0,1) & 1 \\ \phi(1,2) & \phi(1,1) \end{vmatrix} = \begin{vmatrix} \frac{1}{a-1} & 1 \\ \frac{a^2}{(a-1)(a^2-1)} & \frac{a}{a-1} \end{vmatrix}$$
$$= \frac{a}{(a-1)^2} - \frac{a^2}{(a-1)(a^2-1)} = \frac{a}{(a-1)(a^2-1)}.$$

In (6), put $a_n = a^n$ and $r = 2$. Then

$$C_2(1, a, \dots a^{n-1})$$
$$= (\sum_0^n a^n)^2 = \sum_0^n a^n \frac{a^n-1}{a-1}$$
$$= \frac{a^{2n}}{(a-1)(a^2-1)} - \frac{a^n}{(a-1)^2} - \frac{1}{(a-1)(a^2-1)} + \frac{1}{(a-1)^2}$$
$$= \frac{a^{2n} - a^n(a+1) + a}{(a-1)(a^2-1)}$$
$$= \frac{(a^n-1)(a^n-a)}{(a-1)(a^2-1)}.$$

In (7), put $a_n = a^n$ and $r=2$. Then

$$C_2(1, a, \dots a^{n-1})$$

$$= \sum_1^n a^n \sum_0^n a^n = \sum_1^n a^n \frac{a^n-1}{a-1}$$

$$= \frac{a^{2n}}{(a-1)(a^2-1)} - \frac{a^n}{(a-1)^2} - \frac{a^2}{(a-1)(a^2-1)} + \frac{a}{(a-1)^2}$$

$$= \frac{(a^n-1)(a^n-a)}{(a-1)(a^2-1)} \text{ as before.}$$

In (9), put $n=\infty$, and suppose a to be fractional.
Then the limit of $C_r(1, a, a^2, \dots a^{n-1})$ when n is infinite is

$$c_r = (-)^r \frac{a^{\frac{r(r-1)}{2}}}{(a-1)(a^2-1)\dots(a^r-1)},$$

which agrees with (8).

In (9), put $a = e^{\theta\sqrt{-1}}$. Then

$$C_r(1, e^{\theta\sqrt{-1}}, e^{2\theta\sqrt{-1}}, \dots e^{(n-1)\theta\sqrt{-1}})$$

$$= \frac{\sin\frac{n\theta}{2}\sin\frac{n-1}{2}\theta\dots\sin\frac{n-r+1}{2}\theta}{\sin\frac{\theta}{2}\sin\theta\dots\sin\frac{r\theta}{2}} e^{\frac{n-1}{2}r\theta\sqrt{-1}}.$$

But $\qquad C_r(1, e^{\theta\sqrt{-1}}, e^{2\theta\sqrt{-1}}, \dots e^{(n-1)\theta\sqrt{-1}})$

$$= S\{\cos(\alpha+\beta+\dots r\text{ terms})\theta + \sqrt{-1}\sin(\alpha+\beta+\dots r\text{ terms})\theta\},$$

the sum of all terms of the form

$$\cos(\alpha+\beta+\dots)\theta + \sqrt{-1}\sin(\alpha+\beta+\dots)\theta,$$

where $\alpha, \beta, \dots$ are any r different terms of the series $0, 1, 2, \dots n-1$. With this notation,

$$S\cos(\alpha+\beta+\dots)\theta$$

$$= \frac{\sin\frac{n\theta}{2}\sin\frac{n-1}{2}\theta\dots\sin\frac{n-r+1}{2}\theta}{\sin\frac{\theta}{2}\sin\theta\dots\sin\frac{r\theta}{2}}\cos\frac{n-1}{2}r\theta \quad \dots\dots (10);$$

and

$$S\sin(\alpha+\beta+\dots)\theta$$

$$= \frac{\sin\frac{n\theta}{2}\sin\frac{n-1}{2}\theta\dots\sin\frac{n-r+1}{2}\theta}{\sin\frac{\theta}{2}\sin\theta\dots\sin\frac{r\theta}{2}}\sin\frac{n-1}{2}r\theta \quad \dots\dots (11).$$

Let (n) denote a given rational and integral function of n, of the order m; and write $a_n = (n)\,a^n$.

Then

$$\phi(n,1) = \Sigma(n)\,a^n = -\frac{a^n}{1-a}\cdot\frac{1}{1-\frac{a}{1-a}\Delta}(n),$$

$$\phi(n,2) = [\Sigma(n)\,a^n]^2 = \frac{a^{2n}}{1-a\,.\,1-a^2}\cdot\frac{1}{1-\frac{a^2}{1-a^2}\Delta}(n)\,.\,\frac{1}{1-\frac{a}{1-a}\Delta}(n),$$

$$\phi(n,3) = [\Sigma(n)\,a^n]^3 = -\frac{a^{3n}}{1-a\,.\,1-a^2\,.\,1-a^3}\cdot\frac{1}{1-\frac{a^3}{1-a^3}\Delta}(n)$$

$$\times\frac{1}{1-\frac{a^2}{1-a^2}\Delta}(n)\,.\,\frac{1}{1-\frac{a}{1-a}\Delta}(n),$$

... ...

$$\phi(n,r) = \frac{(-)^r a^{rn}}{1-a\,.\,1-a^2\,.\,1-a^r}\cdot\frac{1}{1-\frac{a^r}{1-a^r}\Delta}(n)\,.\,\frac{1}{1-\frac{a^{r-1}}{1-a^{r-1}}\Delta}(n)\,\ldots\ldots$$

$$\ldots\ldots\frac{1}{1-\frac{a}{1-a}\Delta}(n)\quad\ldots\ldots(12);$$

where, in the expressions on the right hand, Δ refers to n [viz., $\Delta(n) = (n+1)-(n)$ as usual]; the fractional symbols represent series $\frac{1}{1-\frac{a}{1-a}\Delta} = 1+\frac{a}{1-a}\Delta+\frac{a^2}{(1-a)^2}\Delta^2+$ &c., and where each power of Δ operates upon everything that follows it.

For instance, in the second formula, if to simplify $(n) = \overline{n+1}$ so that $\Delta(n) = 1$, $\Delta^2(n) = 0$, then omitting the factor $\frac{a^{2n}}{1-a\,.\,1-a^2}$, the remaining factor on the right hand is

$$\left(1+\frac{a^2}{1-a^2}\Delta+\frac{a^4}{(1-a^2)^2}\Delta^2\right)\overline{n+1}\left(1+\frac{a}{1-a}\Delta\right)\overline{n+1},$$

that is,

$$= \left(1+\frac{a^2}{1-a^2}\Delta+\frac{a^4}{(1-a^2)^2}\Delta^2\right)\overline{n+1}\left(\overline{n+1}+\frac{a}{1-a}\right)$$

$$= \left(1+\frac{a^2}{1-a^2}\Delta+\frac{a^4}{(1-a^2)^2}\Delta^2\right)\left(\overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}\right)$$

$$= \overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a} \qquad = \overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}$$

$$+\frac{a^2}{1-a^2}\Delta\left(\overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}\right) \qquad +\frac{a^2}{1-a^2}\left(2n+3+\frac{a}{1-a}\right)$$

$$+\frac{a^4}{(1-a^2)^2}\Delta^2\left(\overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}\right) \qquad +\frac{a^4}{(1-a^2)^2}\,.\,2.$$

Referring to (3) and (4), we see that, in the general case,

$$C_r(a_0, a_1, \dots a_{n-1}) = \begin{vmatrix} \phi(n,r) & \phi(n,r-1) & \phi(n,r-2) & \dots & 1 \\ \phi(0,1) & 1 & 0 & \dots & 0 \\ \phi(0,2) & \phi(0,1) & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \phi(0,r) & \phi(0,r-1) & \phi(0,r-2) & \dots & 1 \end{vmatrix} \quad \dots(13).$$

When $a_n = \overline{n+1}\,.\,a^n$,

$$C_1(1, 2a, \dots na^{n-1}) = \begin{vmatrix} \phi(n,1) & 1 \\ \phi(0,1) & 1 \end{vmatrix}$$

$$C_2(1, 2a, \dots na^{n-1}) = \begin{vmatrix} \phi(n,2) & \phi(n,1) & 1 \\ \phi(0,1) & 1 & 0 \\ \phi(0,2) & \phi(0,1) & 1 \end{vmatrix}$$

where
$$\phi(n,1) = -\frac{a^n}{1-a}\left(\overline{n+1} + \frac{a}{1-a}\right),$$

$$\phi(n,2) = \frac{a^{2n}}{(1-a)(1-a^2)}$$
$$\times\left\{\overline{n+1}^2 + \overline{n+1}\,.\,\frac{a}{1-a} + \frac{a^2}{1-a^2}\left(2n+3+\frac{a}{1-a}\right) + \frac{2a^4}{(1-a^2)^2}\right\}.$$

Hence
$$\phi(0,1) = -\frac{1}{1-a}\left(1+\frac{a}{1-a}\right) = -\frac{1}{(1-a)^2},$$

$$\phi(0,2) = \frac{1}{(1-a)(1-a^2)}\left\{1+\frac{a}{1-a} + \frac{a^2}{1-a^2}\left(3+\frac{a}{1-a}\right) + \frac{2a^4}{(1-a^2)^2}\right\}$$
$$= \frac{1}{(1-a)(1-a^2)}\left(1+\frac{a}{1-a}+\frac{2a^2}{1-a^2}\right)\left(1+\frac{a^2}{1-a^2}\right)$$
$$= \frac{1+a+2a^2}{(1-a)(1-a^2)^3}.$$

$$C_1(1, 2a, \dots na^{n-1}) = \frac{1}{(1-a)^2} - \frac{a^n}{1-a}\left(\overline{n+1} + \frac{a}{1-a}\right)$$
$$= \frac{1}{(1-a)^2}\left(1 - \overline{n+1}\,a^n + na^{n+1}\right) \quad \dots\dots\dots(14),$$

which is easily verified.

$$C_2(1, 2a, \dots na^{n-1}) = \{\phi(0,1)\}^2 - \phi(0,2) - \phi(0,1)\,.\,\phi(n,1) + \phi(n,2)$$
$$= \frac{1}{(1-a)^4} - \frac{1+a+2a^2}{(1-a)(1-a^2)^3} - \frac{a^n}{(1-a)^3}\left(\overline{n+1} + \frac{a}{1-a}\right)$$
$$+ \frac{a^{2n}}{(1-a)(1-a^2)}$$
$$\times\left\{\overline{n+1}^2 + \overline{n+1}\,\frac{a}{1-a} + \frac{a^2}{1-a^2}\left(2n+3+\frac{a}{1-a}\right) + \frac{2a^4}{(1-a^2)^2}\right\}$$

$$= \frac{2a+a^2+a^3}{(1-a)(1-a^2)^3} - \frac{a^n}{(1-a)^4}(n+1-na)$$

$$+ \frac{a^{2n}}{(1-a)(1-a^2)}\left\{n^2+n\left(2+\frac{a}{1-a}+\frac{2a^2}{1-a^2}\right)+\frac{1+a+2a^2}{(1-a^2)^2}\right\}$$

$$= \frac{1}{(1-a)(1-a^2)^3}\left\{2a+a^2+a^3-\overline{n+1}\,.\,a^n-\overline{2n+3}\,.\,a^{n+1}-3a^{n+2}\right.$$
$$+\overline{2n-1}\,.\,a^{n+3}+na^{n+4}+\overline{n+1}^2\,.\,a^{2n}+\overline{n+1}\,.\,a^{2n+1}$$
$$\left.-\overline{2n^2+n-2}\,.\,a^{2n+2}-n\,.\,a^{2n+3}+n\,.\,\overline{n-1}\,.\,a^{2n+4}\right\};$$

e.g., when $n=2$, this is

$$2a = \frac{1}{(1-a)(1-a^2)^3}\left\{2a-2a^2-6a^3+6a^4+6a^5-6a^6-2a^7+2a^8\right\},$$

the truth of which is evident.

When n is infinite and a fractional,

$$C_2(1,\ 2a,\ \dots\ na^{n-1}) = \frac{2a+a^2+a^3}{(1-a)(1-a^2)^3} \quad\dots\dots\dots\dots\dots (15).$$

This result is true approximately whenever na^n is small enough to be neglected.

When $n=10$, $a=\frac{1}{10}$, the result obtained from (15) is

·241,620,824,554,538 ...,

the correct value of $C_2(1,\ 2a,\ \dots\ 10a^9)$ being

·241,620,823,030,380,9.

It may be noticed that the term independent of n in (14) is $\frac{1}{(1-a)^2} = 1+2a+3a^2+\dots$ *ad inf.* by expansion; that the term independent of n in the expression for $C_2(1,\ 2a,\ \dots\ na^{n-1})$ is the limit of that expression when a is fractional and n infinite; and generally, since we have, from (12),

$$\phi(n,\ r) = (a^r)^n\,F(n),$$

where $F(n)$ is a rational integral function of n, when a is fractional, $\phi(\infty,\ r) = 0$ for all values of r, and the limit of $C_r[(0),\ (1)\ a,\ \dots\ (n-1)\ a^{n-1}] = c_r$, the term in its expression independent of n.

The general term of $\phi(n,\ r)$ obtained from (12) by expansion is

$$\frac{(-)^r a^{rn}}{(1-a)(1-a^2)\dots(1-a^r)}\left(\frac{a^r}{1-a^r}\Delta\right)^{\kappa_r}(n)\left(\frac{a^{r-1}}{1-a^{r-1}}\Delta\right)^{\kappa_{r-1}}(n)\dots\dots$$
$$\dots\dots\left(\frac{a}{1-a}\Delta\right)(n)$$

$$= \frac{(-)^r a^{rn+\kappa_1+2\kappa_2+\dots+r\kappa_r}}{(1-a)^{\kappa_1+1}(1-a^2)^{\kappa_2+1}\dots(1-a^r)^{\kappa_r+1}}\,\Delta^{\kappa_r}(n)\,\Delta^{\kappa_{r-1}}(n)\dots\Delta^{\kappa_1}(n).$$

Now

$$\Delta^x u_n v_n = \Delta^x u_n \,.\, v_n + x\Delta^{x-1} u_{n+1}\,.\,\Delta v_n + \frac{x(x-1)}{\underline{|2}}\,\Delta^{x-2} u_{n+2}\,.\,\Delta^2 v_n + \text{\&c.}$$

Hence, putting $x = \kappa_r$, $u_n = (n)$, and writing for shortness

$$\Delta^{\kappa_r}(n) = (n)_{\kappa_r}, \qquad \Delta^{\kappa_r - 1}(n+1) = (n+1)_{\kappa_r - 1},$$

$$\Delta^{\kappa_r - 2}(n+2) = (n+2)_{\kappa_r - 2}, \quad \text{\&c.},$$

we have

$$\Delta^{\kappa_r}(n)\, v_n = (n)_{\kappa_r} v_n + \kappa_r (n+1)_{\kappa_r - 1} \Delta v_n + \frac{\kappa_r(\kappa_r - 1)}{\underline{|2}} (n+2)_{\kappa_r - 2}\, \Delta^2 v_n + \text{\&c.};$$

and substituting for v_n, $\Delta^{\kappa_{r-1}}(n)\,\Delta^{\kappa_{r-2}}(n)\,\ldots\,\Delta^{\kappa_1}(n)$,
the general term of this expression is

$$\frac{\underline{|\kappa_r}}{\underline{|l_{r-1}}\;\underline{|\kappa_r - l_{r-1}}}\,(n + l_{r-1})_{\kappa_r - l_{r-1}}\;\Delta^{\kappa_{r-1} + l_{r-1}}(n)\;\Delta^{\kappa_{r-2}}(n)\;\ldots\;\Delta^{\kappa_1}(n).$$

Continuing this process, the general term of $\phi(n, r)$ is found to be

$$\frac{(-)^r\, a^{rn + \kappa_1 + 2\kappa_2 + \ldots + r\kappa_r}}{(1-a)^{\kappa_1 + 1}(1-a^2)^{\kappa_2 + 1}\ldots(1-a^r)^{\kappa_r + 1}}\cdot\frac{\underline{|\kappa_r}}{\underline{|l_{r-1}}\;\underline{|\kappa_r - l_{r-1}}}$$

$$\times\frac{\underline{|\kappa_{r-1} + l_{r-1}}}{\underline{|l_{r-2}}\;\underline{|\kappa_{r-1} + l_{r-1} - l_{r-2}}}\cdot\frac{\underline{|\kappa_{r-2} + l_{r-2}}}{\underline{|l_{r-3}}\;\underline{|\kappa_{r-2} + l_{r-2} - l_{r-3}}}\cdots\cdots\frac{\underline{|\kappa_2 + l_2}}{\underline{|l_1}\;\underline{|\kappa_2 + l_2 - l_1}}$$

$$\times (n + l_{r-1})_{\kappa_r - l_{r-1}}\cdot(n + l_{r-2})_{\kappa_{r-1} + l_{r-1} - l_{r-2}}\cdot(n + l_{r-3})_{\kappa_{r-2} + l_{r-2} - l_{r-3}}\cdots\cdots$$

$$\cdots\cdots(n + l_1)_{\kappa_2 + l_2 - l_1}\cdot(n)_{\kappa_1 + l_1}.$$

Now put

$$\left.\begin{aligned} \kappa_1 + l_1 &= \iota_1 \\ \kappa_2 + l_2 - l_1 &= \iota_2 \\ \kappa_3 + l_3 - l_2 &= \iota_3 \\ \ldots\quad \ldots\quad &\ldots\quad \ldots \\ \kappa_{r-1} + l_{r-1} - l_{r-2} &= \iota_{r-1} \\ \kappa_r - l_{r-1} &= \iota_r \end{aligned}\right\}.$$

Then

$$\begin{aligned} l_1 &= \iota_1 - \kappa_1, \\ l_2 &= \iota_1 + \iota_2 - \kappa_1 - \kappa_2, \\ \ldots\quad &\ldots\quad \ldots\quad \ldots \\ l_{r-1} &= \iota_1 + \iota_2 + \ldots + \iota_{r-1} - \kappa_1 - \kappa_2 - \ldots - \kappa_{r-1}, \\ \iota_1 + \iota_2 + &\ldots + \iota_r = \kappa_1 + \kappa_2 + \ldots + \kappa_r; \end{aligned}$$

and the general term of $\phi(n, r)$ becomes

$$\frac{(-)^r\, a^{rn + \kappa_1 + 2\kappa_2 + \ldots + r\kappa_r}}{(1-a)^{\kappa_1 + 1}(1-a^2)^{\kappa_2 + 1}\ldots(1-a^r)^{\kappa_r + 1}}\;\frac{\underline{|l_1 + \iota_2}\;\underline{|l_2 + \iota_3}\ldots\underline{|l_{r-1} + \iota_r}}{\underline{|l_1}\;\underline{|\iota_2}\;\underline{|l_2}\;\underline{|\iota_3}\ldots\underline{|l_{r-1}}\;\underline{|\iota_r}}$$

$$\times (n)_{\iota_1}\,(n + l_1)_{\iota_2}\,(n + l_2)_{\iota_3}\ldots(n + l_{r-1})_{\iota_r} \quad\ldots\ldots(16).$$

The complete expression of which (16) is the general term is found by giving each of the quantities $\iota_1, \iota_2, \dots \iota_r, \kappa_1, \kappa_2, \dots \kappa_r$ all possible values [$l_1, l_2, \dots l_{r-1}$ being given in terms of $\iota_1, \iota_2, \dots \iota_r$ and $\kappa_1, \kappa_2, \dots \kappa_r$], and adding the results.

Each of these quantities is either zero or a positive integer; and since (n) is a rational integral function of n of degree m, the greatest value of any one of the quantities $\iota_1, \iota_2, \dots \iota_r$ is m.

Hence the greatest possible value of κ_1 is m, the greatest possible value of κ_2 is $2m$, and so on.

Thus each of the quantities $\iota_1, \iota_2, \dots \iota_r$ has all positive integral values from 0 to m inclusive.

κ_1 has all positive integral values from 0 to m inclusive;
κ_2 " " " 0 to $2m$ "
... ...
κ_r " " " 0 to rm "

Those values of $\iota_1, \iota_2, \dots \iota_r$ and $\kappa_1, \kappa_2, \dots \kappa_r$ which make any of the quantities $l_1, l_2, \dots l_{r-1}$ negative are inadmissible and must be rejected, as also those that do not satisfy the relation

$$\kappa_1+\kappa_2+\dots+\kappa_r = \iota_1+\iota_2+\dots+\iota_r.$$

For instance, when $r=2$, the general term of $\phi(n, 2)$ is

$$\frac{a^{2n+\kappa_1+2\kappa_2}}{(1-a)^{\kappa_1+1}(1-a^2)^{\kappa_2+1}} \frac{\lfloor l_1+\iota_2}{\lfloor l_1 \lfloor \iota_2} (n)_{\iota_1} (n+l_1)_{\iota_2};$$

and all the possible values of $\iota_1, \iota_2, \kappa_1, \kappa_2, l_1$ are found from the following table:—

$r=2$

ι_1	ι_2	κ_1	κ_2	l_1	
0	0	0	0	0	
0	1	0	1	0	
1	0	0	1	1	$m=1$
1	0	1	0	0	
1	1	0	2	1	
1	1	1	1	0	
0	2	0	2	0	
1	2	0	3	1	
1	2	1	2	0	$m=2$
2	0	0	2	2	
2	0	1	1	1	
2	0	2	0	0	
2	1	0	3	2	
2	1	1	2	1	
2	1	2	1	0	
2	2	0	4	2	
2	2	1	3	1	
2	2	2	2	0	
...	...	...	...	...	
...	...	...	...	...	
...	...	...	...	...	

Hence

$$\phi(n,2)=\frac{a^{2n}}{(1-a)(1-a^2)}\left\{(n)^2+\frac{a^2}{1-a^2}(n)(n)_1+\frac{a^2}{1-a^2}(n)_1(n+1)\right.$$
$$+\frac{a}{1-a}(n)_1(n)+\frac{a^4}{(1-a^2)^2}\,2(n)_1(n+1)_1$$
$$\left.+\frac{a^3}{(1-a)(1-a^2)}(n)_1^2+\dots\right\}.$$

The terms given are those corresponding to $m=1$.

Since $\quad (n+l)_s=(1+\Delta)^l\Delta^s(n)$
$$=(n)_s+l\,(n)_{s+1}+\frac{l(l-1)}{\lfloor 2}(n)_{s+2}+\dots,$$
it follows that $\phi(n,r)$ is of the form
$$\frac{(-)^r a^{rn}}{(1-a)(1-a^2)\dots(1-a^r)}[(n),(n)_1,(n)_2,\dots(n)_m]^r.$$

For example, in the result given above, put $(n+1)=(n)+(n)_1$, $(n+1)_1=(n)_1+(n)_2$, ...

Then

$$\phi(n,2)=\frac{a^{2n}}{(1-a)(1-a^2)}\left\{(n)^2+(n)(n)_1\left[\frac{2a^2}{1-a^2}+\frac{a}{1-a}\right]\right.$$
$$+(n)_1^2\left[\frac{a^2}{1-a^2}+\frac{2a^4}{(1-a^2)^2}+\frac{a^3}{(1-a)(1-a^2)}\right]$$
$$\left.+\text{ terms involving }(n)_2,(n)_3,\dots\right\}$$
$$=\frac{a^{2n}}{(1-a)(1-a^2)}\left\{(n)^2+(n)(n)_1\frac{a+3a^2}{1-a^2}+(n)_1^2\frac{a^2+a^3+2a^4}{(1-a^2)^2}+\dots\right\}$$
$$=\frac{a^{2n}}{(1-a)(1-a^2)}[(n),(n)_1,(n)_2,\dots(n)_m]^2.$$

The terms given are those corresponding to $m=1$.

When $a=1$, $\phi(n,r)=\infty$, each element of the determinant (13) assumes the form $\pm\infty$, the value found for $C_r[(0),(1),\dots(n-1)]$ is indeterminate, and the foregoing investigation fails.

And generally, when $a^x=1$, where x is any integer from 1 to r inclusive, $\phi(n,x)$, $\phi(n,x+1)$, ... $\phi(n,r)$ are all infinite, and $C_r[(0),(1)a,\dots(n-1)a^{n-1}]$ is indeterminate.

In the simplest case, when $a=1$, the general term of $C_r[(0),(1),\dots(n-1)]$ may be found by the following method:

It will be shown hereafter that when (n) is expressed in the form

$$A_0+A_1n+A_2\frac{[n]^2}{\lfloor 2}+\dots+A_m\frac{[n]^m}{\lfloor m},$$

and $\Sigma^p(n)$ is taken to be

$$A_0\frac{[n]^p}{\lfloor p} + A_1\frac{[n]^{p+1}}{\lfloor p+1} + \dots + A_m\frac{[n]^{m+p}}{\lfloor m+p},$$

the constants of summation all vanish.

In the formula

$$\Sigma u_n v_n = u_{n-1}\,.\,\Sigma v_n - \Delta u_{n-2}\,.\,\Sigma^2 v_n + \Delta^2 u_{n-3}\,.\,\Sigma^3 v_n - \&c.,$$

put $\quad u_n = (n),\quad$ and $\quad v_n = C_{r-1}[(0), (1), \dots (n-1)],$

so that $\quad \Sigma u_n v_n = C_r[(0), (1), \dots (n-1)];$

then, writing shortly

$$\Delta(n-2) = (n-2)_1,\quad \Delta^2(n-3) = (n-3)_2,\quad \dots\dots,$$

$$\begin{aligned}
&C_r[(0), (1), \dots (n-1)]\\
&\quad = \{(n-1)\Sigma - (n-2)_1\Sigma^2 + \dots + (-)^m(n-m-1)_m\Sigma^{m+1}\}\\
&\qquad\qquad\qquad\qquad \times C_{r-1}[(0), (1), \dots (n-1)]\\
&\quad = \{(n-1)\Sigma - (n-2)_1\Sigma^2 + \dots\}^2 C_{r-2}[(0), (1), \dots (n-1)]\\
&\quad = \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\
&\quad = \{(n-1)\Sigma - (n-2)_1\Sigma^2 + \dots + (-)^m(n-m-1)_m\Sigma^{m+1}\}^{r-1}\Sigma(n),
\end{aligned}$$

since $\quad C_1[(0), (1), \dots (n-1)] = \Sigma(n).$

The general term of $C_r[(0), (1), \dots (n-1)]$ is therefore

$$\begin{aligned}
&(-)^{\kappa_1+1}(n-\kappa_1)_{\kappa_1-1}\Sigma^{\kappa_1}(-)^{\kappa_2+1}(n-\kappa_2)_{\kappa_2-1}\Sigma^{\kappa_2}\dots\dots\\
&\qquad\qquad \dots\dots(-)^{\kappa_{r-1}+1}(n-\kappa_{r-1})_{\kappa_{r-1}-1}\Sigma^{\kappa_{r-1}+1}(n),
\end{aligned}$$

where each κ may have any integral value from 1 to $m+1$ inclusive.

It may easily be proved, by the same method as that given in Boole's "Finite Differences" (p. 74, Second Edition), that

$$\Sigma^\kappa u_n v_n = u_{n-\kappa}\,.\,\Sigma^\kappa v_n - \kappa\Delta u_{n-\kappa-1}\,.\,\Sigma^{\kappa+1}v_n + \frac{\kappa(\kappa+1)}{\lfloor 2}\Delta^2 u_{n-\kappa-2}\,.\,\Sigma^{\kappa+2}v_n - \dots,$$

the general term being $\quad (-)^l\dfrac{\lfloor \kappa+l-1}{\lfloor \kappa-1\,\lfloor l}\Delta^l u_{n-\kappa-l}\,.\,\Sigma^{\kappa-l}v_n.$

Thus $C_r[(0), (1), \dots (n-1)]$ consists of a series of terms of the form

$$\begin{aligned}
&(-)^{\kappa_1+1+\kappa_2+l_1+1}(n-\kappa_1)_{\kappa_1-1}\frac{\lfloor \kappa_1+l_1-1}{\lfloor \kappa_1-1\,\lfloor l_1}(n-\kappa_2-\kappa_1-l_1)_{\kappa_2+l_1-1}\\
&\qquad\qquad\qquad \times \Sigma^{\kappa_2+\kappa_1+l_1}(-)^{\kappa_3+1}(n-\kappa_3)_{\kappa_3-1}\dots\dots.
\end{aligned}$$

Proceeding in this manner, the general term of $C_r[(0), (1), \dots (n-1)]$ is found to be

$$(-)^{\kappa_1+\kappa_2+\ldots+\kappa_{r-1}+l_1+l_2+\ldots+l_{r-2}+r-1}$$
$$\times \frac{\lfloor \kappa_1+l_1-1 \; \lfloor \kappa_2+\kappa_1+l_1+l_2-1 \ldots\ldots (r-2) \text{ factors}}{\lfloor \kappa_1-1 \; \lfloor l_1 \; \lfloor \kappa_2+\kappa_1+l_1-1 \; \lfloor l_2 \ldots\ldots \ldots \ldots \ldots}$$
$$\times (n-\kappa_1)_{\kappa_1-1} \cdot (n-\kappa_2-\kappa_1-l_1)_{\kappa_2+l_1-1} \cdot (n-\kappa_3-\kappa_2-\kappa_1-l_1-l_2)_{\kappa_3+l_2-1} \ldots\ldots$$
$$\ldots\ldots (r-1) \text{ factors } \Sigma^{1+\kappa_1+\kappa_2+\ldots\kappa_{r-1}+l_1+l_2+\ldots l_{r-2}}(n).$$

Now put

$$\left.\begin{aligned} \kappa_1 &= \iota_1 \\ \kappa_2+l_1 &= \iota_2 \\ \kappa_3+l_2 &= \iota_3 \\ \ldots \; &\ldots \; \ldots \\ \kappa_{r-1}+l_{r-2} &= \iota_{r-1} \end{aligned}\right\};$$

and let

$$\begin{aligned} I_1 &= \iota_1, \\ I_2 &= \iota_1+\iota_2, \\ I_3 &= \iota_1+\iota_2+\iota_3, \\ \ldots \; &\ldots \; \ldots \; \ldots \\ I_{r-1} &= \iota_1+\iota_2+\ldots+\iota_{r-1}. \end{aligned}$$

Then the general term of $C_r[(0), (1), \ldots (n-1)]$ becomes

$$(-)^{I_{r-1}+r-1} \frac{\lfloor I_1+l_1-1 \; \lfloor I_2+l_2-1 \ldots \lfloor I_{r-2}+l_{r-2}-1}{\lfloor I_1-1 \; \lfloor l_1 \; \lfloor I_2-1 \; \lfloor l_2 \ldots \lfloor I_{r-2}-1 \; \lfloor l_{r-2}}$$
$$\times (n-I_1)_{\iota_1-1} \cdot (n-I_2)_{\iota_2-1} \ldots (n-I_{r-1})_{\iota_{r-1}-1} \cdot \Sigma^{I_{r-1}+1}(n).$$

Different terms for which $\iota_1, \iota_2, \ldots \iota_{r-1}$ have respectively the same values differ only in their coefficients, and may be added together. The general coefficient of the sum of such terms is found by giving $l_1, l_2, \ldots l_{r-2}$ all possible values in the coefficient of the general term, and adding the results.

Now, since the least value of any of the quantities $\kappa_1, \kappa_2, \ldots \kappa_{r-1}$ is 1, the greatest value of l_1 is ι_2-1, the greatest value of l_2 is ι_3-1, and so on. Thus the coefficient becomes

$$\left\{1+I_1+\frac{I_1(I_1+1)}{\lfloor 2}+\ldots \iota_2 \text{ terms}\right\}\left\{1+I_2+\frac{I_2(I_2+1)}{\lfloor 2}+\ldots \iota_3 \text{ terms}\right\}\ldots$$

which may easily be shewn equal to

$$\frac{\lfloor I_1+\iota_2-1}{\lfloor I_1 \; \lfloor \iota_2-1} \cdot \frac{\lfloor I_2+\iota_3-1}{\lfloor I_2 \; \lfloor \iota_3-1} \ldots = \frac{\lfloor I_2-1}{\lfloor I_1 \; \lfloor \iota_2-1} \cdot \frac{\lfloor I_3-1}{\lfloor I_2 \; \lfloor \iota_3-1} \ldots \frac{\lfloor I_{r-1}-1}{\lfloor I_{r-2} \; \lfloor \iota_{r-1}-1}.$$

And since there are $r-1$ of the quantities $\iota_1, \iota_2, \ldots \iota_{r-1}$, each of which may have any positive integral value from 1 to $m+1$ inclusive.

$C_r[(0), (1), \dots (n-1)]$ consists of $(m+1)^{r-1}$ terms of the form

$$(-)^{I_{r-1}+r-1}\frac{\lfloor I_2-1}{\lfloor I_1 \lfloor \iota_2-1}\cdot\frac{\lfloor I_3-1}{\lfloor I_2 \lfloor \iota_3-1}\cdots\cdots\frac{\lfloor I_{r-1}-1}{\lfloor I_{r-2} \lfloor \iota_{r-1}-1}$$

$$\times(n-I_1)_{\iota_1-1}\cdot(n-I_2)_{\iota_2-1}\dots(n-I_{r-1})_{\iota_{r-1}-1}\cdot\Sigma^{I_{r-1}+1}(n)\dots.(17).$$

The degree of each of these terms is

$$(m-\iota_1+1)+(m-\iota_2+1)+\dots+(m-\iota_{r-1}+1)+(m+I_{r-1}+1)=(m+1)r.$$

The constants of summation, being found from the condition that $C_r(a_0, a_1, \dots a_{n-1}) = 0$, when $n=0, 1, \dots r-1$; all vanish. For each term contains a factor of the form $\Sigma^{I_{r-1}+1}(n)$, the least value of I_{r-1} being $r-1$; and by hypothesis

$$\Sigma^p(n) = A_0\frac{[n]^p}{\lfloor p} + A_1\frac{[n]^{p+1}}{\lfloor p+1} + \dots\dots,$$

which, when $p=r$, and for all higher values of p, vanishes when n is put $= 0, 1, \dots r-1$.

For the same reason the constants of summation all vanish when

$$(n) = A_0+A_1[n+1]+A_2\frac{[n+1]^2}{\lfloor 2} + \dots\dots,$$

$$(n) = A_0+A_1[n+1]+A_2\frac{[n+2]^2}{\lfloor 2} + \dots\dots.$$

Provided that $\Sigma^p(n)$ is taken to be

$$A_0[n]^p+A_1\frac{[n+1]^{p+1}}{\lfloor p+1} + A_2\frac{[n+1]^{p+2}}{\lfloor p+2} + \dots\dots,$$

$$A_0[n]^p+A_1\frac{[n+1]^{p+1}}{\lfloor p+1} + A_2\frac{[n+2]^{p+2}}{\lfloor p+2} + \dots\dots.$$

Let $r=2$, then the general term of $C_2[(0), (1), \dots (n-1)]$ is

$$(-)^{\iota_1+1}(n-\iota_1)_{\iota_1-1}\cdot\Sigma^{\iota_1+1}(n).$$

Thus $$C_2[(0), (1), \dots (n-1)]$$

$$=(n-1)\Sigma^2(n)-(n-2)_1\Sigma^3(n)+\dots+(-)^m(n-m-1)_m\Sigma^{m+1}(n)\dots(18);$$

this result is a particular case of the well known formula for the sum of the product of two functions; viz., the functions are (n) and $\Sigma(n)$.

In the case $(n)=n+1$,

$$C_2(1, 2, \dots n) = n\cdot\frac{[n+1]^3}{\lfloor 3} - 1\cdot\frac{[n+1]^4}{\lfloor 4}$$

$$=\frac{n+1\cdot n\cdot n-1\cdot 3n+2}{24}\dots\dots\dots\dots(19).$$

In the case $(n) = [n+1]^2 = n+1+\frac{2[n+1]^2}{\underline{|2}}$,

$$C_2(1, 4, \ldots n^2)$$

$$= n^2\left\{\frac{[n+1]^3}{\underline{|3}}+\frac{2[n+1]^4}{\underline{|4}}\right\} - \overline{2n-1}\left\{\frac{[n+1]^4}{\underline{|4}}+\frac{2[n+1]^5}{\underline{|5}}\right\}$$
$$+ 2\left\{\frac{[n+1]^5}{\underline{|5}}+\frac{2[n+1]^6}{\underline{|6}}\right\};$$

which reduces to

$$C_2(1, 4, \ldots n^2) = \frac{n+1 \,.\, n \,.\, n-1 \,.\, 2n+1 \,.\, 2n-1 \,.\, 5n+6}{360} \quad \ldots(20).$$

When $r=3$ the general term becomes

$$(-)^{I_2}\frac{\underline{|I_2-1}}{\underline{|I_1}\,\underline{|\iota_2-1}}(n-I_1)_{\iota_1-1}\,.\,(n-I_2)_{\iota_2-1}\,.\,\Sigma^{I_2+1}(n).$$

Thus, writing down the corresponding terms of $C_3[(0), (1), \ldots (n-1)]$, and values of ι_1, ι_2, I_1, I_2,

ι_1	ι_2	I_1	I_2		
1	1	1	2	$(n-1)\ (n-2)\ \Sigma^3(n)$	
1	2	1	3	$-\ 2(n-1)\ (n-3)_1\Sigma^4(n)$	$= C_3[(0), (1), \ldots (n-1)]$ when $m=1$ ...... (21);
2	1	2	3	$-\ (n-2)_1(n-3)\ \Sigma^4(n)$	
2	2	2	4	$+\ 3(n-2)_1(n-4)_1\Sigma^5(n)$	
1	3	1	4	$+\ 3(n-1)\ (n-4)_2\Sigma^5(n)$	$= C_3[(0), (1), \ldots (n-1)]$ when $m=2$ ...... (22).
3	1	3	4	$+\ (n-3)_2(n-4)\ \Sigma^5(n)$	
2	3	2	5	$-\ 6(n-2)_1(n-5)_2\Sigma^6(n)$	
3	2	3	5	$-\ 4(n-3)_2(n-5)_1\Sigma^6(n)$	
3	3	3	6	$+10(n-3)_2(n-6)_2\Sigma^7(n)$	
...	...	...	...	... ...	
...	...	...	...	... ...	

When $r=4$ the general term becomes

$$(-)^{I_3+1}\frac{\underline{|I_2-1}}{\underline{|I_1}\,\underline{|\iota_2-1}}\cdot\frac{\underline{|I_3-1}}{\underline{|I_2}\,\underline{|\iota_3-1}}$$
$$\times\,(n-I_1)_{\iota_1-1}\,.\,(n-I_2)_{\iota_2-1}\,.\,(n-I_3)_{\iota_3-1}\,.\,\Sigma^{I_3+1}(n).$$

Thus, writing down the corresponding terms of $C_4[(0), (1), \ldots (n-1)]$, and values of ι_1, ι_2, ι_3, I_1, I_2, I_3,

ι_1	ι_2	ι_3	I_1	I_2	I_3	
1	1	1	1	2	3	$(n-1)\,(n-2)\,(n-3)\,\Sigma^4(n)$
1	1	2	1	2	4	$-\,3\,(n-1)\,(n-2)\,(n-4)_1\,\Sigma^5(n)$
1	2	1	1	3	4	$-\,2\,(n-1)\,(n-3)_1\,(n-4)\,\Sigma^5(n)$
2	1	1	2	3	4	$-\,(n-2)_1\,(n-3)\,(n-4)\,\Sigma^5(n)$
1	2	2	1	3	5	$+\,8\,(n-1)\,(n-3)_1\,(n-5)_1\,\Sigma^6(n)$
2	1	2	2	3	5	$+\,4\,(n-2)_1\,(n-3)\,(n-5)_1\,\Sigma^6(n)$
2	2	1	2	4	5	$+\,3\,(n-2)_1\,(n-4)_1\,(n-5)\,\Sigma^6(n)$
2	2	2	2	4	6	$-15\,(n-2)_1\,(n-4)_1\,(n-6)_1\,\Sigma^7(n)$
...	...	...	...	...	...	... ...
...	.	...	...	...	...	... ...

$= C_4[(0), (1), \ldots \ldots (n-1)]$ when $m=1$ .... (23).

The following results are obtained from formulæ (18), (21), (22), and (23). I have not given proofs of them, as they are obtained, like (19) and (20), by common algebraical substitution and reduction; but I have verified them carefully, and believe them to be free from error:

$$C_2(1, 2, \ldots n) = \frac{n+1\,.\,n\,.\,n-1\,.\,3n+2}{24},$$

$$C_3(1, 2, \ldots n) = \frac{(n+1)^2\,n^2\,(n-1)\,(n-2)}{48},$$

$$C_4(1, 2, \ldots n) = \frac{n+1\,.\,n\,.\,n-1\,.\,n-2\,.\,n-3}{8\,\lfloor 6}\,(15n^3+15n^2-10n-8),$$

$$C_2(1, 4, \ldots n^2) = \frac{n+1\,.\,n\,.\,n-1\,.\,2n+1\,.\,2n-1\,.\,5n+6}{3\,\lfloor 5},$$

$$C_3(1, 4, \ldots n^2) = \frac{n+1\,.\,n\,.\,n-1\,.\,n-2\,.\,2n+1\,.\,2n-1\,.\,2n-3}{9\,\lfloor 7} \times (35n^2+91n+60).$$

In the last two formulæ it is remarkable that

$C_2(1, 4, \ldots n^2)$ is divisible by $2n+1\,.\,2n-1$; *i.e.*, by $\Delta n^2\,.\,\Delta(n-1)^2$.

$C_3(1, 4, \ldots n^2)$ is divisible by $\Delta n^2\,.\,\Delta(n-1)^2\,.\,\Delta(n-2)^2$.

It is well known that $C_1(1, 4, \ldots n^2) = \dfrac{n+1\,.\,n\,.\,2n+1}{6}$, which is divisible by Δn^2; but I do not know whether similar results hold for $C_4(1, 4, \ldots n^2)$ ... or not.

Since $C_r(a_0, a_1, \ldots a_{n-1}) = \left(\sum_0^n a_n\right)^r$, $C_r[(0), (1), \ldots (n-1)]$ may be found by a repeated use of the Euler-Maclaurin sum formula.

For example,

$$C_2(1, 2, \dots n) = \left(\sum_0^n \overline{n+1}\right)^2$$

$$= \sum_0^n \overline{n+1}\left(\frac{n^2}{2} + n - \frac{n}{2}\right)$$

$$= \sum_0^n \left(\frac{n^3}{2} + n^2 + \frac{n}{2}\right)$$

$$= \left(\frac{n^4}{8} + \frac{n^3}{3} + \frac{n^2}{4}\right) - \frac{1}{2}\left(\frac{n^3}{2} + n^2 + \frac{n}{2}\right) + \frac{1}{12}\left(\frac{3n^2}{2} + 2n\right)$$

$$= \frac{3n^4 + 2n^3 - 3n^2 - 2n}{24} = \frac{n+1 \,.\, n \,.\, n-1 \,.\, 3n+2}{24}.$$

Since reading this paper I have re-written it in accordance with some valuable suggestions, for which I am indebted to Prof. Cayley, and have added to it considerably.

March 10*th*, 1876.

W. SPOTTISWOODE, Esq., F.R.S., Vice-President, in the Chair.

Messrs Arthur Cockshott and R. T. Wright were elected, and Messrs. E. B. Elliott, C. M. Leudesdorf, and J. W. Russell, were admitted into the Society.

The following communications were made to the Society:—

Prof. Cayley, "On the Bicursal Sextic, and the Problem of Three-Bar Motion."

Prof. Clifford, "On the Classification of Geometric Algebras."

Presents received:—

Extraits de deux lettres adressées à D. B. Boncampagni par M. le Comte Léopold Hugo (extrait des Atti dell' Accademia Pontificia de' Nuovi Lincei, 19 Dec., 1875).

"Sur une classe de points singuliers de surfaces," par H. G. Zeuthen (from the Math. Annalen, ix.), Copenhague, 27 Août, 1875.

Extrait d'une lettre de M. Ch. Hermite à M. L. Königsberger "Sur le développement des fonctions elliptiques suivant les puissances croissantes de la variable" (Crelle, 81 vol.)

Lettre de M. Ch. Hermite à M. Borchardt "Sur la fonction de Jacob Bernoulli" (Crelle, vol. 79).

Extrait d'une lettre de M. Ch. Hermite à M. Borchardt "Sur la transformation des formes quadratiques ternaires en elles-mêmes" (Crelle, 78 vol.)

Extrait d'une lettre de M. Ch. Hermite à M. Borchardt "Sur la réduction des formes quadratiques ternaires" (Crelle, vol. 79).

The same to the same on a property of Bernoulli's numbers (Crelle, vol. 81).

"Sur les développements de la fonction F $(x) = sn^a x \,.\, cn^b x \,.\, dn^c x$, où les exposants sont entiers," par M. Ch. Hermite (Bihang till K. Svenska vet Akad. Handlingar. Band 3, No. 10), Stockholm, 1875.

Extrait d'une lettre de M. Ch. Hermite de Paris à M. L. Fuchs de Gottingue "Sur quelques équations différentielles linéaires" (Crelle, vol. 79).

(*The above presented by M. Ch. Hermite.*)

"Nouvelles propriétés géométriques de la surface de l'onde qui s'interprètent en Optique," par M. A. Mannheim, 7 Février, 1876.

"Démonstration géometrique d'une relation due à M. Laguerre" (the same, 6 Mars, 1876).

"Association Française pour l'avancement des Sciences—Congrès de Nantes, Paris, 1875;" containing "Recherches sur la surface de l'onde (21 Août 1875), "Propriétés des diamètres de la surface de l'onde et interprétation physique de ces propriétés" (25 Août, 1875).

(*The above presented by M. Mannheim.*)

Carte de Visite likeness and photograph, from Dr. H. G. Zeuthen.

On Three-Bar Motion. *By* Prof. Cayley.

[*Read March* 10*th*, 1876.]

The discovery by Mr. Roberts of the triple generation of a Three-Bar Curve, throws a new light on the whole theory, and is a copious source of further developments.* The present paper gives in its most simple form the theorem of the triple generation; it also establishes the relation between the nodes and foci; and it contains other researches. I have made on the subject a further investigation, which I give in a separate paper, "On the Bicursal Sextic"; but the two papers are intimately related, and should be read in connection.

The Three-Bar Curve is derived from the motion of a system of three bars of given lengths pivoted to each other, and to two fixed

* See his paper "On Three-Bar Motion in Plane Space," *ante*, pp. 15—23, which contains more than I had supposed of the results here arrived at. There is no question as to Mr. Roberts' priority in all his results.