On the Direct Determination of Stress in an Elastic Solid, with application to the Theory of Plates. *By* J. H. Michell, M.A.

Read April 13th, 1899. Received, in revised form, September 4th, 1899.

In treating the problem of an elastic solid in equilibrium under given volume- and surface-forces, some of the advantages of a direct determination of the stress are so obvious that it is surprising more attention has not been given to this mode of attack. G. B. Airy,* in 1862, gave a solution of the statical equations of stress in two dimensions in terms of a function which is called by Maxwell "Airy's function of stress in two dimensions." Airy did not consider the differential equation satisfied by his function. This arises from substituting in the identical strain-relation† of St. Venant the values of the strains in terms of the stresses.

Maxwell,‡ in 1869, supplied this equation in an awkward form, and extended Airy's method to three dimensions by means of three functions of stress. Finally, Ibbetson§, in 1886, gave the straightforward process for determining the equations satisfied by Maxwell's functions by substitution in the six identical strain-relations of St. Venant.

In the present paper I begin with a discussion of plane stress in an isotropic body under given volume- and surface-forces. The problem is reduced to the determination of a function ψ satisfying $\nabla_{xy}^4 \psi = 0$ with ψ, and $d\psi/dn$ given over the boundary. It is shown that the stress is independent of the moduli of elasticity if there is no volume-force, and if the body is simply-connected, and that the same is true for a multiply-connected body if the resultant force (not necessarily the couple) over *each* boundary separately vanishes. Reference must here be made to a statement of Maxwell's at the bottom of p. 201 of the paper cited, which, without adequate discussion, partly anticipates this result, but appears to involve more than one oversight. I have

* *Brit. Assoc. Report*, 1862.

† See A. E. H. Love, *Elasticity*, Vol. I., § 66.

‡ *Scientific Papers*, Vol. II., p. 161.

§ *Proc. Lond. Math. Soc.*, Vol. XVII. Reference should also be made to Voigt, Wiedemann, *Annalen*, XVI., 1882.

discussed at some length the conditions to be satisfied by ψ, in order that the displacements may be single-valued, as well as the form and uniqueness of the solution to be obtained for that function.

In the second part of the paper I begin by obtaining the equations of stress in three dimensions, in a form analogous to the ordinary equations of displacement. It is not advantageous to introduce Maxwell's functions of stress, at any rate for the applications I have in view. Here, again, I have considered the surface-conditions to be satisfied by the stresses in order that the displacements deduced may be single-valued.

In the last part of the paper the stress-equations are applied to the theory of plates. A general method of solution for any distribution of force is given. The essential difference between this and previous solutions is that no assumption is here made as to the values of the stresses on planes parallel to the faces of the plate. Instead, it is shown how to begin by determining the value of the normal pressure on such planes without considering the boundary-conditions. The possibility of this rests on the fact, almost intuitive, that any local normal pressure cannot be transmitted along the plate, except to an utterly negligible extent, a distance many times the plate's thickness. It is further shown that each of the tangential stresses on planes parallel to the faces is composed of two terms, one of which depends on the form of the median plane of the plate, and the other is determined directly in terms of the applied forces. The other stresses are then expressed in terms of the curvature and stretch of the median plane and their rates of change, together with the quantities already completely determined. Differential equations of the fourth order are next obtained for the two unknown functions in terms of which the normal displacement and the stretch of the median plane are expressed.

Finally, a method of successive approximation is indicated connecting the solution here given with the ordinary approximation. The elastic solid has throughout been supposed isotropic; the method of extension to anisotropic bodies is perfectly obvious.

Plane Stress.

Under this heading we may, following Maxwell, conveniently treat two problems: (*a*) that of a long cylinder with applied forces perpendicular to its length, and the same at corresponding points along its length; (*b*) that of a thin plate with applied forces in its plane.

Case (*a*).

Adopting the notation of Thomson and Tait's *Natural Philosophy* and Love's *Elasticity*, and taking the axis of z in the direction of the length of the cylinder, we have here

$$S = T = 0,$$

$$g = \text{const.},$$

and u, v functions of x, y only.

The equations of stress become

$$P_x + U_y = V_x,$$

$$U_x + Q_y = V_y,$$

assuming a force-potential and using suffixes to denote differentiation where no doubt can arise as to the meaning.

These equations are satisfied quite generally by

$$P - V = \psi_{yy},$$
$$Q - V = \psi_{xx},$$
$$U = -\psi_{xy},$$

where ψ is Airy's function of stress.

Write $\qquad \Theta \equiv P + Q + R = (3\lambda + 2\mu)(e + f + g).$

Since $\qquad P + Q = 2(\lambda + \mu)(e + f) + 2\lambda g,$

we have $\qquad \Theta = \dfrac{3\lambda + 2\mu}{2(\lambda + \mu)}(P + Q) + \text{const.}$

The strain-relation* $\qquad e_{yy} + f_{xx} = c_{xy}$

becomes $\qquad P_{yy} + Q_{xx} - \dfrac{\lambda}{3\lambda + 2\mu}\nabla^2_{xy}\Theta = 2U_{xy}$

or $\qquad \nabla^4_{xy}\psi + \nabla^2_{xy}V - \dfrac{\lambda}{2(\lambda + \mu)}\nabla^2_{xy}(\nabla^2_{xy}\psi + 2V) = 0,$

that is, $\qquad (\lambda + 2\mu)\nabla^4_{xy}\psi = -2\mu\nabla^2_{xy}V,$

where $\qquad \nabla^2_{xy} \equiv \dfrac{d^2}{dx^2} + \dfrac{d^2}{dy^2}.$

* Love's *Elasticity*, Vol. I., § 66.

We may now drop the suffix, since the coordinate z no longer appears, and write the equation for ψ,

$$(\lambda+2\mu)\nabla^4\psi = -2\mu\nabla^2 V, \tag{1}$$

where

$$\nabla^2 \equiv \frac{d^2}{dx^2}+\frac{d^2}{dy^2}.$$

Case (b).

We adopt for the present the ordinary approximation, viz.: taking the axis of z normal to the plate, we put

$$R = S = T = 0,$$

so that

$$\Theta = P+Q.$$

The stress-equations are solved as in Case (a) and the strain-relation, which is equivalent to

$$P_{yy}+Q_{xx}-\frac{\lambda}{3\lambda+2\mu}\nabla^2_{xy}\Theta = 2U_{xy},$$

as before, now becomes

$$\nabla_{xy}\psi+\nabla_{xy}V-\frac{\lambda}{3\lambda+2\mu}\nabla^2_{xy}(\nabla^2_{xy}\psi+2V) = 0,$$

that is

$$2(\lambda+\mu)\nabla^4_{xy}\psi = -(\lambda+2\mu)\nabla^2_{xy}V,$$

or, dropping the suffixes, the equation for ψ is now

$$2(\lambda+\mu)\nabla^4\psi = -(\lambda+2\mu)\nabla^2 V. \tag{2}$$

Conditions for ψ in a Multiply-connected Body.

In these equations ψ is not in general single-valued if the body is not singly-connected. The second and higher derivatives of ψ must always be single-valued if V is so, since the stresses must be so. Further, ψ must be such that the displacements are single-valued.

Since

$$u_{xy} = e_y,$$
$$u_{yy} = c_y - f_x,$$

we have

$$u_y = {}_0u_y + \int_0 \{e_y\,dx+(c_y-f_x)\,dy\}.$$

The strain-relation makes the integral vanish for a complete reducible circuit; but we must have in addition

$$\int_0^0 \{e_y\,dx+(c_y-f_x)\,dy\} = 0 \tag{3}$$

or each independent irreducible circuit.

Since $u_y + v_x = c$ is single-valued, the condition (3) will ensure that v_x is single-valued, and hence no further conditions are required for the first derivatives of the displacements. We have still to express the conditions that u, v may be single-valued. We must have

$$\int_0^0 (u_x dx + u_y dy) = 0$$

and

$$\int_0^0 (v_x dx + v_y dy) = 0$$

for each irreducible circuit.

Now $$\int_0^0 u_x dx = \left[xu_x\right]_0^0 - \int_0^0 x\,du_x = -\int_0^0 x\,(e_x dx + e_y dy)$$

and $$\int_0^0 u_y dy = \left[yu_y\right]_0^0 - \int_0^0 y\,du_y = -\int_0^0 y\,\{e_y dx + (c_y - f_x)\,dy\},$$

the condition (3) being supposed satisfied.

Hence $$\int_0^0 \left[(xe_x + ye_y)\,dx + \{xe_y + y\,(c_y - f_x)\}\,dy\right] = 0, \tag{4}$$

and, similarly,

$$\int_0^0 \left[\{yf_x + x\,(c_x - e_y)\}\,dx + (xf_x + yf_y)\,dy\right] = 0. \tag{5}$$

We proceed to express these equations in terms of ψ.

In *Case* (*a*), $$2\mu e = P - \frac{\lambda}{3\lambda + 2\mu}\,\Theta$$

$$= P - \frac{\lambda}{2\,(\lambda+\mu)}\,(P+Q) + \text{const.}$$

or $$4\mu\,(\lambda+\mu)\,e = (\lambda + 2\mu)(P+Q) - 2\,(\lambda+\mu)\,Q + \text{const.},$$

and, similarly,

$$4\mu\,(\lambda+\mu)\,f = (\lambda+2\mu)(P+Q) - 2\,(\lambda+\mu)\,P + \text{const.}$$

Hence equation (3) becomes

$$(\lambda+2\mu)\int_0^0 \left\{\frac{d}{dy}(P+Q)\,dx - \frac{d}{dx}(P+Q)\,dy\right\}$$
$$-2\,(\lambda+\mu)\int_0^0 \left\{\frac{d}{dy}(\psi_{xx})\,dx + \frac{d}{dy}(\psi_{xy})\,dy\right\}$$
$$-2\,(\lambda+\mu)\int_0^0 (V_y dx - V_x dy) = 0,$$

that is

$$-(\lambda+2\mu)\int_0^0 \frac{d}{dn}(P+Q)\,ds-2\,(\lambda+\mu)\left[\psi_{xy}\right]_0^0+2\,(\lambda+\mu)\int_0^0 \frac{dV}{dn}\,ds=0,$$

where ds is an element of arc of the boundary of a section z const., dn an element of normal to that boundary.

Since ψ_{xy} is single-valued, this reduces to

$$(\lambda+2\mu)\int_0^0 \frac{d}{dn}(\nabla^2\psi)\,ds+2\mu\int_0^0 \frac{dV}{dn}\,ds=0, \tag{6}$$

which may also be written

$$\frac{\lambda+2\mu}{3\lambda+2\mu}\int_0^0 \frac{d\Theta}{dn}\,ds=\int_0^0 \frac{dV}{dn}\,ds,$$

or, again,
$$(\lambda+2\mu)\int_0^0 \frac{d\theta}{dn}\,ds=\int_0^0 \frac{dV}{dn}\,ds,$$

in which form it is a simple deduction from the displacement-equations

$$(\lambda+2\mu)\,\theta_x-2\mu\varpi_y=V_x,$$

$$(\lambda+2\mu)\,\theta_y+2\mu\varpi_x=V_y,$$

viz., we deduce
$$(\lambda+2\mu)\frac{d\theta}{dn}=\frac{dV}{dn}+2\mu\frac{d\varpi}{ds};$$

and, integrating around a boundary and remembering that ϖ is single-valued, the equation at once follows.

If there is no volume-force,

$$\int_0^0 \frac{d\theta}{dn}\,ds=0$$

over each boundary.

The existence of the corresponding equation

$$\int \frac{d\theta}{dn}\,dS=0$$

over *each* bounding surface in a three-dimensional solid under no volume-force may here be noted. It is of importance in connexion with solutions in which a determination of θ is the first step. For example, in the problem of the stress of a cylindrical or a spherical boiler under uniform pressure it shows at once that θ is constant.

The equation (4) becomes

$$2(\lambda+\mu)\int_0^0 d\psi_x + (\lambda+2\mu)\int_0^0 \left\{ x\frac{d}{ds}(P+Q) - y\frac{d}{dn}(P+Q) \right\} ds$$

$$-2(\lambda+\mu)\int_0^0 \left(x\frac{dV}{ds} - y\frac{dV}{dn} \right) ds = 0,$$

or

$$2(\lambda+\mu)\left[\psi_x\right]_0^0 + (\lambda+2\mu)\int_0^0 \left(x\frac{d}{ds}\nabla^2\psi - y\frac{d}{dn}\nabla^2\psi \right) ds$$

$$+2\mu\int_0^0 \left(x\frac{dV}{ds} - y\frac{dV}{dn} \right) ds = 0, \qquad (7)$$

and, similarly, equation (5) becomes

$$2(\lambda+\mu)\left[\psi_y\right] + (\lambda+2\mu)\int_0^0 \left(y\frac{d}{ds}\nabla^2\psi + x\frac{d}{dn}\nabla^2\psi \right) ds$$

$$+2\mu\int_0^0 \left(y\frac{dV}{ds} + x\frac{dV}{dn} \right) ds = 0. \qquad (8)$$

In *Case* (*b*), $\quad 2\mu e = P - \dfrac{\lambda}{3\lambda+2\mu}(P+Q),$

or $\quad 2\mu(3\lambda+2\mu)\, e = 2(\lambda+\mu)(P+Q) - (3\lambda+2\mu)\, Q$

and $\quad 2\mu(3\lambda+2\mu) f = 2(\lambda+\mu)(P+Q) - (3\lambda+2\mu)\, P,$

so that the appropriate equations in this case are derived from those of *Case* (*a*) by substituting for λ according to the equation

$$\frac{\lambda'}{3\lambda'+2\mu} = \frac{\lambda}{2(\lambda+\mu)},$$

that is, putting $\quad \lambda+2\mu = \dfrac{4\mu(\lambda'+\mu)}{\lambda'+2\mu},$

$$\lambda+\ \mu = \frac{\mu(3\lambda'+2\mu)}{\lambda'+2\mu}.$$

The meanings of $\left[\psi_x\right]_0^0$, $\left[\psi_y\right]_0^0$ will appear from the next section.

The Stresses at the Boundary.

With the usual notation,

$$lP + mU = F,$$

$$lU + mQ = G,$$

where $\quad l = dy/ds, \quad m = -dx/ds,$

and F, G are the x, y components of the external stress.

Hence* $$F = \frac{dy}{ds}(\psi_{yy}+V) + \frac{dx}{ds}\psi_{xy} = \frac{d}{ds}(\psi_y) + V\frac{dy}{ds}$$

and $$G = -\frac{d}{ds}(\psi_x) - V\frac{dx}{ds},$$

so that $$\left.\begin{aligned} \psi_x &= -\int_0^s G\,ds - \int_0^s V\frac{dx}{ds}\,ds + a \\ \psi_y &= \int_0^s F\,ds - \int_0^s V\frac{dy}{ds}\,ds + \beta \end{aligned}\right\}, \tag{9}$$

where a, β are constants.

We may therefore write

$$\psi_x = H + a,$$
$$\psi_y = K + \beta,$$

where H, K are known functions for each boundary, and a, β are unknown constants, different of course, in general, for each boundary. Hence

$$\left.\begin{aligned} \psi &= \int_0^s \left(H\frac{dx}{ds} + K\frac{dy}{ds}\right) ds + ax + \beta y + \gamma \\ \frac{d\psi}{dn} &= H\frac{dy}{ds} - K\frac{dx}{ds} + a\frac{dy}{ds} - \beta\frac{dx}{ds} \end{aligned}\right\}, \tag{10}$$

where γ is another unknown constant to be determined for each boundary.

If the body is singly-connected, the values of a, β, γ for the single boundary do not affect the result, for the addition of a solution

$$\psi = -ax - \beta y - \gamma,$$

which does not affect the stresses, makes the constants disappear. The same is no longer true if the body is multiply-connected. We must then apply the equations (6), (7), (8) to determine the *three* constants corresponding to each boundary. Of course, the three constants corresponding to one of the boundaries are still arbitrary.

It appears from equations (9) that, if there is no volume force,

$$\left[\psi_x\right]_0^0 = -\int_0^0 G\,ds,$$

$$\left[\psi_y\right]_0^0 = \int_0^0 F\,ds,$$

* *Cf.* Maxwell, *loc. cit.*, p. 193. Maxwell's process is, I think, erroneous.

so that the quantities on the left are the resultant forces in the directions of the two axes, on the boundary considered. If these resultants vanish for each boundary, the function ψ is determined by the following conditions :—

$$\nabla^4\psi = 0 \text{ throughout,} \tag{11}$$

$$\left.\begin{aligned} \psi &= \int_0^s (H\,dx + K\,dy) + \alpha x + \beta y + \gamma \\ \frac{d\psi}{dn} &= H\frac{dy}{ds} - K\frac{dx}{ds} + \alpha\frac{dy}{ds} - \beta\frac{dx}{ds} \end{aligned}\right\} \tag{12}$$

at each point of each boundary, and

$$\left.\begin{aligned} &\int_0^0 \frac{d}{dn}(\nabla^2\psi)\,ds = 0 \\ &\int_0^0 \left\{ x\frac{d}{ds}(\nabla^2\psi) - y\frac{d}{dn}(\nabla^2\psi) \right\} ds = 0 \\ &\int_0^0 \left\{ y\frac{d}{ds}(\nabla^2\psi) + x\frac{d}{dn}(\nabla^2\psi) \right\} ds = 0 \end{aligned}\right\} \tag{13}$$

for each boundary.

Form of Solution.

If the constants α, β, γ are fixed, the equations (11), (12) will determine ψ uniquely. The properties of the solutions of such equations have been discussed by Mathieu* in connexion with other problems. Mathieu considers only singly-connected regions or single-valued functions, but we can easily extend his result in the present connexion. If ψ_1, ψ_2 are two solutions, $\phi = \psi_1 - \psi_2$ will satisfy $\nabla^4\phi = 0$ and make $\phi = 0$, $d\phi/dn = 0$ over the boundaries. The function ϕ is therefore a single-valued function† in the region, and Mathieu's proof shows that $\phi = 0$, or that the two assumed solutions are identical. The solution having been obtained with arbitrary constants α, β, γ, their actual values are found by substituting the solution in equations (6), (7), (8), or, in the particular case above, in equations (13). Now the moduli of elasticity do not appear in

---

* *Journal de Math.*, T. XIV., 2^{me} sér., 1869, p. 391.

† Make the region simply-connected by cross-cuts. The first derivatives of ϕ have the same value (zero) at an end of a cross-cut on its two sides, and the second derivatives are single-valued; therefore the first derivatives have the same values on the two sides along the whole cross-cut. Similarly, ϕ itself is the same on the two sides of the cross-cut.

equations (13). *Hence the stresses are independent of the moduli, provided the resultant force on each boundary vanishes.* This condition is, of course, always satisfied where there is only a single boundary, so that the stresses are always independent of the moduli in this case. It must be remembered that we have assumed the absence of volume-force.

Supposing, now, that the resultant forces on the boundaries do not vanish, we can add any convenient type of stress over the boundaries to reduce the resultants to zero, so that, if there are n boundaries, the function ψ will be of the form

$$c_1\psi_1+c_2\psi_2+\ldots+c_{2n-2}\psi_{2n-2}+\psi',$$

where $\psi_1, \psi_2, \ldots, \psi_{2n-2}$ involve the moduli of elasticity and depend only on the form of the body and the types of stress added, not on the given distribution of surface stress, while $c_1, c_2, \ldots, c_{2n-2}$ are constants depending only on the magnitudes of the resultant forces on the boundaries. Finally, ψ' is independent of the moduli. We can proceed further in this direction. Add any $(n-1)$ convenient types of stress which reduce the couples on the boundaries also to zero. We may then write

$$\psi = c_1\psi_1+\ldots+c_{3n-3}\psi_{3n-3}+\psi'',$$

where ψ'' is now single-valued, and $\psi_{2n-1} \ldots \psi_{3n-3}$ are functions depending on the types of stress chosen and independent of the moduli. To prove this it is only necessary to show that, if the forces on each boundary equilibrate, ψ is single-valued. Now, since in this case

$$[\psi_x]_0^0 = -\int_0^0 G\,ds = 0,$$

$$[\psi_y]_0^0 = \int_0^0 F\,ds = 0,$$

ψ_x, ψ_y are single-valued. And

$$\begin{aligned}[\psi]_0^0 &= \int_0^0 (H\,dx+K\,dy)\\ &= [Hx+Ky]_0^0-\int_0^0\left(x\frac{dH}{ds}+y\frac{dK}{ds}\right)ds\\ &= \int_0^0 (Gx-Fy)\,ds\\ &= 0,\end{aligned}$$

since the couple of the stresses on s vanishes. Hence ψ is single-valued in this case.

Introduction of Curvilinear Coordinates.

For application to curved boundaries the expressions for the stresses in curvilinear coordinates are required. These may be readily obtained as follows. We have seen that the components of stress, in the directions of the axes, across any element of arc ds, are

$$F = \frac{d}{ds}(\psi_y),$$

$$G = -\frac{d}{ds}(\psi_x),$$

giving for the stress normal to the arc

$$\begin{aligned} N &= \frac{dy}{ds}\frac{d}{ds}(\psi_y) + \frac{dx}{ds}\frac{d}{ds}(\psi_x) \\ &= \frac{d}{ds}\left(\frac{dy}{ds}\psi_y + \frac{dx}{ds}\psi_x\right) - \left(\frac{d^2y}{ds^2}\psi_y + \frac{d^2x}{ds^2}\psi_x\right) \\ &= \frac{d^2\psi}{ds^2} + \kappa\frac{d\psi}{dn}, \end{aligned}$$

where κ is the curvature of the arc and dn is an element of normal to it, in the direction opposite to that in which the curvature is measured.

The tangential stress is

$$\begin{aligned} M &= \frac{dx}{ds}\frac{d}{ds}(\psi_y) - \frac{dy}{ds}\frac{d}{ds}(\psi_x) \\ &= \frac{d}{ds}\left(\frac{dx}{ds}\psi_y - \frac{dy}{ds}\psi_x\right) - \left(\frac{d^2x}{ds^2}\psi_y - \frac{d^2y}{ds^2}\psi_x\right) \\ &= -\frac{d}{ds}\left(\frac{d\psi}{dn}\right) + \kappa\frac{d\psi}{ds}. \end{aligned}$$

Introducing orthogonal curvilinears, so that

$$\begin{aligned} ds &= h_2 d\eta, \\ dn &= h_1 d\xi, \\ \kappa &= \frac{1}{h_1 h_2}\frac{dh_2}{d\xi}, \end{aligned}$$

these expressions give at once for the corresponding elements of stress

$$\left.\begin{aligned} P' &= \frac{1}{h_2^2}\frac{d^2\psi}{d\eta^2} - \frac{1}{h_2^3}\frac{dh_2}{d\eta}\frac{d\psi}{d\eta} + \frac{1}{h_1^2 h_2}\frac{dh_2}{d\xi}\frac{d\psi}{d\xi} \\ Q' &= \frac{1}{h_1^2}\frac{d^2\psi}{d\xi^2} - \frac{1}{h_1}\frac{dh_1}{d\xi}\frac{d\psi}{d\xi} + \frac{1}{h_2^2 h_1}\frac{dh_1}{d\eta}\frac{d\psi}{d\eta} \\ U' &= -\frac{1}{h_1 h_2}\frac{d^2\psi}{d\xi\, d\eta} + \frac{1}{h_1^2 h_2}\frac{dh_1}{d\eta}\frac{d\psi}{d\xi} + \frac{1}{h_2^2 h_1}\frac{dh_2}{d\xi}\frac{d\psi}{d\eta} \end{aligned}\right\}. \tag{14}$$

For example, in plane polars,

$$P' = \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + \frac{1}{r}\frac{d\psi}{dr},$$

$$Q' = \frac{d^2\psi}{dr^2},$$

$$U' = -\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{d\theta}\right),$$

and the general form of solution for a cylindrical shell is

$$\begin{aligned} \psi = {} & A_0 r^2 + B_0 r^2 (\log r - 1) + C_0 \log r + D_0 \theta \\ & + (A_1 r + B_1 r^{-1} + B_1' \theta r + C_1 r^3 + D_1 r \log r)\cos\theta \\ & + (E_1 r + F_1 r^{-1} + F_1' \theta r + G_1 r^3 + H_1 r \log r)\sin\theta \\ & + \sum_2^\infty (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2})\cos n\theta \\ & + \sum_2^\infty (E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2})\sin n\theta. \end{aligned}$$

I will conclude this part of the paper by remarking that Betti's "Reciprocal Theorems"* here take the form of generalized Green theorems for ψ, some of which are given by Mathieu.† The particular case of a circular boundary would, I think, repay a detailed treatment.‡

Stress Equations in Three Dimensions.

Writing the displacement equations in the form

$$\lambda + \mu)\,\theta_x + \mu\nabla^2 u = -X, \text{ \&c.}, \tag{15}$$

* Love, *Elasticity*, Vol. I., ch. viii. † *Loc. cit.*

‡ See Borchardt, "Ueber Deformationen elasticher isotroper Körper," &c., *Berlin Monatsberichte*, 1873; Hertz, *Miscellaneous Papers*, p. 261.

we have
$$(\lambda+2\mu)\,\nabla^2\theta = -(X_x+Y_y+Z_z)$$
$$= -\Lambda, \text{ say.}$$

Since
$$P = \lambda\theta+2\mu e,$$
$$\nabla^2 P = 2\mu\nabla^2 e-\Lambda\lambda/(\lambda+2\mu).$$

Differentiating (15) with respect to x,
$$(\lambda+\mu)\,\theta_{xx}+\mu\nabla^2 e = -X_x$$

therefore
$$\nabla^2 P+2(\lambda+\mu)\,\theta_{xx} = -2X_x-\Lambda\lambda/(\lambda+2\mu),$$

or, similarly,
$$\left.\begin{aligned}\nabla^2 P+\kappa\Theta_{xx} &= -2X_x-\nu\Lambda\\ \nabla^2 Q+\kappa\Theta_{yy} &= -2Y_y-\nu\Lambda\\ \nabla^2 R+\kappa\Theta_{zz} &= -2Z_z-\nu\Lambda\end{aligned}\right\},\tag{16}$$

where
$$\kappa = 2(\lambda+\mu)/(3\lambda+2\mu),$$
$$\nu = \lambda/(\lambda+2\mu).$$

Equations (16) involve
$$\nabla^2\Theta = -(2\nu+1)\,\Lambda.\tag{17}$$

Again, from (15),
$$2(\lambda+\mu)\,\theta_{yz}+\mu\nabla^2(w_y+v_z) = -Z_y-Y_z,$$

that is, similarly,
$$\left.\begin{aligned}\nabla^2 S+\kappa\Theta_{yz} &= -Z_y-Y_z\\ \nabla^2 T+\kappa\Theta_{zx} &= -X_z-Z_x\\ \nabla^2 U+\kappa\Theta_{xy} &= -Y_x-X_y\end{aligned}\right\},\tag{18}$$

We proceed to show that equations (16) and (18), together with
$$\left.\begin{aligned}P_x+U_y+T_z &= -X\\ U_x+Q_y+S_z &= -Y\\ T_x+S_y+R_z &= -Z\end{aligned}\right\},\tag{19}$$

imply the existence of the identical strain-relations. Take first the type
$$e_{yy}+f_{xx} = c_{xy}.\tag{20}$$

From (19), we have
$$2U_{xy}+P_{xx}+Q_{yy}-R_{zz} = 2Z_z-\Lambda,$$

so that (20) becomes

$$2\mu e_{yy} + 2\mu f_{xx} + P_{xx} + Q_{yy} - R_{zz} = 2Z_z - \Lambda,$$

or
$$P_{yy} + Q_{xx} - \lambda(\theta_{xx} + \theta_{yy}) + P_{xx} + Q_{yy} - R_{zz} = 2Z_z - \Lambda,$$

or
$$\nabla^2(P+Q) - \Theta_{zz} - \lambda(\theta_{xx} + \theta_{yy}) = 2Z_z - \Lambda,$$

which, using (16), becomes

$$-\kappa(\Theta_{xx} + \Theta_{yy}) - \Theta_{zz} - (1-\kappa)(\Theta_{xx} + \Theta_{yy}) = 2Z_z - \Lambda + 2(X_x + Y_y) + 2\nu\Lambda$$
$$= (2\nu + 1)\Lambda,$$

so that (20) becomes $\quad \nabla^2\Theta = -(2\nu+1)\Lambda,$
which is (17).

Take now the second type of strain-relation

$$2e_{yz} + a_{zx} = b_{xy} + c_{zx}, \tag{21}$$

or
$$P_{yz} - \lambda\theta_{yz} + S_{xx} = T_{xy} + U_{zx}.$$

Now, from (19),

$$T_{xy} + U_{zx} + S_{yy} + S_{zz} + R_{yz} + Q_{yz} = -Z_y - Y_z,$$

so that (21) becomes

$$P_{yz} - (1-\kappa)\Theta_{yz} + \nabla^2 S + Q_{yz} + R_{yz} = -Z_y - Y_z,$$

that is
$$\nabla^2 S + \kappa\Theta_{yz} = -Z_y - Y_z,$$

which is one of equations (18).

The strain-relations are therefore satisfied, and conversely they lead by a reversal of the above proofs to the stress-equations obtained. Hence the *complete* stress-equations are

$$\begin{aligned}
\nabla^2 P + \kappa\Theta_{xx} &= -2X_x - \nu\Lambda,\\
\nabla^2 Q + \kappa\Theta_{yy} &= -2Y_y - \nu\Lambda,\\
\nabla^2 R + \kappa\Theta_{zz} &= -2Z_z - \nu\Lambda,\\
\nabla^2 S + \kappa\Theta_{yz} &= -\ Z_y - Y_z,\\
\nabla^2 T + \kappa\Theta_{zx} &= -\ X_z - Z_x,\\
\nabla^2 U + \kappa\Theta_{xy} &= -\ Y_x - X_y,
\end{aligned}$$

$$\begin{aligned}
P_x + U_y + T_z &= -X,\\
U_x + Q_y + S_z &= -Y,\\
T_x + S_y + R_z &= -Z,
\end{aligned}$$

where $$\Theta = P+Q+R.$$

Of course similar equations can be obtained for anisotropic bodies.

As in the case of plane-stress, the satisfaction of the strain-relations does not ensure that the displacements and their first derivatives are single-valued in the case of a multiply-connected solid. The additional conditions are obtained as in two dimensions. We have

$$du_y = e_y dx + (c_y - f_z)\, dy + \tfrac{1}{2}(c_z + b_y - a_x)\, dz.$$

Hence for each irreducible circuit

$$\int_0^0 \{e_y dx + (c_y - f_x)\, dy + \tfrac{1}{2}(c_z + b_y - a_x)\, dz\} = 0.$$

There are two similar conditions derived from a consideration of v_z and w_x.

Further $$du = u_x dx + u_y dy + u_z dz.$$

Hence $$\int_0^0 (u_x dx + u_y dy + u_z dz) = 0$$

for each irreducible circuit. This is equivalent to

$$\int_0^0 (x\, du_x + y\, du_y + z\, du_z) = 0,$$

if the conditions already obtained are satisfied. Hence

$$\int^0 \Big[(xe_x + ye_y + ze_z)\, dx + \{xe_y + y(c_y - f_x) + \tfrac{1}{2}z(c_z + b_y - a_x)\}\, dy \\ + \{xe_z + \tfrac{1}{2}y(c_z + b_y - a_z) + z(b_z - g_x)\}\, dz\Big] = 0.$$

There are two similar equations derived from v and w.

We have therefore *six* conditions which can be at once expressed as stress-conditions by substitution for the strains.

Theory of Plates.

Let the faces of the plate be $z = \pm h$, and suppose at first that it extends to infinity in all directions. We may then find the stresses in the plate in the following manner. From the equations

$$\nabla^2 R + \kappa\Theta_{zz} = -2Z_z - \nu\Lambda,$$

$$\nabla^2\Theta = -(2\nu+1)\Lambda,$$

we obtain $$\nabla^4 R = -2\nabla^2 Z_z - \nu\nabla^2\Lambda + (\nu+1)\Lambda_{zz}, \qquad (22)$$

remembering that $$2\nu+1 = \frac{3\lambda+2\mu}{\lambda+2\mu},$$

so that $$(2\nu+1)\,\kappa = \frac{2(\lambda+\mu)}{\lambda+2\mu} = \nu+1.$$

Over the faces $z = \pm h$ we have R, S, T given; let them be H, F, G over $z = h$, and H', F', G' over $z = -h$. Since, throughout the plate,

$$T_x + S_y + R_z = -Z,$$

we have $$R_z = -Z - T_x - S_y,$$

also known over $$z = \pm h.$$

Now R is completely determined by (22) when R and dR/dz are given over the faces of the plate, and its value can be written down, *e.g.*, by means of Fourier-integrals. But, without entering into the different ways of obtaining the solution, we may now assume R known. This is the fundamental point of the present theory. It may be noted that, if there is no volume-force, or if, more generally,

$$-2\nabla^2 Z_z - \nu\nabla^2\Lambda + (\nu+1)\,\Lambda_{zz} = 0 \tag{23}$$

throughout, the stress R is independent of the moduli of elasticity. If (23) holds and if R and $-Z-T_x-S_y$ vanish over the faces, then $R=0$ throughout, and *these are the necessary and sufficient conditions that R may vanish throughout the plate*, a point about which there has been so much discussion.

Taking now $$\kappa\Theta_{zz} = -2Z_z - \nu\Lambda - \nabla^2 R,$$

we have $$\kappa\Theta = -\int_0^z\int_0^z (2Z_z + \nu\Lambda + \nabla^2 R)\,dz^2 + \kappa z\,{}_0\Theta_z + \kappa\Theta_0, \tag{24}$$

where ${}_0\Theta_z$, Θ_0 are the values of Θ_z, Θ at $(x, y, 0)$. Also

$$\nabla^2\Theta = -(2\nu+1)\,\Lambda,$$

and therefore $$\nabla^2\Theta_z = -(2\nu+1)\,\Lambda_z,$$

so that
$$\begin{aligned}\nabla^2_{xy}\Theta_z &= -(2\nu+1)\,\Lambda_z - \Theta_{zzz}\\ &= -(2\nu+1)\,\Lambda_z + \frac{2}{\kappa}Z_{zz} + \frac{\nu}{\kappa}\Lambda_z + \frac{1}{\kappa}\nabla^2 R_z\\ &= -\frac{1}{\kappa}\Lambda_z + \frac{2}{\kappa}Z_{zz} + \frac{1}{\kappa}\nabla^2 R_z.\end{aligned}$$

Hence $$\kappa\nabla^2_{xy}\,{}_0\Theta_z = 2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0, \tag{25}$$

and, similarly, $$\kappa\nabla^2_{xy}\Theta_0 = 2\,{}_0Z_z - \Lambda_0 + (\nabla^2 R)_0. \tag{26}$$

Further $$\nabla^2 S + \kappa\Theta_{yz} = -Y_z - Z_y;$$

therefore $$\nabla^2(\nabla^2_{xy}S) + \kappa(\nabla^2_{xy}\Theta)_{yz} = -\nabla^2_{xy}(Y_z + Z_y);$$

so that $$\nabla^2(\nabla^2_{xy}S) = \Lambda_{yz} - 2Z_{yzz} - \nabla^2 R_{yz} - \nabla^2_{xy}(Y_z + Z_y), \tag{27}$$

and, since $\nabla^2_{xy}S$ is known over $z = \pm h$, being obtained by differentiation of the known values of S, equation (27) determines $\nabla^2_{xy}S$ throughout the plate. Also

$$\nabla^2 S_z + \kappa\Theta_{yzz} = -Y_{zz} - Z_{yz}$$

and $$\kappa\Theta_{yzz} = -2Z_{yz} - \nu\Lambda_y - \nabla^2 R_y,$$

so that $$S_{zzz} = -\nabla^2_{xy}S_z + \nu\Lambda_y + \nabla^2 R_y - Y_{zz} + Z_{yz};$$

and therefore S_{zzz} is known throughout the plate. Hence

$$S = A + Bz + Cz^2 + \int_0^z\int_0^z\int_0^z S_{zzz}\,dz^3$$

$$= A + Bz + Cz^2 + S', \text{ say,}$$

where $$F = A + Bh + Ch^2 + S'(h),$$

$$F' = A - Bh + Ch^2 + S'(-h)$$

are known, giving

$$Bh = \tfrac{1}{2}\{F - F' - S'(h) + S'(-h)\}$$

and $$A + Ch^2 = \tfrac{1}{2}\{F + F' - S'(h) - S'(-h)\},$$

so that we may write

$$S = \sigma(z^2 - h^2) + S'',$$

where S'' is known, but σ is an unknown function of x, y.

Similarly, $$T = \tau(z^2 - h^2) + T'',$$

where T'' is known, but τ is an unknown function of x, y.

Substituting these values in

$$\nabla^2 S + \kappa\Theta_{yz} = -Y_z - Z_y,$$

$$\nabla^2 T + \kappa\Theta_{zx} = -Z_x - X_z,$$

we get $$2\sigma = -\kappa\Theta_{yz} - Y_z - Z_y - S''_{zz} - \nabla^2_{xy}S;$$

or, putting $z = 0$, $$\sigma = -\tfrac{1}{2}\kappa({}_0\Theta_z)_y + \sigma',$$

where σ' is a known function of x, y.

Similarly, $\tau = -\frac{1}{2}\kappa\,({}_0\Theta_z)_x + \tau'$,

where τ' is known. Hence

$$\left.\begin{aligned} S &= -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_y\,(z^2-h^2)+S''' \\ T &= -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_x\,(z^2-h^2)+T''' \end{aligned}\right\}, \tag{28}$$

where

$$S''' = S''+\sigma'\,(z^2-h^2),$$
$$T''' = T''+\tau'\,(z^2-h^2),$$

and S''', T''' are known.

Now, take the equation

$$\{R-(1-\kappa)\,\Theta\}_{xx}+\{P-(1-\kappa)\,\Theta\}_{zz} = 2T_{zx},$$

which is the strain-relation

$$g_{xx}+c_{zz} = b_{zx},$$

expressed in terms of the stresses.

It gives, using equation (24),

$$P_{zz} = -2\kappa\,({}_0\Theta_z)_{xx}z+(1-\kappa)({}_0\Theta_z)_{xx}z+(1-\kappa)(\Theta_0)_{xx}+P'_{zz},$$

where P'_{zz} is known. Hence

$$P = (1-3\kappa)\frac{z^3}{6}\,({}_0\Theta_z)_{xx}+(1-\kappa)\frac{z^2}{2}\,(\Theta_0)_{xx}+z\,{}_0P_z+P_0+P'', \tag{29}$$

where

$$P'' = \int_0^z\!\int_0^z P'_{zz}\,dz^2,$$

but ${}_0P_z$ and P_0 are undetermined functions of x, y.

Similarly,

$$\left.\begin{aligned} Q &= (1-3\kappa)\frac{z^3}{6}\,({}_0\Theta_z)_{yy}+(1-\kappa)\frac{z^2}{2}\,(\Theta_0)_{yy}+z\,{}_0Q_z+Q_0+Q'' \\ U &= (1-3\kappa)\frac{z^3}{6}\,({}_0\Theta_z)_{xy}+(1-\kappa)\frac{z^2}{2}\,(\Theta_0)_{xy}+z\,{}_0U_z+U_0+U'' \end{aligned}\right\}, \tag{29}$$

the last being derived from the equation

$$\{R-(1-\kappa)\,\Theta\}_{xy}+U_{zz}-S_{zx}-T_{zy} = 0,$$

which is the equivalent of the strain-relation

$$2g_{xy}+c_{zz}-a_{zx}-b_{zy} = 0.$$

It remains to connect and find differential equations for the unknowns ${}_0\Theta_z$, Θ_0, ${}_0P_z$, P_0, ${}_0Q_z$, Q_0, ${}_0U_z$, U_0.

Now, first, from the equations

$$P_x + U_y + T_z = -X,$$

$$U_x + Q_y + S_z = -Y,$$

we have

$$(P_0)_x + (U_0)_y = -X_0 - {}_0T_z''',$$

$$(U_0)_x + (Q_0)_y = -Y_0 - {}_0S_z''';$$

and hence

$$P_0 = \psi_{yy} + V,$$

$$Q_0 = \psi_{xx} + W,$$

$$U_0 = -\psi_{xy};$$

and therefore

$$\Theta_0 = \nabla_{xy}^2 \psi + V + W + R_0,$$

where V, W are known.

Substituting in $\kappa \nabla_{xy}^2 \Theta_0 = 2\,{}_0Z_z - \Lambda_0 + (\nabla^2 R)_0$,

we have

$$\kappa \nabla_{xy}^4 \psi = 2\,{}_0Z_z - \Lambda_0 + (\nabla^2 R)_0 - \kappa \nabla_{xy}^2 (V + W + R_0), \qquad (30)$$

which is the differential equation for ψ.

Similarly,

$$P_x + U_y + T_z = -X,$$

$$U_x + Q_y + S_z = -Y$$

give

$$({}_0P_z)_x + ({}_0U_z)_y = -{}_0X_z - {}_0T_{zz}''' + \kappa ({}_0\Theta_z)_x,$$

$$({}_0U_z)_x + ({}_0Q_z)_y = -{}_0Y_z - {}_0S_{zz}''' + \kappa ({}_0\Theta_z)_y.$$

Hence

$${}_0P_z - \kappa\, {}_0\Theta_z = \phi_{yy} + V',$$

$${}_0Q_z - \kappa\, {}_0\Theta_z = \phi_{xx} + W',$$

$${}_0U_z = -\phi_{xy};$$

and therefore

$$(1 - 2\kappa)\, {}_0\Theta_z = \nabla_{xy}^2 \phi + V' + W' + {}_0R_z,$$

where V', W' are known.

Substituting in the equation

$$\kappa \nabla_{xy}^2\, {}_0\Theta_z = 2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0,$$

we obtain

$$\frac{\kappa}{1-2\kappa} \nabla_{xy}^4 \phi = 2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0 - \frac{\kappa}{1-2\kappa} \nabla_{xy}^2 (V' + W' + {}_0R_z) \quad (31)$$

or $(\nu+1) \nabla_{xy}^4 \phi = -2\,{}_0Z_{zz} + {}_0\Lambda_z - (\nabla^2 R_z)_0 - (\nu+1) \nabla_{xy}^2 (V' + W' + {}_0R_z)$.

The unknown functions are now reduced to two functions ϕ, ψ of x, y for which the differential equations (30), (31) hold. The differential equation for the normal displacement of the median plane is derived at once.

We have $$w_{xx} = (u_z + w_x)_x - e_z.$$

Hence $$2\mu w_{xx} = 2T_x - \{P - (1-\kappa)\Theta\},$$

and $$2\mu\, {}_0w_{xx} = +\kappa h^2 ({}_0\Theta_z)_{xx} + 2\,{}_0T_x''' + (1-2\kappa)\,{}_0\Theta_z - \phi_{yy} - V'$$
$$= \kappa h^2 ({}_0\Theta_z)_{xx} + 2\,{}_0T_x''' + \phi_{xx} + W' + {}_0R_z.$$

Similarly,
$$2\mu\, {}_0w_{yy} = \kappa h^2 ({}_0\Theta_z)_{yy} + 2\,{}_0S_y''' + \phi_{yy} + V' + {}_0R_z,$$
$$2\mu\, {}_0w_{xy} = \kappa h^2 ({}_0\Theta_z)_{xy} + {}_0S_x''' + {}_0T_y''' + \phi_{xy}.$$

Therefore $$2\mu\, w_0 = \kappa h^2\, {}_0\Theta_z + \phi + \Omega$$
$$= \frac{\kappa h^2}{1-2\kappa}(\nabla_{xy}^2\phi + V' + W' + {}_0R_z) + \phi + \Omega,$$

where Ω is a known function of x, y; a linear function of x, y denoting a rigid body displacement of the median plane being neglected.

Also
$$2\mu\nabla_{xy}^2 w_0 = \kappa h^2 \nabla_{xy}^2({}_0\Theta_z) + 2\,({}_0T_x''' + {}_0S_y''') + \nabla_{xy}^2\phi + V' + W' + 2\,{}_0R_z.$$

This simplifies very much by use of the equation
$$T_x + S_y + R_z = -Z,$$

which gives $${}_0T_x''' + {}_0S_y''' + {}_0R_z + \tfrac{1}{2}\kappa h^2 \nabla_{xy}^2({}_0\Theta_z) = -Z_0.$$

Hence $$2\mu\nabla_{xy}^2 w_0 = \nabla_{xy}^2\phi + V' + W' - 2Z_0$$
$$= (1-2\kappa)\,{}_0\Theta_z - {}_0R_z - 2Z_0,$$

so that $\phi = 2\mu w_0 + 2\mu(\nu+1)\,h^2\nabla_{xy}^2 w_0 + (\nu+1)\,h^2({}_0R_z + 2Z_0) - \Omega$,

and w_0 is known in terms of ϕ and conversely. The differential equation for w_0 is therefore
$$2\mu\kappa\nabla_{xy}^4 w_0 = (1-2\kappa)\left\{2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0\right\} - \kappa\nabla_{xy}^2({}_0R_z + 2Z_0).$$

This result is more easily obtained from the equation
$$(\lambda+\mu)\,\theta_z + \mu\nabla^2 w = -Z$$

or $$\kappa\Theta_z + 2\mu\nabla_{xy}^2 w + R_z - (1-\kappa)\,\Theta_z = -Z.$$

Putting $z = 0$, we have
$$2\mu\nabla_{xy}^2 w_0 = (1-2\kappa)\,{}_0\Theta_z - {}_0R_z - 2Z_0,$$

the same equation as before.

Application to Finite Plates.

If the plate is finite, the equation

$$\nabla^4 R = -2\nabla^2 Z_z - \nu\nabla^2\Lambda + (2\nu+1)\,\kappa\Lambda_{zz},$$

with R and R_z given over $z = \pm h$, does not theoretically completely determine R. The problem is of exactly the same nature as that of the ordinary plate condenser. The potential between the plates is not completely determined by $\nabla^2 V = 0$, with V given over $z = \pm h$. In each case R or V is practically determined with great accuracy by the conditions, except for points at a distance less than a small multiple of the thickness $2h$ from the edge. The same remark applies to the determination of $\nabla^2_{xy} S$ and $\nabla^2_{xy} T$. The above solution then applies without modification to the finite plate. The edge conditions for ϕ, ψ are written down as in the Thomson-Boussinesq theory, using the principle of equipollent loads, and there is no occasion to enter on their discussion here.

The nature of the transmission of stress in the plate will perhaps appear more plainly if we suppose that there is no applied force on part of it. The solutions for R, $\nabla^2_{xy} S$, $\nabla^2_{xy} T$ in that part are then simply

$$R = \nabla^2_{xy} S = \nabla^2_{xy} T = 0$$

throughout. Hence

$$S = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_y\,(z^2-h^2),$$

$$T = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_x\,(z^2-h^2),$$

$$P = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{xx} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{xx} + z\,(\kappa\,{}_0\Theta_z + \phi_{yy}) + \psi_{yy},$$

$$Q = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{yy} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{yy} + z\,(\kappa\,{}_0\Theta_z + \phi_{xx}) + \psi_{xx},$$

$$U = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{xy} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{xy} - z\phi_{xy} - \psi_{xy},$$

where

$$(1-2\kappa)\,{}_0\Theta_z = \nabla^2_{xy}\phi,$$

$$\Theta_0 = \nabla^2_{xy}\psi,$$

$$\nabla^2_{xy}\,{}_0\Theta_z = \nabla^2_{xy}\Theta_0 = 0,$$

$$\phi = 2\mu w_0 + 2\mu\,(\nu+1)\,h^2\nabla^2_{xy} w_0,$$

$$\nabla^4_{xy} w_0 = 0,$$

and hence $$\nabla^2_{xy}\phi = 2\mu\nabla^2_{xy}w_0.$$

Introducing the curvatures ϖ_1, ϖ_2 and the twist τ, we have

$$\varpi_1 = {}_0w_{xx},$$

$$\varpi_2 = {}_0w_{yy},$$

$$\tau = {}_0w_{xy},$$

$$\nabla^2_{xy}(\varpi_1+\varpi_2) = 0,$$

and hence $$S = \mu(\nu+1)(\varpi_1+\varpi_2)_y(z^2-h^2),$$

$$T = \mu(\nu+1)(\varpi_1+\varpi_2)_x(z^2-h^2),$$

$$\begin{aligned} P &= 2\mu\frac{(1-3\kappa)}{1-2\kappa}\frac{z^3}{6}(\varpi_1+\varpi_2)_{xx}-2\mu z\{(\nu+1)\varpi_1+\nu\varpi_2\} \\ &\qquad +2\mu(\nu+1)zh^2(\varpi_1+\varpi_2)_{yy} \\ &\qquad +(1-\kappa)\frac{z^2}{2}(P_0+Q_0)_{xx}+P \\ &= -2\mu z\{(\nu+1)\varpi_1+\nu\varpi_2\} \\ &\quad +2\mu\left\{(\nu+2)\frac{z^3}{6}-(\nu+1)zh^2\right\}(\varpi_1+\varpi_2)_{xx} \\ &\quad +P_0+(1-\kappa)\frac{z^2}{2}(P_0+Q_0)_{xx}. \end{aligned}$$

Similarly for Q, and

$$\begin{aligned} U = &-2\mu z\tau+2\mu(\nu+2)\frac{z^3}{6}(\varpi_1+\varpi_2)_{xy}-2\mu(\nu+1)zh^2\nabla^2_{xy}\tau \\ &+U_0+(1-\kappa)\frac{z^2}{2}(P_0+Q_0)_{xy}. \end{aligned}$$

Approximate Solution.

Taking the unit of length as of the same order as the thickness of the plate, if we assume that the rates of variation of the functions of the forces in the equations and conditions for R, $\nabla^2_{xy}S$, $\nabla^2_{xy}T$, parallel to the plate, are small, we may apply the method of successive approximations to find those quantities to any order of accuracy. To keep the algebra within bounds, let us consider the particular case in which

$$X = Y = 0, \quad Z_z = 0,$$

$$F = F' = G = G' = 0,$$

so that the plate is under normal pressures on its faces, and normal volume force. For a first approximation, we write

$$\nabla^4 R = \frac{d^4 R}{dz^4},$$

and hence in this case $$\frac{d^4 R}{dz^4} = 0,$$

$$R = A + Bz + Cz^2 + Dz^3,$$

where
$$H = A + Bh + Ch^2 + Dh^3,$$
$$H = A - Bh + Ch^2 - Dh^3,$$
$$R_z(h) = -Z = B + 2Ch + 3Dh^2,$$
$$R_z(-h) = -Z = B - 2Ch + 3Dh^2.$$

Hence
$$C = 0,$$
$$B + 3Dh^2 = -Z,$$
$$2A = H + H',$$
$$2Bh + 2Dh^3 = H - H',$$
$$4Bh = 3(H - H') + 2hZ,$$
$$4Dh^3 = -(H - H') - 2hZ,$$

and $$R = \tfrac{1}{2}(H + H') + \frac{1}{4h^3}(H - H')(3zh^2 - z^3) + \frac{1}{2h^3} Z (zh^2 - z^3).$$

This is the first approximation for R; for a further approximation we must substitute this value in the terms of $\nabla^4 R$ previously neglected, and repeat the above process. Contenting ourselves with the above value of R, we have

$$\nabla^2 (\nabla^2_{xy} S) = -\nabla^2 Z_y - \nabla^2 R_{yz}$$

or $$\nabla^2 (\nabla^2_{xy} S + Z_y + R_{yz}) = 0.$$

Now, $$S = 0, \quad R_z + Z = 0,$$

when $$z = \pm h,$$

so that $$\nabla^2_{xy} S + Z_y + R_{yz} = 0,$$

when $$z = \pm h.$$

Hence the equation gives

$$\nabla^2_{xy} S + Z_y + R_{yz} = 0$$

throughout, and this is true whenever

$$X = Y = F = F' = G = G' = 0,$$

so that in this case the trouble of successive approximations is connected with the determination of R alone.

Since $$\nabla^2_{xy} S_z + R_{yzz} = 0,$$

we have $$\frac{d^3 S}{dz^3} = +2R_{yzz} = -\frac{3}{h^3}(H_y - H'_y)\, z - \frac{6}{h^3} Z_y z,$$

neglecting $$\nabla^2_{xy} R_y.$$

Hence $$S = A + Bz + Cz^2 - \frac{1}{8h^3}(H_y - H'_y)\, z^4 - \frac{1}{4h^3} Z_y z^4,$$

so that $$S = \sigma(z^2 - h^2) - \frac{1}{8h^3}(H_y - H'_y)(z^4 - h^4) - \frac{1}{4h^3} Z_y (z^4 - h^4),$$

and, similarly,

$$T = \tau(z^2 - h^2) - \frac{1}{8h^3}(H_x - H'_x)(z^4 - h^4) - \frac{1}{4h^3} Z_x (z^4 - h^4),$$

where
$$\begin{aligned} 2\sigma &= -\kappa\, ({}_0\Theta_z)_y - Z_y - \nabla^2_{xy} S_0 \\ &= -\kappa\, ({}_0\Theta_z)_y + ({}_0R_z)_y \\ &= -\kappa\, ({}_0\Theta_z)_y + \frac{3}{4h}(H_y - H'_y) + \tfrac{1}{2} Z_y, \end{aligned}$$

and, similarly, $$2\tau = -\kappa\, ({}_0\Theta_z)_x + \frac{3}{4h}(H_x - H'_x) + \tfrac{1}{2} Z_x,$$

so that $$\begin{aligned} S = -\tfrac{1}{2}\kappa\, ({}_0\Theta_z)_y (z^2 - h^2) &+ \frac{1}{8h^3}(H_y - H'_y)(z^2 - h^2)(2h^2 - z^2) \\ &- \frac{1}{4h^3} Z_y (z^2 - h^2)\, z^2, \end{aligned}$$

and $$\begin{aligned} T = -\tfrac{1}{2}\kappa\, ({}_0\Theta_z)_x (z^2 - h^2) &+ \frac{1}{8h^3}(H_x - H'_x)(z^2 - h^2)(2h^2 - z^2) \\ &- \frac{1}{4h^3} Z_x (z^2 - h^2)\, z^2. \end{aligned}$$

The differential equation for w_0 is, therefore, neglecting the second and higher differential coefficients of the applied forces with respect to x, y,

$$2\mu\kappa \nabla^4_{xy} w_0 = (1 - 2\kappa)\frac{d^3 R}{dz^3} = -(1 - 2\kappa)\left\{\frac{3}{h^3} Z + \frac{3}{2h^3}(H - H')\right\},$$

or $$4\mu(\nu + 1)\, h^3 \nabla^4_{xy} w_0 = 3\{2hZ + H - H'\}.$$

Since $$\nu+1=\frac{2(\lambda+\mu)}{\lambda+2\mu},$$

this is the equation which would be obtained according to the ordinary approximation, $2hZ+H-H'$ being the normal force per unit area of plate.

The processes here developed are obviously well suited to the treatment of the problem of Cerutti and Boussinesq, viz., to determine the stress through an infinite solid, bounded by an infinite plane, on which the forces are given. As a matter of fact, the processes lead directly to a potential-solution of the form given by the authors named, but the present is hardly a fit occasion to give a mere revision of this famous problem.

The Stress in a Rotating Lamina. By J. H. MICHELL, M.A.

Read April 13th, 1899. Received, in revised form, September 4th, 1899.

In a paper* recently presented to the Society I have given a general theory of plates under any forces. I propose, in the present note, to apply the theory to the case of a lamina rotating about a fixed axis perpendicular to its plane. The notation is that of the paper referred to.

Taking the axis of z normal to the lamina, let ϖ be the angular velocity of the lamina, ρ its density, so that the component forces are

$$X=\rho\varpi^2x,$$

$$Y=\rho\varpi^2y,$$

and hence $$X_x+Y_y=\Lambda=2\rho\varpi^2,$$

so that $$\nabla^4R=0.$$

Since R and dR/dz vanish on the faces $z=\pm h$, supposed free from stress, it follows that $R=0$ throughout the plate (except, of course, at points close to the edges, as explained in the former paper).

---

* "On the Direct Determination of Stress in an Elastic Solid, &c.," pp. 100-124.