Some Formulæ in Elimination. *By* F. S. MACAULAY.

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1. The object of the following paper is to investigate the properties of the determinants which arise in the theory of elimination when conducted according to the methods of Bezout, and, in particular, to find a simple expression for the resultant. The equations are supposed homogeneous, of different orders, and *general*, that is, complete in all their terms with unconnected literal coefficients.

Cayley* has given (without proof) an extremely general expression for the resultant of n equations in the form $D/D_1/\dots/D_{n-2}$, or $DD_2\dots/D_1D_3\dots$, where D is any non-vanishing determinant of the complete matrix corresponding to the function $C_1S_1+\dots+C_nS_n$ of order t_n (*cf.* § 3 below), and $D_1, D_2, \dots, D_{n-2}$ are other determinants. The simpler, but less general, expression for the resultant found below is D/Δ, where D is a determinant selected arbitrarily in accordance with a certain rule (§ 6*a*) from the same matrix, and Δ is a minor of D.

For three equations it can be verified that the two results D/D_1 and D/Δ are the same; D_1 and Δ are not, however, composed entirely of the same elements for the same D, but each is independent of the elements in which they differ. To verify the identity of the two results for more than three equations would be difficult, and of little use. The advantage of the simpler form D/Δ lies in the fact that Δ can be at once written down from D, whereas $D_1, D_2, \dots$ are only obtained by a complicated process, which Cayley does not fully explain.

The theory suggested by Cayley has been developed in considerable detail by K. Bes.† He discusses at length the case of three equations, from which he infers the result for n equations. He does not prove that $D, D_1, \dots$ can be so chosen that no one of them vanishes identically; and he is scarcely justified in describing his method as a new process, since it does not appear to differ in any essential feature from that of Cayley.

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* The method is described generally in the *Camb. and Dub. Math. Jour.*, Vol. III., 1848, p. 116, and is explained more in detail in Salmon's *Higher Algebra* (4th edition, 1885), p. 87.

† "Théorie générale de l'Elimination, d'après la méthode Bezout, suivant un nouveau procédé," *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam* (Sectie 1), Deel VI., No. 7, 1899, 8vo, pp. 1–121.

H. Laurent* has also given a supposed explicit expression for the resultant, but an incorrect one.

The resultant of n *general* equations may be defined as an integral function of the coefficients, without repeated factors, whose vanishing is the necessary and sufficient condition that the equations should have a common solution. In the case of n equations containing

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* "L'Elimination," *Scientia, Phys.-Math.*, No. 7, 1900, pp. 1–75. This monograph, although curious and interesting, is rendered practically valueless in what relates to equations in several unknowns by its unreliable methods and conclusions. The resultant is nowhere defined and is regarded as an indefinite fractional expression. The following are some of the principal omissions and errors :—

(1) The proof (§ 15) of the theorem that, when two (non-homogeneous) polynomials in two variables are given as moduli, one variable can be expressed as an integral function of the other is incomplete, since two general assumptions are made proofs of which are not supplied.

(2) The proof (§ 16) that the z-eliminant C of three equations in x, y, z of orders m, n, p is of order mnp in z is faulty, since the author's method for expressing C leads to a fraction instead of an integral function of z. The same error appears still more prominently in § 17.

(3) The proof (§ 18) of Bezout's reduced form of a given polynomial with respect to other given polynomials as moduli completely fails when it passes beyond reduction in one variable.

(4) The statement (§ 20) that $\Omega^2/\Pi J$ [the author uses D for J; *cf.* § 10 (14) of this paper] is a determinant with all its elements to the left of the diagonal zero is an error, but an unimportant one, since it breaks up into a product of determinants in the diagonal. The statement that $\Omega^2/\Pi J$ depends only on the coefficients of the terms of highest order in the several equations is correct, but the proof is lacking. The author's proof of the same result in the *Nouv. Ann. de Math.*, Series 3, Vol. II., 1883, p. 147, is not valid. In the same place, p. 149, he is in error in stating that Ω cannot vanish unless the equations have a double solution, from which he deduces incorrect conclusions. Again, in § 20 of the monograph, the author states that $\Omega^2/\Pi J$ is independent of the roots of the equations. He does not explain what the statement means; but it is certainly untrue. If it were true, then the ratio of Ω to any other expression Ω' formed in like manner would also be independent of the roots, which can easily be tested and found incorrect for the case of a linear and a quadratic equation in two unknowns. Netto, in referring to Laurent, says that $\Omega^2/\Pi J$ is a *constant*, without further explanation (*Encyklopädie d. Math. Wiss.*, Teil I., Band I., Heft 3, 1899, p. 274). It would seem that both writers have been misled by an assumed, but false, analogy with an equation in a single unknown.

(5) In § 23 is contained the so-called explicit expression for the resultant referred to above; but the author is in error in supposing this expression "indépendant des a_{ij}," and in supposing it to be the resultant, or to contain the resultant as a factor.

(6) In § 26 the author implies that in order to calculate the resultant of n homogeneous equations in n unknowns it is of advantage to make the orders equal by multiplying the equations of inferior order by powers of the same unknown, overlooking the fact that, if two of the equations have a common factor, the resultant vanishes identically. Multiplying by powers of different unknowns is also of no advantage.

In contrast with the above we may mention § 19, which gives a proof of Jacobi's theorem, and § 22, which proves that, if the vanishing points (or solutions) of n given polynomials f in n variables are distinct, finite, and complete, then any polynomial which vanishes at all these points is of the form $\Sigma\phi f$, *i.e.*, vanishes identically with respect to the f's as moduli.

more than n unknowns the resultant of elimination of $(n-1)$ of the unknowns is called the eliminant in the remaining unknowns. The most usual form of expression for the resultant is by means of the Poisson product. Select any one of the equations $C=0$; solve the remaining $(n-1)$ equations, after putting one of the unknowns x equal to 1; substitute all the solutions in C, and take the product ΠC; then the numerator R of ΠC (when reduced to its lowest terms) is the resultant of the equations. R is an integral function of the coefficients, being the numerator of a symmetric function of the roots of the $(n-1)$ equations; and the vanishing of R is a necessary and sufficient condition that the n equations have a common solution. The degree of R in the coefficients of any one of the equations is equal to the product of the orders of the remaining equations, and the weight of every term in R is equal to the product of the orders of all the equations. The denominator of the Poisson product is the m-th power of the resultant of the $(n-1)$ equations when $x=0$, m being the order of C. R is independent of the particular choice of the unknown x and the equation $C=0$;* this also follows from § 4 and (18) of § 10 in this paper. R is non-factorisable.† Thus R satisfies all the conditions required by the definition of the resultant. In this paper the resultant is regarded from a different point of view, viz., as a factor of a determinant; but it is identified with R by means of its properties, and also actually identified in § 10 (18).

2. Notation.—Let $C_1^{(n)}, C_2^{(n)}, \ldots, C_n^{(n)}$, or $C_1, C_2, \ldots, C_n$, be n given homogeneous *general* polynomials in n variables $x_1, x_2, \ldots, x_n$, of orders $m_1, m_2, \ldots, m_n$ respectively.

Let $C_r^{(l)}$ denote the value of $C_r^{(n)}$ $(r=1, 2, \ldots, n)$ when $x_{l+1}, x_{l+2}, \ldots, x_n$ are all zero; so that $C_r^{(l)}$ is a homogeneous polynomial in l variables $x_1, x_2, \ldots, x_l$.

We imagine a correspondence to exist between the variables $x_1, x_2, \ldots, x_n$ and the polynomials $C_1, C_2, \ldots, C_n$ respectively. Thus we may, if we like, regard C_r as a polynomial in x_r whose coefficients are polynomials in $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$.

A polynomial containing no arguments $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ divisible by $x_1^{m_1}$ is said to be reduced in x_1; if, further, it contains no arguments

* Hadamard, "Mémoire sur l'Elimination," *Acta Math.*, Vol. xx., 1897, p. 201.
† Netto, *Algebra*, Bd. i., 1896, p. 169, and Bd. ii., 1898, p. 79.

divisible by $x_2^{m_2}$, it is said to be reduced in x_1 and x_2; and so on. When the variables are not specified, a reduced argument or polynomial means one which is reduced in all the variables.

In the function $C_1S^{(0)}+C_2S^{(1)}+C_3S^{(2)}+\ldots$ of order t, it is to be understood that $S^{(0)}$ is a general polynomial of order $t-m_1$, with all its coefficients at disposal, $S^{(1)}$ a general polynomial of order $t-m_2$ reduced in x_1, $S^{(2)}$ a general polynomial of order $t-m_3$ reduced in x_1, x_2, and so on. In the function $C_pS^{(0)}+C_qS^{(1)}+C_rS^{(2)}+\ldots$, where C_p, C_q, C_r, ... are chosen from C_1, C_2, ..., C_n, it is supposed that $S^{(1)}$ is reduced in x_p, $S^{(2)}$ in x_p, x_q, and so on; so that the significance of $S^{(1)}$, $S^{(2)}$, ... depends on the order in which C_1, C_2, ... appear in the constituent terms of the function.

THEOREM.—It is a known theorem* that any polynomial C of order t can be expressed uniquely in the form

$$C_pS^{(0)}+C_qS^{(1)}+\ldots+C_rS^{(l-1)}+S^{(l)},$$

where l ($\leqslant n$) is the number of the given *general* polynomials C_p, ..., C_r.

In order to prove this, we have to show that $S^{(0)}$, $S^{(1)}$, ..., $S^{(l)}$ can be chosen in one and only one way so as to satisfy the identity

$$C_pS^{(0)}+C_qS^{(1)}+\ldots+C_rS^{(l-1)}+S^{(l)}=C.$$

Equate coefficients of the arguments on the two sides of the identity. The number of equations is equal to the number of arguments of order t; this is equal to the number of the unknowns, viz., the coefficients of $S^{(0)}$, $S^{(1)}$, ..., $S^{(l)}$, as may be seen by considering the polynomial

$$x_p^{m_p}S^{(0)}+x_q^{m_q}S^{(1)}+\ldots+x_r^{m_r}S^{(l-1)}+S^{(l)},$$

in which each argument of order t comes in once and once only. Again, the determinant of the coefficients of the unknowns in the equations is not zero; for, if it were, then the identity

$$C_pS^{(0)}+C_qS^{(1)}+\ldots+C_rS^{(l-1)}+S^{(l)}=0$$

could be satisfied without $S^{(0)}$, $S^{(1)}$, ..., $S^{(l)}$ all vanishing identically;

* This theorem is a fundamental one in Bezout's method, and is probably contained in his *Théorie générale des Equations Algébriques* (Paris, 1779, 4to, 471 pp.), which I have not been able to consult.

and this is not possible, since the identity

$$x_p^{m_p} S^{(0)} + x_q^{m_q} S^{(1)} + \ldots + x_r^{m_r} S^{(l-1)} + S^{(l)} = 0$$

cannot be so satisfied. Hence the theorem is proved. In the last step we make use of the fundamental hypothesis that $C_1, C_2, \ldots, C_n$ are *general*, from which we are entitled to assume that, since a certain function of the coefficients of $C_1, C_2, \ldots$ does not vanish when $C_1, C_2, \ldots$ have the particular values $x_1^{m_1}, x_2^{m_2}, \ldots$, it cannot vanish for the actual values of $C_1, C_2, \ldots$.

From this theorem follows another of special importance, viz., that a homogeneous polynomial of the form $C_p S_1 + C_q S_2 + \ldots + C_r S_l$ can be expressed uniquely in any one of the standard forms

$$C_{p'} S^{(0)} + C_{q'} S^{(1)} + \ldots + C_{r'} S^{(l-1)},$$

where $C_{p'}, C_{q'}, \ldots, C_{r'}$ are the l polynomials $C_p, C_q, \ldots, C_r$ taken in any order we please. For $C_p S_1 + C_q S_2 + \ldots + C_r S_l$ can clearly be written in the form

$$C_{p'} S_1' + C_{q'} S_2' + \ldots + C_{r'} (S_l' + C_{p'} S^{(0)} + C_{q'} S^{(1)} + \ldots),$$

which, by a proper choice of $S^{(0)}, S^{(1)}, \ldots, S^{(l-2)}$, can be made of the form

$$C_{p'} S_1' + C_{q'} S_2' + \ldots + C_{r'} S^{(l-1)};$$

and this can clearly be brought step by step to the form

$$C_{p'} S^{(0)} + C_{q'} S^{(1)} + \ldots + C_{r'} S^{(l-1)}.$$

Similar reasoning leads to the theorem that, if $C_1 S_1 + \ldots + C_l S_l$, of order t, vanishes identically, then

$$S_q = \sum_{p=1}^{p=l} C_p S_{pq} \quad (q = 1, 2, \ldots, l),$$

where the polynomials S_{pq} are of assignable orders, and satisfy the relations $S_{pq} = -S_{qp}$ and $S_{pp} = 0$. The same result holds if $C_1, \ldots, C_l$ are *any* given polynomials, provided that certain functions of the coefficients do not vanish.

3. Notation.—The matrix corresponding to a homogeneous integral function $C_1 S_1 + C_2 S_2 + \ldots + C_n S_n$ of order t (also called a matrix of the coefficients of $C_1, C_2, \ldots, C_n$) is formed as follows. Write down horizontally all the arguments $\omega_1, \omega_2, \ldots, \omega_\mu$ of order t. Multiply C_p by any argument ω of S_p, and write the coefficients of ωC_p under their corresponding arguments $\omega_1, \omega_2, \ldots, \omega_\mu$, thus giving a row of the matrix. Write to the left of the row the coefficient of ω in S_p. If

$\lambda_1, \lambda_2, \ldots, \lambda_\rho$ are all the coefficients of $S_1, S_2, \ldots, S_n$, we thus obtain a matrix with μ columns and ρ rows, viz.,

$$\begin{array}{c|cccc} & \omega_1 & \omega_2 & \ldots & \omega_\mu \\ \hline \lambda_1 & a_{11} & a_{12} & \ldots & a_{1\mu} \\ \lambda_2 & a_{21} & a_{22} & \ldots & a_{2\mu} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_\rho & a_{\rho 1} & a_{\rho 2} & \ldots & a_{\rho\mu} \end{array}$$

This (bordered) matrix is a diagrammatic representation of the function $C_1 S_1 + \ldots + C_n S_n$, viz., the function is the sum of the products of every element of the matrix by the two corresponding elements in the border. It represents also the whole coefficient of each argument in the function, viz., the coefficient of ω_q is $\sum_{p=1}^{p=\rho} \lambda_p a_{pq}$.

$D(n, t)$ denotes the determinant whose vanishing is the condition that the identity

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)} = S^{(n)},$$

of order t, can be satisfied. Thus $D(n, t)$ is obtained from the matrix corresponding to $C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)}$ by omitting the columns corresponding to arguments contained in $S^{(n)}$, that is, all columns corresponding to reduced arguments. We take $D(n, t) = 1$ when t is less than the least of $m_1, m_2, \ldots, m_n$.

$R(n, t)$ denotes the H.C.F. of the $n!$ determinants formed in a similar way to $D(n, t)$ when $C_1, C_2, \ldots, C_n$ are arranged in any order. $R(n, t) = D(n, t)$ when t is less than the sum of the least two of $m_1, m_2, \ldots, m_n$; otherwise $R(n, t) < D(n, t)$.

$D(l, t)$, $l \leqslant n$, is the determinant whose vanishing is the condition that the identity

$$C_1^{(l)} S^{(0)} + C_2^{(l)} S^{(1} + \ldots + C_l^{(l)} S^{(l-1)} = S^{(l)},$$

of order t, can be satisfied. Here l is the number of the variables (§ 2), and also the number of the given polynomials C. $R(l, t)$ denotes the H.C.F. of the $l!$ determinants like $D(l, t)$.

We take t_n to stand for $m_1 + m_2 + \ldots + m_n - n + 1$, and t_l for $m_1 + \ldots + m_l - l + 1$.

THEOREM.—The resultant of $C_1, C_2, \ldots, C_n$ is $R(n, t_n)$.

The matrix from which $D(n, t_n)$ is obtained, viz.,

$$\begin{array}{c|cccc} & \omega_1 & \omega_2 & \ldots & \omega_\mu \\ \hline \lambda_1 & a_{11} & a_{12} & \ldots & a_{1\mu} \\ \lambda_2 & a_{21} & a_{22} & \ldots & a_{2\mu} \\ \ldots & \ldots & \ldots & & \ldots \\ \lambda_\mu & a_{\mu 1} & a_{\mu 2} & \ldots & a_{\mu\mu} \end{array}$$

is the determinant $D(n, t_n)$ itself; for there are no reduced arguments of order t_n (since the reduced argument of highest order is $x_1^{m_1-1} x_2^{m_2-1} \ldots x_n^{m_n-1}$ which is of order t_n-1), and consequently there are no columns to be omitted. It is then evident that $D(n, t_n)$ vanishes if the equations

$$\begin{aligned} a_{11}\omega_1 + a_{12}\omega_2 + \ldots + a_{1\mu}\omega_\mu &= 0, \\ a_{21}\omega_1 + a_{22}\omega_2 + \ldots + a_{2\mu}\omega_\mu &= 0, \\ \ldots \quad \ldots \quad \ldots \quad & \ldots \\ a_{\mu 1}\omega_1 + a_{\mu 2}\omega_2 + \ldots + a_{\mu\mu}\omega_\mu &= 0 \end{aligned}$$

can be satisfied. But these equations are satisfied, when the resultant vanishes, by giving to $\omega_1, \omega_2, \ldots, \omega_\mu$ the values which they have for the common solution of the equations $C_1 = 0$, $C_2 = 0$, ..., $C_n = 0$. Hence the resultant is a factor of $D(n, t_n)$, and of all the $n!$ determinants like $D(n, t_n)$; therefore it is a factor of their H.C.F., viz., $R(n, t_n)$. Also $R(n, t_n)$ is of the same degree as the resultant in the coefficients of each of the polynomials $C_1, C_2, \ldots C_n$ (proved in § 4). Hence $R(n, t_n)$ is the resultant.

4. THEOREM.—The degree of $R(n, t)$ in the coefficients of C_r is equal to the number of arguments of order $t-m_r$ which are reduced in all the variables except x_r, *i.e.*, it is equal to the coefficient of x^t in $\dfrac{x^{m_r}}{1-x^{m_r}} \prod_{p=1}^{p=n} \dfrac{1-x^{m_p}}{1-x}$.

Let $D'(n, t)$ be the determinant like $D(n, t)$ for a different order of the polynomials, viz., for the order $C_1, \ldots, C_p, C_q, C_r$.

$D'(n, t)$ is a determinant of the matrix corresponding to

$$C_1 S^{(0)} + \ldots + C_q S^{(n-2)} + C_r S^{(n-1)}.$$

Hence the degree of $D'(n, t)$ in the coefficients of C_r is equal to the number of arguments in $S^{(n-1)}$ of order $t-m_r$. But $R(n, t)$ is a factor of $D'(n, t)$, and therefore cannot contain the coefficients of C_r to a higher degree than the number of arguments of order $t-m_r$ reduced in all the variables except x_r.

Now the polynomial $C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)}$ is represented by

$$\begin{array}{c|cccc} & \omega_1 & \omega_2 & \ldots & \omega_\mu \\ \hline \lambda_1 & a_{11} & a_{12} & \ldots & a_{1\mu} \\ \lambda_2 & a_{21} & a_{22} & \ldots & a_{2\mu} \\ \ldots & \ldots & \ldots & & \ldots \\ \lambda_\rho & a_{\rho 1} & a_{\rho 2} & \ldots & a_{\rho\mu} \end{array}$$

and (§ 2) this can be brought identically to the form

$$C_l S'^{(0)} + \ldots + C_q S'^{(n-2)} + C_r S'^{(n-1)},$$

which is represented by

$$\begin{array}{c|cccc} & \omega_1 & \omega_2 & \ldots & \omega_\mu \\ \hline \lambda'_1 & a'_{11} & a'_{12} & \ldots & a'_{1\mu} \\ \lambda'_2 & a'_{21} & a'_{22} & \ldots & a'_{2\mu} \\ \ldots & \ldots & \ldots & & \ldots \\ \lambda'_\rho & a'_{\rho 1} & a'_{\rho 2} & \ldots & a'_{\rho\mu} \end{array}$$

Hence $$\sum_{p=1}^{p=\rho} \lambda_p a_{pr} = \sum_{p=1}^{p=\rho} \lambda'_p a'_{pr} \quad (r = 1, 2, \ldots, \mu).$$

Taking any ρ values for r, it follows that the ratio of any determinant in the first matrix to the corresponding determinant in the second is equal to the transformation determinant $\left(\frac{\lambda'}{\lambda}\right)$ derived from the identical expression of $\lambda'_1, \lambda'_2, \ldots, \lambda'_\rho$ as linear functions of $\lambda_1, \lambda_2, \ldots, \lambda_\rho$. In particular, we have

$$\frac{D(n, t)}{D'(n, t)} = \left(\frac{\lambda'}{\lambda}\right).$$

We examine then how the quantities λ' are expressed as linear functions of the quantities λ, or, what comes to the same thing, how

I. $$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)}$$

is changed to the form

II. $$C_l S'^{(0)} + \ldots + C_q S'^{(n-2)} + C_r S'^{(n-1)}.$$

The first step is to write I. in the form

$$\text{III.} \qquad C_l\,(S^{(l-1)}+C_r S''^{(0)})+\ldots+C_q\,(S^{(q-1)}+C_r S''^{(n-2)}) \\ +C_r\,(S^{(r-1)}-C_l S''^{(0)}-\ldots-C_q S''^{(n-2)}),$$

and equate the co-factors of C_r in II. and III. We thus have

$$C_l S''^{(0)}+\ldots+C_q S''^{(n-2)}+S'^{(n-1)}=S^{(r-1)}.$$

This identity determines uniquely all the coefficients of $S''^{(0)}, \ldots, S''^{(n-2)}$, $S'^{(n-1)}$ in terms of the coefficients λ in $S^{(r-1)}$, by § 2. The coefficients λ' in $S'^{(n-1)}$ are therefore linear functions of the coefficients λ in $S^{(r-1)}$, and the coefficients of the expressions for these quantities λ' in terms of the quantities λ are fractional functions of the coefficients of $C_l, \ldots, C_p, C_q$, *i.e., they are independent of the coefficients of* C_r.

The next step is to change the co-factor of C_q in III., keeping the co-factor of C_r unchanged, and requires the identity

$$C_l S'''^{(0)}+\ldots+C_p S'''^{(n-3)}+S'^{(n-2)}=S^{(q-1)}+C_r S''^{(n-2)}$$

to be satisfied, $S''^{(n-2)}$ being already determined from the first step. From this we see that the coefficients λ' in $S'^{(n-2)}$ are linear functions of the coefficients λ in $S^{(q-1)}$ and $S^{(r-1)}$, and that *the denominators of the coefficients of the expressions for these quantities λ' in terms of the quantities λ are independent of the coefficients of C_r.* This last property clearly holds for all the quantities λ' when expressed in terms of the quantities λ. Hence the denominator of $\left(\dfrac{\lambda'}{\lambda}\right)$ is independent of the coefficients of C_r.

$D(n, t)$ is therefore divisible by all the factors of $D'(n, t)$ which contain the coefficients of C_r; and similarly each one of the $n!$ determinants like $D(n, t)$ is divisible by the same factors. Hence $R(n, t)$ is divisible by the same factors, and therefore $R(n, t)$ is of the degree stated above in the coefficients of C_r. The degree of $R(n, t)$ in all the coefficients combined is equal to the coefficient of x^t in $\Sigma \dfrac{x^{m_p}}{1-x^{m_p}} \times \Pi \dfrac{1-x^{m_p}}{1-x}$.

It easily follows that the degree of $R(n, t_n)$ in the coefficients of C_r is $\Pi m/m_r$. This completes the proof that $R(n, t_n)$ is the resultant.

The continued ratio of the determinants of the first matrix above is the same as for any one of the $n!$ matrices formed in a similar way when $C_1, C_2, \ldots, C_n$ are arranged in any order, and is equal to the continued ratio of the H.C.F.'s of the sets of $n!$ corresponding determinants. Hence, since $R(n, t)$ is the H.C.F. of the set corresponding to $D(n, t)$, it follows that all the determinants of the first matrix are divisible by $D(n, t) / R(n, t)$. Also $C_1 S^{(0)} + \ldots + C_n S^{(n-1)}$ can be changed identically to $C_1 S_1 + \ldots + C_n S_n$, where $S_1, \ldots, S_n$ contain any the same number ρ of arguments in all as $S^{(0)}, \ldots, S^{(n-1)}$, provided only that the determinants of the matrix corresponding to $C_1 S_1 + \ldots + C_n S_n$ do not all vanish identically. Hence, since corresponding determinants will still remain proportional, it follows that the determinants of the matrix corresponding to $C_1 S_1 + \ldots + C_n S_n$ will have a common factor of the same degree as $D(n, t) / R(n, t)$ in the coefficients of all the polynomials $C_1, C_2, \ldots, C_n$ combined. Similar results hold for the matrices corresponding to

$$C_1 S^{(0)} + \ldots + C_l S^{(l-1)} \quad \text{and} \quad C_1 S_1 + \ldots + C_l S_l.$$

5. Theorem.—To prove that, neglecting sign,

$$\frac{D(n, t)}{R(n, t)} = \frac{D(n-1, t)}{R(n-1, t)} \frac{D(n-1, t-1)}{R(n-1, t-1)} \cdots \frac{D(n-1, t-m_n+1)}{R(n-1, t-m_n+1)}$$

$$\times D(n-1, t-m_n) D(n-1, t-m_n-1) \ldots D(n-1, 1).$$

$R(n, t)$ is a factor of $D(n, t)$, and the remaining factors of $D(n, t)$ are independent of the coefficients of C_n (§ 4). Let a_n be the coefficient of $x_n^{m_n}$ in C_n, and r the number of arguments in $S^{(n-1)}$ (of order $t-m_n$) which are severally used as multipliers of C_n in forming the r rows which correspond to C_n in $D(n, t)$. The element a_n appears in all these r rows of $D(n, t)$, and occupies the columns corresponding to the arguments of $x_n^{m_n} S^{(n-1)}$, the only columns absent from $D(n, t)$ being those which correspond to reduced arguments (§ 3), or arguments comprised in $S^{(n)}$. The remaining columns of $D(n, t)$ are those corresponding to all arguments of order t which are not comprised in $x_n^{m_n} S^{(n-1)} + S^{(n)}$, *i.e.*, $S^{(n-1)}$. Hence the coefficient of a_n^r in the expansion of $D(n, t)$ is the determinant whose vanishing is the condition that the identity

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_{n-1} S^{(n-2)} = S^{(n-1)},$$

of order t, can be satisfied. To find this determinant, assume the identity satisfied, and put $x_n = 0$; then C_q becomes $C_q^{(n-1)}$, and if $S^{(q)}$ becomes $S'^{(q)}$, we have the identity

$$C_1^{(n-1)} S'^{(0)} + C_2^{(n-1)} S'^{(1)} + \ldots + C_{n-1}^{(n-1)} S'^{(n-2)} = S'^{(n-1)},$$

of order t. Hence either $D(n-1, t) = 0$, or $S'^{(0)}, S'^{(1)}, \ldots, S'^{(n-1)}$ all vanish identically. In the latter case, $S^{(0)}, S^{(1)}, \ldots, S^{(n-1)}$ are all divisible by x_n, and on dividing it out we have the identity

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_{n-1} S^{(n-2)} = S^{(n-1)},$$

of order $t-1$. Hence $D(n-1, t-1) = 0$, or a similar identity holds of order $t-2$. It follows that the determinant sought is

$$\prod_{p=0}^{p=t-1} D(n-1, t-p).$$

Hence $$D(n, t) = a_n^r \prod_{p=0}^{p=t-1} D(n-1, t-p) + \ldots .$$

We next find the coefficient of a_n^r in $R(n, t)$. Consider the determinant like $D(n, t)$ when the order of $C_1, C_2, \ldots, C_n$ is changed to $C_n, C_1, \ldots, C_{n-1}$, that is, the determinant whose vanishing is the condition that the identity

$$C_n S^{(0)} + C_1 S^{(1)} + \ldots + C_{n-1} S^{(n-1)} = S^{(n)},$$

of order t, can be satisfied. The element a_n appears in all the r' rows corresponding to C_n, and occupies the columns corresponding to the arguments of $x^{m_n} S^{(0)}$; hence, on expanding, the coefficient of $a_n^{r'}$ is the determinant whose vanishing is the condition that the identity

$$C_1 S^{(1)} + \ldots + C_{n-1} S^{(n-1)} = S^{(n)} + x_n^{m_n} S^{(0)},$$

of order t, can be satisfied. The coefficient of $a_n^{r'}$ is therefore

$$\prod_{p=0}^{p=m_n-1} D(n-1, t-p),$$

by a similar proof to the above. Keeping now C_n fixed, while the order of $C_1, C_2, \ldots, C_{n-1}$ is altered in all possible ways, the H.C.F. of the coefficients of $a_n^{r'}$ in the several expansions is

$$\prod_{p=0}^{p=m_n-1} R(n-1, t-p).$$

This is the coefficient of a_n^r in $R(n, t)$; for it is easily seen to be of the same degree as $R(n, t)$ in the coefficients of $C_1, \ldots, C_{n-1}$. Hence

$$R(n, t) = a_n^r \prod_{p=0}^{p=m_n-1} R(n-1, t-p) + \ldots .$$

Also the ratio of $D(n, t)$ to $R(n, t)$ is equal to the ratio of their first terms when expanded in this way; thus we have the theorem. When $t \geqslant t_n$,

$$R(n, t) = R(n, t_n) = a_n^{m_1 \ldots m_{n-1}} R(n-1, t_{n-1})^{m_n} + \ldots .$$

6. THEOREM.—$R(n, t)$ is the quotient of $D(n, t)$ by the minor of $D(n, t)$ obtained by omitting the columns corresponding to all arguments reduced in $(n-1)$ of the variables $x_1, x_2, \ldots, x_n$, and the rows corresponding to C_r $(r = 1, 2, \ldots, n-1)$ for all multipliers reduced in $x_{r+1}, \ldots, x_n$.*

The resultant $R(n, t_n)$ of $C_1, C_2, \ldots, C_n$ is consequently the quotient of $D(n, t_n)$ by the corresponding minor of $D(n, t_n)$.

Let $\Delta(n, t)$ denote the minor of $D(n, t)$ mentioned above. To prove the theorem, viz., to prove that $\Delta(n, t) = D(n, t)/R(n, t)$, it will be sufficient (§ 5) to show that

$$\Delta(n, t) = \prod_{p=0}^{p=m_n-1} \Delta(n-1, t-p) \prod_{p=m_n}^{p=t-1} D(n-1, t-p),$$

and to verify that

$$\Delta(2, t) = D(2, t)/R(2, t).$$

Now $\Delta(n, t)$ is the determinant whose vanishing is the condition that the identity

$$\text{I.} \qquad C_1 \Sigma^{(0)} + C_2 \Sigma^{(1)} + \ldots + C_{n-1} \Sigma^{(n-2)} = \Sigma,$$

of order t, can be satisfied; where $\Sigma^{(0)}$ is a polynomial whose arguments are non-reduced in at least one of the variables $x_2, x_3, \ldots, x_n$; $\Sigma^{(1)}$ a polynomial whose arguments are reduced in x_1, but non-reduced in at least one of the variables $x_3, \ldots, x_n$; and similarly for $\Sigma^{(2)}, \ldots, \Sigma^{(n-2)}$ (*the last consequently divisible by* $x_n^{m_n}$); and finally Σ a polynomial whose arguments are reduced in at least $n-1$ of the variables. The number of coefficients in $\Sigma^{(0)}, \Sigma^{(1)}, \ldots, \Sigma^{(n-2)}$ is equal to the number of the equations they have to satisfy, and $\Delta(n, t)$ is not identically zero. This is seen by considering the polynomial

$$x_1^{m_1} \Sigma^{(0)} + x_2^{m_2} \Sigma^{(1)} + \ldots + x_{n-1}^{m_{n-1}} \Sigma^{(n-2)} + \Sigma,$$

in which every argument of order t occurs once and once only (*cf.* § 2).

Putting $x_n = 0$, and writing Σ' for the value each Σ then takes, we see that identity I. becomes

$$C_1^{(n-1)} \Sigma'^{(0)} + C_2^{(n-1)} \Sigma'^{(1)} + \ldots + C_{n-2}^{(n-1)} \Sigma'^{(n-3)} = \Sigma',$$

---

* It is to be remembered that the multipliers of C_r are also reduced in $x_1, \ldots, x_{r-1}$. The columns to be omitted are those which contain the elements $a_1, a_2, \ldots, a_n$ in the omitted rows, where a_r is the coefficient of $x_r^{m_r}$ in C_r.

of order t. Hence $\Delta(n-1, t) = 0$, or $\Sigma'^{(0)}, \Sigma'^{(1)}, \ldots, \Sigma'^{(n-3)}, \Sigma'$ all vanish identically. In the latter case x_n divides out of each Σ in I., and we have

$$\text{II.} \qquad C_1\Sigma^{(0)} + C_2\Sigma^{(1)} + \ldots + C_{n-1}\Sigma^{(n-2)} = \Sigma,$$

of order $t-1$. Since II. is obtained by dividing x_n out of I., the part played by $x_n^{m_n}$ in I. is now taken by $x_n^{m_n-1}$; so that in II., for any argument to be reduced or non-reduced in x_n means that it is non-divisible or divisible by $x_n^{m_n-1}$.

Putting $x_n = 0$ in II., we see that $\Delta(n-1, t-1) = 0$, or else identity I. still holds with t, $x_n^{m_n}$ changed to $t-2$, $x_n^{m_n-2}$. Proceeding in this way, we find that

$$\Delta(n, t) = \Delta(n-1, t)\,\Delta(n-1, t-1)\ \ldots\ \Delta(n-1, t-m_n+1)\,\Delta',$$

where Δ' is the determinant whose vanishing is the condition that identity I. holds when t, $x_n^{m_n}$ are changed to $t-m_n$, x_n^0. Thus each Σ is now necessarily non-reduced in x_n, and consequently $\Sigma^{(r)}$ takes the form $S^{(r)}$; while Σ is reduced in $x_1, x_2, \ldots, x_{n-1}$, and takes the form $S^{(n-1)}$. Hence identity I. becomes

$$C_1S^{(0)} + C_2S^{(1)} + \ldots + C_{n-1}S^{(n-2)} = S^{(n-1)}, \ldots$$

of order $t-m_n$. Hence (§ 5)

$$\Delta' = \prod_{p=m_n}^{p=t-1} D(n-1, t-p),$$

which proves the theorem.

Thus definite expressions have now been found for $R(n, t)$, $R(n, t_n)$, viz., $R(n, t) = D(n, t)/\Delta(n, t)$, $R(n, t_n) = D(n, t_n)/\Delta(n, t_n)$.

Another way of expressing the rule for obtaining $\Delta(n, t)$ from $D(n, t)$ is the following:—$\Delta(n, t)$ is the determinant formed by the elements of $D(n, t)$ occurring in all the columns corresponding to arguments non-reduced in two or more variables, and all the rows corresponding to C_r $(r = 1, 2, \ldots, n-1)$ for multipliers non-reduced in one or more variables leaving x_r out of account.

*6*a*. A more general way of forming an expression for the resultant of $C_1, C_2, \ldots, C_n$ is the following:—Form the complete matrix

* This section was added in a revision of the paper, June 17th, 1902. It supplies a further proof of the theorem of § 6.

(having more rows than columns) corresponding to the function $C_1S_1+C_2S_2+\ldots+C_nS_n$ of order $t_n=1+\Sigma(m-1)$. Call he rows which make up $D(n, t_n)$, that is, the rows corresponding to

$$C_1S^{(0)}+C_2S^{(1)}+\ldots+C_nS^{(n-1)},$$

the primary rows, and the rest the supplementary rows. Select any $\Sigma\mu$ of the primary rows, of which μ_r correspond to C_r, where

$$\mu_r m_r=\Pi m \quad (r=1, 2, \ldots, n);$$

and add to them any supplementary rows, so as to form a determinant of the complete matrix which does not vanish identically. Then the resultant is the quotient of this determinant by the minor obtained from it by omitting the $\Sigma\mu$ rows, and the columns which contain the elements $a_1, a_2, \ldots, a_n$ in the $\Sigma\mu$ rows, where a_r is the coefficient of $x_r^{m_r}$ in C_r. Observe that there is one element $a_1, a_2, \ldots, a_n$ in each row and each column of the complete set of primary rows.

The theorem is also true for $R(n, t)$, when $\mu_1, \mu_2, \ldots, \mu_n$ are given their proper values; but for this case the proof given below requires amplification in one or two details.

Let $\rho_1, \rho_2, \ldots, \rho_n$ denote the numbers of the arguments of order t_n of the form $x_1^{m_1}\omega^{(0)}, x_2^{m_2}\omega^{(1)}, \ldots, x_n^{m_n}\omega^{(n-1)}$ respectively, so that $\Sigma\rho$ is the number of columns of the matrix, and $\rho_n=\mu_n$. Let $S_{(q)}^{(p)}$ denote a polynomial, and $\omega_{(q)}^{(p)}$ an argument, which is reduced in the first p and last q of the variables $x_1, x_2, \ldots, x_n$.

The complete set of supplementary rows forms a matrix of rank $\Sigma(\rho-\mu)$, which is the number of supplementary rows in any one of the determinants above; for the matrix corresponds to

$$C_2(x_1^{m_1}S)+C_3(x_1^{m_1}S+x_2^{m_2}S)+\ldots+C_n(x_1^{m_1}S+x_2^{m_2}S+\ldots+x_{n-1}^{m_{n-1}}S)$$

$$=x_1^{m_1}(C_2S_{(n-2)}+C_3S_{(n-3)}+\ldots+C_nS_{(0)})+x_2^{m_2}(C_3S_{(n-3)}+\ldots+C_nS_{(0)})+\ldots$$
$$\ldots+x_{n-1}^{m_{n-1}}(C_nS_{(0)})$$

$$=C_2(x_1^{m_1}S_{(n-2)}^{(0)})+C_3(x_1^{m_1}S_{(n-3)}^{(0)}+x_2^{m_2}S_{(n-3)}^{(1)})+\ldots$$
$$\ldots+C_n(x_1^{m_1}S_{(0)}^{(0)}+x_2^{m_2}S_{(0)}^{(1)}+\ldots+x_{n-1}^{m_{n-1}}S_{(0)}^{(n-2)})$$

$$=x_1^{m_1}(C_2S_{(n-2)}^{(0)}+C_3S_{(n-3)}^{(0)}+\ldots+C_nS_{(0)}^{(0)})$$
$$+x_2^{m_2}(C_3S_{(n-3)}^{(1)}+C_4S_{(n-4)}^{(1)}+\ldots+C_nS_{(0)}^{(1)})+\ldots+x_{n-1}^{m_{n-1}}(C_nS_{(0)}^{(n-2)}),$$

which contains $\Sigma(\rho-\mu)$ parameters only. The number of parameters, for example, in the second bracket of the last line equals the number of arguments in the function

$$x_3^{m_3} S^{(1)}_{(n-3)} + x_4^{m_4} S^{(1)}_{(n-4)} + \ldots + x_n^{m_n} S^{(1)}_{(0)}$$

of order $t_n - m_2$, that is, the total number of arguments $\omega^{(1)}$ of order $t_n - m_2$ less the number in $x_2^{m_2} S^{(1)}_{(n-2)} + S^{(n)}_{(0)}$, or $S^{(1)}_{(n-2)}$, which equals $\rho_2 - \mu_2$.

Hence, if two determinants are chosen having the same primary rows, but different supplementary rows, then the determinants in one set of supplementary rows will be proportional to those in the other (§ 4), and the two original determinants will be in the same proportion. The theorem is therefore true for any set of supplementary rows, if it is true for one set.

The proof of the remaining part of the theorem will be sufficiently indicated by taking $n = 5$. Consider the determinant D arranged in columns and rows as in the following diagram:—

	ρ_1 columns $x_1^{m_1}\omega$	ρ_2 columns $x_2^{m_2}\omega^{(1)}$	ρ_3 columns $x_3^{m_3}\omega^{(2)}$	ρ_4 columns $x_4^{m_4}\omega^{(3)}$	$\rho_5 (= \mu_5)$ cols. $x_5^{m_5}\omega^{(4)}$
μ_1 rows ωC_1					
μ_2 rows $\omega^{(1)} C_2$		D_1			D_1
μ_3 rows $\omega^{(2)} C_3$			D_2		D_2
μ_4 rows $\omega^{(3)} C_4$				D_3	D_3
$\mu_5 (= \rho_5)$ rows $\omega^{(4)} C_5$					D_4 D_4 D_4 D_4
$\rho_4 - \mu_4$ rows $x_4^{m_4}\omega^{(3)}_{(0)} C_5$				Δ_4 D_3	D_3
$\rho_3 - \mu_3$ rows $x_3^{m_3}(\omega^{(2)}_{(1)} C_4, \omega^{(2)}_{(0)} C_5)$			Δ_3 D_2		D_2
$\rho_2 - \mu_2$ rows $x_2^{m_2}(\omega^{(1)}_{(2)} C_3, \omega^{(1)}_{(1)} C_4, \omega^{(1)}_{(0)} C_5)$		Δ_2 D_1			D_1
$\rho_1 - \mu_1$ rows $x_1^{m_1}(\omega^{(0)}_{(3)} C_2, \omega^{(0)}_{(2)} C_3, \omega^{(0)}_{(1)} C_4, \omega^{(0)}_{(0)} C_5)$	Δ_1				

The $\Sigma\mu$ primary rows correspond to $C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_5 S^{(4)}$ where $S^{(r-1)}$ $(r = 1, 2, 3, 4, 5)$ is an incomplete polynomial containing any μ_r arguments $\omega^{(r-1)}$. The $\Sigma(\rho-\mu)$ supplementary rows are chosen in a particular way, each of the arguments $\omega^{(p)}_{(q)}$ in the diagram being given all the values of which it is capable. The elements to the right of the dotted line are all zeros. D_r $(r = 1, 2, 3, 4)$ is the determinant marked in the diagram by writing D_r at the four corners. Δ_r $(r = 1, 2, 3, 4)$ is the determinant marked in the diagram on the supposition that the μ_r columns containing the element a_r in the primary rows are omitted. Let R be the resultant of $C_1, C_2, \ldots, C_5$, and R_r $(r = 1, 2, 3, 4)$ the resultant of $C_{r+1}, \ldots, C_5$ when $x_1, \ldots, x_r$ are made zero, so that $R_4 = a_5$. Then D_r is divisible by $R_r^{m_1 \ldots m_r}$. We shall prove this for the case $r = 2$, by the method of §§ 4, 5.

Let $C_3 S_3^{(2)} + C_4 S_4^{(2)} + C_5 S_5^{(2)}$ be the function whose matrix has D_2 for a determinant, $S_3^{(2)}$, $S_4^{(2)}$, $S_5^{(2)}$ being the sums of the multipliers of C_3, C_4, C_5 in D_2 affected with arbitrary coefficients λ. We shall be able to compare D_2 with a standard determinant of a similar type by bringing $C_3 S_3^{(2)} + C_4 S_4^{(2)} + C_5 S_5^{(2)}$ to a standard form, viz., the function $C_3 S_{(2)}^{(2)} + C_4 S_{(1)}^{(2)} + C_5 S_{(0)}^{(2)}$, with respect to the arguments to which the columns of D_2 correspond. These are the arguments of the type $x_3^{m_3}\omega^{(2)}$, $x_4^{m_4}\omega^{(3)}$, $x_5^{m_5}\omega^{(4)}$, or $\omega^{(2)}$, or $x_3^{m_3}\omega_{(2)}^{(2)}$, $x_4^{m_4}\omega_{(1)}^{(2)}$, $x_5^{m_5}\omega_{(0)}^{(2)}$, from which we see that the standard function contains just the necessary number of parameters. Now the following identities, regarding the functions on the left hand as the unknowns,

$$(1) \quad S_{(2)}^{(2)} + C_4 S_{(1)}^{\prime(2)} + C_5 S_{(0)}^{\prime(2)} + (x_1^{m_1} S_1 + x_2^{m_2} S_2^{(1)}) = S_3^{(2)},$$

$$(2) \quad S_{(1)}^{(2)} + C_5 S_{(0)}^{\prime\prime(2)} \qquad + (x_1^{m_1} S_1' + x_2^{m_2} S_1^{\prime(1)}) = S_4^{(2)} + C_3 S_{(1)}^{\prime(2)},$$

$$(3) \quad S_{(0)}^{(2)} \qquad + (x_1^{m_1} S_1'' + x_2^{m_2} S_2^{\prime\prime(1)}) = S_5^{(2)} + C_3 S_{(0)}^{\prime(2)} + C_4 S_{(0)}^{\prime\prime(2)},$$

can be satisfied in succession uniquely, and on multiplying by C_3, C_4, C_5, and adding, it is seen that the required transformation has been effected.

From (1), (2), (3) we see that the parameters λ' of $S_{(2)}^{(2)}$, $S_{(1)}^{(2)}$, $S_{(0)}^{(2)}$, when expressed in terms of the arbitrary parameters λ of $S_3^{(2)}$, $S_4^{(2)}$, $S_5^{(2)}$, contain the coefficients of C_3 only in the numerator. Hence the determinant D_2 contains all the factors of the standard determinant corresponding to $C_3 S_{(2)}^{(2)} + C_4 S_{(1)}^{(2)} + C_5 S_{(0)}^{(2)}$ which involve the coefficients of C_3 (§ 4). Now the standard determinant, which comes from equating to zero all terms in $C_3 S_{(2)}^{(2)} + C_4 S_{(1)}^{(2)} + C_5 S_{(0)}^{(2)}$ containing

$x_1^{a_1}x_2^{a_2}$ ($a_1 = 0, 1, \ldots, m_1-1$; $a_2=0, 1, \ldots, m_2-1$), breaks up into m_1m_2 determinant factors of the type $D(3, t)$, each of which is divisible by R_2. The determinant D_2 is therefore divisible by $R_2^{m_1m_2}$; and similarly D_r is divisible by $R_r^{m_1\ldots m_r}$.

Again, D_2 and $R_2^{m_1m_2}$ are each of degree μ_3 in the coefficients of C_3, while the coefficient of $a_3^{\mu_3}$ in D_2 is $\Delta_3 D_3$, and in $R_2^{m_1m_2}$ is $R_3^{m_1m_2m_3}$ (§ 5); hence

$$\frac{D_2}{R_2^{m_1m_2}} = \frac{\Delta_3 D_3}{R_3^{m_1m_2m_3}}.$$

From this, and similar results, we have

$$\frac{D}{R} = \Delta_1 \frac{D_1}{R_1^{m_1}} = \Delta_1\Delta_2 \frac{D_2}{R_2^{m_1m_2}} = \Delta_1\Delta_2\Delta_3 \frac{D_3}{R_3^{m_1m_2m_3}} = \Delta_1\Delta_2\Delta_3\Delta_4 \frac{D_4}{R_4^{m_1m_2m_3m_4}}$$
$$= \Delta_1\Delta_2\Delta_3\Delta_4;$$

therefore $$R = D/\Delta_1\Delta_2\Delta_3\Delta_4.$$

This proves the theorem for the case $n = 5$, and in a similar way it follows generally.

7. The problem of dividing out the extraneous factor $\Delta(n, t_n)$ from $D(n, t_n)$, so as to bring the resultant $R(n, t_n)$ to an integral form, appears to be a more difficult one than that of merely finding the extraneous factor. Any series of operations for finding the integral form of $R(n, t_n)$ would probably be very long and complicated.

A slight reduction in the magnitude of the extraneous factor is obtained as follows. Taking $m_1 \leqslant m_2 \ldots \leqslant m_n$, the coefficient a_1 of $x_1^{m_1}$ in C_1 raised to the power $(m_2+\ldots+m_{n-1})_{n-1}/(n-1)!$ divides out of both $D(n, t_n)$ and $\Delta(n, t_n)$ at sight. This is not, however, the whole power of a_1 that divides $D(n, t_n)$ unless $m_1, m_2, \ldots, m_n$ are all equal. Among the multipliers of C_1 we may omit all those divisible by $x_1^{m_n}$. This will result in a diminution of $(m_2+\ldots+m_{n-1})_{n-1}/(n-1)!$ in the number of rows of $D(n, t_n)$, and the same diminution in the number of columns, viz., the columns corresponding to all arguments divisible by $x_1^{m_1+m_n}$. The extraneous factor in the reduced determinant D' is a minor of D'. To obtain it we omit all the rows and columns in D' which had to be omitted in $D(n, t_n)$ and which appear in D'. Those not appearing in D' are the rows corresponding to C_1 for multipliers $x_1^{m_n}\omega$, and the columns corresponding to arguments $x_1^{m_1+m_n}\omega$, where ω is any argument of order $t_n-m_1-m_n$ reduced in $x_2, x_3, \ldots, x_n$. We must then omit some other rows and columns of D' in place of those which have disappeared. They may be chosen in several ways. We may, for example, choose the rows corresponding to C_1 for multipliers $x_1^{m_n-m_2}x_2^{m_2}\omega'$, where ω' is the value that ω takes when x_1 and x_2 are interchanged; for it can be proved that each of these rows simply supplies a factor a_1 to the minor $\Delta(n, t_n)$ of $D(n, t_n)$. Since ω is not divisible by $x_2^{m_2}$, ω' is not divisible by $x_1^{m_2}$, and $x_1^{m_n-m_2}x_2^{m_2}\omega'$ is not divisible by $x_1^{m_n}$, and so is a multiplier for C_1 in D'. The extra columns to be omitted from D' are those corresponding to the arguments $x_1^{m_1+m_n-m_2}x_2^{m_2}\omega'$.

A greater reduction in the magnitude of the extraneous factor is suggested by a method of Sylvester's.* Taking $m_1 \leqslant m_2 \ldots \leqslant m_n$ and

$$(a_1-1)+(a_2-1)+\ldots+(a_n-1) = m_1-1,$$

then any argument in C_1 or C_2 .., or C_n is of higher order than $x_1^{a_1-1}x_2^{a_2-1}\ldots x_n^{a_n-1}$, and therefore divisible by $x_1^{a_1}$ or $x_2^{a_2}\ldots$ or $x_n^{a_n}$; so that we can write

$$C_r = x_1^{a_1}A_r+x_2^{a_2}B_r+\ldots+x_n^{a_n}K_r \quad (r = 1, 2, \ldots, n),$$

where $A_r, \ldots, K_r$ are polynomials. The number of solutions of the equation $\Sigma(a-1) = m_1-1$ in positive integral values of $a_1, \ldots, a_n$, excluding zeros, is equal to the number of arguments of order m_1-1; which is also the number of reduced arguments of order t_n-m_1, as may be seen by dividing any argument of order m_1-1 into $x_1^{m_1-1}x_2^{m_2-1}\ldots x_n^{m_n-1}$. There are therefore the same number of polynomials $\Sigma \pm ABC \ldots K$ of order t_n-m_1 as of reduced arguments of order t_n-m_1, taking only one set of polynomials $A, B, \ldots, K$ for each solution of the equation $\Sigma(a-1) = m_1-1$. The determinant D' corresponding to these polynomials and the function

$$C_1S^{(0)}+C_2S^{(1)}+\ldots+C_nS^{(n-1)},$$

of order t_n-m_1, will have the resultant $R(n, t_n)$ as a factor. The proof that D' does not vanish identically, provided that only one polynomial $\Sigma \pm AB \ldots K$ is chosen for each solution of

$$\Sigma(a-1) = m_1-1,$$

is somewhat complicated, and we omit it. It is clear that D' is divisible by the common factor of the determinants of the matrix

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* Salmon's *Higher Algebra*, 1885, p. 86, and *Camb. and Dub. Math. Jour.*, Vol. VII., 1852, p. 68.

corresponding to $C_1S^{(0)}+\ldots+C_nS^{(n-1)}$; hence (end of §4) D' is divisible by $\Delta(n, t_n-m_1)$. Also the quotient is of the same degree as $R(n, t_n)$ in the coefficients of $C_1, C_2, \ldots, C_n$, and is therefore identical with $R(n, t_n)$. Thus the extraneous factor in D' is $\Delta(n, t_n-m_1)$, which is obtainable from D' by the same rule (end of §6) as $\Delta(n, t_n)$ from $D(n, t_n)$.

This method has the greatest effect in reducing the extraneous factor when $m_1, m_2, \ldots, m_n$ are all equal. When $n=3$ and $m_1=m_2=m_3$, it gets rid of the extraneous factor altogether; and when $n=3$ and $m_1 \leqslant m_2 \leqslant m_3$, it reduces the extraneous factor to a_1 raised to the power

$$\tfrac{1}{2}(m_2-m_1)(m_2-m_1-1)+\tfrac{1}{2}(m_3-m_1)(m_3-m_1-1),$$

of which a_1 to the power $\frac{1}{2}(m_2-m_1)(m_2-m_1-1)$ can be divided out, leaving a_1 to the power $\frac{1}{2}(m_3-m_1)(m_3-m_1-1)$ as an extraneous factor which does not divide out at sight.

8. We add a further list of formulæ without entering into details of proof. The formulæ of the present article are proved by methods that have been already employed. In §§ 9, 10 some indications are given as to how the results are obtained.

(1) $$D(n, t) = D(n-1, t)\, D(n, t-1), \text{ when } t > t_n,$$

(2) $$D(n, t) = \prod_{l=1}^{l=n} D(l, t_l)^{p_l}, \text{ when } t > t_n,$$

where

$$p_l = -\begin{vmatrix} \binom{t_n-t_{l+1}}{1} & 1 & 0 & \ldots\ldots & 0 \\ \binom{t_n-t_{l+2}}{2} & \binom{t_n-t_{l+2}}{1} & 1 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \binom{t_n-t_{n-1}}{n-l-1} & \binom{t_n-t_{n-1}}{n-l-2} & \binom{t_n-t_{n-1}}{n-l-3} & \ldots\ldots & 1 \\ \binom{t_n-t}{n-l} & \binom{t_n-t}{n-l-1} & \binom{t_n-t}{n-l-2} & \ldots\ldots & \binom{t_n-t}{1} \end{vmatrix},$$

$\binom{t}{r}$ denoting the coefficient of x^r in $(1+x)^t$; t_n-t in the last row of the determinant is a negative number.

(3) $$R(n, t) = R(n, t_n), \text{ when } t > t_n.$$

$R(n, t_n)$ is a prime, *i.e.*, non-factorisable (Netto, *l.c.*, p. 5). It is also probable that $R(n, t)$ is a prime for all values of t when $m_1, m_2, \ldots, m_n$ are all equal. When $m_1, m_2, \ldots, m_n$ are not all equal it is probable that $R(n, t)$ has only one factor containing the coefficients of all the C's. In this case, if m_n is not less than any other m, the factor is the numerator of

$$R(n, t) \Big/ \prod_{p=0}^{p=m_n-1} R(n-1, t-p)$$

when reduced to its lowest terms.

If m_1 is less than any other m, then the coefficient of $x_1^{m_1}$ in C_1 is a factor of $R(n, t)$ when

$$t = m_1 + (m_2 - m_1 - 1) + \ldots + (m_n - m_1 - 1) = t_n - (n-1)\, m_1\,;$$

for $C_1 \times x_2^{m_2-m_1-1} \ldots x_n^{m_n-m_1-1}$ is of the form $S^{(n)}$ when this coefficient of C_1 vanishes.

If $m_1, \ldots, m_r$ are equal and less than any other m, then the determinant of the coefficients of the highest powers of $x_1, x_2, \ldots, x_r$ in $C_1, C_2, \ldots, C_r$ is a factor of $R(n, t)$ when

$$t = m_1 + (m_{r+1} - m_1 - 1) + \ldots + (m_n - m_1 - 1) = t_n - (n-1)\, m_1 + (r-1)\,;$$

for $(\lambda_1 C_1 + \ldots + \lambda_r C_r)\, x_{r+1}^{m_{r+1}-m_1-1} \ldots x_n^{m_n-m_1-1}$ can be made of the form $S^{(n)}$ when this determinant vanishes.

Let $D(l, t)^{(n)}$ denote the determinant whose vanishing is the condition that $C_1^{(n)} S^{(0)} + C_2^{(n)} S^{(1)} + \ldots + C_l^{(n)} S^{(l-1)} + S^{(l)}$ can be made identically zero; so that $D(n, t)^{(n)}$ is the same as $D(n, t)$. Then $(n > l)$

(4) $$D(l, t)^{(n)} = D(l, t)^{(n-1)}\, D(l, t-1)^{(n)}$$
$$= \prod_{p=0}^{p=t-1} D(l, t-p)^{p+n-l-1!/p!\,n-l-1!}.$$

Thus $D(l, t)^{(n)}$ is independent of the coefficients of all terms in $C_1, C_2, \ldots, C_l$ which contain any variable other than $x_1, x_2, \ldots, x_l$.

Let $R(l, t)^{(n)}$ be the H.C.F. of the $l!$ determinants similar to $D(l, t)^{(n)}$ for the $l!$ different permutations of $C_1, C_2, \ldots, C_l$. Then $(n > l)$

(5) $$R(l, t)^{(n)} = R(l, t)^{(n-1)}\, R(l, t-1)^{(n)}$$
$$= \prod_{p=0}^{p=t-1} R(l, t-p)^{(n-1)}$$
$$= \prod_{p=0}^{p=t-1} R(l, t-p)^{p+n-l-1!/p!\,n-l-1!}.$$

By induction we have $(n \geqslant l)$

$$(6) \qquad \frac{D\,(l,\,t)^{(n)}}{R\,(l,\,t)^{(n)}} = \frac{D\,(l-1,\,t)^{(n)}}{R\,(l-1,\,t)^{(n)}} R\,(l-1,\,t-m_l)^{(n)},$$

the case for $n = l$ having been already proved (§ 5).

From (6) we have

$$(7) \quad D\,(n,\,t) = R\,(n,\,t)\,R\,(n-1,\,t-m_n)^{(n)}\,R\,(n-2,\,t-m_{n-1})^{(n)} \ldots$$
$$\ldots\, R\,(1,\,t-m_2)^{(n)},$$

from which $D\,(n,\,t)$ is expressed in prime factors when the m's are all equal, by (5) and (3).

The number of different prime factors involved in all the expressions $D\,(l,\,t)^{(n)}$ is $1+t_1+t_2+\ldots+t_{n-1}$ when $m_1,\,m_2,\,\ldots,\,m_n$ are in ascending order of magnitude; in other cases it is equal to or less than this number.

9. From identities of the type $\omega C_l = \omega C_l$ of order t, by writing C_l in full on the left hand, and solving for the arguments, we obtain

$$(8) \quad D\,(n,\,t)\,\omega_q = D_{q1}\omega_1 + \ldots + D_{qr}\omega_r + C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)},$$

where $\omega_1,\,\ldots,\,\omega_r$ are the arguments of order t reduced in all the variables, and $\omega_q\ (q > r)$ is any non-reduced argument. D_{qp} is a contraction for $D\,(n,\,t)_{qp}$, and may be regarded as being obtained from $D\,(n,\,t)$ by replacing the column corresponding to ω_q by the column corresponding to ω_p out of the same matrix, and then altering the sign. Identity (8) is written

$$(9) \qquad D\,(n,\,t)\,\omega_q \equiv \sum_{p=1}^{p=r} D_{qp}\omega_p \quad (\text{mod } C_1,\,C_2,\,\ldots,\,C_n).$$

Dividing out the common factor $D\,(n,\,t)\,/\,R\,(n,\,t)$, we have

$$(10) \qquad R\,(n,\,t)\,\omega_q \equiv \sum_{p=1}^{p=r} R_{qp}\omega_p \quad (\text{mod } C_1,\,C_2,\,\ldots,\,C_n),$$

where $R_{q1},\,R_{q2},\,\ldots,\,R_{qr}$ are integral functions of the coefficients. Formulæ (9), (10) are unique, since there is only one expression for ω_q of the kind given.

In a similar way we have

$$(11) \qquad D\,(l,\,t)^{(n)}\,\omega_q \equiv \sum_{p=1}^{p=r} D_{qp}\omega_p \quad (\text{mod } C_1,\,C_2,\,\ldots,\,C_l),$$

where $\omega_1,\,\omega_2,\,\ldots,\,\omega_r$ are *all* the arguments of order t reduced in $x_1,\,x_2,\,\ldots,\,x_l$, and ω_q is any non-reduced argument involving *only* $x_1,\,x_2,\,\ldots,\,x_l$. This equation may be multiplied by any argument involving only $x_{t+1}, \ldots, x_n$, leaving the R's unchanged, but increasing the order of the arguments.

10. NOTATION.—We proceed to give some formulæ involving the roots of $C_1, C_2, \ldots, C_{n-1}$. For this purpose it is convenient to take $x_n = 1$, and regard $C_1, C_2, \ldots, C_{n-1}$ as non-homogeneous polynomials in $x_1, x_2, \ldots, x_{n-1}$. We can reinsert x_n whenever we please. Let μ be the total number of arguments $\omega_1, \omega_2, \ldots, \omega_\mu$ of order $\leqslant t$ (or of order t when x_n is reinserted), r the number of arguments reduced (in $x_1, x_2, \ldots, x_{n-1}$) of order $\leqslant t$, viz., $\omega_1, \ldots, \omega_r$ in ascending order (so that $\omega_1 = 1$), r' the number of arguments reduced of order $\leqslant t-m_n$, viz., $\omega_1, \omega_2, \ldots, \omega_{r'}$, and ρ $(= m_1 \ldots m_{n-1})$ the number of vanishing points common to $C_1, C_2, \ldots, C_{n-1}$. When $t < t_{n-1}-1$ then $r < \rho$, and when $t \geqslant t_{n-1}-1$ then $r = \rho$. Select the ρ points in any order, calling them the 1st, 2nd, ..., ρ-th points. Let C_{ni} denote the value of C_n, and ω_{pi} the value of ω_p, at the i-th point ($p = 1, 2, \ldots, \mu$; $i = 1, 2, \ldots, \rho$).

The matrix corresponding to $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$, where $S_1, \ldots, S_{n-1}$ are complete (or incomplete) polynomials of orders $t-m_1, t-m_2$, &c., is of *rank* $\mu - r$, *i.e.*, any sub-determinant of the matrix containing more than $\mu - r$ rows and columns is identically zero. For the identical vanishing of $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$ requires μ equations of which only $\mu - r$ are independent, since $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$ can be brought to a form $C_1 S^{(0)} + \ldots + C_{n-1} S^{(n-2)}$ involving only $\mu - r$ parameters.

Similarly, the matrix corresponding to $C_1 S_1 + \ldots + C_n S_n$ is of rank $\mu - r + r'$; $r - r'$ is the number of arguments of order t reduced in all the variables when x_n is reinserted.

The matrix of the roots, viz.,

$$\begin{vmatrix} \omega_{11}, & \omega_{21}, & \ldots, & \omega_{\mu 1} \\ \omega_{12}, & \omega_{22}, & \ldots, & \omega_{\mu 2} \\ \ldots & \ldots & \ldots & \ldots \\ \omega_{1\rho}, & \omega_{2\rho}, & \ldots, & \omega_{\mu\rho} \end{vmatrix}$$

is of rank r when $r < \rho$, or $t < t_{n-1}-1$; for $S^{(0)}, S^{(1)}, \ldots, S^{(n-2)}$ can be so chosen that $C_1 S^{(0)} + \ldots + C_{n-1} S^{(n-2)}$ reduces to a polynomial containing only $r+1$ arguments arbitrarily assigned, and this polynomial vanishes at any $r+1$ of the ρ points.

Select $\mu - r$ rows of the matrix corresponding to $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$ so as to form a matrix of the coefficients the determinants of which do not all vanish identically. These $\mu - r$ rows cannot be chosen arbitrarily. Also select (arbitrarily) r rows of the matrix of the roots, viz., the first r rows; for these correspond to any r of the ρ points. The two resulting matrices are "corresponding," *i.e.*, the sum of the products of corresponding elements of any row in the first and any row in the second vanishes. Hence (neglecting sign) the ratio of any determinant of the first to the corresponding or complementary determinant of the second is constant.* This is expressed in the form

$$(12) \qquad \frac{D(\omega_{p_1}, \omega_{p_2}, \ldots, \omega_{p_r})}{\Sigma \pm \omega_{p_1 1} \omega_{p_2 2} \ldots \omega_{p_r r}} = \frac{D(\omega_{q_1}, \omega_{q_2}, \ldots, \omega_{q_r})}{\Sigma \pm \omega_{q_1 1} \omega_{q_2 2} \ldots \omega_{q_r r}}, \quad \ldots$$

where $D(\omega_{p_1}, \omega_{p_2}, \ldots, \omega_{p_r})$ is the determinant obtained from the matrix of the coefficients containing $\mu - r$ rows by omitting the columns corresponding to $\omega_{p_1}, \ldots, \omega_{p_r}$.

NOTATION.—Let Ω stand for the determinant $\Sigma \pm \omega_{11} \omega_{22} \ldots \omega_{\rho\rho}$ (the i-th row $\omega_{1i}, \ldots, \omega_{\rho i}$ corresponding to the i-th point), where $\omega_1, \ldots, \omega_\rho$ are all the arguments reduced (in $x_1, x_2, \ldots, x_{n-1}$), and $\omega_1 = 1$, $\omega_\rho = x_1^{m_1 - 1} \ldots x_{n-1}^{m_{n-1} - 1}$; also let M_i be the co-factor of $\omega_{\rho i}$ in Ω, J the Jacobian of $C_1, C_2, \ldots, C_{n-1}$, and J_i the value of J at the i-th of the ρ points. Then $\sum_{i=1}^{i=\rho} \frac{\omega_{pi}}{J_i} = 0$ when ω_p is of less order than J, viz., $t_{n-1} - 1$ (*Jacobi's theorem*†). Hence it follows that

$$(13) \qquad M_1 J_1 = \ldots = M_\rho J_\rho = \frac{\Omega}{\Sigma \frac{\omega_{\rho i}}{J_i}} = \frac{\Omega R(n-1, t_{n-1})}{R(n-1, t_{n-1} - 1)},$$

omitting a numerical factor in the right-hand expression.

Dividing each element of the i-th row in Ω by J_i ($i = 1, 2, \ldots, \rho$), and multiplying by Ω, the product $\Omega^2/\Pi J$ breaks up into t_{n-1} determinant factors in the diagonal [*cf.* p. 4, footnote (4)]. These can be evaluated and the p-th, counting from either end, is found to be

$$\frac{R(n-1, t_{n-1} - p) R(n-1, p-1)}{R(n-1, t_{n-1})},$$

* Gordan-Kerschensteiner, *Vorlesungen über Invariantentheorie*, Bd. I., 1885, pp. 95 and 110. The result also follows by cross-multiplication.

† Jacobi proves the theorem for two equations, *Crelle, Jour. f. Math.*, Vol. XIV., 1835, p. 281, and states it to be true for three equations, Vol. XV., p. 306; also *Werke*, 1881, &c., Vol. III., pp. 285, 352. Clebsch proves the general theorem, *Crelle, Jour. f. Math.*, Vol. LXIII., 1864, p. 224, and also Laurent, "L'Elimination," *Scientia, Phys.-Math.*, No. 7, 1900, p. 38.

except for a numerical factor. Hence

$$(14) \qquad \frac{\Omega^2}{\Pi J} = \frac{[R(n-1,\ t_{n-1}-1)^{(n)}]^2}{R(n-1,\ t_{n-1})^{t_{n-1}}},$$

omitting a numerical factor.

In a similar way a more general result may be obtained. Let any r' of the ρ points be chosen, where r' is the number of reduced arguments of order $\leqslant t'$. The number of the remaining r'' points is equal to the number of reduced arguments of order $\leqslant t''$, where $t'+t''=t_{n-1}-2$. Let Ω', Ω'' be the values of the determinant $\Sigma \pm \omega_{11}\omega_{22}\ldots\omega_{rr}$ when t is taken equal to t' and t'', and the points selected are the r' and r'' points respectively, and J', J'' the products of the J's for the r' and r'' points. Then, omitting a numerical factor,

$$(15) \qquad \frac{\Omega'}{R(n-1,\ t')^{(n)}} \bigg/ \left[\frac{J'}{R(n-1,\ t_{n-1})^{t'}}\right]^{\frac{1}{2}} = \frac{\Omega''}{R(n-1,\ t'')^{(n)}} \bigg/ \left[\frac{J''}{R(n-1,\ t_{n-1})^{t''}}\right]^{\frac{1}{2}}.$$

This includes (13) and (14).

If $D(n,\ t)$ is multiplied by $\Sigma \pm \omega_{11}\omega_{22}\ldots\omega_{r'r'}$, the product is found to be equal to $D(n-1,\ t)^{(n)}$ multiplied by the determinant

$$\begin{vmatrix} \omega_{11}C_{n1} & \ldots & \omega_{r'1}C_{n1} & \omega_{r'+11} & \ldots & \omega_{r1} \\ \omega_{12}C_{n2} & \ldots & \omega_{r'2}C_{n2} & \omega_{r'+12} & \ldots & \omega_{r2} \\ \ldots & & \ldots & \ldots & & \ldots \\ \omega_{1r}C_{nr} & \ldots & \omega_{r'r}C_{nr} & \omega_{r'+1r} & \ldots & \omega_{rr} \end{vmatrix}.$$

Hence

$$(16) \qquad \frac{D(n,\ t)}{D(n-1,\ t)^{(n)}} = \frac{R(n,\ t)}{\prod\limits_{p=0}^{p=m_n-1} R(n-1,\ t-p)} = \frac{\Sigma \pm \omega_{11}C_{n1}\ldots\omega_{r'r'}C_{nr'}\omega_{r'+1r'+1}\ldots\omega_{rr}}{\Sigma \pm \omega_{11}\omega_{22}\ldots\omega_{rr}}.$$

This may also be generalized. Let D be the matrix formed from any $\mu-r+r'$ rows of the matrix corresponding to $C_1S_1+\ldots+C_nS_n$, there being only r' rows containing the coefficients of C_n, viz., those obtained by multiplying C_n by $\omega_{p_1}, \omega_{p_2}, \ldots, \omega_{p_{r'}}$. Let $D(\omega_{p_{r'+1}}, \ldots, \omega_{p_r})$ be the determinant formed from D by omitting the columns corresponding to $\omega_{p_{r'+1}}, \ldots, \omega_{p_r}$, and D' the matrix formed from D by omitting the r' rows corresponding to C_n. Then

$$(17) \quad \frac{D(\omega_{p_{r'+1}}, \ldots, \omega_{p_r})}{D'(\omega_{q_1}, \ldots, \omega_{q_r})} = \frac{\Sigma \pm \omega_{p_1 1} C_{n1} \ldots \omega_{p_{r'} r'} C_{nr'} \omega_{p_{r'+1} r'+1} \ldots \omega_{p_r r}}{\Sigma \pm \omega_{q_1 1} \omega_{q_2 2} \ldots \omega_{q_r r}},$$

with the sufficient condition that neither numerator nor denominator on the left hand vanishes.

From (16) and (17) we have, when $t \geqslant t_n$, so that $r = r' = \rho$,

$$(18) \quad \frac{D(n, t)}{D(n-1, t)^{(n)}} = \frac{R(n, t_n)}{R(n-1, t_{n-1})^{m_n}} = \prod_{i=1}^{i=\rho} C_{ni},$$

and

$$\frac{D}{D'(\omega_{q_1}, \ldots, \omega_{q_\rho})} = \frac{\Sigma \pm \omega_{p_1 1} \ldots \omega_{p_\rho \rho}}{\Sigma \pm \omega_{q_1 1} \ldots \omega_{q_\rho \rho}} \prod_{i=1}^{i=\rho} C_{ni}.$$

[To express any integral symmetric function of the roots in terms of the coefficients the following method may be adopted. Let ω_p be any argument, of order t. Eliminate $x_1, x_2, \ldots, x_{n-1}$ from the $n-1$ given equations and the additional equation $C_n = \alpha - \omega_p = 0$, obtaining the determinant form $D(n, t_n) = 0$, where $m_n = t$, $a_n = \alpha$, $t_n = t_{n-1} + t - 1$. The multipliers of C_n in $D(n, t_n)$ are the reduced arguments ω_i $(i = 1, 2, \ldots, \rho)$. Expand $D(n, t_n)$ in powers of α (§ 5). The result to two terms is

$$\alpha^\rho D(n-1, t_n)^{(n)} - \alpha^{\rho-1} \sum_{i=1}^{i=\rho} D_{pi} + \ldots = 0,$$

where D_{pi} is a determinant of the matrix to which $D(n-1, t_n)^{(n)}$ belongs, obtained by replacing the column corresponding to $\omega_p \omega_i$ in $D(n-1, t_n)^{(n)}$ by the column corresponding to ω_i and changing the sign. D_{pi} is zero if there is no column $\omega_p \omega_i$ in $D(n-1, t_n)^{(n)}$, *i.e.*, if $\omega_p \omega_i$ is a reduced argument. The roots of the equation in α are the ρ values of ω_p; hence we have

$$(19) \quad D(n-1, t_n)^{(n)} \Sigma \omega_{pi} = \Sigma D_{pi}.$$

When the factor common to both sides of (19) is divided out, $R(n-1, t_{n-1})^t \Sigma \omega_{pi}$ is expressed as an integral function of the coefficients (§ 5). The other symmetric functions are expressible in terms of the functions $\Sigma \omega_{pi}$; *e.g.*, $\Sigma \omega_{pi} \omega_{qj} = \Sigma \omega_{pi} \times \Sigma \omega_{qi} - \Sigma (\omega_p \omega_q)_i$, and $R(n-1, t_{n-1})^t \Sigma \omega_{pi} \omega_{qj}$ is integral in the coefficients if t is the order of the higher of the two arguments ω_p, ω_q.—*October 8th*, 1902.]

On Groups in which every two Conjugate Operations are Permutable. By W. Burnside. Received and read May 8th, 1902.

In a paper published in the *Quarterly Journal of Mathematics* (1902), "On an Unsettled Question in the Theory of Discontinuous Groups," I have determined the order of a group with given generating operations when subject to the condition that the order of every operation shall be 3. If P and Q are any two operations of such a group, the relations

$$P^3 = 1, \quad Q^3 = 1, \quad (PQ)^3 = 1, \quad (PQ^2)^3 = 1$$

lead at once to $$P \,.\, QPQ^{-1} \,.\, Q^2PQ^{-2} = 1$$

and $$P \,.\, Q^2PQ^{-2} \,.\, QPQ^{-1} = 1;$$

so that P and QPQ^{-1} are permutable. The condition that every operation is of order 3 involves therefore that every two conjugate operations are permutable.

In the present paper I have considered the general problem thus presented; viz., the nature of a group generated by a finite number of operations when every two conjugate operations of the group are permutable. It will be seen that the general problem is closely connected with the more special one above referred to. When no further limitation is imposed on the operations, it is found that every operation of the group is given once and only once by a form

$$P^x Q^y \ldots R^z,$$

where P, Q, ..., R are a finite number of operations belonging to the group; and of the indices x, y, z, ... a certain number take all values from $-\infty$ to $+\infty$, while the remainder take the values 0, 1, 2.

The sufficient and necessary conditions that the group shall be of finite order are that the generating operations be of finite order. When this is the case, the group is the direct product of groups whose orders are powers of primes. In general for such a group the commutator of any two operations is a self-conjugate operation; but the case in which the order is a power of 3 is, as might be expected, exceptional.

1. In dealing with groups in which every two conjugate operations are permutable, the following notation will be used.

If $$P_a,\ P_b,\ P_c,\ \ldots$$

are any operations of such a group, the result of transforming P_a by P_b will be written $P_a P_{ab}$, so that

$$P_a^{-1} P_b^{-1} P_a P_b = P_{ab}.$$

Similarly, $$P_{ab}^{-1} P_c^{-1} P_{ab} P_c = P_{abc},$$

$$P_{abc}^{-1} P_d^{-1} P_{abc} P_d = P_{abcd},$$

and so on.

Further, the notation will be extended so that

$$P_{abc}^{-1} P_{de}^{-1} P_{abc} P_{de} = P_{(abc)(de)}.$$

The use of brackets in the suffixes will prevent any ambiguity; thus $P_{(abcd)e}$ is the same as P_{abcde}; but these are not necessarily the same as $P_{a(bcde)}$ or $P_{(ab)(cde)}$. From the definition of P_{ab} it follows that

$$P_{ba} = P_b^{-1} P_a^{-1} P_b P_a = P_{ab}^{-1}.$$

The operation P_{ab} may be regarded as the product of P_a^{-1} and $P_b^{-1} P_a P_b$; or as that of $P_a^{-1} P_b^{-1} P_a$ and P_b. Hence, since every operation is permutable with its conjugates, P_{ab} is permutable with both P_a and P_b.

Now, from $P_b^{-1} P_a P_b = P_a P_{ab}$ and $P_b^{-1} P_{ab} P_b = P_{ab}$,

it follows that $$P_b^{-y} P_a P_b^{y} = P_a P_{ab}^{y},$$

and $$P_b^{-y} P_a^{x} P_b^{y} = P_a^{x} P_{ab}^{xy},$$

or $$P_a^{-x} P_b^{-y} P_a^{x} P_b^{y} = P_{ab}^{xy}.$$

Since P_{ab} is the product of P_a^{-1} and $P_b^{-1} P_a P_b$, it follows that P_{abc}, P_{abcd}, ... are products of powers of operations which are conjugate to P_a. Similarly, P_{abcd} may be expressed as products of powers of operations which are conjugate to P_b (or to P_c, or to P_d). Hence

$$P_{abcda} = 1,$$

$$P_{abcdb} = 1,$$

&c.

Again, P_{abc} and P_{ad} can both be represented as products of powers of operations which are conjugate to P_a. They are therefore permutable, and

$$P_{(abe)(ad)} = 1.$$

Hence, generally, $$P_{(abe\ldots)(de\ldots)(gh\ldots)\ldots} = 1,$$

if in the multiple suffix any simple suffix occurs more than once.

Since P_{ab} is permutable with both P_a and P_b, every substitution of the sub-group generated by P_a and P_b can be represented in the form

$$P_a^x P_b^y P_{ab}^z.$$

In fact $$\begin{aligned} P_a^x P_b^y P_{ab}^z P_a^{x'} P_b^{y'} P_{ab}^{z'} &= P_a^{x+x'} P_a^{-x'} P_b^y P_a^{x'} P_b^{-y'} P_b^{y+y'} P_{ab}^{z+z'} \\ &= P_a^{x+x'} P_{ab}^{-x'y} P_b^{y+y'} P_{ab}^{z+z'} \\ &= P_a^{x+x'} P_b^{y+y'} P_{ab}^{z+z'-x'y}. \end{aligned}$$

Let P_c be any operation which does not belong to this sub-group. Then

$$P_a P_c P_a^{-1} = P_c P_{ca}^{-1},$$

$$P_b P_a P_c P_a^{-1} P_b^{-1} = P_c P_{cb}^{-1} P_{ca}^{-1} P_{cab},$$

$$P_a^{-1} P_b P_a P_c P_a^{-1} P_b^{-1} P_a = P_c P_{cb}^{-1} P_{cba}^{-1} P_{cab};$$

and therefore $$P_{ab}^{-1} P_c P_{ab} = P_c P_{cba}^{-1} P_{cab},$$

or $$P_{c(ab)} = P_{cba}^{-1} P_{cab}.$$

Now $$P_{c(ab)} = P_{abc}^{-1};$$

hence $$P_{abc} P_{cba}^{-1} P_{cab} = 1,$$

or, since $$P_{cba} = P_{bca}^{-1},$$

$$P_{abc} P_{bca} P_{cab} = 1. \qquad \text{(i)}$$

Again, since $P_a P_b$ and $P_c^{-1} P_a P_b P_c$ $(= P_a P_{ac} P_b P_{bc})$ are conjugate, they are permutable. Hence

$$\begin{aligned} P_a P_{ac} P_b P_{bc} &= P_b^{-1} P_a^{-1} (P_a P_{ac} P_b P_{bc}) P_a P_b \\ &= P_b^{-1} (P_a P_{ac} P_b P_{ba} P_{bc} P_{bca}) P_b \\ &= P_a P_{ab} P_{ac} P_{acb} P_b P_{ba} P_{bc} P_{bca}. \end{aligned}$$

But P_a, P_{ab}, P_{ac}, P_{acb} are all permutable, as also are P_b, P_{ba}, P_{bc}, P_{bca}.

Hence $$P_a P_{ac} P_b P_{bc} = P_a P_{ac} P_{acb} P_{ab} P_{ba} P_{bca} P_b P_{bc},$$

or $$P_{acb} P_{bca} = 1. \qquad \text{(ii)}$$

If P_c belongs to the sub-group generated by P_a and P_b, the relations

(i) and (ii) become identities. They are therefore true in any case, and for any permutation of the suffixes. Now, (ii) may be written

$$P_{bca} = P_{cab},$$

or

$$P_{acb} = P_{cba}.$$

Hence from (i) and (ii) together it follows that

$$P_{abc} = P_{bca} = P_{cab} = P^{-1}_{acb} = P^{-1}_{cba} = P^{-1}_{bac}; \tag{iii}$$

and

$$P^3_{abc} = 1. \tag{iv}$$

The relations (iii) are equivalent to the statement that when any permutation of the suffixes is effected in the symbol P_{abc} the operation represented is unaltered or changed into its inverse, according as the permutation is an even one or an odd one. Since

$$P_{ab\ldots def} = P_{(ab\ldots d)ef} = P^{-1}_{(ab\ldots d)fe} = P^{-1}_{ab\ldots dfe},$$

this statement may clearly be extended at once to any such symbol as $P_{ab\ldots def}$. Again,

$$P_{a(ij\ldots k)} = P^{-1}_{(ij\ldots k)a} = P^{-1}_{ij\ldots ka} = P^{(-1)^{r+1}}_{aij\ldots k},$$

where r is the number of suffixes in the set $i, j, \ldots, k$. Hence

$$P_{l\ldots m)(ij\ldots k)} = P^{(-1)^{r+1}}_{l\ldots mij\ldots k};$$

and thus any symbol $P_{(\ )(\ )\ldots}$ can be replaced by one in which there are no brackets in the suffix.

From the relation (iv) it follows that any symbol with three or more letters in its suffix is an operation of order 3, or else is the identical operation. Further, since

$$P^3_{abc} = 1$$

may be written

$$P^{-3}_{ab}P^{-1}_{c}P^3_{ab}P_c = 1,$$

the cube of every P with two letters in the suffix (*i.e.*, the cube of the commutator of any two operations) is a self-conjugate operation of the group. Again,

$$P^3_{abc} = 1$$

may be written

$$P^{-1}_{ab}P^{-3}_{c}P_{ab}P^3_c = 1.$$

Hence the cube of every operation of the group is permutable with every operation whose suffix contains two or more letters; *i.e.*, with every operation of the derived group.

2. Let $$P_1, P_2, \ldots, P_n$$

be n independent operations which generate a group G, and suppose that the only conditions to which they are subject are that every two conjugate operations of G are permutable.

The product of any two operations of the form

$$P_1^a P_2^b \ldots P_n^c P_{12}^d P_{13}^e \ldots P_{n-1,n}^f P_{123}^g \ldots P_{123\ldots n}^h, \tag{v}$$

where every P with a multiple suffix occurs once, while the P's are written in a definite sequence, is another operation of the same form. In fact, from the preceding paragraph,

$$P_{a_1a_2\ldots a_r}^{-y} P_{b_1b_2\ldots b_s}^{x} P_{a_1a_2\ldots a_r}^{y}$$
$$= P_{b_1b_2\ldots b_s}^{x} P_{b_1b_2\ldots b_s a_1a_2\ldots a_r}^{(-1)^{r+1}xy};$$

so that the multiplication can be actually carried out, and in the result the P's can be re-arranged in the original sequence. Hence with suitably chosen indices every operation of G can be represented in the form (v.).

To specify all distinct operations of the group it remains to show under what conditions a symbol of the form (v) represents the identical operation. As the basis of an induction it will be assumed that when there are $n-1$ generating operations the conditions are that (α) the index of each P with a single or double suffix is zero, and (β) the index of each P with a triple or higher suffix is zero or a multiple of 3.

If to the conditions defining G we add

$$P_1 = 1,$$

a new group is defined, which is simply isomorphic with G/H, where H is the self-conjugate sub-group of G generated by P_1 and its conjugate operations. The latter is an Abelian group, and cannot therefore be identical with G.

Now $$P_1 = 1$$

involves $$P_{12} = 1,\ P_{13} = 1,\ \ldots,$$
$$P_{123} = 1,\ \ldots.$$

Hence G/H is simply isomorphic with the group generated by $P_2, P_3, \ldots, P_n$; and this sub-group of G can therefore have no operation, except identity, in common with H. Suppose now that

$$P_1^a P_2^b \dots P_n^c P_{12}^d P_{13}^e P_{23}^f \dots P_{n-1,n}^g P_{123}^h \dots P_{234}^i \dots = 1.$$

By preceding processes the factors on the left may be rearranged so that all the P's containing 1 in the suffix come at the end, the indices of the remaining P's being unaltered. Hence

$$P_2^b \dots P_n^c P_{23}^f \dots P_{n-1,n}^g P_{234}^i \dots = P_1^a P_{12}^{d'} P_{13}^{e'} \dots P_{123}^{h'} \dots .$$

Now the operation on the right belongs to H, and that on the left to $\{P_2, P_3, \dots, P_n\}$. Hence each must be the identical operation, and therefore by the assumption made

$$b = \dots = c = f = \dots = g = 0,$$
$$i = \dots = 0, \text{ or a multiple of } 3.$$

Similar series of results follow by considering the self-conjugate sub-groups arising respectively from the suppositions

$$P_2 = 1, \text{ or } P_3 = 1, \dots .$$

Hence an operation of the form (v) can only represent the identical operation, in a group generated by n operations, when the indices of *all* the P's with less than n symbols in the suffix satisfy the assumed conditions. But when this is so the operation reduces to $P_{123\dots n}^m$. If m is neither zero nor a multiple of 3,

$$P_{123\dots n}^m = 1, \quad \text{and} \quad P_{123\dots n}^3 = 1$$

involve

$$P_{123\dots n} = 1,$$

which would constitute an additional limitation on the group, not contained in the original conditions. Hence m must be zero or a multiple of 3; and the induction is completed.

Finally, therefore, every operation of G is given once, and once only, by the form (v), if the indices of the P's with a single or double suffix take all values from $-\infty$ to $+\infty$, while the indices of the P's with a triple or higher suffix take all values from 0 to 2.

3. So far it has been supposed that the only limitation on the group considered is that every two conjugate operations are permutable. The group under these conditions necessarily contains operations whose order is not finite. It will still have this property under a variety of further limitations. For instance, the condition

$$P_{123} = 1$$

implies that every P with a multiple suffix in which 1, 2, and 3 occur is the identical operation. The generality of the group is thus reduced; but it still contains operations whose order is not finite.

From the form giving the operations of the group it is clear that, if the order of every operation of the group is finite, the group is one of finite order. Moreover, the necessary and sufficient conditions for this are that each one of the generating operations should be of finite order. That these conditions are necessary is clear from the form (v). To show that they are sufficient, it is only necessary to notice that, if P_1 is of finite order m, P_{12} is of finite order, equal to or a factor of m.

Suppose now that G is of finite order, and consider the operation $S^{-1}T^{-1}ST$ of G. As the product of S^{-1} and $T^{-1}ST$ its order must be equal to or a factor of that of S. Similarly its order is equal to or a factor of that of T. Hence, if the orders of S and T are relatively prime,

$$S^{-1}T^{-1}ST = 1,$$

or S and T are permutable. Since any two operations of G whose orders are relatively prime are permutable, G must be the direct product of a number of groups, the order of each of which is the power of a prime. In dealing with groups of finite order with the property considered, it is therefore sufficient to suppose the order to be p^a, where p is a prime.

The case $p = 3$ evidently stands by itself. Suppose first that p is not 3. Then P_{123} is an operation whose order is a power of p. But in any case it has been seen that

$$P_{123}^3 = 1.$$

Hence

$$P_{123} = 1,$$

or the commutator of any two operations of the group is a self-conjugate operation.

Every operation of the derived group is therefore a self-conjugate operation. That this condition is sufficient to ensure that in a group of order p^a every two conjugate operations are permutable is obvious.

If

$$P_1,\ P_2,\ \ldots,\ P_n,$$

of orders

$$p^{a_1},\ p^{a_2},\ \ldots,\ p^{a_n}$$

$$(a_1 \geqslant a_2 \geqslant a_3 \ldots \geqslant a_n),$$

are the generating operations of such a group, and if the only conditions beyond the given orders of the generating operations are that every two conjugate operations are permutable, the order of P_{rs} $(r<s)$ is p^{a_s}. Every operation of the group will then be given once by

$$\Pi P_r^{x_r} \cdot \Pi P_{rs}^{x_{rs}}$$

$$(r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n;\ r<s),$$

where x_r takes all values from 0 to $p^{a_r}-1$ and x_{rs} from 0 to $p^{a_s}-1$. Hence the order of the group is

$$p^{\Sigma r a_r}$$

When p is equal to 3, the only condition that P with a triple or higher suffix is subjected to is

$$P_{abc\ldots}^3 = 1,$$

and it is no longer the case that the commutator of any two operations is a self-conjugate operation. If the group is generated by

$$P_1,\ P_2,\ \ldots,\ P_n,$$

of orders $$3^{a_1},\ 3^{a_2},\ \ldots,\ 3^{a_n}$$

$$(a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n),$$

every operation of the group will be represented once in the form (v), where the index of P_r lies between 0 and $3^{a_r}-1$, the index of P_{rs} $(r<s)$ lies between 0 and $3^{a_s}-1$, and all the other indices lie between 0 and 2; both limits included. The order of the group is therefore

$$3^{\Sigma r a_r + 2^n - 1 - \frac{1}{2}n(n+1)}$$

In particular, if the order of each of the generating operations is 3, the order of the group is $$3^{2^n-1}.$$

This is the case already referred to in the introduction.

4. Groups of finite order p^a, of the kind here considered, possess two general properties in common with Abelian groups. First, the totality of the operations of the group whose orders are equal to or are less than p^r, where r is a given integer, constitute a sub-group. If P_1 and P_2 are any two operations of the group, of orders p^{a_1} and p^{a_2} $(a_1 \geqslant a_2)$, the order of P_{12} is equal to or is less than p^{a_2}, Now, by a repeated use of the formula

$$P_1^{-a} P_2^{-b} P_1^a P_2^b = P_{12}^{ab},$$

it follows that $$(P_1P_2)^x = P_1^x P_2^x P_{12}^{-\frac{1}{2}x(x-1)}.$$

Hence, if $$P_1^x = 1,\quad P_2^x = 1,\quad P_{12}^{\frac{1}{2}x(x-1)} = 1,$$

then $$(P_1P_2)^x = 1.$$

Now, if $p > 2$, the condition

$$P_{12}^{\frac{1}{2}x(x-1)} = 1$$

may be replaced by $$P_{12}^{x(x-1)} = 1.$$

It follows that, if $p>2$, the order of P_1P_2 is equal to or is less than the order of P_1. The sub-group generated by all the operations whose orders are equal to or less than p^r consists therefore of these operations and of no others; for the product of any two of these operations is an operation whose order does not exceed p^r.

The case $p = 2$ clearly forms an exception to the general theorem; for, if P_1, P_2 are two non-permutable operations of the group of order 2, P_{12} is an operation of order 2. But

$$P_{12} = (P_1P_2)^2,$$

and P_1P_2 is therefore an operation of order 4.

Secondly, the totality of the distinct operations which arise, when every operation of the group is raised to the power p^r, where r is a given integer, constitute a sub-group. Consider the case in which $p>3$, and the group is generated by two operations P_1 and P_2. The operations of the sub-group generated by P_1^p, P_2^p, and P_{12}^p are given by the form

$$P_1^{px} P_2^{py} P_{12}^{pz}.$$

The p-th power of any operation $P_1^a P_2^b P_{12}^c$ is given by the relation, obtained as above,

$$(P_1^a P_2^b P_{12}^c)^p = P_1^{pa} P_2^{pb} P_{12}^{pc-\frac{1}{2}abp(p-1)}.$$

Now $\frac{1}{2}(p-1)$ is an integer. Hence

$$P_1^{px} P_2^{py} P_{12}^{pz} = (P_1^x P_2^y P_{12}^{z+\frac{1}{2}(p-1)xy})^p;$$

i.e., every operation of $\{P_1^p, P_2^p, P_{12}^p\}$ is the p-th power of an operation of the group, and the p-th power of every operation of the group is contained in $\{P_1^p, P_2^p, P_{12}^p\}$. For more than two generating operations the proof will proceed on precisely similar lines. The theorem is therefore true when $r = 1$. But it must also be true for the resulting sub-group; so that it is true generally. The case $p = 2$ forms again an exception, since $\frac{1}{2}(p-1)$ is not then an integer. In fact, in this case $\{P_1^2, P_2^2, P_{12}^2\}$ will not contain P_{12}, which is the square of P_1P_2.

The case $p = 3$ requires separate treatment. It may be easily shown that in this case the cube of any operation of the group when expressed in the form (v) contains no P with a triple or higher suffix; and from this the truth of the theorem immediately follows.

Thursday, June 12th, 1902.

Dr. E. W. HOBSON, F.R.S., President, in the Chair.

Fifteen members present.

The President announced that the "De Morgan medal" for 1902 had been awarded to Prof. A. G. Greenhill.

Mr. A. C. Porter was admitted into the Society.

The following paper was communicated by Prof. Love:—

Prof. A. W. Conway: "The Principle of Huygens in a Uniaxal Crystal."

Lieut.-Col. A. Cunningham gave an account of "Some Investigations concerning the repetition of the Sum-Factor Operation."

The following papers were communicated from the Chair:—

Prof. E. Picard: "Sur un théorème fondamental dans la théorie des équations différentielles."

Mr. G. H. Hardy: "Some Arithmetical Theorems."

Prof. M. J. M. Hill: "On a Geometrical Proposition connected with the Continuation of Power Series."

Mr. J. H. Grace: "Types of Perpetuants."

The following presents were made to the Library:—

"Educational Times," June, 1902.

"Indian Engineering," Vol. xxxi., Nos. 16–20; 1902.

"Queen's College, Galway—Calendar for 1901–1902."

Penfield, S. L.—"The Stereographic Projection and its Possibilities from a Graphical Standpoint," 1901.

Penfield, S. L.—"On the Use of the Stereographic Projection for Geographical Maps and Sailing Charts," 1902.

"Periodico di Matematica," Serie 2, Vol. iv., Fasc. 6; Livorno, 1902.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 7; Livorno, 1902.

"L'Enseignement Mathématique," Année IV., No. 3; 1902.

"Mathematical Gazette," Vol. II., No. 33, 1902.

Guldberg, A.—"Ueber Integralinvarianten und Integralparameter bei Berührungs-Transformationsgruppen," 1902.

"Journal de l'Ecole Polytechnique—Hommage rendu à M. le Colonel Mannheim," 1902.

De Morgan, A.—"Theory of Probabilities" (extract from *Encyclopædia Metropolitana*). From Mr. R. Tucker.

Carvallo, E.—"L'Electricité" (*Scientia*, No. 19).

"Mathematical Questions and Solutions from the *Educational Times*," New Series, Vol. I.; 1902.

D. Ocagne, M.—"Sur quelques Travaux récents relatifs à la Nomographie."

The following exchanges were received :—

Académie Royale de Belgique :—

"Annuaire, 1902," Bruxelles.

"Bulletin de la Classe des Sciences," Nos. 1-3; Bruxelles, 1901-1902.

"Mémoires Couronnés," Bruxelles, 1901-1902.

"Mémoires," Tome LIV., Fasc. 1-4; Bruxelles, 1900-1901.

"Mémoires Couronnés et Mémoires des Savants Etrangers," Tome LIX., Fasc. 1, 2; Bruxelles, 1901.

"Proceedings of the Royal Society," Vol. LXX., Nos. 459, 460; 1902.

"Reports to the Evolution Committee of the Royal Society"; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXVI., No. 5; Leipzig, 1902.

"Bulletin de la Société Mathématique de France," Tome XXX., Fasc. 1; Paris, 1902.

"Bulletin of the American Mathematical Society," Vol. VIII., No. 8; "Transactions," Vol. III., No. 2; New York, 1902.

"Bulletin des Sciences Mathématiques," Tome XXV., "Contents," 1901; Tome XXVI., Mars, 1902; Paris.

"Annali di Matematica," Tomo VII., Fasc. 2, 3; Milano, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. XI., Fasc. 8, 9, 10; Roma, 1902.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 1-22; 1902.