Replacing A^8BA^7B by its inverse, and $BA^{-1}B$ by ABA,

$$ABA^{11}BA^{-1}A^{13}BA^4BA^{-4}BA^7BA^8BA^2 = I.$$

Replacing $A^{13}BA^4B$ by its inverse, then $BA^{-1}B$ by ABA, then BAB by $A^{-1}BA^{-1}$, then BA^6BA^9 by its inverse, we get

$$ABA^{12}BA^4BA^{-6}BA^{-1}BA^2 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, and $BA^{+1}B$ by ABA, we obtain an identity. Hence the theorem is proved.

On Perpetuants. *By* J. H. Grace. Communicated December 11th, 1902. Received December 16th, 1902. Revised March 4th, 1903.

1. In a recent paper I proved that perpetuants of unit degree in each quantic involved are of the form

$$(ab)^\lambda (ac)^\mu \ldots (ak)^\rho (ah)^\sigma (al)^\tau,$$

where $\tau \geqslant 1$, $\sigma \geqslant 2$, $\rho \geqslant 4$, ...

It is easily seen by the methods there explained that the form

$$(ab)^\lambda (bc)^\mu \ldots (kh)^\sigma (hl)^\tau,$$

with the same conditions imposed on the exponents, is equally suited to the expression of perpetuants.

In the present paper I shall find the general form of a perpetuant when all the letters do not refer to different quantics. The results lead incidentally to the generating functions discovered by MacMahon and Stroh.

2. *Statement of Results for one Quantic.*

The symbols $a_1, a_2, a_3, \ldots$ all referring to the same quantic, the general form of a perpetuant is

$$(a_1a_2)^{2\alpha_1}(a_2a_3)^{\alpha_2}(a_3a_4)^{\alpha_3}\ldots(a_ra_{r+1})^{\alpha_r},$$

wherein

$$\alpha_1 \geqslant \alpha_2,\quad \alpha_2 \geqslant \alpha_3+2^{r-3},\quad \alpha_3 \geqslant \alpha_4+2^{r-4},\ \ldots,\ \alpha_{r-2} \geqslant \alpha_{r-1}+2,\quad \alpha_{r-1} \geqslant \alpha_r+1,$$

$$\alpha_r > 0.$$

Hence, writing

$$\begin{aligned}
\alpha_r &= 1+\xi_r,\\
\alpha_{r-1} &= 2+\xi_r+\xi_{r-1},\\
\alpha_{r-2} &= 4+\xi_r+\xi_{r-1}+\xi_{r-2},\\
&\ldots\quad \ldots\quad \ldots\quad \ldots\\
\alpha_2 &= 2^{r-2}+\xi_r+\xi_{r-1}+\ldots+\xi_2,\\
2\alpha_1 &= 2^{r-1}+2\,(\xi_r+\xi_{r-1}+\ldots+\xi_2+\xi_1),
\end{aligned}$$

all the ξ's are zero or positive integers.

The weight ϖ is given by

$$\begin{aligned}
\varpi &= 2\alpha_1+\alpha_2+\ldots+\alpha_{r-1}+\alpha_r\\
&= (2^r-1)+2\xi_1+3\xi_2+\ldots+(r+1)\,\xi_r,
\end{aligned}$$

and accordingly the number of perpetuants of weight ϖ and degree i is the coefficient of z^{ϖ} in

$$\frac{z^{2^{i-1}-1}}{(1-z^2)(1-z^3)\ldots(1-z^i)}.$$

3. *Result for Two Quantics.*

If the letters a refer to one form and the letters b to the other, the general perpetuant of degree i in the coefficients of the first and j in the coefficients of the second is

$$(a_1a_2)^{2\alpha_1}(a_2a_3)^{\alpha_2}\ldots(a_{i-1}a_i)^{\alpha_{i-1}}(a_ib_1)^{\alpha_i}(b_1b_2)^{\beta_1}\ldots(b_{j-1}b_j)^{\beta_{j-1}},$$

with the conditions

$$\beta_{j-1} > 0,\quad \beta_{j-2} \geqslant \beta_{j-1}+1,\quad \beta_{j-3} \geqslant \beta_{j-2}+2,\quad \ldots,\quad \beta_2 \geqslant \beta_3+2^{j-4},$$

$$\beta_1 \geqslant \beta_2+2^{j-3},$$

$$\alpha_i > \beta_1+2^{j-2},\quad \alpha_{i-1} \geqslant 2^j,\quad \alpha_{i-2} \geqslant \alpha_{i-1}+2^j,\quad \alpha_{i-3} \geqslant \alpha_{i-2}+2^{j+1},$$

$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

$$\alpha_3 \geqslant \alpha_4+2^{j+i-5},\quad \alpha_2 \geqslant \alpha_3+2^{j+i-4},\quad \alpha_1 \geqslant \alpha_2.$$

We may therefore write

$$
\begin{aligned}
\beta_{j-1} &= 1+\zeta_j, \\
\beta_{j-2} &= 2+\zeta_j+\zeta_{j-1}, \\
&\cdots \quad \cdots \quad \cdots \\
\beta_3 &= 2^{j-4}+\zeta_j+\zeta_{j-1}+\ldots+\zeta_4, \\
\beta_2 &= 2^{j-3}+\zeta_j+\ldots+\zeta_4+\zeta_3, \\
\beta_1 &= 2^{j-2}+\zeta_j+\ldots+\zeta_2, \\
\alpha_i &= 2^{j-1}+\zeta_j+\zeta_{j-1}+\ldots+\zeta_2+\zeta_1, \\
\alpha_{i-1} &= 2^{j}+\xi_{i-1}, \\
\alpha_{i-2} &= 2^{j+1}+\xi_{i-1}+\xi_{i-2}, \\
\alpha_{i-3} &= 2^{j+2}+\xi_{i-1}+\xi_{i-2}+\xi_{i-3}, \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\alpha_4 &= 2^{j+i-5}+\xi_{i-1}+\xi_{i-2}+\ldots+\xi_4, \\
\alpha_3 &= 2^{j+i-4}+\xi_{i-1}+\ldots+\xi_4+\xi_3, \\
\alpha_3 &= 2^{j+i-3}+\xi_{i-1}+\ldots+\xi_4+\xi_3+\xi_2, \\
2\alpha_1 &= 2^{j+i-2}+2\xi_{i-1}+\ldots+2\xi_4+2\xi_3+2\xi_2+2\xi_1.
\end{aligned}
$$

The total weight is then

$$\varpi = (2^{j+i-1}-1)+(2\xi_1+3\xi_2+\ldots+i\xi_{i-1})+(\zeta_1+2\zeta_2+\ldots+j\zeta_j),$$

and, all the ξ's and ζ's being zero or positive integers, the generating function is

$$\frac{z^{2^{j+i-1}-1}}{(1-z^2)(1-z^3)\ldots(1-z^i)(1-z)(1-z^2)\ldots(1-z^j)}.$$

The result for any number of quantities can now be written down at once.

4. The proof of the foregoing statement will now be given. It depends on simple principles which can be best explained by the consideration of covariants of degrees 3 and 4.

As in the previous paper, I shall make frequent use of the fact that, if

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}\ldots$$

be a covariant in n symbols, then

$$(bc)^{\mu}(cd)^{\nu}\ldots$$

is a covariant in $(n-1)$ symbols, and to this apply methods of reduction that are found before the problem of covariants in n symbols is undertaken.

It is, however, to be remarked that, if in the original covariant b and c are equivalent symbols, they are not to be so considered in the form

$$(bc)^{\mu}(cd)^{\nu}\ldots,$$

because the interchange of b and c would alter a factor we are leaving out of account in the subsidiary reduction.

There are two methods of reduction—

(i.) By means of identities of the type

$$(bc)+(ca)+(ab)=0.$$

(ii.) By interchange of two equivalent symbols.

All reductions possible by the aid of (i.) were obtained in the previous paper; accordingly, if equivalent letters are interchanged as much as possible, and the results of the first paper taken into account, it would seem a fair inference that every possible reduction is obtained. However, I shall not insist on this point in the present paper.

To facilitate such interchanging it is essential that all letters belonging to the same quantic should occur together in the symbolical forms considered, viz.,

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}\ldots;$$

for example, in the discussion of covariants of degree 2 in the coefficients of one quantic, and degree 1 in those of a second quantic, it would be inadvisable to take a and c as the equivalent symbols in

$$(ab)^{\lambda}(bc)^{\mu}.$$

The natural procedure is to take the equivalent symbols to be either a and b or b and c, and, in fact, both these cases will be considered.

5. *Covariants of Degree* 3.

Every covariant of degree 3 can be expressed as an aggregate of covariants

$$(ab)^{\lambda}(bc)^{\mu},$$

and various cases arise according as the letters belong to the same or different quantics. These cases will now be treated in order.

(I.) a, b, c refer to different quantics, *i.e.*, $a\neq b\neq c$.

As was shown in the previous paper, one covariant of each weight is reducible in virtue of the fundamental identity; it is convenient to take that one to be that in which the exponent of (ab) is unity, so that the perpetuants are

$$(ab)^{\lambda}(bc)^{\mu},$$

wherein $\mu \geqslant 1$, $\lambda \geqslant 2$.

(II.) $a,\ b = c$.

Here, in addition to using the identity, we have to interchange b and c. Suppose that $\lambda + \mu = w$; then we can express all covariants of weight w in terms of

$$(ab)^{\lambda}(ac)^{\mu},$$

$$(ac)^{\lambda}(ab)^{\mu},$$

wherein $\lambda \geqslant \mu$.

The case $\lambda = \mu$ can only arise when w is even, and then we have an identity expressing $(bc)^{w}$ in terms of the above covariants; so that $(ab)^{\frac{1}{2}w}(ac)^{\frac{1}{2}w}$ can be expressed in terms of the forms

$$(ab)^{\lambda}(ac)^{\mu},$$

$$(ac)^{\lambda}(ab)^{\mu},$$

$$(bc)^{w},$$

where $\lambda > \mu$.

If w be odd, the identity we have used is useless for our present purpose, because, b and c being equivalent symbols, $(bc)^{w}$ vanishes, and on the other side the terms cancel out two and two. In other words, when w is even we need a reducing identity and there is one at hand, while when w is odd we need no reduction and the reducing identity is nugatory—no opportunity of reduction is lost.

Finally, since b and c are equivalent,

$$(ab)^{\lambda}(ac)^{\mu} = (ac)^{\lambda}(ab)^{\mu},$$

and hence the perpetuants are

$$(ab)^{\lambda}(ac)^{\mu},$$

when $\lambda > \mu$. Then, replacing (ac) by $(ab)+(bc)$, we can express

$$(ab)^{\lambda}(ac)^{\mu}$$

in terms of the set $\qquad (ab)^{\lambda}(bc)^{\mu},$

and the inequality $\lambda > \mu$ persists, for (bc) cannot arise to a greater power than (ac) originally occurred.

Thus, when $b = c$ the perpetuants are given by

$$\lambda \geqslant \mu + 1, \quad \mu \geqslant 1.$$

(III.) $a = b$, c.

By interchanging a and b we can express

$$(ab)^{\lambda}(bc)^{\mu}$$

when λ is odd, in terms of forms for which λ is even, and greater than its original value.

Also by means of the identity we may reject the case in which $\lambda = 1$, but this reduction is included in the former; hence the perpetuants are given by $\mu \geqslant 1$, λ even and $\geqslant 2$.

(IV.) $a = b = c$.

By means of Jordan's lemma we can express all products of (bc), (ca), (ab) in terms of products

$$(ab)^{\lambda}(bc)^{\mu},$$
$$(bc)^{\lambda}(ca)^{\mu},$$
$$(ca)^{\lambda}(ab)^{\mu},$$

where $\lambda \geqslant 2\mu$, and, since the symbols are equivalent, the forms

$$(ab)^{\lambda}(bc)^{\mu}, \quad (bc)^{\lambda}(ca)^{\mu}, \quad (ca)^{\lambda}(ab)^{\mu}$$

are identical.

Hence all covariants except

$$(ab)^{\lambda}(bc)^{\mu} \quad (\lambda \geqslant 2\mu)$$

have been reduced.

Again, by the usual method when λ is odd, we can express

$$(ab)^{\lambda}(bc)^{\mu}$$

in terms of products in which λ is even and greater than its original value, so that for an irreducible form λ is even and equal to or greater than 2μ.

6. *Covariants of Degree* 4.

The typical form is now

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu},$$

and we have to discuss seven cases, of which the first two contain the essential point of the present method.

(I.) $a = b = c = d$.

We first reduce the form $(bc)^{\mu}(cd)^{\nu}$,

in which c, d are equivalent to each other but not to b. Thus, by § 5 (II.),

$$\nu \geqslant 1, \quad \mu \geqslant \nu + 1.$$

We shall now show that we may take λ even and equal to or greater than 2μ. For clearness of exposition it is advisable to consider the reduction of forms

$$(ab)^\lambda (bc)^\mu (cd)^\nu$$

in a definite order, viz., $(ab)^\lambda (bc)^\mu (cd)^\nu$

is taken before $(ab)^{\lambda'} (bc)^{\mu'} (cd)^{\nu'}$,

when the first of the differences

$$\lambda-\lambda', \quad (\lambda+\mu)-(\lambda'+\mu'), \quad (\lambda+\mu+\nu)-(\lambda'+\mu'+\nu')$$

which does not vanish is positive.

When λ is odd we can at once express the form in terms of forms previously considered; so we may assume that λ is even in what follows.

Again, $(ab)^\lambda (bc)^\mu$ can be expressed linearly in terms of the sets

$$(ab)^\alpha (bc)^\beta,$$
$$(bc)^\alpha (cd)^\beta,$$
$$(ca)^\alpha (ab)^\beta,$$

where $\alpha \geqslant 2\beta$. Hence $(ab)^\lambda (bc)^\mu (cd)^\nu$

is a linear combination of the sets

$$(ab)^\alpha (bc)^\beta (cd)^\nu,$$
$$(bc)^\alpha (ca)^\beta (ca)^\nu,$$
$$(ca)^\alpha (ab)^\beta (cd)^\nu.$$

In the second of these sets we replace (cd) by $(ca)+(ad)$, and in the third we replace it by $(cb)+(bd)$.

The three sets are then expressed in terms of the sets

$$(ab)^\alpha (bc)^\beta (cd)^\nu,$$
$$(bc)^\alpha (ca)^\beta (ad)^\nu,$$
$$(ca)^\alpha (ab)^\beta (bd)^\nu,$$

and other forms. In these others, however, the sum of the exponent of factors involving a, b, c only is greater than $\alpha+\beta$ or $\lambda+\mu$; so they have been previously considered by hypothesis.

Further, the three sets just written are equivalent, since $a=b=c$, and hence for a perpetuant we must have

$$\nu \geqslant 1, \quad \mu \geqslant \nu+1, \quad \lambda \geqslant 2\mu \text{ and even.}$$

[If the relation $\mu \geqslant \nu+1$ were disturbed in the course of the second reduction, we should reduce $(bc)^{\mu}(cd)^{\nu}$ again, which would lead to μ being increased, and the resulting covariant would have appeared before.]

COR.—The above considerations are sufficient to show that whenever a, b, c are equivalent and

$$(ab)^{\lambda}(bc)^{\mu}P,$$

where P involves other symbols, is a perpetuant then λ is even and equal to or greater than 2μ.

(II.) $a, b = c = d$.

Just as in (I.), we must have

$$\nu \geqslant 1, \quad \mu \geqslant \nu+1.$$

It will now be shown that $\lambda \geqslant \mu+2$.

In fact, if $\lambda+\mu = w$, the product

$$(ab)^{\lambda}(bc)^{\mu}$$

can be expressed linearly in terms of the products

$$\begin{aligned} &(ab)^{w}, \quad (ab)^{w-1}(ac), \quad \ldots, \quad (ab)^{w-r}(ac)^{r}, \\ &(ac)^{w}, \quad (ac)^{w-1}(ab), \quad \ldots, \quad (ac)^{w-r}(ab)^{r}, \\ &(bc)^{w}, \quad (bc)^{w-1}(ac), \quad \ldots, \quad (bc)^{w-p}(ac)^{p}, \end{aligned}$$

provided $$2(r+1)+p+1 = w+1 \text{ or } w+2, \tag{A}$$

the alternative being settled by the fact that, w and p being given, r must be a positive integer.

In the first two of the three sets of products written above the difference between the exponents of (ab) and (ac), being at least equal to $w-2r$, is greater than p in all cases for $w-2r > p$ by the relation (A).

Now, in the form $(bc)^{w-p}(ac)^{p}(ca)^{\nu}$,

unless $p > 1$, the form $(ac)_{p}(ca)^{\nu}$

can be reduced in such a way that the exponent of (ac) becomes two at least—this follows from the results for the third degree—and thus the sum of the exponents of the factors involving a, b, c only has been increased; so the covariants that arise have been discussed already.

Thus we can express the covariant

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}$$

in terms of forms already discussed and the two sets

$$(ab)^{\alpha}(ac)^{\beta}(cd)^{\nu},$$

$$(ac)^{\alpha}(ab)^{\beta}(cd)^{\nu},$$

where $\alpha > \beta+1$.

Next, the second set can be expressed by means of

$$(ca)^{\alpha}(cb)^{\beta}(cd)^{\nu},$$

where we have still $\alpha > \beta+1$, and, using $(cd) = (cb)+(bd)$, this set in turn can be expressed in terms of the set

$$(ca)^{\alpha}(cb)^{\beta}(bd)^{\nu}$$

and forms already considered.

Further, the set $(ab)^{\alpha}(ac)^{\beta}(cd)^{\nu}$

can be expressed by means of

$$(ab)^{\alpha}(bc)^{\beta}(cd)^{\nu}$$

where $\alpha > \beta+1$ still.

Hence the only forms we need retain are

$$(ab)^{\alpha}(bc)^{\beta}(cd)^{\gamma},$$

$$(ca)^{\alpha}(cb)^{\beta}(cd)^{\gamma},$$

wherein $\gamma \geqslant 1$, $\beta \geqslant \gamma+1$, $\alpha \geqslant \beta+2$.

Finally, these two sets are equivalent, since b and c can be interchanged; so for the perpetuants of degree 4 in this case we have

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu},$$

where $\nu \geqslant 1$, $\mu \geqslant \nu+1$, $\lambda \geqslant \mu+2$.

Cor.—By using just the same argument we can show that, if

$$(ab)^{\lambda}(bc)^{\mu}P$$

be a perpetuant, and the symbols b, c are equivalent, then we may take

$$\lambda-\mu \geqslant \kappa,$$

where κ is the smallest possible exponent of (bc) in a perpetuant involving all the letters in

$$(ab)^{\lambda}(bc)^{\mu}P$$

except a, the letters b, c being now regarded as not equivalent, although this, as may be seen from the results, makes no difference to the minimum exponent.

(III.) $a, b, c = d$.

In this case we must have

$$\nu \geqslant 1, \quad \mu \geqslant \nu+1,$$

and a, b, c being all different, the only condition to be satisfied by λ is that for a perpetuant type, viz., $\lambda \geqslant 4$.

(IV.) $a = b$, $c = d$.

Here $\nu \geqslant 1$, $\mu \geqslant \nu+1$, and, finally, λ, in addition to being 4 at least, must be even.

(V.) $a = b, c, d$.

Here $\nu \geqslant 1$, $\mu \geqslant 2$, $\lambda \geqslant 4$ and even.

(VI.) $a = b = c, d$.

Here $\nu \geqslant 1$, $\mu \geqslant 2$, and, by (I.) of this section, Cor., λ must be even and $\geqslant 2\mu$.

(VII.) $a, b = c, d$.

Here $\nu \geqslant 1$, $\mu \geqslant 2$, and, by Cor. (II.) of this section, $\lambda \geqslant \mu+2$.

7. We shall now complete the theory for a single form.

If the covariant be $\quad (ab)^\lambda (bc)^\mu (cd)^\nu P$,

then to reduce $\quad (bc)^\mu (cd)^\nu P$

we have to use the result for one degree lower, and remember that in such reductions b is not equivalent to $c, d, \ldots$.

Thus, for example, consider the covariant

$$(ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho$$

of degree 5, in which $a = b = c = d = e$.

In the reduction of $\quad (bc)^\mu (cd)^\nu (de)^\rho$,

we have $\quad b, c = d = e$.

Thus $\quad \rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2.$

Finally, by § 6, (I.), Cor., λ is even and at least equal to 2μ.

Hence the general type is

$$(ab)^{2\alpha_1} (bc)^{\alpha_2} (ca)^{\alpha_3} (de)^{\alpha_4},$$

wherein $\alpha_4 \geqslant 1$, $\alpha_3 \geqslant \alpha_4+1$, $\alpha_2 \geqslant \alpha_3+2$, $\alpha_1 \geqslant \alpha_2$.

8. Next consider $\quad (ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho$

with $\quad a, b = c = d = e.$

We have, as before,

$$\rho \geqslant 1, \quad \nu = \rho+1, \quad \mu \geqslant \nu+2,$$

and, by § 6, (II.), Cor., $\lambda - \mu \geqslant 4,$

for 4 is the least value of the first exponent in any perpetuant of degree 4.

Hence the type is $(ab)^{a_1}(bc)^{a_2}(cd)^{a_3}(de)^{a_4}$,

wherein $a_4 \geqslant 1$, $a_3 \geqslant a_4+1$, $a_2 \geqslant a_3+2$, $a_1 \geqslant a_2+4$.

9. Take next $(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}(de)^{\rho}(ef)^{\sigma}$,

in which all the symbols are equivalent.

We have first

$$\sigma \geqslant 1, \quad \rho \geqslant \sigma+1, \quad \nu \geqslant \rho+2, \quad \mu \geqslant \nu+4,$$

and, by § 6, (I.), Cor., λ is even and $\geqslant 2\mu$.

In the same case, if a were unlike the other symbols, the condition for λ would be replaced by

$$\lambda \geqslant \mu+8.$$

After this the general theorem for perpetuants of a single form and that for perpetuants of two forms, but unit degree in one, are self-evident.

10. To develop the theory for two forms we proceed step by step with the parallel case of three forms. the degree in the extra form being unity throughout, and the symbol corresponding to that form coming first in the expression.

The method will become evident on consideration of some examples of covariants of degree 5 of the type

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}(de)^{\rho}.$$

(I.) $a = b = c = d$, e.

Here, from § 6, (VII.),

$$\rho \geqslant 1, \quad \nu \geqslant 2, \quad \mu \geqslant \nu+2,$$

and, by § 6, (I.) Cor., λ is even and $\geqslant 2\mu$.

Thus the general type of perpetuant is

$$(ab)^{2a_1}(bc)^{a_2}(cd)^{a_3}(de)^{a_4},$$

where $a_4 \geqslant 1$, $a_3 \geqslant 2$, $a_2 \geqslant a_3+2$, $a_1 \geqslant a_2$.

(II.) $a = b = c,\ d = e$.

Hence, by § 5, (III.),

$$\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant 4,$$

and, by § 6, (I.), Cor., λ is even and $\geqslant 2\mu$.

The type is therefore expressed by

$$\alpha_4 \geqslant 1, \quad \alpha_3 \geqslant \alpha_4+1, \quad \alpha_2 \geqslant 4, \quad \alpha_1 \geqslant \alpha_2.$$

(III.) $a = b = c,\ d,\ e$.

By the type reduction for degree 4,

$$\rho \geqslant 1, \quad \nu \geqslant 2, \quad \mu \geqslant 4.$$

Finally, λ is even and $\geqslant 2\mu$.

The type is given by

$$\alpha_4 \geqslant 1, \quad \alpha_3 \geqslant 2, \quad \alpha_2 \geqslant 4, \quad \alpha_1 \geqslant \alpha_2.$$

(IV.) $a = b,\ c = d = e$.

By § 6, (II.), $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2.$

Finally, λ is even, and by the type reduction at least 8; thus we have as the general perpetuant

$$(ab)^{2\alpha_1} (bc)^{\alpha_2} (cd)^{\alpha_3} (de)^{\alpha_4},$$

wherein $\alpha_4 \geqslant 1,\ \alpha_3 \geqslant \alpha_4+1,\ \alpha_2 \geqslant \alpha_3+2,\ \alpha_1 > 4$.

(V.) $a,\ b = c,\ d = e$.

By § 6, (III.), $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant 4,$

and, by § 6, (II.), Cor., $\lambda-\mu \geqslant 4.$

Hence the complete conditions are

$$\lambda \geqslant \mu+4, \quad \mu \geqslant 4, \quad \nu \geqslant \rho+1, \quad \rho \geqslant 1.$$

(VI.) $a,\ b = c = d = e$.

By § 6, (II.), $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2,$

and, finally, $\lambda \geqslant \mu+4.$

(VII.) $a,\ b = c = d,\ e$.

By § 6, (VII.), $\rho \geqslant 1, \quad \nu \geqslant 2, \quad \mu \geqslant \nu+2,$

and $\lambda \geqslant \mu+4.$

(VIII.) $a, b, c = d = e$.

$$\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2, \quad \lambda \geqslant 8.$$

11. The complete theory for two forms now presents no difficulty, Consider, for example,

$$(ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho (ef)^\sigma,$$

wherein $$a = b = c, \quad d = e = f.$$

By the reduction of

$$(bc)^\mu (cd)^\nu (de)^\rho (ef)^\sigma \quad (b, c, d = e = f),$$

we have $\sigma \geqslant 1, \quad \rho \geqslant \sigma+1, \quad \nu \geqslant \rho+2, \quad \mu \geqslant 8$ (§ 9, VIII.),

and, by § 6, (I.), Cor., λ is even and equal to or greater than 2μ.

Thus the general form is

$$(ab)^{2\alpha_1} (bc)^{\alpha_2} (cd)^{\alpha_3} (de)^{\alpha_4} (ef)^{\alpha_5},$$

where $\alpha_5 \geqslant 1, \; \alpha_4 \geqslant \alpha_5+1, \; \alpha_3 \geqslant \alpha_4+2, \; \alpha_2 \geqslant 8, \; \alpha_1 \geqslant \alpha_2$.

In general for two forms the conditions are like those for perpetuants of unit degree in the second form until we come to the first factor involving two letters belonging to the second form. The exponent of this is determined from the type theory, and then the remaining conditions are written down by the aid of § 6, (II.), Cor., until we come to the first exponent of all.

It will be noticed that the simplest perpetuant of any given degree is of a form which is quite independent of the symbols being alike or different.