tion group, and

$$S = (y_1 y_2 y_3 y_0 y_4 y_5 y_7), \quad T = (y_2 y_3 y_4)(y_5 y_0 y_7), \quad U = (y_1 y_4 y_3 y_2)(y_5 y_7),$$
$$V = (y_2 y_7 y_5 y_4)(y_3 y_0).$$

Hence in the transformed group g' is the sub-group which leaves one symbol y_1 unchanged, while g permutes the symbols in the two transitive sets y_1, y_2, y_3, y_4 and y_5, y_6, y_7. The given linear substitution therefore transforms the transitive representation of G in respect of g into that in respect of g'.

Linear Null Systems of Binary Forms. *By* J. H. Grace.

Read November 14th, 1901. Received November 18th, 1901.

As an exercise on Hilbert's paper "Ueber vollen Invariantensysteme," *Math. Ann.*, Vol. XLI., I propose to investigate the necessary and sufficient condition that all the combinants of three binary forms which are pure invariants should vanish. The method used applies equally well to any number of binary forms.

Suppose the three forms are

$$f = a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \ldots + a_n x_2^n = a_x^n,$$
$$\phi = b_0 x_1^n + n b_1 x_1^{n-1} x_2 + \ldots + b_n x_2^n = b_x^n,$$
$$\psi = c_0 x_1^n + n c_1 x_1^{n-1} x_2 + \ldots + c_n x_2^n = c_x^n;$$

then the combinants in question are such mutual invariants of f, ϕ, ψ as remain unaltered when any of the forms is replaced by a linear combination $lf + m\phi + n\psi$.

The combinants are well known to be rational integral functions of the determinants of the type

$$\begin{vmatrix} a_\alpha & a_\beta & a_\gamma \\ b_\alpha & b_\beta & b_\gamma \\ c_\alpha & c_\beta & c_\gamma \end{vmatrix},$$

whether they involve the variables or not. We shall denote the above determinant by $p_{\alpha\beta\gamma}$.

2. The most important combinant is the covariant

$$J = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2}, & \frac{\partial^2 f}{\partial x_1 \partial x_2}, & \frac{\partial^2 f}{\partial x_2^2} \\ \frac{\partial^2 \phi}{\partial x_1^2}, & \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, & \frac{\partial^2 \phi}{\partial x_2^2} \\ \frac{\partial^2 \psi}{\partial x_1^2}, & \frac{\partial^2 \psi}{\partial x_1 \partial x_2}, & \frac{\partial^2 \psi}{\partial x_2^2} \end{vmatrix}$$

or

$$(bc)(ca)(ab)\, a_x^{n-2} b_x^{n-2} c_x^{n-2}.$$

It is of order $3n-6$, the coefficient of x_1^{3n-6} is p_{012}, and generally the coefficient of $x_1^{3n-6-\rho} x_2^{\rho}$ is a linear combination of such p's as satisfy the condition

$$\alpha+\beta+\gamma = \rho+3.$$

Now every invariant of J is manifestly a combinant of f, ϕ, ψ, and hence a necessary condition is that all the invariants of J should vanish. But Hilbert has shown that, if all the invariants of a binary form of order m vanish, then the form has a factor of multiplicity equal to the integer next greater than $\frac{m}{2}$; hence in our case J has a linear factor of multiplicity greater than $\frac{1}{2}(3n-6)$. Without loss of generality we may suppose this factor to be x_2.

3. Now consider how many times a factor occurs in J when its mode of occurrence in f, ϕ, ψ is given. For perfect generality suppose the factor (x_2) occurs ν times in every form of the system

$$lf+m\phi+n\psi;$$

then there will be two linearly independent forms, say f_1 and f_2, each containing the factor more than ν times—say μ times. Finally, there will be a single form included in

$$l_1 f_1 + l_2 f_2,$$

containing the given factor more than μ times—say λ times. Hence, given the factor, we can choose three forms included in the system—say f', ϕ', ψ'—such that f' contains the factor λ times, ϕ' contains the factor μ times, and ψ' contains the factor ν times $(\lambda>\mu>\nu)$; and, of course, the integers λ, μ, ν are quite determined by the linear factor under consideration.

We shall clearly not lose generality (since the combinants are the same for any three members of the system) if we suppose f, ϕ, ψ to be the same as f', ϕ', ψ' respectively, and further, since we are dealing with invariant properties, we may suppose the given factor to be x_2.

Hence, referring to the definitions of f, ϕ, ψ, we have

$$\left.\begin{array}{ll} c_0,\ c_1,\ \ldots,\ c_{\nu-1} = 0, & c_\nu \neq 0 \\ b_0,\ b_1,\ \ldots,\ b_{\mu-1} = 0, & b_\mu \neq 0 \\ a_0,\ a_1,\ \ldots,\ a_{\lambda-1} = 0, & a_\lambda \neq 0 \end{array}\right\} \quad (\lambda > \mu > \nu).$$

It follows easily that, if $\alpha+\beta+\gamma < \lambda+\mu+\nu$, then

$$p_{\alpha\beta\gamma} = 0.$$

For $(\alpha-\lambda)+(\beta-\mu)+(\gamma-\nu) < 0$, and we may suppose that $\alpha > \beta > \gamma$; therefore either $\alpha < \lambda$ or $\beta < \mu$ or $\gamma < \nu$.

If $\alpha < \lambda$, then $\beta < \lambda$ and $\gamma < \lambda$; therefore

$$a_\alpha = a_\beta = a_\gamma = 0;$$

therefore $$p_{\alpha\beta\gamma} = 0.$$

If $\beta < \mu$, then $\gamma < \mu$, $\beta < \lambda$, $\gamma < \lambda$; therefore

$$a_\beta = a_\gamma = b_\beta = b_\gamma = 0;$$

therefore $$p_{\alpha\beta\gamma} = 0.$$

If $\gamma < \nu$, then $\gamma < \mu$, $\gamma < \lambda$; therefore

$$a_\gamma = b_\gamma = c_\gamma = 0;$$

therefore $$p_{\alpha\beta\gamma} = 0.$$

Again, if $$\alpha+\beta+\gamma = \lambda+\mu+\nu,$$

then $$p_{\alpha\beta\gamma} = 0,$$

unless $$\alpha = \lambda,\quad \beta = \mu,\quad \gamma = \nu,$$

for otherwise $\alpha < \lambda$ or $\beta < \mu$ or $\gamma < \nu$.

As regards $p_{\lambda\mu\nu}$, it is equal to $a_\lambda a_\mu a_\nu$, and is therefore not zero.

But in the expression for J the coefficient of $x_1^{3n-6-\rho} x_2^\rho$ is a linear combination of p's for which

$$\alpha+\beta+\gamma = \rho+3,$$

so that, if $\rho+3 < \lambda+\mu+\nu$, the coefficient of $x_1^{3n-6-\rho} x_2^\rho$ is zero, but, if

$$\rho+3 = \lambda+\mu+\nu,$$

it is certainly not zero; that is, the factor x_2 occurs precisely $(\lambda+\mu+\nu-3)$ times in J.

4. We have seen, however, that the factor x_2 must occur more than $\frac{1}{2}(3n-6)$ times in J, and accordingly we have

$$\lambda+\mu+\nu-3 > \tfrac{1}{2}(3n-6);$$

therefore $$\lambda+\mu+\nu > \frac{3n}{2}.$$

Now $$p_{\alpha\beta\gamma} = 0,$$

if $$\alpha+\beta+\gamma < \lambda+\mu+\nu;$$

therefore when $$p_{\alpha\beta\gamma} \neq 0$$

we must have $$\alpha+\beta+\gamma > \lambda+\mu+\nu > \frac{3n}{2}.$$

5. The weight of a product of powers of a's, b's, and c's is the sum of the suffixes of the various letters in the product. If the weight of an invariant of partial degrees i_1, i_2, i_3 of three forms of orders n_1, n_2, n_3 be w, then we have

$$2w = n_1 i_1 + n_2 i_2 + n_3 i_3.$$

In our case $$n_1 = n_2 = n_3, \quad i_1 = i_2 = i_3 = i,$$

where i is the degree of the invariant in the p's; for every p is linear in the coefficients of each form.

Hence for a combinant we must have

$$2w = 3ni.$$

Now the weight of $p_{\alpha\beta\gamma}$ is $\alpha+\beta+\gamma$, and for a non-vanishing p we must have

$$\alpha+\beta+\gamma > \frac{3n}{2},$$

and hence for a product of p's of degree i which does not vanish we must have

$$w > \frac{3ni}{2},$$

and, inasmuch as for terms of a combinant

$$w = \frac{3ni}{2},$$

it follows immediately that all the combinants vanish.

Hence the necessary and sufficient condition that all the combinants should vanish is that J should have a factor of multiplicity greater than $\frac{1}{2}(3n-6)$.

In accordance with the general reasoning of Hilbert, we infer that all combinants of three binary forms are integral algebraic functions of invariants of J, and therefore, *a fortiori*, of the coefficients of J.

The results for any number of binary forms are exactly the same.

Addition Theorems for Hyperelliptic Integrals. *By* A. L. DIXON.
Received and read November 14th, 1901.

The present communication is a continuation of my paper on "An Addition Theorem for Hyperelliptic Theta-Functions," presented to the Society in December, 1900 (*Proc. Lond. Math. Soc.*, Vol. XXXIII., No. 755).

The method there given of deducing theorems in the theory of hyperelliptic integrals from the geometrical properties of confocals is applied to the investigation of addition theorems for the integrals of the second and third kinds.*

I must record my obligation to a paper by Herr O. Staude, on the "Geometrische Deutung der Additionstheoreme der hyperelliptischen Integrale" (*Math. Ann.*, Bd. XXII., 1883). In particular the fundamental idea of § 4 has been taken from that paper.

References to my first paper are prefixed by the number I.

Integrals of the Second Kind.

1. Taking the equations (11), I., § 2, of the straight lines through the point h_ι, which lie in the surfaces S and T, one of them is given by

$$\frac{\xi_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \frac{\xi_q}{\sqrt{q-s\,.\,q-t\,.\,r-p}} = \frac{\xi_r}{\sqrt{r-s\,.\,r-t\,.\,p-q}},$$

$$\xi_s = 0, \quad \xi_t = 0.$$

Let S be the distance measured along this line from h_ι. Then

$$S = \sqrt{\Sigma\,(x_\iota - h_\iota)^2} = \sqrt{\xi_p^2 + \xi_q^2 + \xi_r^2}\,; \tag{1}$$

and therefore

$$\frac{\iota S}{\sqrt{q-r\,.\,r-p\,.\,p-q}} = \frac{\xi_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \dots \tag{2}$$

---

* A paper on the application of the method to confocal conicoids in ordinary space and the deduction of theorems for elliptic integrals has appeared in the *Quarterly Journal*, No. 131, 1902.