The following twelve memoirs from "Det Kgl. Norske Universitet à Christiania":—

"Om ligningen af 5[te] grad," af Dr. A. S. Guldberg.

"Om ligningen af 3[die] grad," by the same.

"Om forringelsen af modularligningernes grad," af L. Sylow.

"Om den algebraiske ligning af n[te] grad, hvis Rodder representeres, ved formelen $x = R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}}$," af Dr. A. S. Guldberg.

"Om Vœdskers Udvidelse ved konstant Tryk," by the same.

"Bidrag til Legemernes Molekylartheori," af Cato M. Guldberg.

"Sur le mouvement simultané de corps sphériques variables dans un fluide indéfini et incompressible: premier mémoire," par C. A. Bjernkes, Christiania.

"Ny Interpolationsmethode," af J. J. Åstrand.

"Over en classe geometriske Transformationer," af Sophus Lie.

"Ueber eine Classe geometrischer Transformationen," (Fortsetzung): by the same.

"Om den Gruppe af Substitutioner, der tilhrer ligningen for division af Perioderne ved de Elliptiske Funktioner," af L. Sylow.

"Bidrag til Theorien for de ubestemte chemiske Forbindelser," af Cato M. Guldberg.

Some General Theorems relating to Vibrations. By the Hon. J. W. Strutt, M.A. (Lord Rayleigh).

[*Read June 12th*, 1873.]

This paper contains a short account of some general theorems, with which I have lately become acquainted during the preparation of a work on Acoustics. As they seem to possess considerable interest, I take the present opportunity of bringing them before the Society.

Section I.

The natural periods of a conservative system, vibrating freely about a configuration of stable equilibrium, fulfil the stationary condition.

Let the system be referred, in the usual manner, to independent coordinates $\psi_1, \psi_2, \psi_3, \ldots$ whose origin is taken to correspond with the configuration of equilibrium. Then, the square of the motion being neglected, the kinetic and potential energies are expressible in the form

$$T = \tfrac{1}{2}[11]\dot{\psi}_1^2 + \tfrac{1}{2}[22]\dot{\psi}_2^2 + \ldots\ldots + [12]\dot{\psi}_1\dot{\psi}_2 + \ldots\ldots \quad \ldots(1),$$

$$V = \tfrac{1}{2}\{11\}\psi_1^2 + \tfrac{1}{2}\{22\}\psi_2^2 + \ldots\ldots + \{12\}\psi_1\psi_2 + \ldots\ldots \quad \ldots(2),$$

where [11] ..., {11} ... are constants, subject to the condition of making T and V always positive. For the present purpose, it is convenient, though not necessary, to transform the coordinates in the manner explained in Thomson and Tait's "Natural Philosophy," § 337, so as to reduce T and V to a sum of squares;

$$T = \tfrac{1}{2}[1]\,\dot{\phi}_1^2 + \tfrac{1}{2}[2]\,\dot{\phi}_2^2 + \ldots\ldots \qquad\qquad (3),$$

$$V = \tfrac{1}{2}\{1\}\,\phi_1^2 + \tfrac{1}{2}\{2\}\,\phi_2^2 + \ldots\ldots \qquad\qquad (4),$$

where the coefficients are necessarily positive. The natural vibrations of the system are those represented by the separate variation of the coordinates $\phi_1, \phi_2, \ldots$; and the corresponding differential equations obtained by Lagrange's method are of the form

$$[s]\,\ddot{\phi}_s + \{s\}\,\phi_s = 0 \qquad\qquad (5),$$

showing that the period of the natural vibration ϕ_s is given by

$$\tau_s = 2\pi\,[s]^{\frac{1}{2}} \div \{s\}^{\frac{1}{2}} \qquad\qquad (6).$$

Let us now suppose that the system is no longer allowed to choose its type of vibration, but that an arbitrary type is imposed upon it by a suitable constraint, leaving only one degree of freedom. Thus, let

$$\phi_1 = A_1\theta, \quad \phi_2 = A_2\theta \qquad\qquad (7),$$

where $A_1, A_2, \ldots$ are given real coefficients. The expressions for T and V become

$$T = \left\{\tfrac{1}{2}[1]\,A_1^2 + \tfrac{1}{2}[2]\,A_2^2 + \ldots\ldots\right\}\dot{\theta}^2 \qquad\qquad (8),$$

$$V = \left\{\tfrac{1}{2}\{1\}\,A_1^2 + \tfrac{1}{2}\{2\}\,A_2^2 + \ldots\ldots\right\}\theta^2 \qquad\qquad (9),$$

whence, if θ varies as $\cos pt$,

$$p^2 = \frac{\{1\}\,A_1^2 + \{2\}\,A_2^2 + \ldots\ldots}{[1]\,A_1^2 + [2]\,A_2^2 + \ldots\ldots} \qquad\qquad (10).$$

This gives the period of the vibration of constrained type; and it is evident that the period is stationary, when all but one of the coefficients $A_1, A_2, \ldots$ vanish, that is to say, when the type coincides with one of those natural to the system.

By means of this theorem we may prove that an increase in the mass of any part of a vibrating system is attended by a prolongation of all the natural periods, or at any rate that no period can be diminished. Suppose the increment of mass to be infinitesimal. After the alteration, the types of vibration will in general be changed; but, by a suitable constraint, the system may be made to retain any one of the former types. If this be done, it is certain that any vibration which involves a motion of the part whose mass is increased, has its period prolonged. Only as a particular case (as, for example, when a load is placed at the node of a vibrating string) can the period remain unchanged. The theorem now allows us to assert that the removal of the constraint, and the consequent change of type, can only affect the period by a quantity of the second order; and that therefore, in the limit, the free period cannot be less than before the change. By integration we infer that a finite increase of mass must prolong the period of every vibration which involves a motion of the part affected, and that in no case can any period be diminished; but in order to see the correspondence of the two sets of periods, it may be necessary to suppose the alteration made by steps. The converse of this and corresponding theorems relating to an alteration in the potential energy of a system will now be obvious.

A very useful application of the principle may be made to the approximate calculation of the natural periods of a system whose constitution, though complicated, is but slightly different from one of a much simpler nature. The main difficulty of the general problem consists in the determination of the free types, which may involve the solution of a difficult differential equation. We now see that an approximate knowledge of the type may be sufficient for practical purposes, and that, in the class of cases referred to, the adoption of the type natural to the approximate simpler system in the calculation of T and V will entail an error of the second order only in the final result.

To illustrate this question, we may take a case not without interest of its own—namely, the transverse motion of a stretched string of nearly, but not quite, uniform longitudinal density. If the uniformity were exact, the type of the sth component vibration would be

$$y = \phi_s \sin\frac{s\pi x}{l} \qquad (11),$$

where l is the length, x the distance of any particle from one end, and y the transverse displacement. In accordance with the plan proposed, we are to calculate the period for the variable string on the supposition that (11) is also applicable to it. We find

$$\begin{aligned} T &= \tfrac{1}{2}\dot{\phi}_s^2 \int_0^l \rho \sin^2\frac{s\pi x}{l}\,dx \\ &= \frac{\rho_0 l}{4}\dot{\phi}_s^2\left\{1 - 2\int_0^l \frac{\Delta\rho}{\rho_0}\sin^2\frac{s\pi x}{l}\,dx\right\} \qquad (12), \end{aligned}$$

if $\rho = \rho_0 + \Delta\rho$, $\Delta\rho$ being small.

For the potential energy we have (T_1 being the tension) the usual expression

$$V = \tfrac{1}{2}T_1\int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{T_1 l}{4}\cdot\frac{s^2\pi^2}{l^2}\phi_s^2 \qquad (13).$$

Hence, if the solution be

$$\phi_s = A\cos\left(\frac{2\pi t}{\tau_s} - \epsilon\right) \qquad (14),$$

the period r_s is given by

$$r_s^2 = \frac{4\rho_0 l^2}{s^2 T_1}\left\{1-2\int_0 \frac{\Delta\rho}{\rho_0}\sin^2\frac{s\pi x}{l}\,dx\right\} \text{ .... (15).}$$

As might be expected, the effect of an alteration of density vanishes at the nodes, and is a maximum midway between them.

A similar method applies to a great variety of problems, and gives the means of calculating the correction due to the necessary deviation of any actual system, on which experiments can be made, from the ideal simplicity assumed in theory.

Another point of importance with reference to this application has yet to be noticed. It appears from (10) that the period of the vibration corresponding to any hypothetical type is included between the greatest and least of the periods natural to the system. In the case of systems like strings and plates, which are treated as capable of continuous deformation, there is no least natural period; but we may still assert that the period calculated from any hypothetical type cannot exceed that belonging to the gravest normal type. When, therefore, the object is to estimate the longest natural period of a system by calculations founded on an assumed type, we have, *à priori*, the assurance that the result will come out too small. For example, the value for r_1, given in (15), is certainly less than the truth, while the error is of the second order in $\Delta\rho$.

In the choice of a hypothetical type, judgment must be used, the object being to approach the truth as nearly as can be done without too great a sacrifice of simplicity. The type for a string heavily weighted at any point might suitably be taken from the extreme case of an infinite load, that is to say, the two parts of the string might be supposed to be straight. Even with a uniform unloaded string, the result of the above hypothesis is not so very far from the truth. Taking $y = mx\cos pt$ from $x=0$ to $x = \frac{l}{2}$, we find, for the whole string,

$$T = \frac{\rho p^2 m^2 l^3}{24}\sin^2 pt, \quad V = m^2\frac{T_1 l}{2}\cos^2 pt;$$

whence

$$p^2 = \frac{12 T_1}{\rho l^2} \text{ .... (16).}$$

The correct result for a uniform string is

$$p^2 = \frac{\pi^2 T_1}{\rho l^2} \text{ .... (17);}$$

so that the period calculated from the assumed type is too small in the ratio of $\pi : \sqrt{12}$ or $\cdot 907 : 1$.

A much closer approximation would be obtained by the assumption of a parabolic form

$$y = m\left(1 - \frac{4x^2}{l^2}\right) \quad\text{(18).}$$

Proceeding in the same way as before, we should find a period too short in the ratio $\pi : \sqrt{10}$, or $\cdot 9936 : 1$. In order that the natural type should be parabolic, the density of the string would have to vary as $(l^2 - 4x^2)^{-1}$, being thus a minimum in the middle, and becoming infinite at either end.

The gravest tone of a square plate is obtained when the type of vibration is such that the nodal lines form a cross passing through the centre of the plate, and parallel to the edges. The next type in order of importance gives the diagonals for the nodal lines. Chladni found experimentally that the interval between the two tones was about a fifth. It so happens that the second kind of vibration can be completely treated theoretically, being referable to the simpler case of the vibration of bars; but the first has not hitherto been successfully attacked. I find that if we assume for the type of vibration

$$z = xy \cos pt \quad\text{(19),}$$

the nodal lines being taken for axes of x and y, the boundary conditions are satisfied, and the calculated period comes out greater than that corresponding to the diagonal position of the nodal lines in the ratio of $1{\cdot}37 : 1$. Since this ratio is certainly too small, Chladni's result is about what might have been expected from theoretical considerations.

Before leaving the subject of natural vibrations, I wish to direct the attention of mathematicians to a point which does not appear to have been sufficiently considered: I refer to the expansion of arbitrary functions in series of others of specified types. The best known example of such expansions is that generally called after Fourier, in which an arbitrary periodic function is analysed into a series of harmonics, whose periods are sub-multiples of that of the given function. It is well known that the difficulty of the question is confined to the proof of the *possibility* of the expansion; if this be assumed, the determination of the coefficients is easy enough. What I wish now to draw attention to is, that in this, and an immense variety of similar cases, the possibility of the expansion may be inferred from physical considerations.

To fix our ideas, let us consider the small vibrations of a uniform string stretched between two fixed points. We know, from the general theory, that the whole motion, whatever it may be, can be analysed into a series of harmonic functions of the time, representing component vibrations, each of which can exist by itself. If we can discover these normal types, we shall be in a position to represent the most general possible vibration by combining them, each with arbitrary amplitude and phase. The nature of the normal types is given by the solution of the differential equation

$$\frac{d^2y}{dx^2} + \kappa^2 y = 0 \quad \text{......................... (20)},$$

whence it appears that they are expressed by

$$y = \sin\frac{\pi x}{l}, \quad y = \sin\frac{2\pi x}{l}, \quad y = \sin\frac{3\pi x}{l}, \text{ \&c.}$$

We infer that the most general position which the string can assume is capable of representation by a series of the form

$$A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + \ldots\ldots,$$

which is a particular case of Fourier's theorem. There would be no difficulty in proving it in its most general form.

So far the string has been supposed uniform. But we have only to introduce a variable density, or even a single load at one point of the string, in order completely to alter the expansion whose possibility may be inferred from dynamical theory. It is evident that corresponding to any system, whether string, bar, membrane, plate, or what not, there is an appropriate expansion for an arbitrary function of one or more variables. Thus the expansion in La Place's series may be proved by considering the motion of a thin layer of gas between two concentric spherical surfaces, the expansion in Bessel's functions from the vibrations of a circular membrane, or of the air contained within a rigid cylinder, &c. When the difficulty of a direct analytical proof of even these simple cases is considered, the advantage of the physical point of view will be admitted.

The method of definite integration (or summation, if the system have only finite freedom), by which the constants are determined to suit arbitrary initial circumstances, is well known, and has been applied to a great variety of problems, dealing not only with vibrations, but with other physical questions such as the conduction of heat; but I have never seen the reason of its success distinctly stated. It may be said to depend on the characteristic property of the normal coordinates, namely, their power of expressing the energy of the system as a sum of squares only. In the case of a string, for example, we have

$$y = \phi_1 \sin\frac{\pi x}{l} + \phi_2 \sin\frac{2\pi x}{l} + \ldots\ldots,$$

where ϕ_1, ϕ_2 ...... are the normal coordinates. The expression for the energy is

$$\frac{\rho}{2}\int_0^l \dot{y}^2 dx, \quad \text{or} \quad \frac{\rho}{2}\int_0^l dx \left\{ \dot{\phi}_1 \sin\frac{\pi x}{l} + \dot{\phi}_2 \sin\frac{2\pi x}{l} + \ldots\ldots \right\}^2.$$

If by the solution of the differential equation, or otherwise, we have assured ourselves as to the nature of the normal types, we may assume, without further proof, that the products of the coordinates will disappear from the expression for the energy,—in the above instance, that

$$\int_0^l \sin\frac{r\pi x}{l}\sin\frac{s\pi x}{l}\,dx$$

will vanish, if r and s be different.

SECTION II.

The Dissipation Function.

The original equation of motion of a system in rectangular coordinates, as obtained at once by an application of the Principle of Virtual Velocities, is

$$\Sigma m\,(\ddot{x}\delta x+\ddot{y}\delta y+\ddot{z}\delta z) = \Sigma\,(X\delta x+Y\delta y+Z\delta z) \;\ldots\ldots\ldots\; (21).$$

When transformed to independent coordinates, and restricted so as to give the equations of vibratory motion in their simplest form, this becomes

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right) = \Psi_1 \;\&c. \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (22),$$

where $\Psi_1\delta\psi_1+\Psi_2\delta\psi_2+\ldots$ is the transformation of $\Sigma\,(X\delta x+Y\delta y+Z\delta z)$, denoting the work done on the system by the applied forces during the hypothetical displacement.

If we separate from Ψ the forces which depend only on the position of the system, we obtain

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right)+\frac{dV}{d\psi_1} = \Psi_1, \;\&c. \;\ldots\ldots\ldots\ldots\ldots\ldots\; (23).$$

The principal object of the present section is to show that another group of forces may be advantageously treated in a similar manner.

The forces referred to are those which vary in direct proportion to the component velocities of the parts of the system. It is well known that friction, and other sources of dissipation, may be usefully represented as following this law approximately; and even when the true law is different, the principal features of the case will be brought out. The effect of such forces will be to introduce into the original equation terms of the form

$$\Sigma\,(\kappa_x\dot{x}\delta x+\kappa_y\dot{y}\delta y+\kappa_z\dot{z}\delta z) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (24),$$

where κ_x, κ_y, κ_z are the coefficients of friction, parallel to the axes, for the particle xyz. The transformation to the independent coordinates is effected in a similar manner to that of

$$\Sigma m\,(\dot{x}\delta x+\dot{y}\delta y+\dot{z}dz),$$

and gives

$$\frac{dF}{d\dot{\psi}_1}\delta\psi_1+\frac{dF}{d\dot{\psi}_2}\delta\psi_2+\ldots\ldots,$$

where
$$F = \tfrac{1}{2}\Sigma\,(\kappa_x \dot{x}^2 + \kappa_y \dot{y}^2 + \kappa_z \dot{z}^2)$$
$$= \tfrac{1}{2}(11)\,\dot{\psi}_1^2 + \tfrac{1}{2}(22)\,\dot{\psi}_2^2 + \ldots\ldots + (12)\,\dot{\psi}_1\dot{\psi}_2 + \ldots\ldots \quad \ldots(25).$$

F, it will be observed, is, like T and V, a necessarily positive quadratic function of the coordinates, and represents the rate at which energy is dissipated.

The above investigation refers to forces proportional to the absolute velocities; but it is equally important to include such as depend on the *relative* velocities of the parts of a system, and this fortunately can be done without any increase of complication. For example, if a force acts on the particle x_1 proportional to $\dot{x}_1 - \dot{x}_2$, there must be at the same moment, by the law of action and reaction, an equal and opposite force acting on x_2. The additional terms in the fundamental equation will be of the form
$$\kappa\,(\dot{x}_1 - \dot{x}_2)\,\delta x_1 + \kappa\,(\dot{x}_2 - \dot{x}_1)\,\delta x_2,$$
which may be written
$$\kappa\,(\dot{x}_1 - \dot{x}_2)\,\delta\,(x_1 - x_2) = \delta\psi_1 \frac{d}{d\dot{\psi}_1}\left\{\tfrac{1}{2}(\dot{x}_1 - \dot{x}_2)^2\right\} + \ldots\ldots;$$
and so on for any number of pairs of mutually influencing particles. The only effect is the addition of new terms to F, which still appears in the form (25).*

The existence of the function F does not seem to have been recognised hitherto, and indeed is expressly denied in the excellent "Acoustics" of the late Prof. Donkin (p. 101). We shall see that its existence implies certain relations between the coefficients in the generalized equations of motion, which carry with them important consequences.

Lagrange's equation, after the separation from Ψ of the forces proportional to the displacements *and velocities*, whether absolute or relative, becomes
$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right) + \frac{dF}{d\dot{\psi}} + \frac{dV}{d\psi} = \Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$
where
$$\left.\begin{aligned} T &= \tfrac{1}{2}\,[11]\,\dot{\psi}_1^2 + \ldots\ldots + [12]\,\dot{\psi}_1\dot{\psi}_2 + \ldots\ldots \\ F &= \tfrac{1}{2}\,(11)\,\dot{\psi}_1^2 + \ldots\ldots + (12)\,\dot{\psi}_1\dot{\psi}_2 + \ldots\ldots \\ V &= \tfrac{1}{2}\,\{11\}\,\psi_1^2 + \ldots\ldots + \{12\}\,\psi_1\psi_2 + \ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (27).$$

On substitution, we obtain a system of equations, which may be written:
$$\left.\begin{aligned} \overline{11}\,\psi_1 + \overline{12}\,\psi_2 + \overline{13}\,\psi_3 + \ldots\ldots &= \Psi_1 \\ \overline{21}\,\psi_1 + \overline{22}\,\psi_2 + \overline{23}\,\psi_3 + \ldots\ldots &= \Psi_2 \\ \overline{31}\,\psi_1 + \overline{32}\,\psi_2 + \overline{33}\,\psi_3 + \ldots\ldots &= \Psi_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (28),$$

* The differences referred to in the text may of course pass into differential coefficients in the case of a continuous body.

where a coefficient such as $\overline{rs}$ is an abbreviation for the quadratic operator

$$[rs]\frac{d^2}{dt^2}+(rs)\frac{d}{dt}+\{rs\}.$$

It is to be carefully noticed that since $[rs]=[sr]$, $(rs)=(sr)$, $\{rs\}=\{sr\}$, it follows that $\overline{rs}=\overline{sr}$.

The small vibrations of a system free from dissipative influences can always be analysed into a series of normal components, each of which is similar in character to that of a system possessing but one degree of freedom. It is, in general, otherwise with the vibrations of a dissipative system. These may, indeed, be analysed into components of the quasi-harmonic type (Thomson and Tait, § 343); but these last are different in character from the vibration of a simple dissipative system. For instance, the system, supposed to be animated by one component, does not pass simultaneously through the configuration of equilibrium. The reason of the difference will appear at once. When there is no friction, a suitable transformation of coordinates will always reduce T and V to a sum of squares, and the equations of motion become

$$[1]\ddot{\phi}_1+\{1\}\phi_1=\Phi_1, \ \&c.,$$

the same as for a simple system. The presence of friction will not interfere with the reduction of T and V; but the transformation proper for them will not in general suit also the requirements of F. The equation can then only be reduced to the form

$$[1]\ddot{\phi}_1+(11)\dot{\phi}_1+(12)\dot{\phi}_2+\ldots\ldots+\{1\}\phi_1=\Phi_1 \quad \ldots\ldots\ldots (29),$$

and not to the simple form expressing the vibration of a system of one degree of freedom,

$$[1]\ddot{\phi}_1+(1)\dot{\phi}_1+\{1\}\phi_1=\Phi_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

We may, however, choose which two of the three functions we shall reduce, and the selection would vary according to circumstances.

Cases, however, arise in which, owing to the special character of the system, the same transformation of coordinates will reduce all three functions to a sum of squares, and then the motion possesses an exceptional simplicity. Under this head the most important are probably when F is of the same form as T or V. In the problem of the string, if we assume a direct retarding force proportional to the velocity, we have F proportional to T; if the dissipation is due to viscosity, we might have F proportional to V. The same exceptional reduction is possible when F is a linear function of T and V, or when T is itself of the same form as V. In any of these cases the equations of motion for each component are of the same form as for a dissipative system with one degree of freedom, and the elementary types are characterised by the fact that the whole system passes simultaneously through the configuration of equilibrium. It appears that the law of friction usually assumed for a string is of an exceptional character, and leads to results of, in some respects, delusive simplicity.

Section III.

The present section is devoted to the proof and illustration of a very important law of a reciprocal character, connecting the forces and motions of any two types. Particular cases of it have been noticed by previous writers; but the general theorem is, I believe, new, and indeed could not be proved without the results of the preceding section.

The following partial statement will convey an idea of its nature :—

Let a periodic force Ψ_s, equal to $A_s \cos pt$, act on a system either conservative, or subject to dissipation represented by the function F, giving the forced vibration $\psi_r = \kappa A_s \cos(pt-\epsilon)$, where κ is the coefficient of amplitude, and ϵ the retardation of phase. The theorem asserts that if the system be acted on by the force $\Psi_r = A_r \cos pt$, the corresponding forced motion of type s will be

$$\psi_s = \kappa A_r \cos(pt-\epsilon).$$

The solution of the general equations (28) may be expressed in the form

$$\left.\begin{array}{l} \nabla\psi_1 = \dfrac{d\nabla}{d.\overline{11}}\Psi_1 + \dfrac{d\nabla}{d.\overline{12}}\Psi_2 + \ldots\ldots \\ \nabla\psi_2 = \dfrac{d\nabla}{d.\overline{21}}\Psi_1 + \dfrac{d\nabla}{d.\overline{22}}\Psi_2 + \ldots\ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (31),$$

where ∇ denotes the determinant

$$\left|\begin{array}{llll} \overline{11}, & \overline{12}, & \overline{13}, & \ldots\ldots \\ \overline{21}, & \overline{22}, & \overline{23}, & \ldots\ldots \\ \overline{31}, & \overline{32}, & \overline{33}, & \ldots\ldots \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \end{array}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

and the partial differentiations are made without recognition of the relations $\overline{sr} = \overline{rs}$, &c. By the nature of determinants it follows that,

since $\overline{sr} = \overline{rs}$,
$$\frac{d\nabla}{d.\overline{rs}} = \frac{d\nabla}{d.\overline{sr}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33).$$

Thus the component displacement ψ_r due to a force Ψ_s is given by

$$\nabla\psi_r = \frac{d\nabla}{d.\overline{rs}}\Psi_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34).$$

If, now, we inquire what the effect of a force Ψ_r will be in producing the displacement of type s, we find

$$\nabla\psi_s = \frac{d\nabla}{d\,.\,\overline{sr}}\,\Psi_r \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(35)},$$

so that in virtue of (33) the relation of ψ_s to Ψ_r, in the second case, is the same as the relation of ψ_r to Ψ_s in the first.

Distinguishing the second case by a dash affixed to the corresponding quantities, let us take

$$\Psi_s = A_s\epsilon^{ipt}, \quad \Psi'_r = A'_r\epsilon^{ipt},$$

where the coefficients A_s, A'_r may, without loss of generality, be supposed to be real. The solution may be expressed in the form

$$\left.\begin{aligned} \psi_r &= A_s \frac{d \log \nabla\,(ip)}{d\,.\,\overline{rs}}\,\epsilon^{ipt} \\ \psi'_s &= A'_r \frac{d \log \nabla\,(ip)}{d\,.\,\overline{sr}}\,\epsilon^{ipt} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(36)},$$

where $\frac{d}{dt}$ is replaced by ip in ∇ and its differentials. Hence by (33) we see that

$$A'_r\psi_r = A_s\psi'_s \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(37)},$$

which is the symbolical expression of the reciprocal theorem with respect both to amplitude and phase. If $\Psi'_r = \Psi_s$, then will $\psi'_s = \psi_r$; but it must be remembered that the forces and displacements of different types are not necessarily comparable. The following statement will, however, hold good in all cases :—The force Ψ'_r does as much work on the motion due to Ψ_s, as Ψ_s does on the motion due to Ψ'_r.

There is an important class of cases to which our principle, general as is the proof just given, would not at first sight appear to apply. Among these may be noticed systems in which the cause of the dissipation, or of part of it, is the conduction and radiation of heat. The dissipation cannot always be represented by a function F, which shall be the same in form under all circumstances. I am not at present in a position to discuss this question completely; but there is one consideration which may here be referred to as sufficient to bring a large additional field within the sweep of the demonstration. Since the investigation is concerned only with harmonic motions of period (p), it will be sufficient for the establishment of the theorem if the dissipation function exist *for all vibrations of the given period.*

A few examples may promote the comprehension of a theorem which, on account of its extreme generality, may appear vague.

Let A and B be two points of a uniform or variable stretched string. If a periodic transverse force act at A, the same vibration will be produced at B as would have ensued at A had the force acted at B.

In a space occupied by air, let A and B be two sources of disturbance. The vibration excited at A will have at B the same relative amplitude and phase as if the places were exchanged. Helmholtz (Crelle, Band LVII.) has proved this result in the case of a uniform fluid without friction, in which may be immersed any number of rigid fixed solids; but we are now in a position to assert that the reciprocity will not be interfered with, whatever number of strings, membranes, forks, &c. may be present, even though they are subject to damping.

The theorem includes the optical law, that if one point can be seen from a second, the second can also be seen from the first, whatever reflections or refractions the light may have to undergo on its passage.

A last example may be taken from electricity. Let there be two linear conducting circuits A and B, in whose neighbourhood there may be any number of others (either closed or terminating in condensers), or solid conducting masses. The theorem asserts that an electromotive force acting in A gives the same variable current in B as would be produced in A if the electromotive force were transferred to B.

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*Addition to the Memoir on Geodesic Lines, in particular those of a Quadric Surface.** *By* Prof. Cayley.

[*Read June 12th*, 1873.]

38. In the Memoir above referred to, speaking of the geodesic lines on the skew hyperboloid, I say (No. 35),—"The geodesic of initial direction M1 touches at M the oval curve of curvature M1, and lies wholly above this curve; *it makes an infinity of convolutions round the upper part of the hyperboloid*, cutting all the oval curves of curvature for which p has a positive value greater than p_1 (if p_1 is the value of p corresponding to the oval curve through M), and ascending to infinity." The statement as to the infinity of convolutions is incorrect; I was led to it by the assumption that the geodesic could not touch any hyperbolic curve of curvature. The fact is, that it touches at infinity (has for asymptotes) in general two hyperbolic curves of curvature; viz., the geodesic descending from infinity in the direction of a hyperbolic curve of curvature, so as to touch the oval curve through M, again ascends to infinity in the direction of a hyperbolic curve of curvature (the same as the first-mentioned one, or a different curve),

* See pp. 191—211. The articles are numbered consecutively with those of the original Memoir.