17. It may be noted that

$$\left.\begin{aligned} \angle A\alpha\beta &= \frac{\pi}{2} - B = C\gamma'\beta' \\ \angle B\beta\gamma &= \frac{\pi}{2} - C = A\alpha'\gamma' \\ \angle C\gamma\alpha &= \frac{\pi}{2} - A = B\beta'\alpha' \end{aligned}\right\}.$$

18. To construct the figure, let DEF be the pedal triangle; then its sides are Rp, Rq, Rr.

If DK, EL, FM are the perpendiculars of DEF, then

$$DK = Rqr, \quad EL = Rrp, \quad FM = Rpq. \tag{a}$$

Now
$$\lambda = 4R\Pi\,(\sin A)/\Sigma\,(qr),$$
$$= R\,(p+q+r)/\Sigma\,(qr),$$
$$= R\,\frac{DE+EF+FD}{DK+EL+FM}.$$

Hence the sides of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ (*i.e.*, $\lambda.Rp$, $\lambda.Rq$, $\lambda.Rr$) are known.

[I am indebted to a referee for a suggestion which enables me to considerably simplify the construction, viz.,

$$B\gamma' : \gamma\beta' : \beta'C = \operatorname{cosec} 2B : \operatorname{cosec} 2A : \operatorname{cosec} 2C,$$

i.e., by (a), $$= EL : FM : DK.]$$

On Quantitative Substitutional Analysis. *By* A. Young.

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From any function P of n variables may be obtained $n!$ functions, not necessarily all different, by permuting the variables in P in all possible ways; or, what is the same thing, by operating on P with each of the $n!$ substitutions of the symmetric group of the variables. It frequently happens that between these functions linear relations with constant coefficients exist; such may be written

$$(\lambda_1 + \lambda_2 s_2 + \lambda_3 s_3 + \ldots)\,P = 0,$$

$\lambda_1, \lambda_2, \ldots$ being numbers positive or negative, integral or fractional, and $s_2, s_3, \ldots$ substitutions belonging to the symmetric group of the variables. The words "substitutional relation" will be used to denote a relation such as that just written down; and the expression "substitutional equation" will be used for the same relation when P is an unknown function for which this relation is true.' The simplest form of such a relation is

$$(1-s)\,P = 0,$$

which merely implies that P is unaltered by the substitution s. This is dealt with in the theory of substitutions. The main object of the present paper is the discussion of single equations, such as that written down above, or of simultaneous systems of such equations, with a view to their solution; further, of the discussion of equations of the form

$$(\lambda_1 + \lambda_2 s_2 + \lambda_3 s_3 + \ldots)\,P = R,$$

where $\lambda_1, \lambda_2, \ldots, s_2, s_3, \ldots$ are defined as above, and R is a known function; these equations are also to be included in the term "substitutional equations." It will be seen, moreover, that the right-hand sides of such equations, when a single equation, or else a simultaneous system, is under consideration, are subject to restrictions, in that they have in general to satisfy certain substitutional relations.

The problem proposed is not a purely hypothetical one. In a paper on "The Irreducible Concomitants of any Number of Binary Quartics,"* I have shown that there is one type of concomitant to be discussed for each degree and order; and that such a type satisfies certain substitutional equations, the solution of which enables us to find how many concomitants of that type for a definite number of quartics are irreducible, and which these are. The equations were there discussed, and the irreducible system for any number of quartics was found. Thus, using the notation of that paper, the invariant type degree 6 may be written $(abcdef)$; it satisfies the equations

$$(abcdef) = (bcdefa) = (afedcb),$$

$$(abcdef) + (abcfde) + (abcefd) + (abcdfe) + (abcfed) + (abcedf) = R,$$

$$(abcdef) + (abfcde) + (abefcd) + (abcdfe) + (abecdf) + (abfecd) = R,$$

where R stands for certain reducible terms, with the form of which

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we are not concerned. The other equations satisfied by this type are a necessary consequence of the four written down.

Later, in a paper on "The Invariant Syzygies of Lowest Degree for any Number of Quartics,"* I proved that the substitutional equations satisfied by the quartic types gave all the syzygies between quartic concomitants; but here the form of the reducible terms on the right-hand sides of the equations had to be included in the discussion. The equations for invariant types up to and including degree 7 were discussed, with the result that no invariant syzygies existed of degree less than 7, and that the syzygies of degree 7 could all be obtained from one definite form. Incidentally, the method of discussing the equations with a view simply to finding the irreducible system was somewhat improved; and a theorem connected with substitutional analysis was proved, which has been generalized here, § 8.

The term "substitutional expression" is used to denote an expression of the form

$$\lambda_1+\lambda_2 s_2+\lambda_3 s_3+\ldots+\lambda_p s_p,$$

where λ_1, λ_2, ... are numerical constants (positive or negative) and s_2, s_3, ... are substitutions. It is shown, to start with, that the solution of substitutional equations, so far as rational integral algebraic functions are concerned, may be made to depend on the finding of substitutional expressions which satisfy the equations in virtue of the multiplication table of the group to which all the substitutions belong. The first seven paragraphs of this paper are concerned with substitutional equations; in § 9 some examples are given.

The second part of the paper has to do with two substitutional identities, one of which is proved in § 13, the other in § 15. By means of relations which are established between substitutional and polar operations on functions of a definite kind, from the first of these a proof of Gordon's series is obtained; from the second Capelli's extension of this series, a theorem due to Peano, and some corollaries concerning substitutional equations. An account of the paper which contains Capelli's theorem is also given, § 11, as there exists a fairly close connexion between the analysis of substitutional and polar operators. With this connexion § 12 has to do; it is somewhat further developed in that part of § 17 which has to do with Capelli's theorem.

For convenience, owing to the quantitative use of the symbols, the

* *Proc. Lond. Math. Soc.*, Vol. XXXII., p. 384.

substitutions next a function are regarded as operating on it before those further away, thus

$$s_1 s_2 P = s_1 (s_2 P).$$

To avoid confusion, as the symbol $(abc...)$ is used in two senses, viz., as a substitution and as a concomitant type of a quantic, Roman letters are used when it denotes a substitution, italics when it denotes a type. The usual symbols for a group are used in two senses: first, as a name for the group, and, secondly, to represent the sum of the substitutions of the group. The following notation is made use of:—

$\{s\}$ = the sum of the substitutions of the smallest group including s.

$\{s_1, s_2\}$ = the sum of the substitutions of the smallest group including s_1 and s_2.

$\{G_1, G_2\}$ = the sum of the substitutions of the smallest group having G_1 and G_2 for sub-groups.

$\{\mathrm{abc}...\}$ = the symmetric group of the letters $a, b, c, ...$.

$\{\mathrm{abc}...\}'$ = the sum of the substitutions of the alternating group of the letters $a, b, c, ...$, minus the substitutions of these letters which do not belong to the alternating group.

The expression $\{\mathrm{abc}...\}$ is sometimes referred to as "the positive symmetric group"; while $\{\mathrm{abc}...\}'$ is called "the negative symmetric group."

The paper has been rewritten and greatly enlarged at the request of the referees; my thanks are due to them—particularly to Prof. Burnside—for many valuable criticisms and suggestions.

1. Consider any rational integral algebraic function P of n variables $a_1, a_2, ..., a_n$; its terms may be arranged in sets $P_1, P_2, ..., P_m$, such that each set contains all those terms of P, and only those, which are obtainable from some particular term by means of substitutions and of positive or negative numerical factors. And P may be written

$$P = P_1 + P_2 + ... + P_m.$$

Now, consider any set P_1; let $A_1 a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n}$ be any term of this set, A_1 being a positive or negative numerical coefficient; then

$$P_1 = (A_1 + A_2 s_2 + A_3 s_3 + ...)\, a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n},$$

where $A_1, A_2, ...$ are numerical, and $s_2, s_3, ...$ substitutions belonging to the symmetric group of the n variables. The effects of substitutions on P_1, and consequently all substitutional properties of P_1, depend partly on the substitutional operator $(A_1+A_2s_2+A_3s_3+\ldots)$, partly on the substitutional properties of the term $a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n}$. If in this term all the indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ are different, we obtain by operating on it with the $n!$ substitutions of the symmetric group $\{a_1a_2\ldots a_n\}$ of the variables $n!$ different terms which are connected by no linear relations with constant coefficients. In this case, then, $a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n}$ has no substitutional properties, and all the substitutional properties of P_1 are a consequence of the operator

$$(A_1+A_2s_2+A_3s_3+\ldots).$$

Suppose next that $\alpha_1=\alpha_2=\ldots=\alpha_r=\alpha$, and that $\alpha, \alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_n$ are all different. The substitutional properties of the term

$$a_1^{\alpha}a_2^{\alpha}\ldots a_r^{\alpha}a_{r+1}^{\alpha_{r+1}}a_{r+2}^{\alpha_{r+2}}\ldots a_n^{\alpha_n}$$

consist solely of the fact that this term belongs to the group $\{a_1a_2\ldots a_r\}$. For there result, by operating on it with the $n!$ substitutions of the group $\{a_1a_2\ldots a_n\}$, $\frac{n!}{r!}$ different terms between which no linear relations with constant coefficients can exist. The substitutional properties of this term are then identical with those of

$$\{a_1a_2\ldots a_r\}\, a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n},$$

where all the indices of the a's in $a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n}$ are different. Hence all the substitutional properties of P_1 are, in this case, a consequence of the operator when we write, as may be done,

$$P_1=\frac{1}{r!}\left[(A_1+A_2s_2+A_3s_3+\ldots)\{a_1a_2\ldots a_r\}\right]a_1^{\alpha}a_2^{\alpha}\ldots a_r^{\alpha}a_{r+1}^{\alpha_{r+1}}a_{r+2}^{\alpha_{r+2}}\ldots a_n^{\alpha_n}$$

$$=(B_1+B_2s_2+B_3s_3+\ldots)\,a_1^{\alpha}a_2^{\alpha}\ldots a_r^{\alpha}a_{r+1}^{\alpha_{r+1}}a_{r+2}^{\alpha_{r+2}}\ldots a_n^{\alpha_n},$$

where

$$\frac{1}{r!}(A_1+A_2s_2+A_3s_3+\ldots)\{a_1a_2\ldots a_r\}=(B_1+B_2s_2+B_3s_3+\ldots),$$

the B's being constants. In exactly the same way, whatever be the equalities amongst the indices in the term $a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n}$, a substitutional operator $(B_1+B_2s_2+B_3s_3+\ldots)$ may be obtained, such that

$$P_1=(B_1+B_2s_2+B_3s_3+\ldots)\,a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n},$$

all the substitutional properties of P_1 being a consequence of the operator alone.

Now, owing to the way in which the sets have been chosen, no substitution can change a term of one set into a term of a different set; and there can exist no substitutional relation between different sets. Hence, if P satisfy any substitutional equation

$$(\lambda_1+\lambda_2 s_2+\lambda_3 s_3+\ldots) P = 0,$$

where $\lambda_1, \lambda_2, \ldots$ are constants, each set must independently satisfy this equation. And hence each set possesses all the substitutional properties of P.

Theorem.—Every rational integral algebraic function P of n variables may be written in the form $P = \sum_{i=1}^{i=m} P_i$, where P_i possesses all the substitutional properties of P, and possibly others as well. And P_i may be expressed in the form

$$P_i = (A_1^{(i)}+A_2^{(i)} s_2+A_3^{(i)} s_3+\ldots) F_i,$$

where $A_1^{(i)}, A_2^{(i)}, \ldots$ are positive or negative numerical coefficients, $s_2, s_3, \ldots$ are substitutions belonging to the symmetric group of the n variables, and F_i is a rational integral algebraic function of the variables. Further, the substitutional operator

$$(A_1^{(i)}+A_2^{(i)} s_2+A_3^{(i)} s_3+\ldots)$$

is such that all the substitutional properties of P_i are a direct consequence of it.

For example, take the form

$$P = \tfrac{1}{2}a_2-\tfrac{1}{2}a_3+3a_1^2 a_2-\tfrac{1}{5}a_2^2 a_3-3a_1^2 a_3+\tfrac{1}{5}a_2 a_3^2;$$

then
$$\begin{aligned} P_1 &= \tfrac{1}{2}a_2-\tfrac{1}{2}a_3 = \tfrac{1}{2}\{\mathfrak{a}_2\mathfrak{a}_3\}' a_2 = \tfrac{1}{4}\{\mathfrak{a}_2\mathfrak{a}_3\}'\{\mathfrak{a}_1\mathfrak{a}_3\} a_2 \\ &= \tfrac{1}{4}\left[1-(\mathfrak{a}_2\mathfrak{a}_3)+(\mathfrak{a}_1\mathfrak{a}_3)-(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3)\right] a_2, \\ P_2 &= 3a_1^2 a_2-\tfrac{1}{5}a_2^2 a_3-3a_1^2 a_3+\tfrac{1}{5}a_2 a_3^2 \\ &= \left[3-\tfrac{1}{5}(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3)-3(\mathfrak{a}_2\mathfrak{a}_3)+\tfrac{1}{5}(\mathfrak{a}_1\mathfrak{a}_3)\right] a_1^2 a_2, \\ P &= P_1+P_2. \end{aligned}$$

Here P, P_1, P_2 all satisfy the equation

$$\{\mathfrak{a}_2\mathfrak{a}_3\} P = 0;$$

also P_1 satisfies the equation

$$\{a_1 a_2 a_3\}' P_1 = 0.$$

Again, if the substitutional properties of P are completely summed up by saying that P belongs to the group G of order ρ, it is sufficient and more convenient to write

$$P = \frac{1}{\rho} GP,$$

this being, as it is easy to verify, the necessary and sufficient condition that P may belong to the group G of order ρ.

Corollary. — Every rational integral algebraic solution P of a single equation

$$(\lambda_1 + \lambda_2 s_2 + \lambda_3 s_3 + \dots) P = 0,$$

where $\lambda_1, \lambda_2, \dots$ are constants, and $s_2, s_3, \dots$ substitutions belonging to the symmetric group of the variables, of which P is supposed to be a function, or of a simultaneous system of such equations, may be obtained in the form

$$P = \sum_i (A_1^{(i)} + A_2^{(i)} s_2 + A_3^{(i)} s_3 + \dots) F_i;$$

where $A_1^{(i)}, A^{(i)}, \dots$ are constants, and F_i is a rational integral algebraic function of the variables, the substitutional operator of each term being such that

$$(\lambda_1 + \lambda_2 s_2 + \lambda_3 s_3 + \dots)(A_1^{(i)} + A_2^{(i)} s_2 + A_3^{(i)} s_3 + \dots) \equiv 0,$$

in virtue of the multiplication table of the group.

For P may be written in the form

$$P = \sum_i P_i,$$

where
$$P_i = (A_1^{(i)} + A_2^{(i)} s_2 + \dots) F_i,$$

all the substitutional properties of P_i being consequences of the operator, and, further, where P_i possesses all the substitutional properties of P, and hence is a solution of the equation, or system of equations, of which we are supposing P to be a solution. But, since every substitutional property of P_i is a consequence of the operator $(A_1^{(i)} + A_2^{(i)} s_2 + \dots)$, it follows that

$$(\lambda_1 + \lambda_2 s_2 + \dots)(A_1^{(i)} + A_2^{(i)} s_2 + \dots) \equiv 0.$$

2. The applications of our theory at present required are entirely to functions rational integral algebraic in the variables. Consequently,

we may restrict ourselves to the discussion of such functions, and will throughout this paper tacitly assume that the functions considered are of this nature. Nevertheless, should the theorem of the preceding article be true for any kind of function—as seems to me probable—no restrictions as to the nature of the functions considered would be necessary.

In consequence of the corollary just proved, it follows that in order to obtain the solutions of a system of equations of the form

$$(\lambda_1+\lambda_2 s_2+\lambda_3 s_3+\ldots)\, P=0$$

it is only necessary to discover the substitutional expressions

$$(A_1+A_2 s_2+A_3 s_3\ldots)$$

which are such that

$$(\lambda_1+\lambda_2 s_2+\lambda_3 s_3+\ldots)(A_1+A_2 s_2+A_3 s_3+\ldots)\equiv 0,$$

in virtue of the multiplication table of the group. The solution is then a matter of relations between substitutional operators only. We may then proceed thus: Take the sum of all the substitutions of the group concerned with arbitrary coefficients; for brevity we write this S. Then expand the various expressions

$$(\lambda_1+\lambda_2 s_2+\ldots)\, S$$

obtained by substituting S for P in the various equations of the simultaneous system, and in the results equate the coefficient of each substitution to zero. A system of simultaneous linear equations is thus obtained for the arbitrary constants in S. As a rule, all the arbitrary constants cannot be definitely determined; but the result of solving these linear equations and substituting their values in S will be expressible in the form

$$\sum_{j=1}^{j=m} C_j S_j,$$

where C_j is an arbitrary constant and S_j is a substitutional expression containing no arbitrary constant, which is such that the result of substituting S_j for P in each of the substitutional equations is zero, in virtue of the multiplication table of the group. Every solution may then be written in the form

$$P=\sum_i\,[\sum_j C_{j,i} S_j]\, F_i=\sum_j S_j\,[\sum_i C_{j,i} F_i]=\sum_j S_j\Phi_j,$$

where $C_{j,i}$ is a definite constant

$$\Phi_j=\sum_i C_{j,i} F_i,$$

and F_i and P are functions of the nature under discussion.

An expression in terms of which every solution can be expressed, such as $\Sigma_j S\ \Phi_j$, we call the complete solution of the system of equations. It will be seen later on that this is not always unique.

It is well to remark that it is not necessary to take S equal to the sum of all the substitutions of the symmetric group of the variables with arbitrary coefficients. It is sufficient that S should contain all the substitutions of the smallest group G which contains all those substitutions which actually occur in the expressions of our equations. For, if $G = 1+s_2+\ldots+s_\rho$, it is well known that it is possible to obtain a table

$$\begin{array}{ccccc} 1, & s_2, & s_3, & \ldots, & s_\rho, \\ \sigma_2, & s_2\sigma_2, & s_3\sigma_2, & \ldots, & s_\rho\sigma_2, \\ \ldots & \ldots & \ldots & & \ldots \\ \sigma_{\rho'}, & s_2\sigma_{\rho'}, & s_3\sigma_{\rho'}, & \ldots, & s_\rho\sigma_{\rho'}, \end{array}$$

such that every substitution of the symmetric group is contained once, and only once, in the table; and, further, that the result of multiplying on the left-hand side any substitution in this table by one of the substitutions in G changes it to another substitution in the same horizontal line. Hence, if S be the sum of all the substitutions of the symmetric group with arbitrary coefficients, the substitutional equations only give relations between the constants in the same horizontal line, and the relations for the various lines are the same.

As an example, consider the equation

$$\{(\text{abcd})\}\,P = 0$$

$$S = A_1 + A_2\,(\text{abcd}) + A_3\,(\text{ac})(\text{bd}) + A_4\,(\text{adcb}).$$

Equating the coefficients in $\{(\text{abcd})\}\,S$ to zero, we obtain

$$A_1 + A_2 + A_3 + A_4 = 0.$$

Hence

$$\begin{aligned} S &= -A_2 - A_3 - A_4 + A_2\,(\text{abcd}) + A_3\,(\text{abcd})^2 + A_4\,(\text{abcd})^3 \\ &= \big[1-(\text{abcd})\big]\big[-A_2 - A_3\{1+(\text{abcd})\} - A_4\{1+(\text{abcd})+(\text{abcd})^2\}\big]. \end{aligned}$$

And the complete solution is

$$[1-(\text{abcd})]\,F.$$

3. Consider now a single equation, or a system of equations, of the form

$$[\lambda_1 + \lambda_2 s_2 + \lambda_3 s_3 + \ldots]\,P = R,$$

where, as before, λ_1, λ_2, ... are constants, s_2, s_3, ... substitutions, and R

is a given rational integral algebraic function of the variables. It is, in the first place, to be noticed that the above equation in general implies a restriction on R, viz., that R can be written in the form $[\lambda_1+\lambda_2 s_2+\ldots]\,F$, and, as a consequence, satisfies certain substitutional equations. Thus, if $\lambda_1+\lambda_2 s_2+\ldots = G$ the sum of the substitutions of a group, R must belong to the group G. Let P_1 be any solution of the equations; then, if P_2 be another solution,

$$[\lambda_1+\lambda_2 s_2+\ldots]\,(P_1-P_2)=0.$$

Hence, as in linear differential equations, the work of solution may be divided into two parts. First, any particular solution P_1 is found; and then—what corresponds to the complementary function—the complete solution Q of the system

$$[\lambda_1+\lambda_2 s_2+\ldots]\,Q=0$$

The complete solution—that is, the solution in terms of which every other can be expressed—is then

$$P = P_1+Q.$$

It will be seen later on, in the applications made to the quadratic and quartic invariants, that, in general, R is subject to more conditions than that implied by

$$R = [\lambda_1+\lambda_2 s_2+\ldots]\,F$$

when a simultaneous system of such equations is under discussion.

4. It may happen that the only solution of an equation

$$[\lambda_1+\lambda_2 s_2+\ldots]\,P=0 \qquad \text{(I.)}$$

is $P=0$. Let G be the group of the substitutions which appear in this equation; then, if s be any substitution of G,

$$s\,[\lambda_1+\lambda_2 s_2+\ldots]\,P = 0.$$

Operating, then, on (I.) with each of the ρ substitutions of G, where ρ is the order of G, we obtain ρ linear equations with constant coefficients between the ρ quantities

$$P,\ s_2 P,\ \ldots$$

regarded as independent variables. The necessary and sufficient condition that there may be a solution other than zero is then expressed by the vanishing of a determinant of ρ columns and rows.

If $$\lambda_1+\lambda_2 s_2+\ldots = G = 1+s_2+s_3+\ldots+s_\rho,$$

the sum of the substitutions of a group G of order ρ, the complete solution of (I.) is of the form

$$P = \Sigma (A_1 + A_2 s_2 + \ldots + A_\rho s_\rho) F,$$

where $$G (A_1 + A_2 s_2 + \ldots + A_\rho s_\rho) \equiv 0.$$

This gives $$A_1 + A_2 + \ldots + A_\rho = 0.$$

Hence

$$A_1 + A_2 s_2 + \ldots + A_\rho s_\rho = A_2 (s_2 - 1) + A_3 (s_3 - 1) + \ldots + A_\rho (s_\rho - 1).$$

Now, let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be any substitutions of G which are not all contained in one of its sub-groups, and hence are sufficient to generate G. Then every substitution s of G can be expressed in the form

$$s = \sigma_{r_1}^{a_1} \sigma_{r_2}^{a_2} \ldots \sigma_{r_k}^{a_k},$$

where $r_1, r_2, \ldots, r_k$ are some of the numbers 1, 2, ..., m, not necessarily all different. But

$$s - 1 = \sigma_{r_1}^{a_1} s' - 1 = (\sigma_{r_1}^{a_1} - 1) s' + (s' - 1),$$

where $$s' = \sigma_{r_2}^{a_2} \ldots \sigma_{r_k}^{a_k},$$

and hence

$$s - 1 = (\sigma_{r_1}^{a_1} - 1) \sigma_{r_2}^{a_2} \ldots \sigma_{r_k}^{a_k} + (\sigma_{r_2}^{a_2} - 1) \sigma_{r_3}^{a_3} \ldots \sigma_{r_k}^{a_k} + \ldots + (\sigma_{r_k}^{a_k} - 1)$$
$$= (\sigma_1 - 1) S_1 + (\sigma_2 - 1) S_2 + \ldots + (\sigma_m - 1) S_m,$$

where $S_1, S_2, \ldots, S_m$ are substitutional expressions, some of which may be zero, or merely numerical.

Hence

$$A_1 + A_2 s_2 + \ldots + A_\rho s_\rho = (\sigma_1 - 1) T_1 + (\sigma_2 - 1) T_2 + \ldots + (\sigma_m - 1) T_m,$$

where $T_1, T_2, \ldots, T_m$ are substitutional expressions containing the arbitrary constants $A_2, A_3, \ldots, A_\rho$.

Moreover $$G (\sigma - 1) = 0\,;$$

hence the complete solution of the equation

$$GP = 0$$

may be written

$$P = (\sigma_1 - 1) F_1 + (\sigma_2 - 1) F_2 \ldots + (\sigma_m - 1) F_m,$$

$F_1, F_2, \ldots, F_m$ being arbitrary functions.

Similarly, the complete solution of the equation

$$GP = R,$$

R necessarily belonging to the group G, is

$$P = \frac{R}{\rho} + (\sigma_1 - 1) F_1 + (\sigma_2 - 1) F_2 + \dots + (\sigma_m - 1) F_m,$$

for

$$\frac{1}{\rho} GR = R,$$

and consequently $\frac{R}{\rho}$ is a particular solution.

If, for instance, $G = \{a_1 a_2 \dots a_n\}$,

any one of the three following expressions may be taken as the complete solution :—

$$\{a_1 a_2\}' F_1 + \{a_1 a_3\}' F_2 + \dots + \{a_1 a_n\}' F_n,$$

$$\{a_1 a_2\}' F_1 + \{a_2 a_3\}' F_2 + \dots + \{a_{n-1} a_n\}' F_n,$$

$$\{a_1 a_2\}' F_1 + [1 - (a_1 a_2 a_3 \dots a_n)] F_2 :$$

an illustration of the remark already made, that it would be found that the complete solution was not always unique.

It follows from the above that the solution of

$$\{G_1, G_2\} P = 0$$

may be written

$$P = P_1 + P_2,$$

where

$$G_1 P_1 = 0 \quad \text{and} \quad G_2 P_2 = 0.$$

For we may choose substitutions $\sigma_1, \sigma_2, \dots, \sigma_h$ which generate G_1, and substitutions $\sigma_{h+1}, \sigma_{h+2}, \dots, \sigma_m$ which generate G_2; these substitutions will then together generate $\{G_1, G_2\}$. The solution of

$$\{G_1, G_2\} P = 0$$

may then be written

$$P = (\sigma_1 - 1) F_1 + \dots + (\sigma_h - 1) F_h + (\sigma_{h+1} - 1) F_{h+1} + \dots + (\sigma_m - 1) F_m$$

$$= P_1 + P_2,$$

where

$$P_1 = (\sigma_1 - 1) F_1 + \dots + (\sigma_h - 1) F_h$$

and

$$P_2 = (\sigma_{h+1} - 1) F_{h+1} + \dots + (\sigma_m - 1) F_m,$$

and consequently $G_1 P_1 = 0$ and $G_2 P_2 = 0$.

5. When all the substitutions are powers of a single substitution the equations are easy to solve. Consider a single equation, the most general of its kind,

$$\phi(s) P = (A_0 + A_1 s + A_2 s^2 + \ldots + A_{n-1} s^{n-1}) P = 0,$$

where s is a substitution of order n.

We require to find the most general expression

$$\psi(s) = (B_0 + B_1 s + B_2 s^2 + \ldots + B_{n-1} s^{n-1}),$$

which is such that $\qquad \phi(s)\,\psi(s) \equiv 0.$

Now $\phi(x)\,\psi(x)$ only vanishes when $\phi(x) = 0$, or when $\psi(x) = 0$. Neither of these cases need be discussed here; then the product $\phi(s)\,\psi(s)$ must vanish solely in consequence of the equation

$$s^n = 1.$$

Hence $\qquad \phi(x)\,\psi(x) = (x^n - 1)\,\chi(x).$

To find ψ, we then obtain the H.C.F. of $x^n - 1$ and $\phi(x)$, say $\phi_1(x)$; then

$$\psi(x) = \frac{x^n - 1}{\phi_1(x)}.$$

Now, if a is not a root of $\quad x^n - 1 = 0,$

any function Q may be written in the form

$$Q = (s^{n-1} + as^{n-2} + \ldots + a^{n-1})\,F = \left[\frac{s^n - a^n}{s - a}\right] F = \left[\frac{1 - a^n}{s - a}\right] F,$$

for $\qquad Q = \dfrac{s^n - a^n}{1 - a^n}\,Q = (s^{n-1} + as^{n-2} + \ldots + a^{n-1})\left(\dfrac{s - a}{1 - a^n}\,Q\right).$

Hence, if $\qquad \phi(x) = \phi_1(x)(x - a_1)(x - a_2)\ldots(x - a_r),$

any solution P of the equation may be written

$$\begin{aligned} P = \psi(s)\,F &= \left[\frac{s^n - 1}{\phi_1(s)}\right] F \\ &= \left[\frac{s^n - 1}{\phi_1(s)}\right]\left[\frac{1}{(s - a_1)(s - a_2)\ldots(s - a_r)}\right] E \\ &= \left[\frac{s^n - 1}{\phi(s)}\right] E, \end{aligned}$$

where it is to be understood that the expression $\dfrac{1}{s - a_1}$ is equivalent to $\dfrac{s^{n-1} + as^{n-2} + \ldots + a^{n-1}}{1 - a_1^n}$, when a_1^n is not equal to unity.

It has been tacitly assumed that $\phi(s)$ has no squared factor which is also a factor of s^n-1; if such should occur, we may remove it by adding to $\phi(s)$ a multiple of s^n-1, which is in actual value zero, and then proceed as before. If $\phi(s)$ has no common factor with s^n-1, then $P=0$.

To find a particular solution of

$$\phi(s)\,P = R.$$

The restriction imposed on R by this equation is

$$\frac{s^n-1}{\phi_1(s)}\,R = 0.$$

Hence $$R = \phi_1(s)\,R'.$$

If there is no difficulty in finding R' from this, the particular solution

$$P = \frac{\phi_1(s)}{\phi(s)}\,R'$$

may be taken.

If the form of R' is not at once obvious, the particular solution may be found thus:—

$$\phi_1(s)\,\phi_1(s) = \phi_1(s)\,\phi_1(s) + \lambda\,(s^n-1) = \phi_1(s)\,\phi_2(s),$$

where s^n-1 and $\phi_2(s)$ have no common factor. Then

$$\phi_1(s)\,\phi(s)\,P = \phi_1(s)\,R = \phi_1(s)\left[\frac{\phi(s)}{\phi_1(s)}\,\phi_2(s)\right]P,$$

and $$\left[\frac{\phi_1(s)}{\phi(s)\,\phi_2(s)}\right]R$$

is a solution.

The extension to any set of simultaneous equations involving only powers of s is obvious.

Also it may be seen, in the same way, that the solution of any set of Abelian equations is a matter only of algebra.

Single equations which are not merely formed by the sum of the substitutions of a group, and in which the substitutions are not all contained in an Abelian group, may frequently be solved with the help of the solutions in these two cases. Thus, the solution of the

equation $$\{ab\}\,[1+(abcd)+(abcd)^2]\,P = R$$

—which occurs in the reduction of the quartic invariant types—is

$$P = \tfrac{1}{3}\,[1-2\,(abcd)+(abcd)^2+(abcd)^3]\left[\frac{R}{2}+\{ab\}'\,F\right],$$

and R must satisfy the equation

$$\{ab\}' R = 0.$$

6. Consider now two simultaneous equations

$$\{s\} P = 0, \quad \{\sigma\} P = 0.$$

Then, if

$$s\sigma \equiv \tau_1, \quad \sigma s \equiv \tau_2,$$

$$\sigma\tau_1^s = \tau_2^s\sigma, \quad \tau_1^s = s\tau_2^{s-1}\sigma.$$

Hence, if m be the order of τ_1,

$$\tau_2^m = \sigma\tau_1^m\sigma^{-1} = 1,$$

and the orders of τ_1 and τ_2 must be identical.

Also the expression $(1-\sigma)\{\tau_1\} = (s-1)\{\tau_2\}\sigma$.

Hence

$$P = (1-\sigma)\{\tau_1\} F$$

is a solution of the equations.

Unfortunately this is not always the complete solution, for suppose that

$$s\sigma = \sigma s, \quad s^2 = 1, \quad \sigma^3 = 1\,;$$

then the complete solution may be written

$$P = (1-s)(1-\sigma) F;$$

but the expression $(1-\sigma)\{\tau_1\}$ vanishes identically, for $\{\tau_1\}$ is here equal to $\{s, \sigma\}$.

Again, whenever the substitutions s, σ are permutable, the solution

$$P = (1-\sigma)\{\tau_1\} F,$$

in addition to satisfying the two equations

$$\{s\} P = 0, \quad \{\sigma\} P = 0,$$

belongs to the group $\{\tau_1\}$, which is not in general the case with the complete solution

$$P = (1-\sigma)(1-s) F.$$

However, whenever

$$s^2 = 1 = \sigma^2,$$

the complete solution may be written

$$P = (1-\sigma)\{\tau_1\} F,$$

for

$$\{s, \sigma\} = \{\sigma, \tau_1\} = \{\sigma\}\{\tau_1\},$$

since

$$\tau_1\sigma = s = \sigma\tau_1^{-1}.$$

Hence we may write $S = \Sigma (1+A_a\sigma) B_a\tau_1^a$,

and find S, so that $\{s\} S = 0$ and $\{\sigma\} S = 0$.

The second equation gives $A_a = -1$.

$$\text{Hence}\quad S = (1-\sigma)(B_0+B_1\tau_1+\ldots+B_{m-1}\tau_1^{m-1})$$
$$= \left[s\,(B_0\tau_2^{m-1}+B_1+B_2\tau_2+\ldots+B_{m-1}\tau_2^{m-2}) - (B_0+B_1\tau_2+\ldots+B_{m-1}\tau_2^{m-1})\right]\sigma.$$

The equation $\{s\} S = 0$

then gives $B_0 = B_1 = \ldots = B_{m-1}$.

Hence the complete solution is as stated.

A solution of any number of equations

$$\{s_1\} P = 0,\ \{s_2\} P = 0,\ \ldots,\ \{s_n\} P = 0$$

may then be seen to be

$$P = (1-s_1)\{s_2s_1,\ s_3s_1,\ \ldots,\ s_ns_1\} F.$$

If each of the substitutions $s_1, s_2, \ldots, s_n$ is of order 2, this is the complete solution. For it can be written in the form

$$P = (1-s_1) E,$$

where E is a rational integral algebraic function of the variables, since

$$\{s_1\} P = 0;$$

and by what we have seen above E must belong to the group $\{s_2s_1\}$, if

$$\{s_2\} P = 0.$$

Hence E must belong to the smallest group containing $s_2s_1, s_3s_1, \ldots, s_ns_1$.

7. It frequently happens that a function is given as belonging to a certain group, besides satisfying certain substitutional equations. Thus, the invariant type degree 5 of a quartic belongs to the group $\{(abcde), (ae)(bd)\}$, and satisfies the equation

$$\{abc\} I_5 = E,$$

the other equations which it satisfies being consequences of these facts. Further, in the case of irreducible invariants, we really only require to find the number of invariants of the form I_5 in terms of which the rest can be linearly expressed. In respect to this, we shall prove that:

If M be the number of arbitrary constants in the most general substitutional expression S_1, which may contain all the $n!$ substitutions of the symmetric group of the n variables under consideration, which satisfies the equations

$$G_1 S_1 \equiv 0, \quad G_2 S_1 \equiv r_2 S_1,$$

G_1 and G_2 being groups of orders r_1 and r_2 respectively, and if N be the number of arbitrary constants in the most general substitutional expression S_2 which satisfies the equations

$$G_2 S_2 \equiv 0, \quad G_1 S_2 \equiv r_1 S_2,$$

then
$$M - N = n! \left\{ \frac{1}{r_2} - \frac{1}{r_1} \right\}.$$

Consider S_1, and suppose that at first all the coefficients are arbitrary. Let A_s be the coefficient of s; then the equation

$$G_1 S_1 \equiv 0$$

gives $\frac{n!}{r_1}$ equations of the form

$$\Sigma A_s = 0, \tag{I.}$$

and in no two of these equations does the same coefficient occur. Now, if σ be any substitution of G_2, it follows that, since S_1 has G_2 for a factor,

$$A_{\sigma s} = A_s.$$

Owing to this, there are only $\frac{n!}{r_2}$ different coefficients; and, if this be taken into account, the equations (I.) are not all independent. Let $T' = 0$ be any relation between these equations written out in full; then this is an identity solely on account of the equations $A_{\sigma s} = A_s$. Hence, if substitutions applied to T' be supposed to operate on the suffixes of the A's, we have the equation

$$G_2 T' = 0.$$

And, further, from the form of equations (I.),

$$G_1 T = r_1 T',$$

for $T = 0$ is a relation between different equations (I.). If, then, T'

be what T becomes when for each A_s we write s, T' will satisfy the equations for S_2. Hence, for every relation between the $\frac{n!}{r_1}$ equations to determine the $\frac{n!}{r_2}$ unknown constants in S_1, there is an expression of exactly the same form which satisfies the equations for S_2. Conversely, every solution of the equations for S_2 will give such a relation between the equations for the unknown constants in S_1. Hence the number of independent relations between the equations (I.) is N; consequently, the number of arbitrary constants left in S_1 when all the equations are satisfied is

$$M = \frac{n!}{r_2} - \left(\frac{n!}{r_1} - N\right);$$

and therefore $$M - N = n!\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

Further, the number of those functions obtained from P by permuting the n variables, in terms of which the $n!$ possible functions thus obtained from P may be linearly expressed when P belongs to the group G_2 and satisfies

$$G_1 P = 0,$$

is equal to M, the number of arbitrary constants in the most general substitutional expression S_1 for which

$$G_1 S_1 \equiv 0, \quad G_2 S_1 \equiv r_2 S_1.$$

For, if P_s be the function obtained from P by operating on it with the substitution s, exactly the same linear equations exist between the functions P_s as between the coefficients A_s in S_1. Hence the number of linearly independent functions P_s is the same as the number of arbitrary coefficients in S_1.

8. If a function P satisfy each of the equations

$$\{a_1 a_2\} P = 0, \quad \{a_1 a_3\} P = 0, \quad \ldots, \quad \{a_1 a_n\} P = 0,$$

it is merely changed in sign when operated upon by any transposition of the letters $a_1, a_2, \ldots, a_n$. The complete solution of these equations is then

$$P = \{a_1 a_2 \ldots a_n\}' F.$$

The function P is an alternating function, and may be written, as is well known,

$$P = \sqrt{\Delta}\,\{a_1 a_2 \ldots a_n\} F',$$

where Δ is the product of the squares of the differences of the letters $a_1, a_2, \ldots, a_n$.

Hence, if P is of degree less than $n-1$ in any one letter, it must be zero. Hence also, if Q be any rational integral function of degree $< n-1$ in each of its variables $a_2, a_2, \ldots, a_n$, it satisfies the equation

$$\{a_1 a_2 \ldots a_n\}' Q = 0.$$

In this connection should be mentioned the following propositions already given for the quartic in my paper "On the Invariant Syzygies of Lowest Degree for any Number of Binary Quartics," viz.,

If P be a rational integral function homogeneous and linear in the coefficients of m binary n-ics,

$$(A_0^{(1)}, A_2^{(1)}, \ldots, A_n^{(1)} \between x_1, x_2)^n \ldots (A_0^{(m)}, A_1^{(m)}, \ldots, A_n^{(m)} \between x_1, x_2)^n,$$

m being greater than $n+1$, then

$$\{A^{(1)} A^{(2)} \ldots A^{(n+2)}\}' P = 0, \qquad \text{(i.)}$$

$$\{A^{(1)} A^{(2)} \ldots A^{(n+1)}\}' P = |\, A^{(1)} A^{(2)} \ldots A^{(n+1)} \,|\, P_1, \qquad \text{(ii.)}$$

$$\{A^{(1)} A^{(2)} \ldots A^{(n)}\}' P \;=\; |\, A^{(1)} A^{(2)} \ldots A^{(n)} Q \,|\,, \qquad \text{(iii.)}$$

where a substitution $(A^{(\alpha)} A^{(\beta)})$ operating on P is regarded as interchanging (α) and (β) in all the indices in P; in fact it interchanges the positions held by the coefficients of the two quantics

$$(A_0^{(\alpha)}, A_1^{(\alpha)}, \ldots, A_n^{(\alpha)} \between x_1, x_2)^n, \quad (A_0^{(\beta)}, A_1^{(\beta)}, \ldots, A_n^{\beta} \between x_1, x_2)^n$$

in P; or else it may be regarded as an abbreviation for

$$(A_0^{(\alpha)} A_0^{(\beta)})(A_1^{(\alpha)} A_1^{(\beta)}) \ldots (A_n^{(\alpha)} A_n^{(\beta)}).$$

And $|\, A^{(1)} A^{(2)} \ldots A^{(n+1)} \,|$ is the determinant of $n+1$ rows and columns formed by the coefficients of the $n+1$ quantics concerned; $|\, A^{(1)} A^{(2)} \ldots A^{(n)} Q \,|$ is the same determinant with functions $Q_0, Q_1, \ldots, Q_n$ of the coefficients of the quantics represented by $A^{(n+1)}, A^{(n+2)}, \ldots, A^{(m)}$ of the same character as P, substituted for the coefficients $A_0^{(n+1)}, A_1^{(n+1)}, \ldots, A_n^{(n+1)}$; and P_1 is a function, having the same character as

P, of the coefficients of the quantics represented by $A^{(n+2)} \dots A^{(m)}$. To prove (i.) we observe that P may be written in the form

$$P = \Sigma A^{(1)}_{r_1} A^{(2)}_{r_2} \dots A^{(n+2)}_{r_{n+2}} P',$$

where each of the suffixes $r_1, r_2, \dots, r_{n+2}$ is one of the $n+1$ numbers $0, 1, 2, \dots, n$; hence in any case two suffixes must be equal, and consequently

$$\{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n+2)}\}' P = 0.$$

For (ii.) we write
$$P = \Sigma A^{(1)}_{r_1} A^{(2)}_{r_2} \dots A^{(n+1)}_{r_{n+1}} P',$$

and here it is possible for the suffixes to be all different; if this is so,

$$\begin{aligned} \{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n+1)}\}' A^{(1)}_{r_1} A^{(2)}_{r_2} \dots A^{(n+1)}_{r_{n+1}} & \\ &= \pm \{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n+1)}\}' A^{(1)}_0 A^{(2)}_1 \dots A^{(n+1)}_{n+1} \\ &= \pm \mid A^{(1)} A^{(2)} \dots A^{(n+1)} \mid ; \end{aligned}$$

and therefore

$$\{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n+1)}\}' P = \mid A^{(1)} A^{(2)} \dots A^{(n+1)} \mid P_1.$$

As regards (iii.) we write

$$P = \Sigma A^{(1)}_{r_1} A^{(2)}_{r_2} \dots A^{(n)}_{r_n} P,$$

and distinguish the following cases:—first, terms R' in which two of the suffixes are equal; then terms R_0 in which the suffixes $r_1, r_2, \dots, r_n$ are the numbers $1, 2, \dots, n$ in some order; then terms R_1 in which the suffixes are the numbers $0, 2, 3, \dots, n$ in some order, and so on; finally, terms R_n in which the suffixes are $0, 1, 2, \dots, n-1$ in some order. Now operate with $\{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n)}\}'$; then

$$\{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n)}\}' R' = 0,$$

$$\{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n)}\}' R_0 = \begin{vmatrix} A^{(1)}_1 A^{(1)}_2 \dots A^{(1)}_n \\ A^{(2)}_1 A^{(2)}_2 \dots A^{(2)}_n \\ \dots \quad \dots \quad \dots \\ A^{(n)}_1 A^{(n)}_2 \dots A^{(n)}_n \end{vmatrix} Q_0.$$

The other terms are found in the same way; so that, taking the sum,

$$\{\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(n)}\}' P = \mid A^{(1)} A^{(2)} \dots A^{(n)} Q \mid .$$

9. As an example, consider the invariants of any number of binary quadratics

$$a_x^2,\ b_x^2,\ \dots .$$

The possible invariant forms are

$$(ab)^2,\ (ab)(bc)(ca),\ (ab)(bc)(cd)(da),\ \dots;$$

then $\{\mathrm{bc}\}\,(ab)(bc)(cd) = (ab)(bc)(cd)-(ac)(bc)(bd) = -(bc)^2(ad)$;

so that, if b, c be any pair of consecutive letters in an invariant I, $\{\mathrm{bc}\}\ I$ is reducible.

Again,

$$\{\mathrm{bd}\}'(ab)(bc)(cd)(de) = (ab)(bc)(cd)(de)-(ad)(dc)(cb)(be)$$
$$= (bc)(cd)(db)(ae).$$

Similarly, any other interchange of letters may be dealt with. The number of irreducible invariants I of any degree n is equal to the number of linearly independent functions obtained from the function P by permuting the letters which it contains, when P satisfies the equations

$$\{\mathrm{ab}\}\,P=0,\ \{\mathrm{bc}\}\,P=0,\ \dots,\ \{\mathrm{ac}\}'\,P=0,\ \dots,$$

and, in fact, all the equations which I satisfies, with the right-hand side of each replaced by zero [I being supposed $= (ab)(bc)(cd)\dots(ha)$].

If n, the degree of I, be greater than 3, then by the last article

$$\{\mathrm{abcd}\}'\,I = 0.$$

Since $\qquad \{\mathrm{ab}\}\,P=0,\ \{\mathrm{bc}\}\,P=0,\ \dots,\ \{\mathrm{ha}\}\,P=0,$

$$P = \{\mathrm{abc}\dots\mathrm{h}\}'\,F = \frac{1}{n!}\{\mathrm{abc}\dots\mathrm{h}\}'\,P = 0,$$

and I is reducible when $n>3$. If the actual solution of the equations for I be carried out, it will be found that in general the expressions on the right-hand side have to satisfy relations; these relations will be the syzygies degree n for the quadratic invariant types. In regard to these equations, it should be noticed that in each separate equation for quadratic types, of the form

$$[\lambda_1+\lambda_2 s_2+\lambda_3 s_3+\dots]\,I = R,$$

where R is a given reducible expression, it is obviously true that R possesses the substitutional properties involved in the operator on the left. The syzygies arise from the fact that I satisfies more than one equation of this kind. Hence R is subject, owing to the system of equations, to more conditions than those implied by the operator on the left-hand side. Exactly the same remark applies to the equations for quartic invariant types of degree greater than 6. The equations in their complete form for degree 7 are given in my paper, "On the Invariant Syzygies of Lowest Degree for any Number of Binary Quartics," already quoted.

As has been pointed out at the commencement of this paper, the invariants of any number of quartics give another illustration of substitutional equations. Thus, the invariant type $(abcde)$, degree 5, satisfies the equation

$$\{\mathrm{abc}\}\,(abcde) = R,$$

and is of group $\{(\mathrm{abcde}),\ (\mathrm{be})(\mathrm{cd})\}$. It has been shown that there are only six independent irreducible forms $(abcde)$. If, now, the theorem of § 7 be applied, we find that, if M be the number of the functions obtained from $[abcde]$ by interchanging the variables in terms of which all the functions obtained by every possible interchange can be linearly expressed, where $[abcde]$ is defined as being of group $\{\mathrm{abc}\}$ and as satisfying the equation

$$\{(\mathrm{abcde}),\ (\mathrm{be})(\mathrm{cd})\}\,[abcde] = 0$$

then $$M-6 = 5!\,(\tfrac{1}{8}-\tfrac{1}{10}) = 8$$

and $$M = 14.$$

10. In what follows repeated use will be made of the symmetric group; it is convenient, then, to note that the sum of its substitutions may be factorized in a variety of ways. For instance,

$$\begin{aligned}\{\mathrm{a}_1\mathrm{a}_2\ldots\mathrm{a}_n\} &= \{(\mathrm{a}_1\mathrm{a}_2\ldots\mathrm{a}_n)\}\,\{\mathrm{a}_1\mathrm{a}_2\ldots\mathrm{a}_{n-1}\}\\ &= \left[1+(\mathrm{a}_1\mathrm{a}_n)+(\mathrm{a}_2\mathrm{a}_n)+\ldots+(\mathrm{a}_{n-1}\mathrm{a}_n)\right]\{\mathrm{a}_1\mathrm{a}_2\ldots\mathrm{a}_{n-1}\}\\ &= \{\mathrm{a}_1\mathrm{a}_2\}\,G_n,\end{aligned}$$

where G_n is the alternating group of the n letters.

Now, any purely formal relation between functions of substitutions will still hold good if the sign of every transposition be changed, Hence the negative symmetric group may be factorized in the same way, thus

$$\{\mathrm{a}_1\mathrm{a}_2\ldots\mathrm{a}_n\}' = \left[1-(\mathrm{a}_1\mathrm{a}_n)-(\mathrm{a}_2\mathrm{a}_n)-\ldots-(\mathrm{a}_{n-1}\mathrm{a}_n)\right]\{\mathrm{a}_1\mathrm{a}_n\ldots\mathrm{a}_{n-1}\}'.$$

Again, the product of a group by itself is the group multiplied by a constant factor equal to its order. The product of a group by a sub-group is equal to the whole group multiplied by the order of the sub-group; for, if G be the whole group, and S a substitution belonging to the sub-group G_1, then

$$G.s = G.$$

Again, if $\{a_1 a_2 a_3 \dots a_n\}$ be any positive symmetric group, and $\{a_1 a_2 b_3 \dots b_m\}'$ a negative symmetric group,

$$\begin{aligned} &\{a_1 a_2 a_3 \dots a_n\}\{a_1 a_2 b_3 \dots b_m\}' \\ =\ &\{a_1 a_2 a_3 \dots a_n\}(a_1 a_2)\left[-(a_1 a_2)\{a_1 a_2 b_3 \dots b_m\}'\right] \\ =\ &-\{a_1 a_2 a_3 \dots a_n\}\{a_1 a_2 b_3 \dots b_m\}' = 0. \end{aligned}$$

Let $S[a_1 b_1 b_2 \dots b_m]$ be any substitutional expression affecting the letters $a_1, b_1, b_2, \dots, b_m$, and only these; then

$$\{a_2 a_3 \dots a_n\} S[a_1 b_1 b_2 \dots b_m] = S[a_1 b_1 b_2 \dots b_m]\{a_2 a_3 \dots a_n\}.$$

Hence $\quad \{a_1 a_2 \dots a_n\} S[a_1 b_1 b_2 \dots b_m]\{a_1 a_2 \dots a_n\}$

$$\begin{aligned} &= [1+(a_1 a_2)+(a_1 a_3)+\dots+(a_1 a_n)]\{a_2 a_3 \dots a_n\} S[a_1 b_1 b_2 \dots b_m\} \\ &\qquad\qquad\qquad\qquad \times\{a_1 a_2 \dots a_n\} \\ &= (n-1)!\,[1+(a_1 a_2)+(a_1 a_3)+\dots+(a_1 a_n)] S[a_1 b_1 b_2 \dots b_m]\{a_1 a_2 \dots a_n\} \\ &= (n-1)!\,\big[S[a_1 b_1 b_2 \dots b_m]+S[a_2 b_1 b_2 \dots b_m]+\dots+S[a_n b_1 \dots b_m]\big] \\ &\qquad\qquad\qquad\qquad \times\{a_1 a_2 \dots a_n\}; \end{aligned}$$

or, as may be proved in the same way,

$$\begin{aligned} &= (n-1)!\,\{a_1 a_2 \dots a_n\}\big[S[a_1 b_1 \dots b_m]+S[a_2 b_1 \dots b_m]+\dots \\ &\qquad\qquad\qquad\qquad \dots+S[a_n b_1 \dots b_m]\big]. \end{aligned}$$

11. As certain results, due in the first place to Capelli, are to be obtained in this paper by means of substitutional analysis, some account of the remarkable paper, "Sur les Opérations dans la Théorie des Formes Algébriques,"* in which they occur, is given here. In this paper Capelli considers functions rational, integral, algebraic,

* *Math. Ann.*, Bd. XXXVII., pp. 1–37.

of n sets of variables

$$\begin{array}{cccc} x_1, & x_2, & \ldots, & x_m, \\ y_1, & y_2, & \ldots, & y_m, \\ \ldots & \ldots & & \ldots \\ u_1, & u_2, & \ldots, & u_m, \end{array}$$

there being m variables in each set, and homogeneous in the variables of each set. Such a function is written

$$f(x, y, \ldots, u).$$

He regards the polar operation

$$D_{xy} = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \ldots + y_m \frac{\partial}{\partial x_m}$$

as fundamental, and proceeds in the first section to develop a theory of operations which can be expressed as rational integral functions with constant coefficients of operations of this kind, and proves that, if by Δ be understood some operation which can be thus expressed, every function $f(x, y, \ldots, u)$ of the above sets of variables which is homogeneous and of degree a_i in the variables whose index is i, for all values of i from 1 up to m, can be obtained in the form

$$f(x, y, \ldots, u) = \Delta x_1^{a_1} y_2^{a_2} \ldots v_m^{a_m},$$

there being the same number of sets expressed in the term on which Δ operates as there are variables in each set, Δ depending on the form of f.

His second section is devoted to the discussion of an operation H defined as follows :—

$$\text{If } m = n, \quad H = | xy \ldots u | \left| \frac{\partial}{\partial x} \frac{\partial}{\partial y} \ldots \frac{\partial}{\partial u} \right|,$$

$$\text{if } m > n, \quad H = \sum_i | x_{i_1} y_{i_2} \ldots u_{i_n} | \left| \frac{\partial}{dx_{i_1}} \frac{\partial}{\partial x_{i_2}} \ldots \frac{\partial}{\partial u_{i_n}} \right|,$$

$$\text{is } m < n, \quad H = 0,$$

where $| xy \ldots u |$ is the determinant formed by the variables, and $\left| \frac{\partial}{\partial x} \frac{\partial}{\partial y} \ldots \frac{\partial}{\partial u} \right|$, which is the determinant formed by the first differential operators with respect to the variables, is Cayley's operator Ω.

It is shown that H may be expressed in terms of the operators

D_{xy}, and the form of this expression is found; further, it is proved that H is commutative with all rational integral functions of the operators D_{xy}. It is then proved that, if a function $f(x, y, z, \ldots, t, u)$ of the kind considered, of n sets of variables, there being n variables in each set, is annihilated by each of $D_{xy}, D_{yz}, \ldots, D_{tu}$, it is equal to a power of $| xyz \ldots tu |$ multiplied by a function of the same nature of the sets $y, z, \ldots, t, u$, which is annihilated by $D_{yz}, \ldots, D_{tu}$.

In the third section it is proved that, if two functions of the same number of sets of variables, rational, integral, and homogeneous in the variables of each set, are obtainable from each other by means of a permutation of the sets, they are also obtainable from each other by means of the operators D_{xy}. In other words, an operator which is a rational, integral function of the operators D_{xy} may be always found which will have the same effect on $f(x, y, \ldots, u)$ as any given substitution operating on this function. In view of the importance of this theorem in connection with the present subject, I quote Capelli's illustration. Let $f(x, y, z)$ be any rational, integral function of the variables

$$x_1,\ x_2,\ \ldots,\ x_m,$$
$$y_1,\ y_2,\ \ldots,\ y_m,$$
$$z_1,\ z_2,\ \ldots,\ z_m,$$

homogeneous and of degrees λ, μ, ν respectively in the variables of the three sets. Let

$$\xi_1,\ \xi_2,\ \ldots,\ \xi_m,$$
$$\eta_1,\ \eta_2,\ \ldots,\ \eta_m,$$
$$\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_m$$

be three new sets of variables, independent of each other and of the original sets; then

$$f(\xi, \eta, \zeta) = \frac{1}{\lambda!\,\mu!\,\nu!} D^{\lambda}_{x\xi} D^{\mu}_{y\eta} D^{\nu}_{z\zeta} f(x, y, z),$$

and $$f(y, z, x) = \frac{1}{\lambda!\,\mu!\,\nu!} D^{\lambda}_{\xi y} D^{\mu}_{\eta z} D^{\nu}_{\zeta x} f(\xi, \eta, \zeta);$$

hence $$f(y, z, x) = \left(\frac{1}{\lambda!\,\mu!\,\nu!}\right)^2 D^{\lambda}_{\xi y} D^{\mu}_{\eta z} D^{\nu}_{\zeta x} D^{\lambda}_{x\xi} D^{\mu}_{y\eta} D^{\nu}_{z\xi} f(x, y, z).$$

By means of the methods laid down in the first section of Capelli's paper, it is possible to reduce this to the form $\Delta f(x, y, z)$, where the operators of which Δ is a function only affect the sets x, y, z.

In this section it is also proved that the condition that f should be expressible as a sum of terms each of which is derivable by operations of the kind considered from functions of a smaller number of sets of variables than that contained in f is

$$H.f = 0.$$

In § 4 the following important theorem is proved:

If $f(x, y, \ldots, u)$ is a rational, integral function of n sets of variables, there being n variables in each set, which is homogeneous in the variables of each set, then

$$f(x, y, \ldots, u) = \sum_{\mu, i} | xy\ldots u |^{\mu} . \Delta_i . \phi_i(y, z, \ldots, u),$$

where $$\phi_i(y, z, \ldots, u) = D_{xy}^{\alpha_i} D_{xz}^{\beta_i} \ldots D_{xu}^{\lambda_i} . \Omega^{\mu} f;$$

the Σ extending to all positive integral solutions of

$$\alpha_i + \beta_i + \ldots + \lambda_i + \mu = p,$$

where p is the degree of f in x, and where Δ_i is a rational integral function with constant coefficients of operators of the form D_{xy}, the form of which depends only on the degrees in which the variables occur in f; and, further, the coefficients of different powers of $| xy\ldots u |$ are unique. The last section is devoted to an extension of certain of the results to any analytic function.

12. In what follows substitutions are taken as the fundamental operators, instead of Capelli's operators D_{xy}. Functions $f(a, b, \ldots, k)$ are considered which are rational, integral, homogeneous, and linear in each of n sets of variables

$$\begin{array}{l} a_1, a_2, \ldots, a_m, \\ b_1, b_2, \ldots, b_m, \\ \ldots \quad \ldots \quad \ldots \\ k_1, k_2, \ldots, k_m, \end{array}$$

there being m variables in each set. The letters $a, b, \ldots, k$ are employed, as the applications considered are mainly to concomitant types of quantics. The restriction that f is to be linear in the variables of each set does not in reality restrict the generality of the results obtained; for, if $F(a, b, \ldots, k)$ be a function rational, integral, homogeneous, and of degrees $\alpha, \beta, \ldots, \kappa$ in the variables of the different sets, we may obtain a function f, such that

$$f(a^{(1)}, a^{(2)}, \ldots, a^{(\alpha)}, b^{(1)}, \ldots, b^{(\beta)}, \ldots, k^{(\kappa)})$$
$$= \frac{1}{\alpha!\,\beta!\,\ldots\,\kappa!} D_{aa^{(1)}} D_{aa^{(2)}} \ldots D_{aa^{(\alpha)}} D_{bb^{(1)}} \ldots D_{kk^{(\kappa)}} F(a, b, \ldots, k),$$

and consider, instead of F, the function

$$\frac{1}{\alpha!\,\beta!\,\ldots\,\kappa!} \{\mathrm{a}^{(1)}\mathrm{a}^{(2)} \ldots \mathrm{a}^{(\alpha)}\} \{\mathrm{b}^{(1)} \ldots \mathrm{b}^{(\beta)}\} \ldots \{\mathrm{k}^{(1)} \ldots \mathrm{k}^{(\kappa)}\} f,$$

For, if we write

$$a^{(1)} = a^{(2)} = \ldots = a, \quad b^{(1)} = \ldots = b, \quad \ldots, \quad k^{(1)} = \ldots = k,$$

this becomes F once more. There is a fairly close connexion between the theory of substitutional and of polar operators. Thus any function $f(a, b, \ldots, k)$ of n sets of variables, there being m variables in each set, which is homogeneous and linear in the variables of each set, and homogeneous and of degree α_i in the variables whose index is i, for all values of i from 1 up to m, may be expressed in the form

$$f(a, b, \ldots, k) = S a_1^{(1)} \ldots a_1^{(\alpha_1)} b_2^{(1)} \ldots b_2^{(\alpha_2)} \ldots k_m^{(\alpha_m)},$$

where S is a substitutional operator with constant coefficients. This follows at once from § 1; for there is only one kind of term which can occur here.

The operator H may be expressed as a substitutional operator thus:—We first suppose that H is to operate on a function homogeneous and linear in the variables of each of n sets, there being n variables in each set; then

$$H = |\, ab \ldots k \,|\ \left| \frac{\partial}{\partial a} \frac{\partial}{\partial b} \ldots \frac{\partial}{\partial k} \right|.$$

But in this case

$$|\, ab \ldots k \,|\ \left| \frac{\partial}{\partial a} \frac{\partial}{\partial b} \ldots \frac{\partial}{\partial k} \right| f = \{\mathrm{ab} \ldots \mathrm{k}\}' f.$$

For, if $A a_{i_1} b_{i_2} \ldots k_{i_m}$ be any term of f, the effect of both operators is zero, unless all the indices are different, and, if this is so, both operators give $A \,|\, ab \ldots k \,|$ as the result, the rule for determining the sign being the same in each case.

If f is still linear in the variables of each set, but the number of variables in a set is m, greater than the number n of sets, then H is still equivalent to $\{\mathrm{ab} \dots \mathrm{k}\}'$, for, if $Aa_{i_1} b_{i_2} \dots k_{i_m}$ be any term of f, then

$$\begin{aligned} \{\mathrm{ab} \dots \mathrm{k}\}\, Aa_{i_1} b_{i_2} \dots k_{i_m} &= \mid a_{i_1} b_{i_2} \dots k_{i_m} \mid \left| \frac{\partial}{\partial a_{i_1}} \frac{\partial}{\partial b_{i_2}} \dots \frac{\partial}{\partial k_{i_m}} \right| Aa_{i_2} \dots k_{i_m} \\ &= \left[\sum_j \mid a_{j_1} b_{j_2} \dots k_{j_m} \mid \left| \frac{\partial}{\partial a_{j_1}} \frac{\partial}{\partial b_{j_2}} \dots \frac{\partial}{\partial k_{j_m}} \right| \right] Aa_{i_1} \dots k_{i_m}; \end{aligned}$$

for all terms of the Σ, except the one first quoted, give zero when operating on the term chosen.

If $m<n$, $H=0$, and $\{\mathrm{ab} \dots \mathrm{k}\}'f=0$; for every term of f must contain at least two variables with the same indices.

Now consider any function F homogeneous but no longer linear in the variables of each set, having n sets and m variables in each set. Then we form from F a function f, as shown above, such that we may consider, instead of F,

$$P = \frac{1}{\alpha!\,\beta!\,\dots\,\kappa!} \{\mathrm{a}^{(1)} \mathrm{a}^{(2)} \dots \mathrm{a}^{(\alpha)}\} \{\mathrm{b}^{(1)} \dots \mathrm{b}^{(\beta)}\} \dots$$
$$\dots \{\mathrm{k}^{(1)} \dots \mathrm{k}^{(\kappa)}\} f(a^{(1)}, a^{(2)}, \dots, a^{(\alpha)}, b^{(1)}, \dots, k^{(\kappa)}).$$

Then, if $H_{a^{(1)} b^{(1)} \dots k^{(1)}}$ be what H becomes when we write in it the sets $a^{(1)}, b^{(1)}, \dots, k^{(1)}$ instead of the sets $a, b, \dots, k$, HF becomes

$$\sum_{\alpha_i=1}^{\alpha_i=\alpha} \sum_{\beta_i=1}^{\beta_i=\beta} \dots \sum_{\kappa_i=1}^{\kappa_i=\kappa} H_{a^{(\alpha_i)} b^{(\beta_i)} \dots k^{(\kappa_i)}} P;$$

this last expression being $= HF$ when we write

$$\begin{aligned} a^{(1)} &= a^{(2)} = \dots = a^{(\alpha)} = a, \\ b^{(1)} &= b^{(2)} = \dots = b^{(\beta)} = b, \\ &\dots \quad \dots \quad \dots \\ k^{(1)} &= k^{(2)} = \dots = k^{(\kappa)} = k. \end{aligned}$$

But $$H_{a^{(\alpha_i)} b^{(\beta_i)} \dots k^{(\kappa_i)}} P = \{\mathrm{a}^{(\alpha_i)} \mathrm{b}^{(\beta_i)} \dots \mathrm{k}^{(\kappa_i)}\} P,$$

as we have already seen; hence in this case

$$\sum_{\alpha_i, \beta_i, \dots, \kappa_i} \{\mathrm{a}^{(\alpha_i)} \mathrm{b}^{(\beta_i)} \dots \mathrm{k}^{(\kappa_i)}\}'$$

is the equivalent of H.

Now, in the substitutional equivalent of H it is assumed that there is a substitutional operator

$$\{a^{(1)}a^{(2)}\dots a^{(\alpha)}\}\{b^{(1)}\dots b^{(\beta)}\}\dots\{k^{(1}\dots k^{(\kappa)}\},$$

of definite form applied to the operand. The same operator may then be attached to this equivalent of H, without affecting the result except as regards a constant. Hence we may write

$$H=\frac{1}{\alpha!\,\beta!\,\dots\,\kappa!}\Sigma\,\{a^{(a_i)}b^{(\beta_i)}\dots k^{(\kappa_i)}\}'\{a^{(1)}\dots a^{(\alpha)}\}\{b^{(1)}\dots b^{(\beta)}\}\dots\{k^{(1)}\dots k^{(\kappa)}\}$$

$$=\frac{1}{\alpha!\,\beta!\dots\kappa!\,(\alpha-1)!\,(\beta-1)!\dots(\kappa-1)!}$$

$$\times\{a^{(1)}\dots a^{(\alpha)}\}\{b^{(1)}\dots b^{(\beta)}\}\dots\{k^{(1)}\dots k^{(\kappa)}\}$$

$$\times\{a^{(1)}b^{(1)}\dots k^{(1)}\}'\{a^{(1)}\dots a^{(\alpha)}\}\{b^{(1)}\dots b^{(\beta)}\}\dots\{k^{(1)}\dots k^{(\kappa)}\}.$$

For $$\{a^{(1)}\dots a^{(\alpha)}\}\{a^{(1)}b^{(1)}\dots k^{(1)}\}'\{a^{(1)}\dots a^{(\alpha)}\}$$

$$=\left[1+(a^{(1)}a^{(2)})+(a^{(1)}a^{(3)})+\dots+(a^{(1)}a^{(\alpha)})\right]$$

$$\times\{a^{(2)}\dots a^{(\alpha)}\}\{a^{(1)}b^{(1)}\dots k^{(1)}\}'\{a^{(1)}\dots a^{(\alpha)}\}$$

$$=(\alpha-1)!\left[1+(a^{(1)}a^{(2)})+\dots+(a^{(1)}a^{(\alpha)})\right]\{a^{(1)}b^{(1)}\dots k^{(1)}\}'\{a^{(1)}\dots a^{(\alpha)}\}$$

$$=(\alpha-1)!\sum_{a_i=1}^{a_i=\alpha}\{a^{(a_i)}b^{(1)}\dots k^{(1)}\}\{a^{(1)}\dots a^{(\alpha)}\}.$$

Capelli has shown in the general case how a substitution may be expressed in terms of polar operators; in the case of functions homogeneous and linear in the variables of each set, the effect of a transposition may be obtained thus,

$$D_{ba}D_{ab}f(a,b,c,\dots)=D_{ba}f(b,b,c,\dots)=\{ab\}f(a,b,c,\dots);$$

hence $$(ab)f(a,b,c,\dots)=(D_{ba}D_{ab}-1)f(a,b,c,\dots).$$

Any other substitution operating on f may be expressed as a product of transpositions, and so as a function of polar operators. The converse theorem is also true; for let D_{ab} be a polar operator, operating on a function F of degree α in the variables of the set a, and β in those of the set b; then we consider instead of F the function P

defined as above. The effect of the operator D_{ab} on F is the same as that of

$$\frac{1}{(\beta+1)!}\{b^{(1)}b^{(2)}\dots b^{(\beta+1)}\}\left[D_{a^{(1)}b^{(\beta+1)}}+D_{a^{(2)}b^{(\beta+1)}}+\dots+D_{a^{(\alpha)}b^{(\beta+1)}}\right]$$

on P. For each of the sets $a^{(1)}, a^{(2)}, \dots, a^{(\alpha)}$ in P is in reality equivalent to a, and each of the sets $b^{(1)}, b^{(2)}, \dots$ equivalent to b. Since P does not contain $b^{(\beta+1)}$,

$$D_{a^{(1)}b^{(\beta+1)}}P = (a^{(1)}b^{(\beta+1)})\,P,$$

the right-hand side being no longer a function of $a^{(1)}$.

Now, P is symmetric in the sets $a^{(1)}, \dots, a^{(\alpha)}$; hence the function $(a^{(2)}b^{(\beta+1)})\,P$ is the same as $(a^{(1)}b^{(\beta+1)})\,P$, except that $a^{(1)}$ and $a^{(2)}$ are interchanged; hence the function $D_{ab}F$ is equivalent to

$$\frac{\alpha}{(\beta+1)!}\{b^{(1)}b^{(2)}\dots b^{(\beta+1)}\}\,(a^{(\alpha)}b^{(\beta+1)})\,P,$$

which does not contain the set $a^{(\alpha)}$. In this the new set $b^{(\beta+1)}$ may be replaced by the old set $a^{(\alpha)}$ by operating with $(a^{(\alpha)}b^{(\beta+1)})$, and the result becomes

$$\frac{\alpha}{(\beta+1)!}\{b^{(1)}b^{(2)}\dots b^{(\beta)}a^{(\alpha)}\}\,P,$$

where now $a^{(\alpha)}$ is to be regarded as equivalent to b.

13. If $T_{\alpha,0} \equiv S\{a_1b_1\}'\{a_2b_2\}'\dots\{a_\alpha b_\alpha\}'\{b_1b_2\dots b_m\}\,S$

and $\beta>0$, $T_{\alpha,\beta} \equiv S\{a_1b_1\}'\{a_2b_2\}'\dots\{a_\alpha b_\alpha\}'\{a_{n-\beta+1}\dots a_nb_1\dots b_m\}\,S$,

where $S \equiv \{a_1a_2\dots a_n\}\{b_1b_2\dots b_m\}$,

then $T_{0,0} = A_{0,n}T_{0,n}+A_{1,n-1}T_{1,n-1}+\dots+A_{n,0}T_{n,0}$,

if $m \not< n$; but, if $m<n$, the series must stop with $A_{m,n-m}T_{m,n-m}$, and the coefficients A are given by

$$A_{\alpha,\beta} = \binom{\alpha+\beta}{\beta}\frac{m!\,(m+1+\beta-\alpha)}{(m+\beta+1)!}.$$

The theorem to be established is purely formal,- an identity between certain substitutional expressions.

When $\alpha<h<n-\beta+1$, the expression

$$S\{a_1b_1\}'\{a_2b_2\}'\dots\{a_\alpha b_\alpha\}'(a_hb_1)\{a_{n-\beta+1}\dots a_nb_1b_2\dots b_m\}\,S$$
$$= S\,(a_hb_1)\{a_1a_h\}'\{a_2b_2\}'\dots\{a_\alpha b_\alpha\}'\{a_{n-\beta+1}\dots a_nb_1b_2\dots b_m\}\,S,$$

for it is well known that, if s be any substitution, $(a_hb_1)\,s\,(a_hb_1)$ is the same as that substitution obtained from s by the interchange of a_h and b_1; and hence, if U be any substitutional expression,

$$(a_h b_1)\, U\, (a_h b_1) = U_1,$$

the expression obtained from U by the interchange of a_h and b_1; and hence

$$U\,(a_h b_1) = (a_h b_1)\, U_1.$$

Now, no one of the factors $\{a_2 b_2\}' \ldots \{a_a b_a\}' \{a_{n-\beta+1} \ldots a_n b_1 b_2 \ldots b_m\}$ contains either of the letters a_1 or a_h; hence

$$\begin{aligned} S\,(a_h b_1)\,\{a_1 a_h\}'\,\{a_2 b_2\}' \ldots \{a_a b_a\}'\,\{a_{n-\beta+1} \ldots a_n b_1 b_2 \ldots b_m\}\, S \\ = S\,(a_h b_1)\,\{a_2 b_2\}' \ldots \{a_a b_a\}'\,\{a_{n-\beta+1} \ldots a_n b_1 b_2 \ldots b_m\}\,\{a_1 a_h\}'\, S \\ = 0; \end{aligned}$$

for
$$\{a_1 a_h\}'\,\{a_1 a_2 \ldots a_n\} = 0;$$

and therefore

$$S\,\{a_1 b_1\}'\,\{a_2 b_2\}' \ldots \{a_a b_a\}'\,(a_h b_1)\,\{a_{n-\beta+1} \ldots a_n b_1 b_2 \ldots b_m\}\, S = 0.$$

Hence

$$\begin{aligned} T_{a,\beta} &= S\,\{a_1 b_1\}'\,\{a_2 b_2\}' \ldots \{a_a b_a\}'\,\{a_{n-\beta+1} \ldots a_n b_1 b_2 \ldots b_m\}\, S \\ &= S\,\{a_1 b_1\}' \ldots \{a_a b_a\}'\,\big[1 + (a_{n-\beta+1} a_{n-\beta+2}) + (a_{n-\beta+1} a_{n-\beta+3}) + \ldots \\ &\qquad + (a_{n-\beta+1} a_n) + (a_{n-\beta+1} b_1) + \ldots + (a_{n-\beta+1} b_m)\big] \\ &\qquad \times \{a_{n-\beta+2} \ldots a_n b_1 b_2 \ldots b_m\}\, S \\ &= S\,\{a_1 b_1\}' \ldots \{a_a b_a\}'\,\big[\beta + (a_{n-\beta+1} b_{a+1}) + \ldots + (a_{n-\beta+1} b_m)\big] \\ &\qquad \times \{a_{n-\beta+2} \ldots a_n b_1 b_2 \ldots b_m\}\, S \\ &= S\,\{a_1 b_1\}' \ldots \{a_a b_a\}'\,\big[-(m-a)\,\{a_{n-\beta+1} b_{a+1}\}' + (m+\beta-a)\big] \\ &\qquad \times \{a_{n-\beta+2} \ldots a_n b_1 b_2 \ldots b_m\}\, S \\ &= -(m-a)\, T_{a+1,\beta-1} + (m+\beta-a)\, T_{a,\beta-1}; \end{aligned}$$

therefore
$$T_{a,\beta} = \frac{1}{m+\beta-a+1}\, T_{a,\beta+1} + \frac{m-a}{m+\beta-a+1}\, T_{a+1,\beta}.$$

By repeated application of this formula, we obtain

$$\begin{aligned} T_{00} &= \frac{1}{m+1}\, T_{0,1} + \frac{m}{m+1}\, T_{1,0} \\ &= \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ &= A_{0,i}\, T_{0,i} + A_{1,i-1}\, T_{1,i-1} + \ldots + A_{i,0}\, T_{i,0}, \end{aligned}$$

except when $i > m$, in which case the series ends with $A_{m,i-m} T_{m,i-m}$; i being supposed to be not greater than n, and the A's being numerical coefficients.

To find these coefficients a recurrence formula is obtained by proceeding from the last line written down a step further. The coefficient of $T_{j,i-j+1}$ in this will be

$$A_{j,i-j+1} = \frac{m-j+1}{m+i-2j+3} A_{j-1,i-j+1} + \frac{1}{m+i-2j+1} A_{j,i-j}.$$

Hence $$A_{\alpha,\beta} = \frac{m-\alpha+1}{m+\beta-\alpha+2} A_{\alpha-1,\beta} + \frac{1}{m+\beta-\alpha} A_{\alpha,\beta-1}.$$

It follows from this that, if

$$A_{\alpha,\beta} = \binom{\alpha+\beta}{\beta} \frac{m!\,(m+1+\beta-\alpha)}{(m+\beta+1)!},$$

when $\alpha < \alpha_1$, and also when $\alpha = \alpha_1$ so long as $\beta < \beta_1$, it is true when $\alpha = \alpha_1$ and $\beta = \beta_1$. Hence, on this hypothesis it is true whenever $\alpha < \alpha_1 + 1$. But this form of $A_{\alpha,\beta}$ is correct, as it is easy to verify, when $\alpha = 0$, and also when $\beta = 0$ and $\alpha < n+1$. Hence it is true always when $\alpha < n+1$. And the theorem is proved.

14. As has been pointed out, the theorem just proved is merely a substitutional identity. If the two sides of the identity be made to operate on the same function, the results must be equal. This operand may be taken to be any function of $m+n$ variables

$$a_1, a_2, \ldots, a_n, \quad b_1, b_2, \ldots, b_m;$$

or else any function of $m+n$ sets of variables

$$\begin{array}{llll} a_{1,1}, & a_{1,2}, & a_{1,3}, \ldots, & a_{1,p}, \\ a_{2,1}, & a_{2,2}, & \ldots\ldots\ldots, & a_{2,p}, \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,1}, & a_{n,2}, & \ldots\ldots\ldots, & a_{n,p}, \\ b_{1,1}, & b_{1,2}, & \ldots\ldots\ldots, & b_{1,p}, \\ \ldots & \ldots & \ldots & \ldots \\ b_{m,1}, & b_{m,2}, & \ldots\ldots\ldots, & b_{m,p}, \end{array}$$

a substitution $(a_i b_j)$ interchanging two sets, just as in §§ 10 and 11 a substitution on the functions there discussed interchanged two sets. In this case, as has been seen in § 11, when the operand F is linear and homogeneous in the variables of each set, the expression $\{a_1 b_1\}'$ is equivalent to H_{a_1, b_1}, where

$$H_{a_1, b_1} = \Sigma \mid a_{1, t_1} \; b_{1, t_2} \mid \left| \frac{\partial}{\partial a_{1, t_1}} \; \frac{\partial}{\partial b_{1, t_2}} \right|.$$

(1) Let us take for operand

$$F = a_{1_x} a_{2_x} \dots a_{n_x} b_{1_y} b_{2_y} \dots b_{m_y},$$

where the factors of F are binary symbolical factors, thus

$$a_{1_x} = a_{1,1} x_1 + a_{1,2} x_2,$$
$$b_{1_y} = b_{1,1} y_1 + b_{1,2} y_2.$$

Then $T_{0,0} F = \{a_1 a_2 \dots a_n\} \{b_1 b_2 \dots b_m\} \{b_1 \dots b_m\} \{a_1 \dots a_n\} \{b_1 \dots b_m\} F'$

$$= (n!)^2 (m!)^3 F.$$

Denote by D and Δ polar operators, such that, ϕ being homogeneous and of order n in x_1, x_2, and homogeneous and of order m in y_1, y_2, then

$$D\phi = \frac{1}{m}\left(x_1 \frac{\partial\phi}{\partial y_1} + x_2 \frac{\partial\phi}{\partial y_2}\right),$$

$$\Delta\phi = \frac{1}{n}\left(y_1 \frac{\partial\phi}{\partial x_1} + y_2 \frac{\partial\phi}{\partial x_2}\right),$$

also let

$$\Omega\phi = \frac{1}{mn}\left(\frac{\partial^2\phi}{\partial x_1 \partial y_2} - \frac{\partial^2\phi}{\partial x_2 \partial y_1}\right);$$

these being the operators used by Clebsch (*Binaren Formen*, pp. 13, 14, *et seq.*).

Then $$D^m F = a_{1_x} a_{2_x} \dots a_{n_x} b_{1_x} b_{2_x} \dots b_{m_x},$$

and the effect of operating with Δ^m on this function is to change it to the sum of all possible terms obtained from $D^m F$ by writing y for x in m of its factors, divided by their number. But this is the same as

$$\frac{1}{(m+n)!} \{a_1 a_2 \dots a_n b_1 b_2 \dots b_m\} \, a_{1_x} a_{2_x} \dots a_{n_x} b_{1_y} b_{2_y} \dots b_{m_y}.$$

Hence

$$T_{0,n} F = \{a_1 \dots a_n\} \{b_1 \dots b_m\} \{a_1 \dots a_n b_1 \dots b_m\} \{a_1 \dots a_n\} \{b_1 \dots b_m\} F$$
$$= (n!)^2 (m!)^2 \{a_1 \dots a_n b_1 \dots b_m\} F$$
$$= (n!)^3 (m!)^2 (m+n)! \, \Delta^m D^m F.$$

Again,

$$\Omega F=\frac{1}{mn}\sum_{i=1}^{i=n}\sum_{j=1}^{j=m}(a_ib_j)\,a_{1_x}\dots a_{i-1_x}a_{i+1_x}\dots a_{n_x}b_{1_y}\dots b_{j-1_y}b_{j+1_y}\dots b_{m_y};$$

and $$D^{m-1}\Omega F=\frac{1}{mn}\sum_i\sum_j(a_ib_j)\,a_{1_x}\dots a_{i-1_x}a_{i+1_x}a_{n_x}b_{1_x}\dots b_{j-1_x}b_{j+1_x}\dots b_{m_x}.$$

And $\Delta^{m-1}D^{m-1}\Omega F$ is equal to the sum of all possible terms obtained by substituting y for x in $m-1$ of the factors of each term of $D^{m-1}\Omega F$ divided by their number

$$=\frac{1}{(m+n-2)!}\,\frac{1}{mn}\sum_{i,j}(a_ib)\,\{\mathrm{a}_1\dots\mathrm{a}_{i-1}\mathrm{a}_{i+1}\dots\mathrm{a}_n\mathrm{b}_1\dots\mathrm{b}_{j-1}\mathrm{b}_{j+1}\dots\mathrm{b}_m\}$$
$$\times\,a_{1_x}\dots a_{i-1_x}a_{i+1_x}\dots a_{n_x}b_{1_y}\dots b_{j-1_y}b_{j+1_y}\dots b_{m_y}$$

$$=\frac{1}{(m+n-2)!}\,\frac{1}{m!\,n!}\{\mathrm{a}_1\mathrm{a}_2\dots\mathrm{a}_n\}\{\mathrm{b}_1\mathrm{b}_2\dots\mathrm{b}_m\}(a_1b_1)\{\mathrm{a}_2\dots\mathrm{a}_n\,\mathrm{b}_2\dots\mathrm{b}_m\}$$
$$\times\,a_2\dots a_{n_x}b_{2_y}\dots b_{m_y}.$$

Hence

$$T_{1,n-1}F=S\{\mathrm{a}_1\mathrm{b}_1\}'\{\mathrm{a}_2\dots\mathrm{a}_n\,\mathrm{b}_1\dots b_m\}SF$$
$$=n!\,m!\,S\{\mathrm{a}_1\mathrm{b}_1\}'\{\mathrm{a}_2\dots\mathrm{a}_n\,\mathrm{b}_1\dots\mathrm{b}_m\}\,a_{1_x}\dots a_{n_x}b_{1_y}\dots b_{m_y}$$
$$=n!\,m!\,S\{\mathrm{a}_1\mathrm{b}_1\}'\big[ma_{1_x}b_{1_y}\{\mathrm{a}_2\dots\mathrm{a}_n\,\mathrm{b}_2\dots\mathrm{b}_m\}\,a_{2_x}\dots a_{n_x}b_{2_y}\dots b_{m_y}$$
$$+(n-1)\,a_{1_x}b_{1_x}\{\mathrm{a}_2\dots\mathrm{a}_n\,\mathrm{b}_2\dots\mathrm{b}_m\}\,a_{2_x}\dots a_{(n-1)_x}a_{n_y}b_{2_y}\dots b_{m_y}\big]$$
$$=n!\,m!\,mS\,(a_1b_1)\,(xy)\,\{\mathrm{a}_2\dots\mathrm{a}_n\,\mathrm{b}_2\dots\mathrm{b}_m\}\,a_{2_x}\dots a_{n_x}b_2\dots b_{m_y}$$
$$=(xy)\,(n!)^2\,(m!)^2\,(m+n-2)!\,m\Delta^{m-1}D^{m-1}\Omega F.$$

Proceeding in the same way,

$$\Omega^hF=\frac{(m-h)!\,(n-h)!}{m!\,n!}\sum(a_1b_1)(a_2b_2)\dots(a_hb_h)\,a_{h+1_x}\dots a_{n_x}b_{h+1_y}\dots b_{m_y},$$

$D^{m-h}\Omega^hF$

$$=\frac{(m-h)!\,(n-h)!}{m!\,n!}\sum(a_1b_1)(a_2b_2)\dots(a_hb_h)\,a_{h+1_x}\dots a_{n_x}b_{h+1_y}\dots b_{m_x},$$

$\Delta^{m-h}D^{m-h}\Omega^hF$

$$=\frac{1}{(m+n-2h)!}\,\frac{1}{m!\,n!}\{\mathrm{a}_1\mathrm{a}_2\dots\mathrm{a}_n\}\{\mathrm{b}_1\dots\mathrm{b}_m\}(a_1b_1)(a_2b_2)\dots(a_hb_h)$$
$$\times\{\mathrm{a}_{h+1}\dots\mathrm{a}_n\mathrm{b}_{h+1}\dots\mathrm{b}_m\}\,a_{h+1_x}\dots a_{n_x}b_{h+1_y}\dots b_{m_y}.$$

And hence

$$T'_{h,n-h}F = S\{a_1 b_1\}'\{a_2 b_2\}'\ldots\{a_h b_h\}'\{a_{h+1}\ldots a_n\, b_1\ldots b_m\}SF$$

$$= n!\, m!\, S\{a_1 b_1\}'\ldots\{a_h b_h\}'\Big[\frac{m!}{(m-h)!}a_{1_x}\ldots a_{h_x} b_{1_y}\ldots b_{h_y}$$

$$\times\{a_{h+1}\ldots a_n\, b_{h+1}\ldots b_m\}a_{h+1_x}\ldots a_{n_x} b_{h+1_y}\ldots b_{m_y}+P\Big],$$

where P contains the product $a_{i_x}b_{i_x}$ for some value of i between 1 and h inclusive, and hence is annihilated by the product $\{a_1 b_1\}'\ldots\{a_h b_h\}'$; therefore

$$T_{h,n-h}F = (xy)^h n!\, m!\frac{m!}{(m-h)!}S(a_1 b_1)\ldots(a_h b_h)\{a_{h+1}\ldots a_n\, b_{h+1}\ldots b_m\}$$

$$\times a_{h+1_x}\ldots a_{n_x} b_{h+1_y}\ldots b_{m_y}$$

$$= (xy)^h (n!)^2 (m!)^2 (m+n-2h)!\frac{m!}{(m-h)!}\Delta^{m-h}D^{m-h}\Omega^h F;$$

and hence $A_{h,n-h}T_{h,n-h}F$

$$= (xy)^h\binom{n}{h}\frac{m!\,(m+1+n-2h)}{(m+n-h+1)!}(n!)^2(m!)^2(m+n-2h)!\frac{m!}{(m-h)!}$$

$$\times\Delta^{m-h}D^{m-h}\Omega^h F$$

$$= (n!)^2(m!)^3\frac{\binom{n}{h}\binom{m}{h}}{\binom{m+n-h+1}{h}}(xy)^h\Delta^{m-h}D^{m-h}\Omega^h F.$$

Hence we obtain Gordan's series

$$F = \sum_{h=0}^{h=m}\frac{\binom{n}{h}\binom{m}{h}}{\binom{m+n-h+1}{h}}(xy)^h\Delta^{m-h}D^{m-h}\Omega^h F,$$

if $n \geqslant m$; if $n<m$, the summation must be taken from $h=0$ to $h=n$.

(2) Let F as before $= a_{1_x}a_{2_x}\ldots a_{n_x}b_{1_y}b_{2_y}\ldots b_{m_y}$, where the factors of F are now ternary symbolical factors, thus

$$a_{1_x} = a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3.$$

Then $$T_{0,0}F = (n!)^2(m!)^3 F;$$

and $$T_{0,n}F = (n!)^2(m!)^2(m+n)!\,\Delta^m D^m F,$$

just as when the factors of F were binary; the definition of Δ and D being that, if ϕ is a function homogeneous and of order n in x_1, x_2, x_3, and homogeneous and of order m in y_1, y_2, y_3, then

$$D\phi = \frac{1}{m}\left(x_1\frac{\partial\phi}{\partial y_1}+x_2\frac{\partial\phi}{\partial y_2}+x_3\frac{\partial\phi}{\partial y_3}\right),$$

$$\Delta\phi = \frac{1}{n}\left(y_1\frac{\partial\phi}{\partial x_1}+y_2\frac{\partial\phi}{\partial x_2}+y_3\frac{\partial\phi}{\partial x_3}\right).$$

Let u_1, u_2, u_3 be three quantities defined by

$$u_1 = x_2y_3-x_3y_2,$$
$$u_2 = x_3y_1-x_1y_3,$$
$$u_3 = x_1y_2-x_2y_1;$$

then
$$a_xb_y-a_yb_x = (abu);$$

and, just as in the former case,

$$T_{h,n-h}F = n!\,m!\,S\{\mathrm{a}_1\mathrm{b}_1\}'\ldots\{\mathrm{a}_h\mathrm{b}_h\}'\frac{m!}{(m-h)!}a_{1_x}\ldots a_{h_x}b_{1_y}\ldots b_{h_y}$$
$$\times\{\mathrm{a}_{h+1}\ldots\mathrm{a}_n\,\mathrm{b}_{h+1}\ldots\mathrm{b}_m\}\,a_{h+1_x}\ldots a_{n_x}b_{h+1_y}\ldots b_{m_y}$$
$$= n!\,m!\frac{m!}{(m-h)!}S(a_1b_1u)\ldots(a_hb_hu)(m+n-2h)!\,\Delta^{m-h}D^{m-h}$$
$$\times a_{h+1_x}\ldots a_{n_x}b_{h+1_y}\ldots b_{m_y}.$$

Let us now suppose that $F = a_x^n b_y^m$, and that we may write

$$a_1 = a_2 = \ldots = a_n = a,\quad b_1 = b_2 = \ldots = b_m = b,$$

after all the substitutional operations have been performed on F; then

$$T_{h,n-h}F = (n!)^2(m!)^2\frac{m!}{(m-h)!}(m+n-2h)!\,(abu)^h\Delta^{m-h}D^{m-h}a_x^{n-h}b_y^{m-h},$$

and, as in the case of Gordan's series, we obtain

$$a_x^nb_y^m = \sum_{h=0}^{h=m}\frac{\binom{n}{h}\binom{m}{h}}{\binom{m+n-h+1}{h}}(abu)^h\Delta^{m-h}D^{m-h}a_x^{n-h}b_y^{m-h},$$

if $n \geqslant m$; if $n<m$, the summation must be taken from $h=0$ to $h=n$. The same series may be established in exactly the same way if a_x, b_y are p-ary symbolical factors, provided we write instead of (abu) the difference $a_xb_y-a_yb_x$.

(3) The series furnishes information concerning those functions F which satisfy the substitutional equations

$$\{a_1 b_1 b_2 \dots b_m\} F = 0,$$
$$\{a_2 b_1 b_2 \dots b_m\} F = 0,$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\{a_n b_1 b_2 \dots b_m\} F = 0.$$

For in this case $T_{\alpha, \beta} F = 0$, provided $\beta > 0$.

Hence

$$n!\,(m!)^2 \{a_1 a_2 \dots a_n\} \{b_1 b_2 \dots b_m\} F = T_{0,0} F = \frac{m+1-n}{m+1} T_{n,0} F \quad \text{or} = 0$$

according as $m+1$ is or is not greater than n. When n is greater than m,

$$\{a_1 a_2 \dots a_{m+1}\} \{b_1 b_2 \dots b_m\} F = 0.$$

15. Let the letters $a_1, a_2, \dots, a_n$ be arranged in any manner in horizontal rows, so that each row has its first letter in the same vertical column, its second letter in a second vertical column, and so on, and so that no row contains more letters than any row above it; then form the substitutional expression

$$S = \Gamma_1' \Gamma_2' \dots \Gamma_h' G_1 G_2 \dots G_k,$$

such that Γ_1' is the negative symmetric group of the letters of the first row, Γ_2' that of the letters of the second row, and so on, Γ_h' being that of the letters of the last row; and that G_1 is the positive symmetric group of the letters of the first column, G_2 that of the letters of the second column, and so on, G_k being that of the letters of the last column (it being understood, in case a row or column contains only one letter, that the positive or negative symmetric group of a single letter is unity). Then, if $T_{\alpha_1, \alpha_2, \dots, \alpha_h}$ be the sum of all expressions S formed as above from all possible tabular arrangements of the letters, so that there are α_1 letters in the first row, α_2 in the second, and so on, the α's satisfying

$$\alpha_1 + \alpha_2 + \dots + \alpha_h = n,$$

and

$$\alpha_1 \nless \alpha_2 \nless \alpha_3 \dots \nless \alpha_h,$$

it is possible to uniquely determine numerical coefficients $A_{\alpha_1, \alpha_2, \dots, \alpha_h}$ so that

$$1 = \Sigma A_{\alpha_1, \alpha_2, \dots, \alpha_h} T_{\alpha_1, \alpha_2, \dots, \alpha_h},$$

where the Σ extends to all possible positive integral values of $a_1, a_2, \ldots, a_h$ which satisfy the two conditions just laid down, the number h of a's not being fixed.

Let us suppose the terms T to be arranged in order, so that $T_{a_1, a_2, \ldots, a_h}$ will come before $T_{\beta_1, \beta_2, \ldots, \beta_{h'}}$, if $a_1 < \beta_1$, or if $a_1 = \beta_1$, but $a_2 < \beta_2$, or if $a_1 = \beta_1$, $a_2 = \beta_2$, ..., $a_{i-1} = \beta_{i-1}$, but $a_i < \beta_i$.

Consider one of the expressions S of which $T_{a_1, a_2, \ldots, a_h}$ is the sum, and the table of letters from which S is formed. Let N be the product of the negative symmetric groups of S, and P the product of its positive symmetric groups, so that

$$S = NP.$$

The degrees of the groups in N are $a_1, a_2, \ldots, a_h$; the degrees of the groups in P depend solely on these numbers, as may be seen from the table, for these groups are formed by the vertical columns in the table. Thus there are only h rows, so that there cannot be more than h elements in any column; in the first a_h columns there are exactly h elements, since the number of letters a_h in the last row is not greater than that in any row above. Next, there are $a_{h-1} - a_h$ columns containing exactly $h-1$ elements, and so on. Hence in P there are first a_h groups of degree h, then $a_{h-1} - a_h$ groups of degree $h-1$, and so on, there being a_1 groups altogether.

Let Γ' be any negative symmetric group which contains a pair of letters out of some one column in the table for S; then $P\Gamma' = 0$, for P contains this pair of letters in a positive symmetric group; and always, as has been seen in § 10,

$$\{abcd\ldots\}\{abc'd'\ldots\} = 0.$$

Again, if Γ' be of degree greater than a_1, then it must contain a pair of letters out of some one column in the table for S, for there are only a_1 different columns. Hence, if the degree of Γ' is greater than a_1, $P\Gamma' = 0$.

Now, let S_1 be one of the expressions of which $T_{\beta_1, \beta_2, \ldots, \beta_{h'}}$ is the sum, where $T_{\beta_1, \beta_2, \ldots, \beta_{h'}}$ is a term which comes after the term $T_{a_1, a_2, \ldots, a_h}$ when these terms are arranged in order; and let

$$S_1 = N_1 P_1,$$

where N_1 is the product of the negative symmetric groups of S_1, and P_1 that of the positive symmetric groups. Then

$$PN_1 = 0.$$

For, if $\beta_1 > \alpha_1$, N_1 contains a negative symmetric group Γ' of degree greater than α_1; and hence, as we have seen,

$$P\Gamma' = 0,$$

and therefore $$PN_1 = 0.$$

Now, since $T_{\beta_1, \beta_2, \ldots, \beta_{h'}}$ comes after $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$, then $\beta_1 > \alpha_1$; or $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$; or $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, ..., $\beta_{i-1} = \alpha_{i-1}$, but $\beta_i > \alpha_i$.

Let $$N_1 = \Gamma'_{\beta_1}\Gamma'_{\beta_2} \ldots \Gamma'_{\beta_{h'}},$$

the degrees of the different groups being equal to their suffixes. Suppose that $\beta_1 = \alpha_1$, and that Γ'_{β_1} contains no pair of letters which occur in any one column of the table for S (otherwise $PN_1 = 0$), and that $\beta_2 > \alpha_2$. Then Γ'_{β_1} contains one letter belonging to each of the columns, that is, one letter belonging to each of the $\beta_1 = \alpha_1$ groups of P. We will for the moment suppress all these letters belonging to Γ'_{β_1}. When this is done, let P become P', N_1 become N'_1; then P' and N'_1 are related in exactly the same way as P and N_1 are. Thus there are only α_2 groups in P', and α_2 columns in the table which gives P', for one letter from each group or column has been suppressed, and thus $\alpha_1 - \alpha_2$ groups have gone altogether. But all the β_2 letters of Γ'_{β_2} occur in the table for P'; and, since $\beta_2 > \alpha_2$, some one of the α_2 columns of P' must contain more than one of the letters of Γ'_{β_2}; hence

$$P'\Gamma'_{\beta_2} = 0.$$

But P is obtained from P' by adding α_1 new letters to its groups; and hence, if one of the groups of P' has a pair of letters in common with Γ'_{β_2}, the same is true for P; and therefore

$$P\Gamma'_{\beta_2} = 0,$$

and $$PN_1 = 0.$$

The argument is exactly the same in the general case

$$\beta_1 = \alpha_1,\ \beta_2 = \alpha_2,\ \ldots,\ \beta_{i-1} = \alpha_{i-1},\ \beta_i > \alpha_i.$$

The letters of each of the groups $\Gamma'_{\beta_1}, \Gamma'_{\beta_2}, \ldots, \Gamma'_{\beta_{i-1}}$ are suppressed, it being supposed that

$$P\Gamma_{\beta_1}\Gamma'_{\beta_2} \ldots \Gamma'_{\beta_{i-1}}$$

does not vanish. Then, if P and N_1 become P' and N'_1, these products are related to each other in the same way as P and N_1, and the necessary consequence of $\beta_i > \alpha_i$ becomes

$$P'N'_1 = 0,$$

for there is a group in N_1' which contains more letters than there are different columns in the table for P', and hence it must contain a pair of letters from the same column. Then P and N are obtained from P' and N_1' by adding new letters and new groups; but the letters in P' and N_1' are left undisturbed. Hence, if

$$P'N_1' = 0,$$

then
$$PN_1 = 0.$$

Hence, provided $T_{\beta_1, \beta_2, \ldots, \beta_{h'}}$ comes after $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$, when the terms are arranged in order,

$$PN_1 = 0;$$

and hence
$$PS_1 = PN_1P_1 = 0,$$

and
$$PT_{\beta_1, \beta_2, \ldots, \beta_{h'}} = P.\Sigma S_1 = 0;$$

therefore
$$NP.T_{\beta_1, \beta_2, \ldots, \beta_{h'}} = 0,$$

and
$$T_{\alpha_1, \alpha_2, \ldots, \alpha_h} T_{\beta_1, \beta_2, \ldots, \beta_{h'}} = (\Sigma NP)\, T_{\beta_1, \beta_2, \ldots, \beta_{h'}} = 0.$$

Let $t_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ represent the sum of all those substitutions of the group $\{a_1 a_2 \ldots a_n\}$ which are formed of h cycles of orders $\alpha_1, \alpha_2, \ldots, \alpha_h$ respectively. Then, from the way in which $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ is formed, viz., as the sum of the expressions obtained when the letters in the table are permuted in any way, but so that the number of letters in any row or column is unchanged, it follows that $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ is a function of the expressions $t_{\beta_1, \beta_2, \ldots, \beta_{h'}}$ only. That is, if it contains any one substitution s multiplied by some constant, it contains every substitution similar to s multiplied by the same constant. Hence

$$T_{\alpha_1, \alpha_2, \ldots, \alpha_h} = \Sigma \lambda_{\beta_1, \beta_2, \ldots, \beta_{h'}} t_{\beta_1, \beta_2, \ldots, \beta_{h'}},$$

where the λ's are constants.

Consider the coefficient of the identical substitution in the product

$$T_{\alpha_1, \alpha_2, \ldots, \alpha_h} . T_{\alpha_1, \alpha_2, \ldots, \alpha_h} \equiv T^2_{\alpha_1, \alpha_2, \ldots, \alpha_h}.$$

To obtain it we have to multiply each term λs of the first T by the term $\lambda' s^{-1}$, involving the inverse substitution, in the second factor. But every substitution is similar to its own inverse, and therefore, if s is a term of $t_{\beta_1, \beta_2, \ldots, \beta_{h'}}$, s^{-1} is also a term of this expression. It follows from the form just found for $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ that the coefficients of s and s^{-1} are the same. Consequently the coefficient of the identical substitution in $T^2_{a_1, a_2, \ldots, a_h}$ is

$$\Sigma \mu \lambda^2_{\beta_1, \beta_2, \ldots, \beta_{h'}},$$

where μ is the number of different substitutions in the sum

$$t_{\beta_1, \beta_2, \ldots, \beta_{h'}}.$$

Now, every term of $\Sigma \mu \lambda^2_{\beta_1, \beta_2, \ldots, \beta_{h'}}$ is essentially positive, for no unreal quantities can occur in the formation of T; this coefficient cannot then be zero. Consequently $T^2_{a_1, a_2, \ldots, a_h}$ does not vanish identically.

We can now prove that no relation exists between the T's; for, suppose that one such exists, of which the first term when the T's are arranged according to their proper order is $\lambda T_{a_1, a_2, \ldots, a_h}$. Multiply this equation by $T_{a_1, a_2, \ldots, a_h}$; then every term but the first vanishes: for $TT' = 0$ if T'' comes after T. Hence

$$\lambda T^2_{a_1, a_2, \ldots, a_h} = 0;$$

and therefore, by what we have just proved, $\lambda = 0$. Hence $T_{a_1, a_2, \ldots, a_h}$ cannot be the first term, and the relation is impossible.

The expression $t_{a_1, a_2, \ldots, a_h}$ has been defined as the sum of all the substitutions of the group $\{a_1 a_2 \ldots a_n\}$ which are formed of cycles whose orders are $a_1, a_2, \ldots, a_h$ respectively; if cycles order 1 are taken into consideration, the condition

$$a_1 + a_2 + \ldots + a_h$$

may be introduced. Further, the order of the a's in the suffixes of $t_{a_1, a_2, \ldots, a_h}$ is immaterial, so that they may be supposed to be in descending order of magnitude. Then $t_{a_1, a_2, \ldots, a_h}$ thus defined depends on exactly the same numbers as $T_{a_1, a_2, \ldots, a_h}$; hence there are the same number of expressions t as expressions T. Moreover, every T can be expressed in terms of the t's, and no relation can exist between the T's alone; so that we have the same number of independent linear equations as unknown quantities $t_{a_1, a_2, \ldots, a_h}$. It is then possible to solve; hence in general

$$t_{a_1, a_2, \ldots, a_h} = \Sigma \mu_{\beta_1, \beta_2, \ldots, \beta_{h'}} \,.\, T_{\beta_1, \beta_2, \ldots, \beta_{h'}},$$

where μ is numerical; and therefore in particular

$$1 = t_{1, 1, 1, \ldots, 1} = \Sigma A_{a_1, a_2, \ldots, a_h} \,.\, T_{a_1, a_2, \ldots, a_h}.$$

16. For $n = 2, 3, 4$, the work of finding the coefficients of the series by direct calculation is not too laborious: the results are

$$n=2,\quad 1=\tfrac{1}{2}\{a_1a_2\}+\tfrac{1}{2}\{a_1a_2\}';$$

$$n=3,\quad 1=\frac{1}{3!}\{a_1a_2a_3\}+\frac{1}{9}\Sigma\{a_1a_2\}'\{a_1a_3\}+\frac{1}{3!}\{a_1a_2a_3\}';$$

$$n=4,\quad 1=\frac{1}{4!}\{a_1a_2a_3a_4\}+\frac{1}{32}\Sigma\{a_1a_3\}'\{a_2a_3a_4\}$$

$$+\frac{1}{36}\Sigma\{a_1a_2\}'\{a_3a_4\}'\{a_2a_3\}\{a_1a_4\}$$

$$+\frac{1}{32}\Sigma\{a_1a_2a_3\}'\{a_3a_4\}+\frac{1}{4!}\{a_1a_2a_3a_4\}'.$$

It is worthy of remark too that, if N be the product of the negative and P that of the positive symmetric groups of one of the expressions of which T is the sum, then

$$T=\Sigma NP=\Sigma PN.$$

For
$$T=\Sigma NP=\lambda\Sigma PNP=\Sigma PN,$$

since PNP is equal to a numerical multiple of

$$(\Sigma N)\,P,$$

where ΣN is the sum of the different expressions obtained from N by operating on N with all the substitutions of P; for it was shown in § 10 that, if $S\,[a_1b_1b_2\ldots b_m]$ is any substitutional expression affecting the letters $a_1, b_1, b_2, \ldots, b_m$,

$$\{a_1a_2\ldots a_n\}\,S\,[a_1b_1\ldots b_m]\,\{a_1\ldots a_n\}$$
$$=(n-1)!\,\big[\Sigma S\,[a_1, b_1, \ldots b_m]\big]\,\{a_1\ldots a_n\},$$

the result stated here being an extension of this. In the same way, PNP is the same multiple of

$$P\,(\Sigma N).$$

It is easy now to show that, if T and T' be any two different terms of the sum of § 15, then

$$T\,.\,T'=0.$$

For, let
$$T=\Sigma NP,\quad T'=\Sigma N'P';$$

then, if T comes before T' in the series, it has been shown already that

$$T\,.\,T'=0.$$

Suppose, then, that T comes after T'; then

$$T.T' = [\Sigma NP][\Sigma N'P']$$
$$= [\Sigma PN][\Sigma P'N'];$$

but in this case

$$NP' = 0;$$

hence $$TT' = 0,$$

whenever T and T' are different.

Multiply now the series of § 15 by

$$T_{a_1, a_2, \ldots, a_h};$$

we then obtain $T_{a_1, a_2, \ldots, a_h} = A_{a_1, a_2, \ldots, a_h} . T^2_{a_1, a_2, \ldots a_h}$.

17. The theorem of § 15, like that of § 13, is purely a substitutional identity; algebraic theorems may be deduced from it by suitably choosing the operand.

(1) *Capelli's Theorem.*—Let the operand be the function $f(a_1, a_2, \ldots, a_n)$ of the n sets of variables

$$\begin{matrix} a_{1,1}, & a_{1,2}, & \ldots, & a_{1,m}, \\ a_{2,1}, & a_{2,2}, & \ldots, & a_{2,m}, \\ \ldots & \ldots & & \ldots \\ a_{n,1}, & a_{n,2}, & \ldots, & a_{n,m}, \end{matrix}$$

homogeneous and linear in the m variables of each set, such as was under discussion in § 12.

Let $\{a_1 a_2 \ldots a_a\}$ be the positive symmetric group of a of the sets; then

$$\{a_1 a_2 \ldots a_a\} f(a_1, a_2, \ldots, a_n)$$

may be obtained by means of polar operations only from the function

$$f(a_1, a_1, \ldots, a_1, a_{a+1}, a_{a+2}, \ldots, a_n).$$

For, if $\lambda a_{1,r_1} a_{2,r_2} \ldots a_{a,r_a} a_{a+1,r_{a+1}} \ldots a_{n,r_n}$ be any term of f, then

$$\{a_1 a_2 \ldots a_a\} \lambda a_{1,r_1} a_{2,r_2} \ldots a_{a,r_a} \ldots a_{n,r_n}$$
$$= D_{a_1 a_2} D_{a_1 a_3} \ldots D_{a_1 a_a} \lambda a_{1,r_1} a_{1,r_2} \ldots a_{1,r_a} a_{a+1,r_{a+1}} \ldots a_{n,r_n},$$

where D_{xy} is Capelli's operator

$$y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \dots + y_m \frac{\partial}{\partial x_m}.$$

Hence $\{\mathrm{a}_1 \mathrm{a}_2 \dots \mathrm{a}_a\} f = D_{a_1 a_2} D_{a_1 a_3} \dots D_{a_1 a_a} D_{a_2 a_1} D_{a_3 a_1} \dots D_{a_a a_1} f.$

And in the same way, if P be the product of β positive symmetric groups no two of which contain the same letter, and which between them contain all the letters $a_1, a_2, \dots, a_n$, groups of degree unity being taken into account, then Pf is a function which may be obtained by means of polar operations only from a function f_1 which contains only β variables, and f_1 is obtainable by means of polar operations only from f.

Again, it was shown in § 12 that

$$\{\mathrm{a}_1 \mathrm{a}_2 \dots \mathrm{a}_a\}' f = H_{a_1 a_2 \dots a_a} f.$$

Hence
$$T_{a_1, a_2, \dots, a_h} f = \Sigma H_{a_1} H_{a_2} \dots H_{a_h} \Delta f$$
$$= \Sigma \Delta H_{a_1} H_{a_2} \dots H_{a_h} f,$$

where, if
$$T = \Sigma NP,$$

H_{a_1} is that H which affects the letters contained in the negative symmetric group degree a_1 of N, H_{a_2} that which affects the letters of the group degree a_2, and so on, and where Δ is the polar operation corresponding to P the form of which we have shown how to find.

If it is required to expand a function $F(x, y, \dots, u)$ of m sets of variables, there being m variables in each set, which is homogeneous but not linear in the variables of the different sets, we may obtain from this a function

$$f(a_1, a_2, \dots, a_n)$$

homogeneous and linear in the variables of each of n sets, there being m variables in each set, such that, when we put

$$a_1 = a_2 = \dots = a_{\rho_1} = x, \quad a_{\rho_1+1} = \dots = a_{\rho_2} = y, \quad \dots\dots, \quad a_n = u,$$

f becomes F; this was shown in § 12. Now, f may be expanded as we have just seen; in the result, the variables of F may be substituted for those of f, and the expansion becomes that for F. This expansion is the same as that obtained by Capelli, and quoted in § 10.

For, if $a_1 < m$, the function $T_{a_1, a_2, \dots, a_h} f$ may be obtained from f_1 by means of polar operators only, where f_1 is a function of a_1 sets of variables, obtained from f by means of polar operators only. If $a_1 > m$, then

$$T_{a_1, a_2, \dots, a_h} f = 0.$$

And, if $a_1 = a_2 = \dots = a_i = m$, $a_{i+1} < m$, then $T_{a_1, a_2, \dots, a_h} f$ gives rise to a term $| xy \dots u |^i \phi$, where ϕ is a function obtained from a function of not more than $m-1$ variables by means of polar operations only, which is itself to be obtained by means of polar and Ω operations only from either f or F. For, in the expression P, where

$$T_{a_1, a_2, \dots, a_h} f = \Sigma PNf$$
$$= \Sigma P . H_{a_1} H_{a_2} \dots H_{a_h} f,$$

there are only a_{i+1} groups which affect the letters of

$$\Omega_{a_1} \Omega_{a_2} \dots \Omega_{a_i} f,$$

where by Ω_a is understood the Ω operator which affects the letters contained in H_a.

The expansion might otherwise be obtained, viz., by considering the function

$$F' = a_{1_x} a_{2_x} \dots a_{\rho_{1_x}} a_{\rho_1 + 1_y} \dots a_{nu},$$

where the factors of f are m-ary symbolical factors, and then proceeding in a similar manner to that in which Gordan's series was obtained in § 14.

(2) *Peano's Theorem.**—Starting from Capelli's theorem, Peano has proved the following:—"The complete system of concomitants for any number of binary n-ics may be obtained from that for n n-ics by polarization alone; with the one possible exception of that invariant which is the determinant of $n+1$ rows formed by the coefficients of $n+1$ n-ics." He then deduced that the number of concomitant types of a binary n-ic is finite; and proceeded to find the types for a binary cubic, showing that they all give irreducible forms for two cubics because the invariant determinant type referred to above is reducible for the cubic. I have quoted his results for the cubic in my paper, already referred to, on "The Irreducible Concomitants of any Number of Binary Quartics." Peano's theorem may be deduced

* *Atti di Torino*, t. XVII., p. 580.

directly from that of § 15:—Let F be a type of a binary m-ic of degree n, linear in the coefficients of each of n m-ics; then

$$F = \Sigma A_{a_1, a_2, \ldots, a_h} \cdot T_{a_1, a_2, \ldots, a_h} F.$$

If $a_1 > m+1$, $T_{a_1, a_2, \ldots, a_h} F = 0$; if $a_1 = m+1$, $T_{a_1, a_2, \ldots, a_h} F$ is the sum of terms each one of which has for a factor the determinant of $m+1$ rows formed from the coefficients of $m+1$ of the m-ics, and is in consequence reducible. If $a_1 < m+1$, then

$$T_{a_1, a_2, \ldots, a_h} F = [\Sigma PN] F,$$

where P is the product of a_1 positive symmetric groups, no two of which contain a common element, and which between them contain all the letters $a_1, a_1, \ldots, a_n$; it being possible that one or more of these groups is of degree unity. In this case PNF is a function obtained by polarization from a function F_1 of only a_1 sets of variables, where F_1 is a function obtained by polarization from F, as has been proved already. Hence $T_{a_1, a_2, \ldots, a_h} F$ is reducible, unless F gives an irreducible concomitant for a_1 m-ics; for concomitants obtained by polarization from reducible concomitants are themselves reducible. Hence, if F is a type of a binary m-ic, which gives no irreducible concomitant for m m-ics, it is reducible, unless F is the determinant of $m+1$ rows formed by the coefficients of $m+1$ of the m-ics. Now, if $a_1 = m$, then

$$T_{a_1, a_2, \ldots, a_h} F = [\Sigma \Gamma'_{a_1} S_1] F,$$

where Γ'_{a_1} is the negative symmetric group degree a_1 in each of the expressions $\Gamma'_{a_1} S_1$ of which T is the sum. But it has been shown, § 8, that, if Φ be a concomitant type of a binary m-ic, and if $\{a_1, a_2, \ldots, a_m\}'$ be the negative symmetric group of the letters $a_1, a_2, \ldots, a_m$, each letter referring to a different quantic, then

$$\{\mathrm{a}_1 \mathrm{a}_2 \ldots \mathrm{a}_m\}' \Phi = | \, a_1 a_2 \ldots a_m Q \, | ,$$

where Q refers to the coefficients of a concomitant type of order m, viz., $(Q_0, Q_1, \ldots, Q_m \between x_1, x_2)^m$. Hence, as Γ'_{a_1} is a negative symmetric group degree $a_1 = m$, in this case

$$T_{a_1, a_2, \ldots, a_h} F = \Sigma \, | \, a_1 a_2 \ldots a_m Q \, | .$$

And it follows that every rational integral concomitant of any number of m-ics can be expressed as a sum of terms each of which is a product of concomitants of types which give irreducible forms for $m-1$ m-ics, and of types of the form

$$| a_1 a_2 \dots a_m Q | ,$$

where $(Q_0, Q_1, \dots, Q_m \between x_1, x_2)^m$ is a covariant type order m. If $| a_1, a_2, \dots, a_{m+1} |$ is reducible as in the case of the cubic, it follows at once that $| a_1 a_2 \dots a_m Q |$ is reducible; and hence that all types which give no irreducible form for $m-1$ m-ics are reducible.

Similar results follow for ternary forms, and, in fact, for forms with any number of variables. Thus, for types of the ternary m-ic, we suppose, as before, that each letter refers to one m-ic, and that the coefficients of the m-ic a_1 are

$$a_{1,1}\, a_{1,2} \dots a_{1,\frac{1}{2}(m+1)(m+2)}.$$

Thus we are dealing in reality with functions of n sets of variables, there being $\frac{1}{2}(m+1)(m+2)$ variables in each set. Every type which gives no irreducible concomitant for $\frac{1}{2}(m+1)(m+2)-1$ m-ics is reducible, with the single exception of the determinant of $\frac{1}{2}(m+1)(m+2)$ rows formed by the coefficients of this number of m-ics.

Moreover, the proof has nothing to do with the fact that the functions are invariant; except that none of the operations employed destroy the property of invariance. Similar results might be deduced for other kinds of algebraic functions.

Again, if $F=0$ be a syzygy between types of a binary m-ic, then every term of $\Gamma' F$ vanishes when Γ' is a negative symmetric group of degree greater than $m+1$. Hence, expanding F by the theorem of § 15, it follows that every syzygy between types must give at least one syzygy, when not more than $m+1$ m-ics are under discussion, which does not reduce to a mere identity; with the exception of syzygies which are wholly due to the fact that

$$\Gamma' Q = 0,$$

where Γ' is a negative symmetric group degree greater than $m+1$, and Q is any product of m-ic types. For, suppose that $F=0$ is a syzygy which always reduces to an identity when less than $m+2$ binary m-ics are under discussion; then, if $a_1 < m+2$, each of the terms

$$T_{a_1, a_2, \dots, a}\, F$$

is identically zero. Further, if $a_1 > m+1$, each of the terms

$$T_{a_1, a_2, \dots, a_h}\, F$$

is zero, being the sum of terms such as $\Gamma' Q$ mentioned above. Hence

F, which is $= \Sigma A_{a_1, \dots, a_h} . T_{a_1, \dots, a_h} F$, is the sum of such terms, and $F = 0$ is a syzygy of that nature. As an example of a syzygy of this nature we have that between quadratic invariant types

$$[ab] = a_0 b_2 + a_2 b_0 - 2a_1 b_1,$$

viz.,

$$\{\text{bdfh}\}' [ab][cd][ef][gh] = 0.$$

(3) To find the system of concomitants for r binary m-ics. Let F be any type, then, if Γ'_{r+1} be any negative symmetric group degree $r+1$, of the letters $a_1, a_2, \dots, a_n$,

$$\Gamma'_{r+1} F = 0,$$

for there are not more than r different quantics represented by the letters, so that amongst $r+1$ letters at least two must refer to some one quantic. This is necessary; it is also sufficient, for

$$F = \Sigma A_{a_1, \dots, a_h} . T_{a_1, \dots, a_h} F,$$

and, if $a_1 > r_1$,

$$T_{a_1, \dots, a_h} F = 0;$$

but, if a_1 is equal to or less than r, the term is obtainable by polarization from a concomitant of not more than r m-ics. Hence in this case we take the ordinary relations for the type F, coupled with all possible equations of the form

$$\Gamma'_{r+1} F = 0.$$

(4) The complete solution of the simultaneous system of equations

$$\Gamma'_{r+1} F = R,$$

where Γ'_{r+1} is any negative symmetric group of degree $r+1$, of the letters $a_1, a_2, \dots, a_n$, and there is one equation for every combination of these letters $r+1$ at a time, is

$$F = \Sigma G_1 G_2 \dots G_\nu f + R', \quad \nu < r+1,$$

where $G_1, G_2, \dots, G_\nu$ are positive symmetric groups no two of which have a common letter, but which between them contain all the n letters, and R' is a function obtained from the R's by means of substitutions alone. This is evidently a solution, for, provided that R' is chosen so that

$$\Gamma'_{r+1} R' = R,$$

it satisfies each of the equations. Moreover

$$F = \Sigma A_{a_1, a_2, \dots, a_h} . T_{a_1, a_2, \dots, a_h} F,$$

and $T_{a_1, a_2, \ldots, a_h} F = R_1$, if $a_1 > r$, where R_1 is obtained in a definite manner from the given functions R, since $T_{a_1, a_2, \ldots, a_h} = \Sigma P.N$, where N contains as a factor a negative symmetric group degree a_1. And further, if $a_1 \leqslant r_1$,

$$T_{a_1, a_2, \ldots, a_h} = \Sigma PN,$$

where

$$P = G_1 G_2 \ldots G_\nu,$$

ν being $< r+1$, and the groups $G_1, G_2, \ldots, G_\nu$ having the character laid down above. The solution is then the complete one, and we see further that, in each term of the sum $\Sigma G_1 G_2 \ldots G_\nu f$, f is such that it may be obtained from F by means of the operation of the product N of certain definite negative symmetric groups.

Conversely, the complete solution of all possible equations of the form

$$G_1 G_2 \ldots G_\nu F = R, \quad \nu < r+1,$$

where the groups $G_1, G_2, \ldots, G_\nu$ are positive symmetric groups, no two of which contain a common letter, and which between them contain all the letters $a_1, a_2, \ldots, a_n$, is

$$F = \Sigma \Gamma'_{r+1} f + R',$$

where Γ'_{r+1} is a negative symmetric group degree $r+1$, and R' a function obtained from the R's by means of substitutions alone; and, further, the f for each term may be obtained from F by means of substitutions alone.

(5) In exactly the same way it may be shown that, if G_{r+1} is a positive symmetric group degree $r+1$, the solution of all possible equations of the form

$$G_{r+1} F = R$$

is

$$F = \Sigma \Gamma'_1 \Gamma'_2 \ldots \Gamma'_\nu f + R', \quad \nu < r+1.$$

(6) Suppose that G_r is the alternating group of certain r letters, that $G_r^{(1)}$ is the positive symmetric group of the same letters, and that $G_r^{(2)}$ is the negative symmetric group. Then, if

$$G_r F = 0,$$

it follows that $\quad G_r^{(1)} F = 0 \quad$ and $\quad G_r^{(2)} F = 0.$

For, if a and b are any two letters affected by G_r, then

$$G_r^{(1)} = \left[1 + (\mathrm{ab})\right] G_r$$

and

$$G_r^{(2)} = \left[1 - (\mathrm{ab})\right] G_r.$$

Consider, then, the simultaneous system of equations

$$G_r F = 0,$$

where the r letters affected by G_r are chosen in any manner from the letters $a_1, a_2, \ldots, a_n$, there being one equation for each combination of these letters r at a time; then

$$F = \Sigma A_{a_1, a_2, \ldots, a_h} \cdot T_{a_1, a_2, \ldots, a_h} F,$$

and all terms of this expansion vanish in which T possesses either positive or negative symmetric groups of degree $\geqslant r$. Hence, if

$$T_{a_1, a_2, \ldots, a_h} F$$

is not zero, $a_1 < r$ and $h < r$; for a_1 is the degree of the greatest negative symmetric group, and h that of the greatest positive symmetric group contained in T. Now

$$a_1 \not< a_2 \not< a_3 \not< \ldots \not< a_h;$$

and hence $$n = a_1 + a_2 + \ldots + a_h \not> h a_1;$$

and therefore, in order that both h and a_1 may be $< r$, we must have

$$n \not> (r-1)^2.$$

If therefore $n > (r-1)^2$, every term

$$T_{a_1, a_2, \ldots, a_h} F$$

is zero, and F itself is zero.

On Group-Characteristics. By W. Burnside.

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In a series of memoirs published in the *Berliner Sitzungsberichte* ("Über Gruppencharaktere," 1896, pp. 985–1021; "Über die Primfactoren der Gruppendeterminante," 1896, pp. 1343–1382; and others) Herr Frobenius has developed a theory of group-characteristics which must have a far-reaching importance in connexion with groups of finite order. For Abelian groups, an admirable account of the theory will be found in the second volume of Herr Weber's *Lehrbuch der Algebra.* The extension of the theory to non-