and therefore $$X = \frac{18.65}{6} + 8 + 3 = 206,$$

$$Y = 203,$$

agreeing with the values calculated independently.

The Genesis of the Double Gamma Functions. *By* E. W. BARNES, B.A., Fellow of Trinity College, Cambridge. Received December 5th, 1899. Communicated December 14th, 1899.

1. The following paper is the natural sequence of results obtained in two previous papers.

The "Theory of the Gamma Function"* contained a discussion of the function defined by the formula

$$\frac{1}{e^{\gamma z}\Gamma(z+1)} = \prod_{m=1}^{\infty}\left\{\left(1+\frac{z}{m}\right)e^{-\frac{z}{m}}\right\},$$

and it is evident that the expression on the right-hand side of this equality may be regarded as the positive half of the product expression for $\sin \pi z$; we may, in fact, term it the "halb-sinus" with Betti.†

Again, in the "Theory of the G Function,"‡ it was shown that

$$G(z) = e^{r(z)} z \prod_{m}{}_0^{\infty} \prod_{n}{}' \left\{\left(1+\frac{z}{m+n}\right)e^{-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}}\right\},$$

where $r(z)$ is a quadratic function of z.

If now we can associate with the letter m, each time that it occurs in this product, a complex constant τ which is not real and negative, we shall obtain a product which may be regarded as the positive quarter of the product expression for Weierstrass's function $\sigma(z)$, and which will be therefore a natural extension of the Γ function.

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* *Messenger of Mathematics*, Vol. XXIX., pp. 64 *et seq.*

† Klein (quoting Betti), *Ueber die hypergeometrische Function* (1894), p. 126.

‡ *Quarterly Journal of Mathematics*, Vol. XXXI., pp. 264 *et seq.*

Such a product we call a double gamma function $G(z \mid \tau)$. It is such that by suitable choice of the associated exponential factor, it satisfies the difference equation

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z).$$

It is evident that, when $\tau = 1$, the function reduces to the G function $G(z)$.

The existence of such functions has been surmised by Méray,* and indicated by Pincherle,† while Alexeiewsky‡ appears to have investigated some of their properties. The first two have not considered in detail any of the properties of these functions; and the last, so far as his results are accessible to me, does not appear to have entered into the essentials of the theory. He makes, for example, no mention of the gamma modular constants $C(\tau)$ and $D(\tau)$. The present notation was adopted before I had seen Alexeiewsky's paper, and his function $H(x, a)$ would be written $G(z \mid \tau)$ in the notation of this paper.

As indicated in the title, I only consider in the present paper the genesis of the double gamma functions. Several different product expressions are given for $G(z \mid \tau)$; the gamma modular constants are shown to be transcendental functions of τ; it is shown that $G(z \mid \tau)$ satisfies the two difference equations

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z),$$

$$f(z+\tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{-z+\frac{1}{2}} \Gamma(z) f(\tau);$$

and, finally, the connexion is indicated between these functions, Appell's generalization of the Eulerian functions, and the theta functions.

In a subsequent paper I propose to give in complete detail a symmetric theory of double gamma functions, in which τ is replaced by parameters ω_1 and ω_2, as in the theory of elliptic functions.

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* Méray, *L'Analyse Infinitesimale*, Deuxième Partie, concluding pages.

† Pincherle, *Comptes Rendus*, Tome CVI., p. 266.

‡ Alexeiewsky, *Ann. de l'Imp. Univ. de Charkow*, 1889, as quoted in the *Jahrbuch über die Fortschritte der Mathematik*, Vol. XXII., p. 439. A synopsis of this paper appears in the *Leipzig Berichte*, 1894, Vol. XLVI., pp. 268–295.

2. We will first take the product

$$G(z \mid \tau) = Ae^{a\frac{z}{\tau} + b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \prod_m{}_0^{\infty} \prod_n{}' \left\{ \left(1 + \frac{z}{m\tau + n}\right) e^{-\frac{z}{m\tau+n} + \frac{z^2}{2(m\tau+n)^2}} \right\},$$

in which τ is any constant, real or complex, which is not real and negative, and in which a and b are functions of τ only.

We notice that each term of the product is of Weierstrass's form, and that, by Eisenstein's theorem, the product is absolutely convergent.

We proceed to transform this product into one of different form. Since we may group the terms of the product as we please, we have

$$G(z \mid \tau) = Ae^{a\frac{z}{\tau} + b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \prod_{m=1}^{\infty} \left\{ \left(1 + \frac{z}{m\tau}\right) e^{-\frac{z}{m\tau} + \frac{z^2}{2m^2\tau^2}} \right\}$$

$$\times \prod_{m=0}^{\infty} \prod_{n=1}^{\infty} \left\{ \frac{\left(1 + \frac{z+m\tau}{n}\right)}{\left(1 + \frac{m\tau}{n}\right)} \frac{e^{-\frac{z+m\tau}{n}}}{e^{-\frac{m\tau}{n}}} e^{-\frac{z}{m\tau+n} + \frac{z}{n} + \frac{1}{2}\frac{z^2}{(m\tau+n)^2}} \right\},$$

and hence, remembering that

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{m=1}^{\infty} \left\{ \left(1 + \frac{z}{m}\right) e^{-\frac{z}{m}} \right\},$$

we see that

$$G(z \mid \tau) = Ae^{a\frac{z}{\tau} + b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \frac{e^{-\gamma\frac{z}{\tau}}}{\Gamma\left(\frac{z}{\tau} + 1\right)} e^{\frac{z^2}{2\tau^2} \sum_{m=1}^{\infty} \frac{1}{m^2}}$$

$$\times \prod_{m=0}^{\infty} \left\{ \frac{\Gamma(1+m\tau)}{\Gamma(1+z+m\tau)} e^{-\gamma z + z \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+m\tau}\right) + \frac{z^2}{2} \sum_{n=0}^{\infty} \frac{1}{(m\tau+n)^2}} \right\}$$

$$= Ae^{(a-\gamma)\frac{z}{\tau} + \frac{z^2}{2\tau^2}\left(b + \frac{\pi^2}{6}\right)} \Gamma^{-1}\left(\frac{z}{\tau}\right)$$

$$\times \prod_{m=0}^{\infty} \left\{ \frac{\Gamma(1+m\tau)}{\Gamma(1+z+m\tau)} e^{-\gamma z + z \sum_{n=1}^{\infty} \frac{m\tau}{n(n+m\tau)} + \frac{z^2}{2} \sum_{n=0}^{\infty} \frac{1}{(m\tau+n)^2}} \right\}.$$

But, from the product expression

$$\Gamma^{-1}(1+m\tau) = e^{\gamma m\tau} \prod_{m=1}^{\infty} \left\{ \left(1 + \frac{m\tau}{n}\right) e^{-\frac{m\tau}{n}} \right\},$$

we obtain, if we put

$$\psi(x) = \frac{d}{dx} \log \Gamma(x)$$

in conformity with Gauss's notation,

$$-\psi(1+m\tau) = \gamma + \sum_{n=1}^{\infty} \left\{ \frac{1}{m\tau+n} - \frac{1}{n} \right\},$$

and, if

$$\psi'(x) = \frac{d}{dx}\psi(x),$$

$$\psi'(1+m\tau) = \sum_{n=1}^{\infty} \frac{1}{(m\tau+n)^2}.$$

Thus we may write

$$\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau) = Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)}$$
$$\times \prod_{m=0}^{\infty} \left\{ \frac{\Gamma(1+m\tau)}{\Gamma(z+1+m\tau)} e^{z\psi(1+m\tau)+\frac{z^2}{2}\psi'(1+m\tau)} \right\}.$$

A form equivalent to this is incorrectly given by Alexeiewsky.

We may conveniently modify this expression slightly.

Since

$$\Gamma(z+1) = z\Gamma(z),$$

$$\psi(1) = -\gamma \quad \text{and} \quad \psi'(1) = \frac{\pi^2}{6},$$

we shall have

$$\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau) = Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} \frac{1}{\Gamma(z+1)} e^{-\gamma z+\frac{z^2}{2}\frac{\pi^2}{6}}$$
$$\times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} \frac{m\tau}{z+m\tau} e^{\frac{z}{\tau m}+z\psi(\tau m)-\frac{z^2}{2\tau^2 m^2}+\frac{z^2}{2}\psi'(\tau m)} \right\},$$

and thus

$$\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau) = Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} e^{-\gamma z+\frac{z^2}{2}\frac{\pi^2}{6}-\frac{z^2}{2\tau^2}\frac{\pi^2}{6}}$$
$$\times \Gamma\left(\frac{z}{\tau}+1\right) e^{\gamma\frac{z}{\tau}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau)+\frac{z^2}{2}\psi'(m\tau)} \right\};$$

or, finally,

$$G(z \mid \tau) = \frac{A}{\Gamma(z+1)} e^{(a-\gamma)z+\frac{z^2}{2}\left(\frac{b}{\tau^2}+\frac{\pi^2}{6}\right)} \frac{z}{\tau} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau)+\frac{z^2}{2}\psi'(m\tau)} \right\}.$$

It will be noted that this last product, as all employed in the transformation, is absolutely convergent.

Before we proceed to show that, for suitable values of a and b, the function $G(z \mid \tau)$ satisfies the difference equation

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z),$$

it is necessary to interpolate two algebraical limit theorems analogous to those considered in the "Theory of the G Function," §§ 6 and 7.

3. We will first show that, when τ is not real and negative,

$$\underset{m=\infty}{\text{Lt}} \{\psi(\tau) + \psi(2\tau) + \ldots + \psi(m\tau)\}$$
$$= C(\tau) + (m + \tfrac{1}{2} - \tfrac{1}{2}\tau) \log m - m + \frac{1-2\tau}{12\tau^2 m} + \ldots,$$

where $C(\tau)$ is a definite finite function of τ, independent of m. Unless the contrary is explicitly stated, the logarithms have their principal values, in which the imaginary part lies between $\pm\pi i$.

As there is no formula to express $\psi(\overline{m+1}\,\tau)$ in terms of $\psi(m\tau)$ for general values of τ, we cannot extend the method formerly employed. We therefore use the Maclaurin sum-formula,*

$$\Sigma u_x = C + \int u_x dx + \tfrac{1}{2} u_x + \tfrac{1}{12} \frac{du_x}{dx} - \tfrac{1}{720} \frac{d^3 u_x}{dx^3} + \ldots.$$

Put $u_x = \psi(\tau x)$, and we obtain

$$\sum_{x=1}^{m} \psi(\tau x) = C' + \frac{1}{\tau} \log \Gamma(\tau m) + \tfrac{1}{2}\psi(\tau m) + \frac{\tau}{12}\psi'(\tau m) + \ldots.$$

Now, provided τ be not real and negative, we have, by Stieltjes' theorem,† when m is very large, the asymptotic equality

$$\log \Gamma(\tau m) = \tfrac{1}{2} \log 2\pi + (\tau m - \tfrac{1}{2}) \log \tau m - \tau m + \frac{1}{12\tau m} - \ldots;$$

and therefore the derived asymptotic equalities

$$\psi(\tau m) = \log \tau m - \frac{1}{2\tau m} - \frac{1}{12\tau^2 m^2} \ldots,$$

$$\psi'(\tau m) = \frac{1}{\tau m} + \frac{1}{2\tau^2 m^2} + \ldots.$$

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* Boole, *Finite Differences*, § 2, p. 90.

† "Theory of the Gamma Function," Part IV.

Hence we have, when m is very large,

$$\psi(\tau)+\psi(2\tau)+\ldots+\psi(m\tau)$$

$$= C'+\log\frac{\sqrt{2\pi}}{\tau}+\frac{\tau m-\frac{1}{2}}{\tau}\log\tau m+\tfrac{1}{2}\log\tau m+\frac{1}{12\tau^2 m}-\frac{1}{4\tau m}+\frac{1}{12\tau m}+\ldots$$

$$= C(\tau)+\left(m+\tfrac{1}{2}-\frac{1}{2\tau}\right)\log\tau m-m+\frac{1-2\tau}{12\tau^2 m}+\ldots,$$

where $C(\tau)$ is a definite function of τ independent of m.

From the "Theory of the G Function," § 6, we see that

$$C(1)=\tfrac{1}{2}.$$

4. We will next show that, when τ is not real and negative, and m is a large positive integer,

$$\psi'(\tau)+\psi'(2\tau)+\ldots\ \psi'(m\tau)=D(\tau)+\frac{1}{\tau}\log\tau m+\frac{\tau-1}{2\tau^2 m}+\ldots,$$

where $D(\tau)$ is a definite function of τ independent of m.

On putting $u_x=\psi'(\tau x)$ in the Maclaurin sum formula, we have at once

$$\psi'(\tau)+\psi'(2\tau)+\ldots+\psi'(m\tau)$$

$$=D(\tau)+\frac{1}{\tau}\,\psi(m\tau)+\tfrac{1}{2}\psi'(m\tau)+\frac{\tau}{12}\psi''(m\tau)+\ldots.$$

Hence, using the asymptotic equalities,

$$\psi(\tau m)=\log\tau m-\frac{1}{2\tau m}-\frac{1}{12\tau^2 m^2}\ldots,$$

$$\psi'(\tau m)=\frac{1}{\tau m}+\frac{1}{2\tau^2 m^2}+\ldots,$$

we find

$$\psi'(\tau)+\psi'(2\tau)+\ldots+\psi'(m\tau)$$

$$=D(\tau)+\frac{1}{\tau}\log\tau m-\frac{1}{2\tau^2 m}+\ldots+\frac{1}{2\tau m}+\frac{1}{4\tau^2 m^2}+\ldots$$

$$=D(\tau)+\frac{1}{\tau}\log\tau m+\frac{\tau-1}{2\tau^2 m}+\ldots.$$

On making $\tau=1$, we see that

$$D(1)=1+\gamma.$$

5. The forms $C(\tau)$ and $D(\tau)$ will enter into the theory of double gamma functions from whatever side we may approach it. From the value which $D(\tau)$ assumes when $\tau = 1$, it might be anticipated that this function cannot be expressed in finite form or in terms of elementary transcendents. We proceed to show that this is actually the case; by an analogous process the same theorem might be proved to hold with regard to $C(\tau)$.

Suppose that m and n are large positive integers, and that $\frac{m}{n}$ is very small, and consider the function

$$\sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(m_1\tau+n_1)^2},$$

the accent denoting that the term in the summation for which $\left.\begin{matrix} m_1 = 0 \\ n_1 = 0 \end{matrix}\right\}$ is to be omitted.

We have seen that

$$\psi'(1+m_1\tau) = \sum_{n_1=1}^{\infty} \frac{1}{(m_1\tau+n_1)^2} = \sum_{n_1=1}^{n} \frac{1}{(m_1\tau+n_1)^2} + \sum_{n_1=1}^{\infty} \frac{1}{(m_1\tau+n+n_1)^2},$$

and hence

$$\sum_{n_1=1}^{n} \frac{1}{(m_1\tau+n_1)^2} = \psi'(1+m_1\tau) - \psi'(1+m_1\tau+n),$$

so that, by the asymptotic equality used in § 3,

$$\sum_{n_1=1}^{n} \frac{1}{(m_1\tau+n_1)^2} = \psi'(1+m_1\tau) - \frac{1}{1+m_1\tau+n} + \dots$$

$$= \psi'(1+m_1\tau) - \frac{1}{n} + \frac{(\dots)}{n^2} + \dots,$$

since m_1 is small compared with m. Thus

$$\sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(m_1\tau+n_1)^2}$$

$$= \sum_{m_1=1}^{m} \frac{1}{(m_1\tau)^2} + \sum_{n_1=1}^{n} \frac{1}{n_1^2} + \sum_{m_1=1}^{m} \left\{ \psi'(1+m_1\tau) - \frac{1}{n} + \frac{(\dots)}{n^2} + \dots \right\}$$

$$= \frac{\pi^2}{6} - \frac{1}{n} + \dots + \sum_{m_1=1}^{m} \left\{ \psi'(m_1\tau) - \frac{1}{n} + \frac{(\dots)}{n^2} + \dots \right\}$$

$$= \frac{\pi^2}{6} + D(\tau) + \frac{1}{\tau}\log(\tau m) - \frac{m+1}{n} + \frac{\tau-1}{2\tau^2 m} + \dots .$$

Now

$$\sum_{m_1=-m}^{m} \sum_{n_1=-n}^{n}{}' \frac{1}{(m_1\tau+n_1)^2}$$

$$= 2\sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(m_1\tau+n_1)^2} + 2\sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(-m_1\tau+n_1)^2}$$

$$- \sum_{m_1=1}^{m} \frac{1}{(m_1\tau)^2} - \sum_{n_1=1}^{n} \frac{1}{n^2},$$

as a graphical representation of the terms of the series will readily show.

Hence, when m and n are large positive integers, and τ is any complex quantity,

$$\sum_{m_1=-m}^{m} \sum_{n_1=-n}^{n}{}' \frac{1}{(m_1\tau+n_1)^2}$$

$$= 2\left\{D(\tau)+\frac{\pi^2}{6}+\frac{1}{\tau}\log \tau m - \frac{\tau-1}{2\tau^2 m} - \frac{m+1}{n}\ldots\right\}$$

$$+2\left\{D(-\tau)+\frac{\pi^2}{6}-\frac{1}{\tau}\log(\tau m) \mp \frac{\pi\iota}{\tau} + \frac{\tau+1}{2\tau^2 m} - \frac{m+1}{n}+\ldots\right\}$$

$$-\frac{\pi^2}{6}\left(1+\frac{1}{\tau^2}\right) - \frac{1}{m\tau^2} - \frac{1}{n} - \ldots$$

$$= 2\{D(\tau)+D(-\tau)\} + \frac{\pi^2}{6}\left(1-\frac{1}{\tau^2}\right) \mp \frac{2\pi\iota}{\tau}$$

+ terms which vanish when m and n become infinite.

The upper or lower sign must be taken as $R(\iota\tau)$ is positive or negative. Now, by a theorem* due to Forsyth, when m and n become infinite, $\frac{m}{n}$ being small,

$$\text{Lt} \sum_{m_1=-m}^{m} \sum_{n_1=-n}^{n}{}' \frac{1}{\left(m_1\frac{\iota K'}{K}+n_1\right)^2} = \left\{\frac{E}{K}-\tfrac{1}{3}(1+k^2)\right\} 4K^2,$$

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* Forsyth, "Some Doubly Infinite Converging Series," *Quarterly Journal of Mathematics*, Vol. XXI., p. 263.

in the usual notation of elliptic functions. Hence, if

$$\tau = \frac{\iota K'}{K},$$

so that $R(\iota\tau)$ is negative,

$$D(\tau) + D(-\tau) = -\frac{\pi^2}{12} - \frac{\pi i}{\tau} + \frac{\pi^2}{12\tau^2} + 2EK - \tfrac{2}{3}K^2(1+k'^2),$$

an expression which is usually called a modular function of τ, and which does not in general admit of representation in finite form by elementary transcendents.

We propose then to call $C(\tau)$ and $D(\tau)$ double gamma modular functions of τ.

We shall subsequently express equivalent symmetrical functions as definite integrals.

6. We are now in a position to prove that, for suitable values of the τ-functions a and b,

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau).$$

We have established in § 2 that, if

$$a' = a - \gamma\tau,$$

$$b' = b + \frac{\pi^2\tau^2}{6},$$

$$G(z \mid \tau) = \frac{A}{\tau\Gamma(z)} e^{a'\frac{z}{\tau} + b'\frac{z^2}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + \frac{z^2}{2}\psi'(m\tau)} \right\}.$$

Hence

$$\frac{G(z+1 \mid \tau)}{G(z \mid \tau)} = \frac{1}{z} e^{\frac{a'}{\tau} + b'\frac{2\tau+1}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{1}{z+m\tau} e^{\psi(m\tau) + \frac{2z+1}{2}\psi'(m\tau)} \right\}$$

$$= \frac{1}{z} e^{\frac{a'}{\tau} + b'\frac{2z+1}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{1}{\left(1+\frac{z}{m\tau}\right) e^{-\frac{z}{m\tau}}} \frac{e^{\frac{z}{m\tau} + \psi(m\tau) + \frac{2z+1}{2}\psi'(m\tau)}}{m\tau} \right\}$$

$$= \frac{1}{z} e^{\frac{a'}{\tau} + \frac{b'(2z+1)}{2\tau^2}} \Gamma\left(\frac{z}{\tau}+1\right) e^{\gamma\frac{z}{\tau}}$$

$$\times \operatorname*{Lt}_{m=\infty} \left\{ \frac{m^{-\frac{z}{\tau}}}{m!\,\tau^m} e^{-\frac{\gamma z}{\tau} + \sum_{r=1}^{m} \left[\psi(r\tau) + \frac{2z+1}{2}\psi'(r\tau)\right]} \right\},$$

and thus

$$\frac{G(z+1 \mid \tau)}{\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)} = \frac{1}{\tau} e^{\frac{a'}{\tau}+\frac{\gamma z}{\tau}+\frac{b'}{2\tau^2}(2z+1)}$$

$$\times \operatorname*{Lt}_{m=\infty} \left\{ \frac{1}{(2\pi)^{\frac{1}{2}}} m^{-m-\frac{1}{2}} \tau^{-m} e^{m-\frac{\gamma z}{\tau}} m^{-\frac{z}{\tau}} \right\}$$

$$\times \operatorname*{Lt}_{m=\infty} \left\{ e^{C(\tau)+\left(m+\frac{1}{2}-\frac{1}{2\tau}\right) \log m\tau - m + (z+\frac{1}{2}) [D(\tau)+\frac{1}{\tau} \log \tau m]} \right\},$$

on using the limits which have been investigated in §§ 3 and 4. Hence

$$\frac{G(z+1 \mid \tau)}{\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)} = \frac{\tau^{\frac{z}{\tau}-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} e^{C(\tau)+\frac{1}{2}D(\tau)+\frac{a'}{\tau}+\frac{b'}{2\tau^2}+z\left[D(\tau)+\frac{b'}{\tau^2}\right]},$$

and thus we shall have for $G(z \mid \tau)$ the difference equation

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)$$

provided we choose a' and b' so that

$$D(\tau) + \frac{b'}{\tau^2} + \frac{1}{\tau} \log \tau = 0$$

and

$$C(\tau) + \tfrac{1}{2} D(\tau) + \frac{a'}{\tau} + \frac{b'}{2\tau^2} = \tfrac{1}{2} \log (2\pi\tau)$$

and thus we must take

$$a' = \frac{\tau}{2} \log (2\pi\tau) + \tfrac{1}{2} \log \tau - \tau C(\tau),$$

$$b' = -\tau \log \tau - \tau^2 D(\tau);$$

or, finally,

$$a = \frac{\tau}{2} \log (2\pi\tau) + \tfrac{1}{2} \log \tau - \tau C(\tau) + \gamma\tau,$$

$$b = -\tau \log \tau - \tau^2 D(\tau) - \frac{\pi^2\tau^2}{6}.$$

We have now, by § 2,

$$G(z \mid \tau) = \frac{Az}{\tau} e^{-zC(\tau)-\frac{z^2}{2}D(\tau)+\gamma z-\frac{\pi^2 z^2}{12}}$$

$$\times (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} \prod_{m=0}^{\infty} \prod_{n=0}^{\infty}{}' \left\{ \left(1+\frac{z}{\Omega}\right) e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}} \right\},$$

where

$$\Omega = m\tau + n.$$

When $r = 1$, we have, using the values of $C(1)$ and $D(1)$ given in §§ 3 and 4, respectively,

$$a = \gamma - \tfrac{1}{2} + \tfrac{1}{2}\log 2\pi,$$

$$b = -\left(\frac{\pi^2}{b} + 1 + \gamma\right),$$

and hence, when $r = 1$, we have

$$G(z \mid 1) = Az\,(2\pi)^{\frac{z}{2}}\, e^{z(\gamma - \frac{1}{2}) - \frac{z^2}{2}\left(\frac{\pi^2}{6} + 1 + \gamma\right)}$$

$$\times \prod_{m=0}^{\infty} \prod_{n=0}^{\infty}{}' \left\{ \left(1 + \frac{z}{m+n}\right) e^{-\frac{z}{m+n} + \frac{z^2}{2(m+n)^2}} \right\},$$

an expression which agrees with that previously found for $G(z)$.

As a *corollary*, note that we have incidentally proved that

$$\prod_{m=1}^{\infty} \left\{ \frac{1}{z + m\tau} e^{\psi(m\tau) + \frac{2z+1}{2}\psi'(m\tau)} \right\} = \Gamma\left(\frac{z}{\tau} + 1\right) \frac{\tau^{\frac{z}{\tau} + \frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} e^{C(\tau) + (z + \frac{1}{2}) D(\tau)}.$$

7. We now proceed to determine the constant A by assigning the condition that

$$G(1 \mid \tau) = 1.$$

We have, from § 2,

$$G(z \mid \tau) = \frac{A}{\tau\Gamma(z)} e^{a'\frac{z}{\tau} + b'\frac{z^2}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z + m\tau)} e^{z\psi(m\tau) + \frac{z^2}{2}\psi'(m\tau)} \right\},$$

and it has just been seen in § 6 that

$$a' = \frac{\tau}{2}\log 2\pi\tau + \tfrac{1}{2}\log\tau - \tau C(\tau),$$

$$b' = -\tau\log\tau - \tau^2 D(\tau).$$

Hence, if we make $z = 1$, and assign the condition $G(1 \mid \tau) = 1$, we find

$$1 = \frac{A}{\tau} e^{\frac{a'}{\tau} + \frac{b'}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{1}{m\tau} e^{\psi(m\tau) + \frac{1}{2}\psi'(m\tau)} \right\}.$$

But, if we put $z = 0$ in the corollary to § 6, we have

$$\prod_{m=1}^{\infty} \left\{ \frac{1}{m\tau} e^{\psi(m\tau) + \frac{1}{2}\psi'(m\tau)} \right\} = \frac{1}{(2\pi)^{\frac{1}{2}}} \tau^{\frac{1}{2}} e^{C(\tau) + \frac{1}{2}D(\tau)}.$$

We thus find $A = 1$.

8. It is possible to give a third product expression for $G(z \mid \tau)$. To obtain this expression we take the formula of § 2,

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \prod_{m=0}^{\infty} \prod_{n=0}^{\infty}{}' \left\{ \left(1+\frac{z}{m\tau+n}\right) e^{-\frac{z}{m\tau+n}+\frac{z^2}{2(m\tau+n)^2}} \right\},$$

where (§ 6) $\qquad a = \frac{\tau}{2} \log 2\pi\tau + \frac{1}{2}\log\tau + \gamma\tau - C(\tau),$

$$b = -\tau\log\tau - \tau^2 D(\tau) - \frac{\pi^2\tau^2}{6},$$

and we write it in the form

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \prod_{n=1}^{\infty} \left\{ \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}+\frac{z^2}{2n^2}} \right\}$$

$$\times \prod_{n=0}^{\infty} \prod_{m=1}^{\infty} \left\{ \frac{1+\frac{z+n}{m\tau}}{1+\frac{n}{m\tau}} \frac{e^{-\frac{z+n}{m\tau}}}{e^{-\frac{n}{m\tau}}} e^{\frac{z}{m\tau}-\frac{z}{m\tau+n}+\frac{z^2}{2(m+\tau n)^2}} \right\},$$

as we may obviously do, since each term is of Weierstrass's form; and now we have

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{z}{\tau} e^{-\gamma z} \frac{1}{z\Gamma(z)} e^{\frac{\pi z^2}{12}}$$

$$\times \prod_{n=0}^{\infty} \left\{ \frac{\Gamma\left(1+\frac{n}{\tau}\right)}{\Gamma\left(1+\frac{z+n}{\tau}\right)} e^{-\gamma\frac{z}{\tau}+z\sum_{m=1}^{\infty}\left(\frac{1}{m\tau}-\frac{1}{m\tau+n}\right)+\frac{z^2}{2}\sum_{m=1}^{\infty}\frac{1}{(m\tau+n)^2}} \right\}.$$

But, if, as usual, $\qquad \psi(z) = \frac{d}{dz}\log\Gamma(z),$

$$\psi'(z) = \frac{d^2}{dz^2}\log\Gamma(z),$$

we have $\qquad -\psi\left(1+\frac{n}{\tau}\right) = \gamma + \sum_{m=1}^{\infty}\left(\frac{\tau}{n+m\tau}-\frac{1}{m}\right),$

$$\psi'\left(1+\frac{n}{\tau}\right) = \sum_{m=1}^{\infty}\frac{\tau^2}{(n+m\tau)^2},$$

and hence

$$G(z \mid \tau) = e^{a\frac{z}{\tau} + b\frac{z^2}{2\tau^2}} \frac{1}{\tau\Gamma(z)} e^{-\gamma z + \frac{\pi^2 z^2}{12}}$$

$$\times \prod_{n=0}^{\infty} \left\{ \frac{\Gamma\left(1+\frac{n}{\tau}\right)}{\Gamma\left(1+\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi\left(1+\frac{n}{\tau}\right) + \frac{z^2}{2\tau^2}\psi'\left(1+\frac{n}{\tau}\right)} \right\}$$

We may slightly modify this expression by writing

$$G(z \mid \tau) = e^{z\left(\frac{a}{\tau}-\gamma\right) + \frac{z^2}{2}\left(\frac{b}{\tau^2}+\frac{\pi^2}{6}\right)} \frac{1}{\tau\Gamma(z)\,\Gamma\left(1+\frac{z}{\tau}\right)} e^{\frac{z}{\tau}\psi(1) + \frac{z^2}{2\tau^2}\psi'(1)}$$

$$\times \prod_{n=1}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} \frac{n}{z+n} e^{\frac{z}{\tau}\psi\left(\frac{n}{\tau}\right) + \frac{z^2}{2\tau^2}\psi'\left(\frac{n}{\tau}\right) + \frac{z}{n} - \frac{z^2}{2n^2}} \right\},$$

and now, since

$$\psi(1) = -\gamma,$$

$$\psi'(1) = \frac{\pi^2}{6},$$

and

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau),$$

we obtain, finally,

$$G(z+1 \mid \tau) = e^{z\left(\frac{a-\gamma}{\tau}\right) + \frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} \prod_{n=1}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi\left(\frac{n}{\tau}\right) + \frac{z^2}{2\tau^2}\psi'(n)} \right\},$$

which yields, on substituting the values a and b,

$$G(z+1 \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{z\left\{\gamma - \frac{\gamma}{\tau} - C(\tau)\right\} + \frac{z^2}{2}\left\{\frac{\pi^2}{6}(1-\tau^2) - D(\tau)\right\}}$$

$$\times \prod_{n=}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi\left(\frac{n}{\tau}\right) + \frac{z^2}{2\tau^2}\psi'\left(\frac{n}{\tau}\right)} \right\}.$$

9. Recapitulating the results which have now been obtained, we see that a solution of

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z), \qquad (1)$$

with the condition $$f(1) = 1,$$

is given by

$$G(z \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{z\{\gamma - C(\tau)\} - \frac{z^2}{2}\{\frac{\pi^2}{6} + D(\tau)\}} \frac{z}{\tau}$$

$$\times \prod_{m}^{\infty}{}_0 \prod_{n=0}^{\infty}{}' \left\{ \left(1 + \frac{z}{m\tau + n}\right) e^{-\frac{z}{m\tau+n} + \frac{z^2}{2(m\tau+n)^2}} \right\},$$

where $C(\tau)$ and $D(\tau)$ are certain double gamma modular constants.

The general solution of the difference equation (1) is

$$G(z \mid \tau) \times F(e^{2\pi i z}),$$

where $F(e^{2\pi i z})$ is any function of z simply periodic of period unity.

The function $G(z \mid \tau)$ may also be expressed as an infinite product of gamma functions of arguments differing by multiples of τ in the form

$$G(z \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{-zC(\tau) - \frac{z^2}{2} D(\tau)} \frac{1}{\tau\Gamma(z)}$$

$$\times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + \frac{z^2}{2}\psi'(m\tau)} \right\},$$

and again as an infinite product of gamma functions of arguments differing by multiples of $\frac{1}{\tau}$ in the form

$$G(z+1 \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{z\{\gamma - \frac{\gamma}{\tau} - C(\tau)\} + \frac{z^2}{2}\{\frac{\pi^2}{6}(1-\tau^2) - D(\tau)\}}$$

$$\times \prod_{n=1}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi(n\tau) + \frac{z^2}{2\tau^2}\psi'\left(\frac{n}{\tau}\right)} \right\}.$$

We might at this stage obtain the first terms of the value to which $G(z \mid \tau)$ tends, as z tends to real positive infinity, employing a method similar to that used in the theory of the G function, §§ 3 and 4. The results of such an investigation would, however, be incomplete, and it is therefore more convenient to employ the more powerful methods which will subsequently be adopted.

10. It is now possible for us to prove the fundamentally important theorem

$$G(z+\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{-z+\frac{1}{2}} \Gamma(z)\, G(z \mid \tau),$$

where
$$\tau^{-z+\frac{1}{2}} = e^{(-z+\frac{1}{2})\log\tau},$$

the principal value of the logarithm being taken. This theorem might be expected *a priori*; for we have seen that $G(z \mid \tau)$ satisfies the difference equation

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau),$$

and we have also seen that $G(z \mid \tau)$ can be expressed as products of factors essentially characterized by $\Gamma\left(\frac{z+n}{\tau}\right)$ and $\Gamma(z+m\tau)$ respectively. And the former type bears the same relation to the second difference equation as does the latter type to the difference equation which we proceed to investigate.

Take the formula

$$G(z+\tau \mid \tau) = \frac{1}{\tau\Gamma(z+\tau)} e^{a'\frac{z+\tau}{\tau}+b'\frac{(z+\tau)^2}{2\tau^2}} \times \prod_{m=1}^{\infty}\left\{\frac{\Gamma(m\tau)}{\Gamma(z+\tau+m\tau)} e^{(z+\tau)\psi(m\tau)+\frac{(z+\tau)^2}{2}\psi'(m\tau)}\right\},$$

and write it in the form

$$G(z+\tau \mid \tau) = \frac{1}{\tau} e^{a'\frac{z}{\tau}+b'\frac{z^2}{2\tau^2}} e^{a'+b'\frac{2z+\tau}{2\tau}} \times \underset{p=\infty}{\mathrm{Lt}} \prod_{m=1}^{p}\left\{\frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau)+\frac{z^2}{2}\psi'(m\tau)}\right\} \times \underset{p=\infty}{\mathrm{Lt}}\left\{\frac{e^{\tau\sum_{m=1}^{p}\psi(m\tau)+\tau\frac{2z+\tau}{2}\sum_{m=1}^{p}\psi'(m\tau)}}{\Gamma(z+\overline{p+1}\,\tau)}\right\}.$$

Then we shall have

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = e^{a'+b'\frac{2z+\tau}{2\tau}} \underset{p=\infty}{\mathrm{Lt}}\left\{\frac{e^{\tau\sum_{m=1}^{p}\psi(m\tau)+\tau\frac{2z+\tau}{2}\sum_{m=1}^{p}\psi'(m\tau)}}{\Gamma(z+\overline{p+1}\,\tau)}\right\},$$

and, with the proviso that τ be not real and negative, which holds throughout the present investigation, the last written limit may be put in the form

$$\underset{p=\infty}{\mathrm{Lt}}\left\{\frac{e^{\tau C(\tau)-p\tau+\tau\frac{2z+\tau}{2}D(\tau)}\,(\tau p)^{\tau p+\frac{\tau-1}{2}+\frac{2z+\tau}{2}}}{(2\pi)^{\frac{1}{2}}\,(z+\tau p+\tau)^{z+\tau p+\tau-\frac{1}{2}}\,e^{-z-\tau p-\tau}}\right\} = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{\tau C(\tau)+\tau\frac{2z+\tau}{2}D(\tau)}.$$

We have therefore

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = \frac{1}{(2\pi)^{\frac{1}{2}}}\, e^{b'\frac{z}{\tau} + a' + \frac{b'}{2} + \tau C(\tau) + \tau \frac{2z+\tau}{2} D(\tau)},$$

and, on utilizing the values of a' and b' given in § 6, we find, finally,

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = \tau^{-z+\frac{1}{2}}\, (2\pi)^{\frac{\tau-1}{2}},$$

the result stated.

We note that the transcendental double gamma modular constants have disappeared from the final equation.

11. We proceed now to find the value of $G(\tau, \tau)$, and obtain Alexeiewsky's form of the second difference equation for $G(z \mid \tau)$.

Make $z = 0$ in the expression for $G(z \mid \tau)$ as a double product, and we have

$$\underset{z=0}{\text{Lt}} \left\{ \frac{G(z \mid \tau)}{z} \right\} = \frac{1}{\tau};$$

and therefore

$$\underset{z=0}{\text{Lt}} \left\{ G(z \mid \tau)\, \Gamma(z) \right\} = \frac{1}{\tau}.$$

Make now $z = 0$ in the identity

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = \tau^{-z+\frac{1}{2}}\, (2\pi)^{\frac{\tau-1}{2}},$$

and we have

$$\tau\, G(\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}}\, \tau^{\frac{1}{2}},$$

so that

$$G(\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}}\, \tau^{-\frac{1}{2}}.$$

Substitute this value, and we have Alexeiewsky's formula

$$G(z+\tau \mid \tau) = \Gamma(z)\, G(z \mid \tau)\, \frac{G(\tau \mid \tau)}{\tau^{z-1}}.$$

We note that, when

$$\tau = 1,$$

we have

$$G(\tau, \tau) = 1,$$

and the equation just written becomes

$$G(z+1) = \Gamma(z)\, G(z).$$

12. We now see that $G(z \mid \tau)$ satisfies *two* difference equations

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z)$$

and $$f(z+\tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{-z+\frac{1}{2}} \Gamma(z) f(z \mid \tau).$$

It is this fact which leads to an entirely new conception of double gamma functions.

For, if we write $$\Psi(z \mid \tau) = \frac{d}{dz} \log G(z \mid \tau),$$

$$\Psi'(z \mid \tau) = \frac{d}{dz} \Psi(z \mid \tau),$$

we shall have for $\Psi(z \mid \tau)$ the two difference equations

$$f(z+1) = \frac{1}{\tau} \psi\left(\frac{z}{\tau}\right) + f(z),$$

$$f(z+\tau) = \psi(z) + f(z) - \log \tau,$$

where, as usual, $$\psi(z) = \frac{d}{dz} \log \Gamma(z),$$

and for $\Psi'(z \mid \tau)$ we have the difference equations

$$f(z+1) = f(z) + \frac{d^2}{dz^2} \log \Gamma\left(\frac{z}{\tau}\right),$$

$$f(z+\tau) = f(z) + \frac{d^2}{dz^2} \log \Gamma(z).$$

The symmetry of these equations suggests that we write

$$\tau = \frac{\omega_2}{\omega_1},$$

and take absolutely symmetrical difference equations

$$f(z+\omega_1) = f(z) - \psi_1^{(1)}(z \mid \omega_1),$$

$$f(z+\omega_2) = f(z) - \psi_1^{(1)}(z \mid \omega_2),$$

[where $$\psi_1^{(1)}(z \mid \omega_1) = \frac{d^2}{dz^2} \log \Gamma_1(z \mid \omega_1),$$

in the notation of the "Theory of the Gamma Function,"] from which to build up a symmetrical double gamma function. It is on such lines that I propose to develop the theory of the function in a subsequent paper.

13. It is advisable, however, while still retaining the present notation to connect the double gamma function with certain functions already introduced into analysis. With this object in

view we will consider the function

$$T(z \mid \tau) = G(z+1 \mid \tau)\, G(-z \mid -\tau).$$

We have the difference equation

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau),$$

and hence we derive

$$G(-z+1 \mid -\tau) = \Gamma\left(\frac{z}{\tau}\right) G(-z \mid -\tau).$$

Thus $$T(z+1 \mid \tau) = T(z \mid \tau),$$

so that $T(z \mid \tau)$ is a function of z simply periodic of period unity.

Take next the second difference equation

$$G(z+\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}}\, \tau^{-z+\frac{1}{2}}\, \Gamma(z)\, G(z \mid \tau).$$

We obtain at once

$$G(-z-\tau \mid -\tau) = (2\pi)^{\frac{-\tau-1}{2}}\, (-\tau)^{-z+\frac{1}{2}}\, \Gamma(-z)\, G(-z \mid -\tau).$$

Remembering that their principal values are always to be assigned to the many valued functions involved, we see that

$$\frac{T(z+\tau \mid \tau)}{T(z \mid \tau)} = \frac{1}{2\pi}\, \frac{\pi}{\sin\pi(z+1)}\, e^{\pm\pi\iota(z+\frac{1}{2})},$$

the upper or lower sign being taken as $R(\iota\tau)$ is positive or negative. Therefore

$$\frac{T(z+\tau \mid \tau)}{T(z \mid \tau)} = \frac{1}{1-e^{\mp 2\pi\iota z}},$$

with the same determination of the signs.

A simply periodic solution (of period unity) of this equation is

$$\prod_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\},$$

and therefore $T(z \mid \tau)$ is included among the functions

$$P(z) \prod_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\},$$

where $P(z)$ is an arbitrary doubly periodic function of z of periods 1 and τ.

Now $G(z \mid \tau)$ is an integral transcendental function of z with zeroes given by

$$z = -(m\tau+n), \quad \begin{cases} m = 0, 1, \ldots, \infty, \\ n = 0, 1, \ldots, \infty; \end{cases}$$

and therefore $T(z \mid \tau) = G(z+1 \mid \tau)\, G(-z \mid -\tau)$

is a transcendental integral function of z with zeroes given by

$$z = -(m\tau+n), \begin{cases} m = 0, 1, \ldots, \infty, \\ n = -\infty, \ldots, -1, 0, 1, \ldots, \infty, \end{cases}$$

as may be at once seen from a graphical representation of the zeroes of its factors.

But $\prod_{m=0}^{\infty} \{1-e^{\mp 2\pi\iota(z+m\tau)}\}$ is a transcendental integral function with exactly these zeroes. And hence $P(z)$ is a doubly periodic function with no zeroes, and is therefore a constant.

Hence we may write

$$T(z \mid \tau) = K \frac{\prod_{m=0}^{\infty} \{1-e^{\mp 2\pi i(z+m\tau)}\}}{\prod_{m=1}^{\infty} \{1-e^{\mp 2\pi\iota m\tau}\}},$$

where K is independent of z.

Now
$$\underset{z=0}{\mathrm{Lt}} \left\{ \frac{G(z \mid \tau)}{z} \right\} = \frac{1}{\tau}.$$

Hence
$$\underset{z=0}{\mathrm{Lt}} \{T(z \mid \tau)\} = \underset{z=0}{\mathrm{Lt}} \left\{ \frac{\tau}{\tau} \cdot \frac{z}{\tau} \right\} = \underset{z=0}{\mathrm{Lt}} \left\{ \frac{z}{\tau} \right\},$$

and
$$\underset{z=0}{\mathrm{Lt}}\, K \frac{\prod_{m=0}^{\infty} \{1-e^{\mp 2\pi\iota(z+m\tau)}\}}{\prod_{m=1}^{\infty} \{1-e^{\mp 2\pi\iota m\tau}\}} = \underset{z=0}{\mathrm{Lt}}\, K \{1-e^{\mp 2\pi\iota z}\}$$
$$= \mathrm{Lt} \{\pm 2\pi\iota z K\}.$$

Thus
$$K = \pm \frac{1}{2\pi\iota\tau},$$

and hence we obtain, finally,

$$T(z \mid \tau) = \pm \frac{1}{2\pi\iota\tau} \frac{\prod_{m=0}^{\infty} \{1-e^{\mp 2\pi\iota(z+m\tau)}\}}{\prod_{m=1}^{\infty} \{1-e^{\mp 2\pi\iota m\tau}\}}.$$

In the notation of Appell's generalized Eulerian functions,* we may

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* Appell, "Generalisation des fonctions Eulériennes," *Math. Ann.*, Bd. XIX., pp. 84–102.

write this in the form

$$T(z \mid \tau) = \frac{1}{\tau} \frac{O(\mp z \mid 1, \mp\tau)}{\left\{ \frac{d}{dz} O(\mp z \mid 1, \mp\tau) \right\}_{z=0}},$$

either the upper or the lower signs being taken throughout as $R(\iota\tau)$ is positive or negative, where

$$O(z \mid 1, \tau) = \prod_{m=0}^{\infty} \left\{1 - e^{2\pi i (z + m\tau)}\right\}.$$

The theorem just proved is the natural extension to double gamma functions of the relation

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin z\pi}.$$

Note that the product

$$O(z \mid 1, \tau) = \prod_{m=0}^{\infty} \left\{1 - e^{2\pi i (z + m\tau)}\right\}$$

is only convergent, provided $R(\iota\tau)$ is positive and $|\tau| < 1$, or provided $R(\iota\tau)$ is negative and $|\tau| > 1$; while $T(z \mid \tau)$ expressed as a product of two double gamma functions is always convergent provided τ be complex.

14. We may, however, give a single infinite product for $T(z \mid \tau)$ which shall be valid for all complex values of τ.

For this purpose we take the expression

$$G(z \mid \tau) = \frac{1}{\tau\Gamma(z)} e^{a' \frac{z}{\tau} + b' \frac{z^2}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + \frac{z^2}{\tau}\psi'(m\tau)} \right\},$$

where (§ 6)

$$a' = \frac{\tau}{2} \log 2\pi\tau + \tfrac{1}{2} \log \tau - \tau C(\tau),$$

$$b' = -\tau \log \tau - \tau^2 D(\tau),$$

and now, if we write

$$T(z \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)\, G(-z \mid -\tau),$$

we obtain

$$T(z \mid \tau) = \frac{-\Gamma\left(\frac{z}{\tau}\right)}{\Gamma(z)\,\Gamma(-z)} e^{\frac{z}{\tau}[a'(\tau) + a'(-\tau)] + \frac{z^2}{2\tau^2}[b'(\tau) + b'(-\tau)]}$$

$$\times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)\,\Gamma(-m\tau)}{\Gamma(z+m\tau)\,\Gamma(-z-m\tau)} e^{z[\psi(m\tau) - \psi(-m\tau)] + \frac{z^2}{2}[\psi'(m\tau) - \psi'(-m\tau)]} \right\}.$$

Now $$\Gamma(z)\,\Gamma(-z) = \frac{-\pi}{z \sin z\pi},$$

and hence $$\psi(z) - \psi(-z) = -\frac{1}{z} - \pi \cot z\pi,$$

$$\psi'(z) + \psi'(-z) = \frac{1}{z^2} + \frac{\pi^2}{\sin^2 \pi z}.$$

Thus we obtain

$$T(z \mid \tau) = \frac{\pi}{\tau z^2 \sin z\pi}\, e^{\frac{z}{\tau}[a'(\tau)+a'(-\tau)] + \frac{z^2}{2\tau^2}[b'(\tau)+b'(-\tau)] - \gamma\frac{z}{\tau} + \frac{z^2\pi^2}{12\tau^2}}$$
$$\times \prod_{m=1}^{\infty} \left\{ \frac{\sin \pi (z + m\tau)}{\sin \pi m\tau}\, e^{-\pi z \cot m\pi\tau + \frac{\pi^2 z^2}{2 \sin^2 \pi m\tau}} \right\},$$

and, on substituting the values of a' and b', we have, finally,

$$T(z \mid \tau) = \frac{\pi}{z^2 \sin z\pi}\, \tau^{\frac{z}{\tau} - 1}\, e^{\pm\frac{\pi i z}{2}\left(\frac{1}{\tau} - 1\right) - \frac{\gamma z}{\tau} \pm \frac{\pi i z^2}{2\tau} + \frac{z^2\pi^2}{12\tau^2}}$$
$$\times e^{-z[C(\tau) - C(-\tau)] - \frac{z^2}{2}[D(\tau) + D(-\tau)]} \prod_{m=1}^{\infty} \left\{ \frac{\sin \pi(z + m\tau)}{\sin \pi m\tau}\, e^{-\pi z \cot m\pi\tau + \frac{\pi^2 z^2}{2 \sin^2 \pi m\tau}} \right\},$$

an expression for $T(z \mid \tau)$ valid for all values of τ, except those which are entirely real. The upper or lower sign is to be taken as $R(\iota\tau)$ is positive or negative.

15. Let us finally consider the relation of the double gamma functions to the theta functions.

For this purpose we take

$$\Sigma(z \mid \tau) = T(z \mid \tau)\, T(z - \tau \mid -\tau)$$
$$= G(1+z \mid \tau)\, G(-z \mid -\tau)\, G(z - \tau + 1 \mid -\tau)\, G(-z + \tau \mid \tau),$$

a product of four double gamma functions.

We have seen that $T(z \mid \tau)$ is a function of z simply periodic of period unity, and hence the same is true of $\Sigma(z \mid \tau)$. Thus

$$\Sigma(z \mid \tau) = \Sigma(z + 1 \mid \tau).$$

Again, we have $$\Sigma(z + \tau \mid \tau)$$
$$= G(1 + z + \tau \mid \tau)\, G(-z - \tau \mid -\tau)\, G(1 + z \mid -\tau)\, G(-z \mid \tau),$$

and hence
$$\frac{\Sigma(z+\tau \mid \tau)}{\Sigma(z \mid \tau)}$$

$$= \frac{G(1+z+\tau \mid \tau)}{G(1+z \mid \tau)} \, \frac{G(-z \mid \tau)}{G(\tau-z \mid \tau)} \, \frac{G(-z-\tau \mid -\tau)}{G(-z \mid -\tau)} \, \frac{G(1+z \mid -\tau)}{G(1+z-\tau \mid -\tau)}.$$

But we have seen (§ 10) that

$$\frac{G(z+\tau \mid \tau)}{G(z \mid \tau)} = \Gamma(z)\, \tau^{-z+\frac{1}{2}} (2\pi)^{\frac{\tau-1}{2}},$$

and, since their principal values are always assigned to the many valued functions involved,

$$\frac{G(z-\tau \mid -\tau)}{G(z \mid -\tau)} = \Gamma(z)\, e^{\pm(-z+\frac{1}{2})\pi i}\, \tau^{-z+\frac{1}{2}} (2\pi)^{\frac{\tau-1}{2}},$$

the upper or lower sign being taken as $R(\iota\tau)$ is positive or negative. Hence

$$\frac{\Sigma(z+\tau \mid \tau)}{\Sigma(z \mid \tau)} = \frac{\Gamma(1+z)\, \tau^{-z-\frac{1}{2}} (2\pi)^{\frac{\tau-1}{2}}}{\Gamma(-z)\, \tau^{z+\frac{1}{2}} (2\pi)^{\frac{\tau-1}{2}}} \, \frac{\Gamma(-z)(e^{\pm\pi i}\tau)^{z+\frac{1}{2}} (2\pi)^{\frac{-\tau-1}{2}}}{\Gamma(1+z)(e^{\pm\pi i}\tau)^{-z-\frac{1}{2}} (2\pi)^{\frac{-\tau-1}{2}}}$$

$$= e^{\pm 2\pi\iota(z+\frac{1}{2})}$$

$$= -e^{\pm 2\pi i z}.$$

Thus we see that $\Sigma(z \mid \tau)$ satisfies the two difference equations characteristic of the theta functions

$$f(z+1) = f(z),$$
$$f(z+\tau) = -e^{\pm 2\pi\iota z} f(z).$$

Now it has been shown (§ 13) that

$$T(z \mid \tau) = \pm \frac{1}{2\pi\iota\tau} \, \frac{\prod\limits_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\}}{\prod\limits_{m=1}^{\infty} \left\{1-e^{\mp 2\pi\iota m\tau}\right\}}.$$

From the reduction just obtained for $\frac{\Sigma(z+\tau \mid \tau)}{\Sigma(z \mid \tau)}$ we see that the function

$$T(+z-\tau \mid -\tau) = G(z-\tau+1 \mid -\tau)\, G(-z+\tau \mid \tau)$$

is such that
$$\frac{T(z \mid -\tau)}{T(z-\tau \mid -\tau)} = 1-e^{\pm 2\pi i z},$$

a difference relation which can at once be obtained from the relation

$$\frac{T(z \mid \tau)}{T(z+\tau \mid \tau)} = 1-e^{\mp 2\pi i z},$$

by merely changing τ into $-\tau$. For it is evident that such a change involves the opposite prescription for $R(\iota\tau)$.

We may obtain the same result and at the same time a useful verification of our formulæ if we take the difference equations

$$\frac{T(z \mid -\tau)}{T(z-\tau \mid -\tau)} = 1-e^{\pm 2\pi i z},$$

$$T(z-\tau+1 \mid -\tau) = T(z-\tau \mid -\tau),$$

and proceed as in § 13.

We readily find that

$$T(z-\tau \mid -\tau) = \pm \frac{1}{2\pi i\tau} \frac{\prod_{m=1}^{\infty} \left\{1-e^{\pm 2\pi i(z-m\tau)}\right\}}{\prod_{m=1}^{\infty} \left\{1-e^{\mp 2\pi i m\tau}\right\}},$$

for this expression satisfies the requisite functional relations; its zeroes are given by

$$z = m\tau+n, \quad \begin{cases} m = 1, 2, \ldots, \infty, \\ n = -\infty, \ldots, -1, 0, 1, \ldots, \infty, \end{cases}$$

just as are those of $G(-z+\tau \mid \tau)\, G(1+z-\tau \mid -\tau)$; and each side reduces to $\pm \dfrac{1}{2\pi\iota\tau}$ when $z = 0$.

We now have

$$\Sigma(z \mid \tau) = \frac{e^{\mp 2\pi i z}-1}{(2\pi\tau)^2} \frac{\prod_{m=1}^{\infty} \left\{(1-e^{\mp 2\pi i(z+m\tau)})(1-e^{\pm 2\pi i(z-m\tau)})\right\}}{\prod_{m=1}^{\infty} \left\{(1-e^{\mp 2\pi i m\tau})^2\right\}}.$$

Thus, if we put $q = e^{\mp \pi i\tau}$, the upper or lower sign being taken as $R(\iota\tau)$ is positive or negative, we find

$$\Sigma(z \mid \tau) = \frac{e^{\mp 2\pi i z}-1}{(2\pi\tau)^2} \prod_{m=1}^{\infty} \left\{\frac{1-2q^{2m}\cos 2\pi z+q^{4m}}{(1-q^{2m})^2}\right\}.$$

Assume now that $R(\iota\tau)$ is negative; then, with the notation of the theta functions adopted by Tannery and Molk, we have*

$$\vartheta_1(z) = 2q_0 q^{\frac{1}{4}} \sin z\pi \prod_{m=1}^{\infty} \left\{1-2q^{2m}\cos 2z\pi+q^{4m}\right\},$$

where

$$q_0 = \prod_{m=1}^{\infty} \left\{1-q^{2m}\right\}.$$

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* *Fonctions Elliptiques*, Tome II., p. 252.

Hence we see that

$$\Sigma(z \mid \tau) = \frac{e^{2\pi i z}-1}{(2\pi\tau)^2}\,\frac{\vartheta_1(z)}{2q_0^3 q^{\frac{1}{4}} \sin \pi z}$$

$$= \frac{i e^{\pi i z}}{(2\pi\tau)^2}\,\frac{\vartheta_1(z)}{q_0^3 q^{\frac{1}{4}}}$$

$$= \frac{i e^{\pi i z}}{2\pi\tau^2}\,\frac{\vartheta_1(z)}{\vartheta_1'(0)},$$

since*

$$\vartheta_1'(0) = 2\pi q_0^3 q^{\frac{1}{4}}.$$

Finally, then, when $R(\iota\tau)$ is negative,

$$\Sigma(z \mid \tau) = -\frac{\pi i e^{\pi i z}}{2(\log q)^2}\,\frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

In this manner we have expressed $\vartheta_1(z)$ as a product of four double gamma functions. And it is now evident that we may build up all four theta functions by means of the functions $G(z \mid \tau)$. And from quotients of such products of double gamma functions we may form the Jacobian elliptic functions sn z, cn z, and dn z.

At this stage we are naturally conducted to the consideration of the formation of Weierstrass's σ function, which is in essence a theta function symmetrical in ω_1 and ω_2—the two parameters whose quotient is τ. And such considerations lead to the formation of the analogous symmetrical double gamma function which will be discussed in a following paper.

The Theorem of Residuation, being a general treatment of the Intersections of Plane Curves at Multiple Points. *By* F. S. Macaulay. Received and read December 14th, 1899.

I.

1. The following paper contains some developments of a theory which appears to be capable of considerable extension, and which is founded essentially, both as regards methods and applications, on

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* *Fonctions Elliptiques*, Tome II., p. 257.