On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Divisions of a Scale. By W. F. Sheppard, M.A., LL.M.

Read March 10th, 1898. Received, in revised form, May 10th, 1898.

I. General Formulæ.

1\. Let
$$z = f(x)$$
be the equation to a curve which lies wholly on the positive side of the axis of x, and extends continuously from the ordinate $z_0 = f(x_0)$ to the ordinate $z_p = f(x_p)$, and which is subject to the further condition that $f(x)$ is single-valued between $x = x_0$ and $x = x_p$. Let the range $x_p - x_0$ be divided into p equal segments, each equal to h, by points whose abscissæ are $x_1, x_2, \ldots, x_{p-1}$, so that
$$x_r = x_0 + rh;$$
let the ordinates to the curve from the points of division be $z_1, z_2, \ldots, z_{p-1}$, so that
$$z_r = f(x_r);$$
and let A_r denote the area between two consecutive ordinates z_{r-1} and z_r, the curve, and the base $z = 0$, so that
$$A_r = \int_{x_{r-1}}^{x_r} f(x)\,dx.$$

For the mathematical treatment of statistics we require a convenient formula for calculating, approximately, the value of $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$, where—

(i.) $\phi(x)$ is a known function of x, which is single-valued, finite, and continuous from $x = x_0$ to $x = x_p$; and

(ii.) the form of $f(x)$ is unknown, but either

(*a*) the values of the isolated ordinates $z_0 = f(x_0)$, $z_1 = f(x_1)$, …, $z_p = f(x_p)$ are known, or

(*b*) the values of the successive areas
$$A_1 = \int_{x_0}^{x_1} f(x)\,dx, \quad A_2 = \int_{x_1}^{x_2} f(x)\,dx, \quad \ldots, \quad A_p = \int_{x_{p-1}}^{x_p} f(x)\,dx$$
are known.

2. The Euler-Maclaurin formula

$$\int u_x\,dx = C+\Sigma u_x-\tfrac{1}{2}u_x-\frac{B_1}{2!}\frac{du_x}{dx}+\frac{B_2}{4!}\frac{d^3u_x}{dx^3}-\dots \qquad (1)^*$$

may be made to meet both these cases. For the first case we have at once

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$=\tfrac{1}{2}f(x_0)\,\phi(x_0)+f(x_1)\,\phi(x_1)+\dots+f(x_{p-1})\,\phi(x_{p-1})+\tfrac{1}{2}f(x_p)\,\phi(x_p)$$

$$-\Big[\Big\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\dots\Big\}f(x)\,\phi(x)\Big]_{x=x_0}^{x=x_p} \qquad (2)$$

$$=\tfrac{1}{2}z_0\phi(x_0)+z_1\phi(x_1)+\dots+z_{p-1}\phi(x_{p-1})+\tfrac{1}{2}z_p\phi(x_p)$$

$$+\tfrac{1}{12}(\Delta-\tfrac{1}{2}\Delta^2+\tfrac{19}{60}\Delta^3-\dots)f(x_0)\,\phi(x_0)$$

$$+\tfrac{1}{12}(\Delta'-\tfrac{1}{2}\Delta'^2+\tfrac{19}{60}\Delta'^3-\dots)f(x_p)\,\phi(x_p), \qquad (2\text{A})$$

the second set of differences $\Delta', \Delta'^2, \Delta'^3, \dots$ being accented to show that they are taken backwards, *i.e.*, from $x = x_p$ towards $x = x_0$.

For the second case we write

$$F(x) = \int_x^{x_p} f(x)\,dx,$$

so that $F(x)$ denotes the whole area of the curve lying beyond the ordinate $z = f(x)$. Then

$$\left.\begin{aligned} F(x_0) &= A_1+A_2+A_3+\dots+A_p \\ F(x_1) &= \phantom{A_1+{}} A_2+A_3+\dots+A_p \\ &\dots\dots\dots\dots\dots \\ F(x_{p-1}) &= A_p \\ F(x_p) &= 0 \end{aligned}\right\}, \qquad (3)$$

so that $F(x)$ is a function for which the isolated values $F(x_0)$, $F(x_1), \dots, F(x_p)$ are known. By integration by parts,

$$\int_{x_0}^{x_p} f(x)\,\varphi(x)\,dx = F(x_0)\,\varphi(x_0)+\int_{x_0}^{x_p} F(x)\,\varphi'(x)\,dx, \qquad (4)\dagger$$

* Here Σu_x denotes the sum up to and including u_x. If Σ is used as equivalent to Δ^{-1}, Σu_x is the sum up to and including u_{x-1}, and we must replace $-\tfrac{1}{2}u_x$ by $+\tfrac{1}{2}u_x$.

† For instance, let $F(x)$ denote the proportion of individuals alive at age x for every one alive at age x_0; then $f(x)dx$ will denote the proportion who die between ages $x-\tfrac{1}{2}dx$ and $x+\tfrac{1}{2}dx$. If

$$\phi(x) = P(1+i)^{-(x-x_0)},$$

and therefore

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$=\frac{1}{h}F(x_0)\,\phi(x_0)+\tfrac{1}{2}F(x_0)\,\phi'(x_0)+F(x_1)\,\phi'(x_1)+\ldots+F(x_{p-1})\,\phi'(x_{p-1})$$

$$-\left[\left\{\frac{B_1}{2\,!}h\frac{d}{dx}-\frac{B_2}{4\,!}h^3\frac{d^3}{dx^3}+\ldots\right\}F(x)\,\phi'(x)\right]_{x=x_0}^{x=x_p} \qquad (5)$$

$$=\frac{1}{h}F(x_0)\,\phi(x_0)+\tfrac{1}{2}F(x_0)\,\phi'(x_0)+F(x_1)\,\phi'(x_1)+\ldots+F(x_{p-1})\,\phi'(x_{p-1})$$

$$+\tfrac{1}{12}\left(\Delta-\tfrac{1}{2}\Delta^2+\tfrac{19}{80}\Delta^3-\ldots\right)F(x_0)\,\phi'(x_0)$$

$$+\tfrac{1}{12}\left(\Delta-\tfrac{1}{2}\Delta'^2+\tfrac{19}{80}\Delta'^3-\ldots\right)F(x_p)\,\phi'(x_p). \qquad (5\text{A})$$

3. The formulæ (2A) and (5A) are sufficient, but they involve the calculation of successive differences. In the cases which we have specially in view the curve $z=f(x)$ touches the base $z=0$, to a very high order of contact, at the extreme points $x=x_0$ and $x=x_p$. In such a case $f(x_0)$ and $f(x_p)$ and their first few differential coefficients are zero. Hence the expression given by the square brackets in (2) is negligible, and we have, *approximately*,

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx=\tfrac{1}{2}z_0\phi(x_0)+z_1\phi(x_1)+\ldots+z_{p-1}\phi(x_{p-1})+\tfrac{1}{2}z_p\phi(x_p). \qquad (2\text{B})$$

Also the first few differential coefficients of $F(x_0)$ and $F(x_p)$ are zero, and

$$F(x_p)=0;$$

but $F(x_0)$ is not zero. Hence the expression given by the square brackets in (5) is negligible, except as to the part involving $F(x_0)$. Taking this out, we have, approximately,

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$=\tfrac{1}{2}F(x_0)\,\phi'(x_0)+F(x_1)\,\phi'(x_1)+\ldots+F(x_{p-1})\,\phi'(x_{p-1})$$

$$+\frac{1}{h}F(x_0)\left\{\phi(x_0)+\frac{B_1}{2\,!}h^2\phi''(x_0)-\frac{B_2}{4\,!}h^4\phi^{\text{iv}}(x_0)+\ldots\right\} \qquad (5\text{B})$$

$\int_{x_0}^{\infty} f(x)\,\phi(x)\,dx$ will be the present value of a reversion of value P, due on the death of a person aged x_0, interest being reckoned at $100i$ per cent. per annum, while $\int_{x_0}^{\infty} F(x)\,\phi(x)\,dx$ will be the present value of a continuous annuity at the rate of P per annum, payable to a person of that age, and (4) shows that

(present value of reversion) $= P-\log_e(1+i)$. (present value of annuity),

which is a well-known formula.

the expression omitted from the right-hand side being

$$\Big[F(x)\Big\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\dots\Big\}\varphi'(x)$$

$$-\Big\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\dots\Big\}F(x)\varphi'(x)\Big]_{x=x_0}^{x=x_p} \qquad (5\text{C})$$

The formula (2B) is convenient for calculation, but (5B) is obviously inconvenient, on account of its unsymmetrical form. Our object is therefore to obtain a general formula for $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$, which shall be better suited for these particular cases.

4. Let $\xi_1, \xi_2, \dots, \xi_p$ be the abscissæ of the middle points of the bases of the areas $A_1, A_2, \dots, A_p$, so that

$$\xi_r = \tfrac{1}{2}(x_{r-1}+x_r).$$

We wish to obtain the value of $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$ in the form

$$\Sigma A_r\phi_1(\xi_r)+h\left[\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

where $\phi_1(x)$ is a function of x which can be derived from $\phi(x)$, and $\phi_2(x)$ is an expression which can be derived from $f(x)$ and $\phi(x)$, and which may be disregarded when $f(x)$ and its first few differential coefficients are zero. We must therefore express

$$\int_{x_{r-1}}^{x_r} f(x)\,\phi(x)\,dx \equiv \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)\,\phi(\xi_r+\theta)\,d\theta$$

in the form

$$\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)\,d\theta\,.\,\phi_1(\xi_r)+h\left[\phi_2(x)\right]_{x=x_{r-1}}^{x=x_r};$$

in other words, we must find such a value for $\phi_1(\xi)$ that

$$\Phi(\xi)\equiv\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,\phi(\xi+\theta)\,d\theta-\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,d\theta\,.\,\phi_1(\xi) \qquad (6)$$

may be capable of being expressed in the form

$$h\left\{\phi_2(\xi+\tfrac{1}{2}h)-\phi_2(\xi-\tfrac{1}{2}h)\right\}, \qquad (7)$$

where $\phi_2(\xi)$ contains only $f(\xi)$ and $\phi(\xi)$ and their successive differential coefficients.

Now, in order that $\Phi(\xi)$ may be expressible in the form (7), it is necessary and sufficient (subject to certain conditions as to convergence) that $\Phi(\xi)\,d\xi$ should be an exact differential. It is necessary, for if we expand (7) in ascending powers of h we have only differential coefficients of $\phi_2(\xi)$. And it is sufficient, for if $\Phi(\xi)$ is equal to $h\Psi'(\xi)$ we can obviously choose the coefficients $C_1, C_2, \ldots$ so as to make it equal to

$$\{\Psi(\xi+\tfrac{1}{2}h)-\Psi(\xi-\tfrac{1}{2}h)\}+C_1\{\Psi''(\xi+\tfrac{1}{2}h)-\Psi''(\xi-\tfrac{1}{2}h)\}(\tfrac{1}{2}h)^2 + C_2\{\Psi^{\text{iv}}(\xi+\tfrac{1}{2}h)-\Psi^{\text{iv}}(\xi-\tfrac{1}{2}h)\}(\tfrac{1}{2}h)^4+\ldots.$$

The values of these coefficients can be obtained most simply by the method of operators, which shows that

$$\begin{aligned} h\Psi'(\xi) &= \left(e^{\frac{1}{2}h(d/d\xi)}-e^{-\frac{1}{2}h(d/d\xi)}\right)\frac{\tfrac{1}{2}h\dfrac{d}{d\xi}}{\sinh\tfrac{1}{2}h\dfrac{d}{d\xi}}\,\Psi(\xi) \\ &= \{\Psi(\xi+\tfrac{1}{2}h)-\Psi(\xi-\tfrac{1}{2}h)\} \\ &\quad -\frac{P_1}{2!}\{\Psi''(\xi+\tfrac{1}{2}h)-\Psi''(\xi-\tfrac{1}{2}h)\}(\tfrac{1}{2}h)^2 \\ &\quad +\frac{P_2}{4!}\{\Psi^{\text{iv}}(\xi+\tfrac{1}{2}h)-\Psi^{\text{iv}}(\xi-\tfrac{1}{2}h)\}(\tfrac{1}{2}h)^4 \\ &\quad -\ldots, \end{aligned} \tag{8}$$

where $P_1, P_2, \ldots$ are the coefficients in the expansion

$$\frac{\theta}{\sinh\theta}=1-\frac{P_1}{2!}\theta^2+\frac{P_2}{4!}\theta^4-\ldots, \tag{9}$$

so that

$$P_r=(2^{2r}-2)\,B_r. \tag{10}$$

We consider that we are only dealing with cases in which the series (8) is initially convergent.

Now let us assume that

$$\phi_1(\xi)=\psi\left(\tfrac{1}{2}h\frac{d}{d\xi}\right)\phi(\xi). \tag{11}$$

Then if we denote $\tfrac{1}{2}h\dfrac{d}{d\xi}$ operating on $f(\xi)$ by D_1, and $\tfrac{1}{2}h\dfrac{d}{d\xi}$ operating on $\phi(\xi)$ by D_2, $\Phi(\xi)$ is equal to

$$h\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\psi(D_2)\right\}f(\xi)\,\phi(\xi). \tag{12}$$

In order that $\Phi(\xi)\,d\xi$ may be an exact differential, it is clearly necessary that (12) may be capable of being written in the form

$$h(D_1+D_2)\,\chi(D_1,\,D_2)\,f(\xi)\,\phi(\xi), \tag{13}$$

where $\chi(D_1,\,D_2)$ is a function containing only positive integral powers (including the power 0) of D_1 and D_2. It will be seen that

$$\psi(D_2)=\frac{D_2}{\sinh D_2}$$

satisfies these conditions. For

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}$$

can be expanded so as to include only positive integral powers of D_1 and D_2; and it vanishes when $D_1+D_2=0$, so that D_1+D_2 is a factor. Hence we have

$$\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,\phi(\xi+\theta)\,d\theta$$

$$=\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,d\theta\,.\left\{\phi(\xi)-\frac{P_1}{2!}\phi''(\xi)\,(\tfrac{1}{2}h)^2+\frac{P_2}{4!}\phi^{\text{iv}}(\xi)\,(\tfrac{1}{2}h)^4-\dots\right\}$$

$$+h\left\{\phi_2(\xi+\tfrac{1}{2}h)-\phi_2(\xi-\tfrac{1}{2}h)\right\}, \tag{14}$$

where

$$\phi_2(\xi)=\tfrac{1}{2}\left\{\frac{1}{D_1+D_2}-\frac{D_2\sinh D_1}{D_1\sinh D_2\sinh(D_1+D_2)}\right\}f(\xi)\,\phi(\xi)$$

$$=\tfrac{1}{2}\frac{D_2}{D_1}\left\{\left(\coth\overline{D_1+D_2}-\frac{1}{D_1+D_2}\right)-\left(\coth D_2-\frac{1}{D_2}\right)\right\}f(\xi)\,\phi(\xi).$$

Since
$$\tfrac{1}{2}\theta\coth\tfrac{1}{2}\theta=1+\frac{B_1}{2!}\theta^2-\frac{B_2}{4!}\theta^4+\dots, \tag{15}$$

this reduces to

$$\phi_2(\xi)=F(\xi)\left\{\frac{B_1}{2!}h\frac{d}{d\xi}-\frac{B_2}{4!}h^3\frac{d^3}{d\xi^3}+\dots\right\}\phi'(\xi)$$

$$-\left\{\frac{B_1}{2!}h\frac{d}{d\xi}-\frac{B_2}{4!}h^3\frac{d^3}{d\xi^3}+\dots\right\}F(\xi)\,\phi'(\xi). \tag{16}$$

Substituting in (14), writing $\xi=\xi_1,\,\xi_2,\,\dots,\,\xi_p$ successively, and adding together the p equations so obtained, we have, finally,

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=1}^{r=p} A_r \left\{ \phi(\xi_r) - \frac{P_1}{2!}(\tfrac{1}{2}h)^2 \phi''(\xi_r) + \frac{P_2}{4!}(\tfrac{1}{2}h)^4 \phi^{\text{iv}}(\xi_r) - \dots \right\}$$

$$+ h \Big[F(x) \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} \phi'(x)$$

$$- \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} F(x)\,\phi'(x) \Big]_{x=x_0}^{x=x_p}. \qquad (17)$$

This result can be verified by comparing it with (5B) and (5C). To give the expression in square brackets explicitly in terms of $f(x)$ and $\phi(x)$ and their derived functions, we write, for convenience,

$$Q_r = \frac{B_r}{2r}, \qquad (18)$$

and we find that

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=1}^{r=p} A_r \left\{ \phi(\xi_r) - \frac{P_1}{2!}(\tfrac{1}{2}h)^2 \phi''(\xi_r) + \frac{P_2}{4!}(\tfrac{1}{2}h)^4 \phi^{\text{iv}}(\xi_r) - \dots \right\}$$

$$+ h^2 \Big[\left\{ \frac{Q_1}{0!} \phi'(x) - \frac{Q_2}{2!} h^2 \phi'''(x) + \frac{Q_3}{4!} h^4 \phi^{\text{v}}(x) - \dots \right\} f(x)$$

$$- \frac{1}{2!} \left\{ \frac{Q_2}{1!} \phi''(x) - \frac{Q_3}{3!} h^2 \phi^{\text{iv}}(x) + \frac{Q_4}{5!} h^4 \phi^{\text{vi}}(x) - \dots \right\} h^2 f'(x)$$

$$- \frac{1}{3!} \left\{ \frac{Q_2}{0!} \phi'(x) - \frac{Q_3}{2!} h^2 \phi'''(x) + \frac{Q_4}{4!} h^4 \phi^{\text{v}}(x) - \dots \right\} h^2 f''(x)$$

$$+ \frac{1}{4!} \left\{ \frac{Q_3}{1!} \phi''(x) - \frac{Q_4}{3!} h^2 \phi^{\text{iv}}(x) + \frac{Q_5}{5!} h^4 \phi^{\text{vi}}(x) - \dots \right\} h^4 f'''(x)$$

$$+ \&c. \Big]_{x=x_0}^{x=x_p}, \qquad (19)$$

where

$$\left. \begin{array}{lllll} P_1 = \frac{1}{3}, & P_2 = \frac{1}{15}, & P_3 = \frac{31}{21}, & P_4 = \frac{127}{15}, & P_5 = \frac{2555}{33}, \dots \\ Q_1 = \frac{1}{12}, & Q_2 = \frac{1}{120}, & Q_3 = \frac{1}{252}, & Q_4 = \frac{1}{240}, & Q_5 = \frac{1}{132}, \dots \end{array} \right\}. \qquad (20)$$

5. The value of $\phi_1(\xi)$ might have been obtained without using the symbolic method, as follows.

Let us assume that

$$\phi_1(\xi) = \phi(\xi) + \frac{\kappa_1}{2!}(\tfrac{1}{2}h)^2\phi''(\xi) + \frac{\kappa_2}{4!}(\tfrac{1}{2}h)^4\phi^{\text{iv}}(\xi) + \dots.$$

(The odd differential coefficients are omitted, since it is clear that $\phi_1(\xi)$ is not altered by changing $\frac{1}{2}h$ into $-\frac{1}{2}h$). Expand $f(\xi+\theta)$ and $\phi(\xi+\theta)$ in ascending powers of θ by Taylor's theorem, substitute in (6), and perform the integrations; then substitute for $\phi_1(\xi)$ from the above expression, and arrange the result in a series proceeding by ascending powers of h. The successive coefficients will contain products as follows:—

co. h^3	co. h^5	co. h^7	
$f(\xi)\,\phi''(\xi)$,	$f(\xi)\,\phi^{\text{iv}}(\xi)$,	$f(\xi)\,\phi^{\text{vi}}(\xi)$,	...,
$f'(\xi)\,\phi'(\xi)$,	$f'(\xi)\,\phi'''(\xi)$,	$f'(\xi)\,\phi^{\text{v}}(\xi)$,	...,
	$f''(\xi)\,\phi''(\xi)$,	$f''(\xi)\,\phi^{\text{iv}}(\xi)$,	...,
	$f'''(\xi)\,\phi'(\xi)$,	$f'''(\xi)\,\phi'''(\xi)$,	...,
		$f^{\text{iv}}(\xi)\,\phi''(\xi)$,	...,
		$f^{\text{v}}(\xi)\,\phi'(\xi)$,	...,
			....

The first coefficient will involve κ_1, the second κ_1 and κ_2, and so on. Hence the values of $\kappa_1, \kappa_2, \dots$ can be chosen successively so that each term in the expansion of $\Phi(\xi)\,d\xi$ in ascending powers of h shall be an exact differential. The n^{th} term, divided by $\frac{2}{2n+1}(\frac{1}{2}h)^{2n+1}$, is

$$\begin{aligned}\Big\{\frac{1}{0!}\,\frac{1}{(2n)!}\Big(1-\frac{2n+1}{1}\kappa_n\Big)f(\xi)\,\phi^{(2n)}(\xi) + \frac{1}{1!}\,\frac{1}{(2n-1)!}f'(\xi)\,\phi^{(2n-1)}(\xi) \\ + \frac{1}{2!}\,\frac{1}{(2n-2)!}\Big(1-\frac{2n+1}{3}\kappa_{n-1}\Big)f''(\xi)\,\phi^{(2n-2)}(\xi) + \dots \\ \dots + \frac{1}{(2n-2)!}\,\frac{1}{2!}\Big(1-\frac{2n+1}{2n-1}\kappa_1\Big)f^{(2n-2)}(\xi)\,\phi'(\xi) \\ + \frac{1}{(2n-1)!}\,\frac{1}{1!}f^{(2n-1)}(\xi)\,\phi'(\xi)\Big\}\,d\xi.\end{aligned}$$

Let us choose κ_n so as to make this the differential of

$$C_1\frac{1}{1!}\,\frac{1}{(2n-1)!}f(\xi)\,\phi^{(2n-1)}(\xi) + C_2\frac{1}{2!}\,\frac{1}{(2n-2)!}f'(\xi)\,\phi^{(2n-2)}(\xi) + \dots$$
$$\dots + C_{2n-1}\frac{1}{(2n-1)!}\,\frac{1}{1!}f^{(2n-2)}(\xi)\,\phi'(\xi).$$

Differentiating this last expression, and equating coefficients,

$$\begin{array}{lllll}
1-\frac{2n+1}{1}\kappa_n & = & & \frac{2n}{1}C_1, & \\
1 & = C_1 & + \frac{2n-1}{2}C_2, & & \\
1-\frac{2n+1}{3}\kappa_{n-1} & = C_2 & + \frac{2n-2}{3}C_3, & & \\
\dots & \dots & \dots & \dots & \dots \\
1-\frac{2n+1}{2n-1}\kappa_1 & = C_{2n-2} & + \frac{2}{2n-1}C_{2n-1}, & & \\
1 & = C_{2n-1}. & & &
\end{array}$$

Eliminating, and remembering that

$$1-\frac{2n}{1}+\frac{2n\,(2n-1)}{1.2}-\dots+\frac{2n\,(2n-1)\dots 1}{1.2\dots 2n}=(1-1)^{2n}=0,$$

we find that

$$\frac{1}{0!\,(2n+1)!}+\frac{\kappa_1}{2!\,(2n-1)!}+\frac{\kappa_2}{4!\,(2n-3)!}+\dots+\frac{\kappa_n}{(2n)!\,1!}=0.$$

Now let $$\Omega(\theta)=1+\frac{\kappa_1}{2!}\theta^2+\frac{\kappa_2}{4!}\theta^4+\dots .$$

Then the above relation shows that

$$\Omega(\theta)\,.\sinh\theta=\theta,$$

and therefore $$\Omega(\theta)=\frac{\theta}{\sinh\theta},$$

which agrees with (14).

6. In the general formula (17) or (19) the expressions to be summed are given in terms of $\xi_1, \xi_2, \dots, \xi_p$, the abscissæ of the middle points of the bases of the compartments $A_1, A_2, \dots, A_p$. If preferred, we can use $x_0, x_1, \dots, x_p$. For

$$\phi_1(\xi)=\frac{\frac{1}{2}h\frac{d}{d\xi}}{\sinh\frac{1}{2}h\frac{d}{d\xi}}\phi(\xi)$$

$$=\frac{h\frac{d}{d\xi}}{\sinh h\frac{d}{d\xi}}\,\tfrac{1}{2}\left\{\phi(\xi+\tfrac{1}{2}h)+\phi(\xi-\tfrac{1}{2}h)\right\}, \qquad (21)$$

and therefore

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=0}^{r=p} \tfrac{1}{2}(A_r + A_{r+1}) \left\{ \phi(x_r) - \frac{P_1}{2!} h^2 \phi''(x_r) + \frac{P_2}{4!} h^4 \phi^{\text{iv}}(x_r) - \ldots \right\}$$

$$+ h \left[F(x) \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \ldots \right\} \phi'(x) \right.$$

$$\left. - \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \ldots \right\} F(x)\,\phi'(x) \right]_{x=x_0}^{x=x_p}, \qquad (22)$$

where A_0 and A_{p+1} are each zero.

II. Application to Calculation of Σx^m.*

7. When an attribute common to a number of individuals is capable of exact measurement, and of representation by a linear magnitude x, whose variation is continuous, the result of a classification of the individuals according to this attribute is generally given in the form of a table showing the numbers which fall into consecutive classes corresponding to equal increments of x, *e.g.*,

(A) *Weights of 5,552 Englishmen.*†

Weight.	Number.	Weight.	Number.
90 lbs. to 100 lbs.	2	180 lbs. to 190 lbs.	304
100 lbs. to 110 lbs.	26	190 lbs. to 200 lbs.	174
110 lbs. to 120 lbs.	133	200 lbs. to 210 lbs.	75
120 lbs. to 130 lbs.	338	210 lbs. to 220 lbs.	62
130 lbs. to 140 lbs.	694	220 lbs. to 230 lbs.	33
140 lbs. to 150 lbs.	1,240	230 lbs. to 240 lbs.	10
150 lbs. to 160 lbs.	1,075	240 lbs. to 250 lbs.	9
160 lbs. to 170 lbs.	881	250 lbs. to 260 lbs.	3
170 lbs. to 180 lbs.	492	260 lbs. to 270 lbs.	1

* Some of the results of this section have been stated, without proof, in a paper printed in the *Journal of the Royal Statistical Society* (September, 1897, p. 698).

† *Report of the British Association*, 1883, p. 257.

(B) *Examination Marks of* 1,000 *Sandhurst Candidates.* *

Mark.	Number.	Mark.	Number.
0 to 99	15	500 to 599	221
100 to 199	54	600 to 699	115
200 to 299	131	700 to 799	31
300 to 399	227	800 to 899	3
400 to 499	203	900 to 999	0

From the statistical point of view we are not concerned with particular individuals. We regard them—on grounds which need not be discussed here—as having been obtained by random selection from an indefinitely great community; and we investigate the distribution of the values of x in this hypothetical community. Any such distribution can be expressed in terms of the mean values of $x, x^2, x^3, \ldots$ for the community, which we may denote by $[x], [x^2], [x^3], \ldots$; and therefore the problem is to determine, from the data, the most probable combination of values of $[x], [x^2], [x^3], \ldots$.

To do this, when the data take the form of a table such as (A) or (B), we adopt the same process as if we knew the exact value of x for each of the observed individuals. We postulate an indefinite number of communities with different laws of distribution of x, and therefore with different values of $[x], [x^2], [x^3], \ldots$; we suppose that from each of these communities a random selection of n individuals is made N times, where n is the number of individuals actually observed, and N is an indefinitely great number; and from this double aggregate of sets of n individuals, we pick out those sets in which the numbers in the consecutive classes are the same as in the actual data, and consider how the different values of $[x], [x^2], [x^3], \ldots$ are distributed in these particular cases.

8. Let the range of values of x, in the original community, be from x_0 to x_p, this range being divided into p segments, each equal to h; and let the actual data consist of a table giving numbers $n_1, n_2, \ldots n_p$ in the corresponding p classes, where $n_1 + n_2 + \ldots + n_p = n$.

* These are the marks in English Composition (a few candidates being omitted) at the examinations in November, 1894, and June, 1895. The table must not be taken as giving the law of distribution of the marks at either examination separately.

We assume that the data show that both the variation of x and the frequency of the variation may be regarded as continuous; *i.e.*, that the true figure of frequency of x may be regarded as bounded by a continuous curve. Let the equation to this unknown curve be

$$z = f(x),$$

and let the areas between the successive ordinates $f(x_0), f(x_1), \ldots, f(x_p)$, where $x_r = x_0 + xh$, be $A_1, A_2, \ldots, A_p$, the total area of the curve being A. Then our fundamental postulate is that there are an indefinite number of forms of $f(x)$, or, rather, of $f(x) \div A$; and we have to consider the result of a random selection of n individuals when $f(x) \div A$ has any particular form, and thence to determine the distribution of the different values of

$$[x^m] \equiv \int_{x_0}^{x_p} f(x)\, x^m\, dx \div A,$$

when the random selection gives numbers $n_1, n_2, \ldots, n_p$ in the p classes.

If in the expression contained in square brackets in (17) we replace the differential coefficients by differences, and if the resulting series is initially convergent when

$$\phi(x) = x^m,$$

we obtain $\int_{x_0}^{x_p} f(x)\, x^m\, dx$ in the form

$$C_1 A_1 + C_2 A_2 + \ldots + C_p A_p.$$

Let

$$n'_r = nA_r/A,$$

so that $n'_1, n'_2, \ldots, n'_p$ are the numbers (not necessarily integral) that would fall into the p classes if we made a *representative* selection of n individuals. Then

$$n\,[x^m] \equiv n \int_{x_0}^{x_p} f(x)\, x^m dx \div A$$

is of the form

$$C_1 n'_1 + C_2 n'_2 + \ldots + C_p n'_p.$$

By hypothesis, the numbers $n_1, n_2, \ldots, n_p$ are obtained by *random* selection, and they therefore differ from $n'_1, n'_2, \ldots, n'_p$ by amounts which, when n is large, are distributed according to the normal law with mean values zero, and mean squares $n'_1(n-n'_1)/n$, $n'_2(n-n'_2)/n, \ldots, n'_p(n-n'_p)/n$, the p distributions being normally correlated. It follows, by the ordinary methods of inverse probabability, that, if the *a priori* frequency of different values of $[x^m]$, in those cases in which $A_1, A_2, \ldots, A_p$ are nearly proportional to $n_1, n_2, \ldots n_p$, is continuous, the most probable value of $n[x^m]$ as deducible from the data is (approximately) $C_1 n_1 + C_2 n_2 + \ldots + C_p n_p$; and the possible values are distributed normally about this value with mean square $\{\Sigma C_r^2 n_r - (\Sigma C_r n_r)^2\} \div n$. Similarly, it may be shown that the different distributions obtained by giving different values to m are normally correlated; and it follows that the most probable combination of values of $[x], [x^2], [x^3], \ldots$ is to be found by assuming the areas $A_1, A_2, \ldots, A_p$ of the figure of frequency to be proportional to $n_1, n_2, \ldots, n_p$. It is understood, throughout, that n is a large number.

Denote the most probable value of $[x^m]$ by "$[x^m]$". Then, writing $\phi(x) = x^m$ in (17), multiplying both sides of the equation by n/A, and replacing nA_r/A by n_r, we have

$$n\text{“}[x^m]\text{”} = \sum_{r=1}^{r=p} n_r \left(\xi_r^m - \tfrac{1}{4} P_1 \frac{m!}{2!(m-2)!} h^2 \xi_r^{m-2} + \tfrac{1}{16} P_2 \frac{m!}{4!(m-4)!} h^4 \xi_r^{m-4} - \ldots\right)$$

$$- n\left\{B_1 \frac{m!}{2!\,(m-2)!} h^2 x_0^{m-2} - B_2 \frac{m!}{4!\,(m-4)!} h^4 x_0^{m-4} + \ldots\right\}$$

$$- \frac{n}{A}\left[\left\{\frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \ldots\right\} mhx^{m-1} \int_x^{x_p} f(x)\,dx\right]_{x=x_0}^{x=x_p}, \quad (23)$$

where $$\xi_r = \tfrac{1}{2}(x_{r-1} + x_r).$$

To express this in a convenient form, let M_q denote the *rough* value of Σx^q for the n individuals, obtained by massing the individuals in each class at the centre of the range of the class; *i.e.*, by taking every value of x between x_{r-1} and x_r as equal to the arithmetic mean of x_{r-1} and x_r. Thus

$$M_q = \sum_{r=1}^{r=p} n_r \xi_r^q.$$

Also let $$n_r + n_{r+1} + \ldots + n_p = N_{r-1},$$

so that $$N_0 = n, \quad N_p = 0;$$

and let $m(N_0 x_0^{m-1} + N_p x_p^{m-1}) = S_0, \quad m(N_1 x_1^{m-1} + N_{p-1} x_{p-1}^{m-1}) = S_1,$

$$m(N_2 x_2^{m-1} + N_{p-2} x_{p-2}^{m-1}) = S_2, \ldots.$$

Then, if $S_1 - S_0 = \Delta S_0,\ S_2 - S_1 = \Delta S_1,\ \ldots,\ \Delta S_1 - \Delta S_0 = \Delta^2 S_0,\ \ldots,$ (23) becomes

$$n\,``[x^m]" = M_m - \tfrac{1}{4}P_1 \frac{m!}{2!\,(m-2)!} M_{m-2}h^2 + \tfrac{1}{16}P_2 \frac{m!}{4!\,(m-4)!} M_{m-4}h^4 - \ldots$$
$$- n\left\{\tfrac{1}{6}\frac{m!}{2!\,(m-2)!}h^2x_0^{m-2} - \tfrac{1}{80}\frac{m!}{4!\,(m-4)!}h^4x_0^{m-4} + \ldots\right\}$$
$$+ \tfrac{1}{12}h\left\{\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{19}{60}\Delta^3 - \tfrac{9}{40}\Delta^4 + \tfrac{863}{5040}\Delta^5 - \ldots\right\} S_0. \qquad (24)$$

9. The data may be studied by another method, which avoids the difficulties incident to inverse probability, and is therefore incomplete. If we knew the exact value of x for each of the n individuals, we should calculate the values of $\Sigma x \div n$, $\Sigma x^2 \div n$, $\Sigma x^3 \div n$, ...; and these calculated averages would constitute the data for determining the mean values $[x]$, $[x^2]$, $[x^3]$, ... When we only know the numbers $n_1, n_2, \ldots, n_p$ in the p classes, we cannot calculate the value of Σx^m exactly. We may, therefore, leave on one side the question of the amount of information that the value of Σx^m gives as to the frequency-constants of the original community, and consider only what is the most probable value of Σx^m itself, on the assumption that the numbers $n_1, n_2, \ldots, n_p$ are obtained by random selection from some particular (unknown) community, and how, on the same assumption, the different possible values are distributed about this most probable value.

Let
$$z = f(x)$$
be the equation to the curve of frequency in the original community; and, as before, let
$$n'_r = nA_r/A.$$

Since the numbers $n_1, n_2, \ldots, n_p$ are obtained by random selection, the difference $n_r \sim n'_r$ may be regarded as comparable with $\sqrt{n_r(n-n_r)/n}$. To examine the distribution of possible values of Σx^m, we suppose an indefinitely great number of sets of n individuals to be obtained by random selection, and we pick out the cases in which the numbers in the p classes are $n_1, n_2, \ldots, n_p$. Let $\Sigma_r x^m$ denote the portion of Σx^m which is due to the n_r individuals for which x lies between x_{r-1} and x_r. Then it is clear, from the principles of random selection, that the distributions of $\Sigma_1 x^m$, $\Sigma_2 x^m$, ..., $\Sigma_p x^m$, and of their sum Σx^m, in the special cases considered, are the same as if we made *independent* random selections of $n_1, n_2, \ldots, n_p$ individuals from the p classes of the original community.

Let $(x^m)_r$ denote the mean value of x^m for those individuals in the original community for which x lies between x_{r-1} and x_r, so that

$$(x^m)_r = \int_{x_{r-1}}^{x_r} f(x)\, x^m dx \div \int_{r-1}^{x_r} f(x)\, dx = \int_{x_{r-1}}^{x_r} f(x)\, x^m dx \div A_r.$$

Then the theory of error shows that, if n_r individuals were taken at random from this particular class, the mean value of $\Sigma_r x^m$ would be $n_r (x^m)_r$, and the different values would be distributed about this mean with mean square $n_r \left[(x^{2m})_r - \{(x^m)_r\}^2 \right]$. It follows from the last paragraph that the different values of Σx^m are distributed about a mean value $\sum_{r=1}^{r=p} n_r (x^m)_r$ with mean square $\sum_{r=1}^{r=p} n_r \left[(x^{2m})_r - \{(x^m)_r\}^2 \right]$. This is true whatever the value of n may be. When n is large, the distribution is approximately normal, and the most probable value of Σx^m is the same as the mean value $\sum_{r=1}^{r=p} n_r (x^m)_r$, while the actual value may be regarded as differing from this by an amount comparable with

$$\sqrt{\sum_{r=1}^{r=p} n_r \left[(x^{2m})_r - \{(x^m)_r\}^2 \right]},$$

i.e., by an amount comparable with

$$\sqrt{n} \sqrt{\frac{\Sigma x^{2m}}{n} - \sum_{r=1}^{r=p} \frac{n_r'}{n} \{(x^m)_r\}^2}.$$

Let "Σx^m" denote the most probable value. Then, from (14), with the same notation as before,

$$\begin{aligned}
\text{“}\Sigma x^m\text{”} &= \sum_{r=1}^{r=p} n_r (x^m)_r \\
&= \sum_{r=1}^{r=p} \frac{n_r}{A_r} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)(\xi_r+\theta)^m d\theta \\
&= \sum_{r=1}^{r=p} n_r \left\{ \xi_r^m - \tfrac{1}{4} P_1 \frac{m!}{2!(m-2)!} h^2 \xi_r^{m-2} + \tfrac{1}{16} P_2 \frac{m!}{4!(m-4)!} h^4 \xi_r^{m-4} - \dots \right\} \\
&\quad + \sum_{r=1}^{r=p} \frac{n_r}{A_r} \left[h\phi_2(x) \right]_{x=x_{r-1}}^{x=x_r} \\
&= M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{16} P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \dots \\
&\quad + \frac{n}{A} \left[h\phi_2(x) \right]_{x=x_0}^{x=x_p} \\
&\quad + \sum_{r=1}^{r=p} \frac{n_r - n_r'}{A_r} \left[h\phi_2(x) \right]_{x=x_{r-1}}^{x=x_r}, \qquad (25)
\end{aligned}$$

where

$$h\phi_2(x) = \left\{ B_1 \frac{m!}{2!\,(m-2)!} h^2 x^{m-2} - B_2 \frac{m!}{4!\,(m-4)!} h^4 x^{m-4} + \dots \right\} \int_x^{x_p} f(x)\,dx$$

$$- \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} mhx^{m-1} \int_x^{x_p} f(x)\,dx. \quad (26)$$

The value of "Σx^m", as given by (25), and the value of n"$[x^m]$", as given by (23) or (24), agree in the terms

$$M_m - \tfrac{1}{4}P_1 \frac{m!}{2!\,(m-2)!} M_{m-2}h^2 + \tfrac{1}{16}P_2 \frac{m!}{4!\,(m-4)!} M_{m-2}h^4 - \dots,$$

but the remaining terms are different. The difference, however, is a linear function of the differences $n_1 - n_1'$, $n_2 - n_2'$, ..., $n_p - n_p'$; and these are small in comparison with n. Hence n"$[x^m]$" and "Σx^m" may be regarded as substantially the same, the value of either being given by (24).

10. The formula (24) applies to any distribution which satisfies the condition as to continuity. In the special cases we are considering, the extreme values $n_0, n_1, \dots,$ and $n_p, n_{p-1}, \dots$ are very small. We have then, approximately,

$$n\text{“}[x^m]\text{”} = \text{“}\Sigma x^m\text{”}$$
$$= M_m - \tfrac{1}{4}P_1 \frac{m!}{2!\,(m-2)!} M_{m-2}h^2 + \tfrac{1}{16}P_2 \frac{m!}{4!\,(m-4)!} M_{m-4}h^4 - \dots, \quad (27)$$

which gives, as particular cases,

$$\left.\begin{aligned} n\text{“}[x]\text{”} &= \text{“}\Sigma x\text{”} = M_1 \\ n\text{“}[x^2]\text{”} &= \text{“}\Sigma x^2\text{”} = M_2 - \tfrac{1}{12}nh^2 \\ n\text{“}[x^3]\text{”} &= \text{“}\Sigma x^3\text{”} = M_3 - \tfrac{1}{4}M_1h^2 \\ n\text{“}[x^4]\text{”} &= \text{“}\Sigma x^4\text{”} = M_4 - \tfrac{1}{2}M_2h^2 + \tfrac{7}{240}nh^4 \\ n\text{“}[x^5]\text{”} &= \text{“}\Sigma x^5\text{”} = M_5 - \tfrac{5}{6}M_3h^2 + \tfrac{7}{48}M_1h^4 \\ &\&c. \end{aligned}\right\}, \quad (28)$$

since, of course, $M_0 = n$.

This is the most convenient form when we are able to take the zero at the middle point of one of the divisions of the range, so that $\xi_1, \xi_2, \dots, \xi_p$ may be integral multiples of h. In some cases it may be desirable to take the zero at a point of division, so that $x_0, x_1, x_2, \dots, x_p$

will be integral multiples of h. Let $M'_1, M'_2, \ldots, M'_m$ denote the rough values of $\Sigma x, \Sigma x^2, \ldots, \Sigma x^m$ obtained by massing $\frac{1}{2}n_1$ individuals at x_0, $\frac{1}{2}(n_1+n_2)$ at x_1, $\frac{1}{2}(n_2+n_3)$ at $x_2, \ldots, \frac{1}{2}n_p$ at x_p. Then, by (22),

$$\left.\begin{aligned} n\text{"}[x]\text{"} &= \text{"}\Sigma x\text{"} = M'_1 \\ n\text{"}[x^2]\text{"} &= \text{"}\Sigma x^2\text{"} = M'_2 - \tfrac{1}{8}nh^2 \\ n\text{"}[x^3]\text{"} &= \text{"}\Sigma x^3\text{"} = M'_3 - M'_1h^2 \\ n\text{"}[x^4]\text{"} &= \text{"}\Sigma x^4\text{"} = M'_4 - 2M'_2h^2 + \tfrac{7}{15}nh^4 \\ n\text{"}[x^5]\text{"} &= \text{"}\Sigma x^5\text{"} = M'_5 - \tfrac{10}{3}M'_3h^2 + \tfrac{7}{3}M'_1h^4 \\ &\&c. \end{aligned}\right\}. \qquad (29)$$

If we use the rough values $M_1, M_2, \ldots$ to determine $\mu_2, \mu_3, \ldots$, the mean square, mean cube, ... of the deviation from the mean, and if the common difference h is the unit of measurement, we have, from (28), μ_1 being zero,

$$\left.\begin{aligned} \text{corrected value of } \mu_2 &= \mu_2 - \tfrac{1}{12} \\ &= \mu_2 - \cdot 08\dot{3} \\ \text{corrected value of } \mu_4 &= \mu_4 - \tfrac{1}{2}\mu_2 + \tfrac{7}{240} \\ &= \mu_4 - \tfrac{1}{2}\mu_2 + \cdot 0291\dot{6} \\ \text{corrected value of } \mu_5 &= \mu_5 - \tfrac{5}{6}\mu_3 \end{aligned}\right\}, \qquad (30)$$

the value of μ_3 being unaltered. Similarly, if $\mu'_2, \mu'_3, \ldots$ denote the values of $\mu_2, \mu_3, \ldots$ as deduced from $M'_1, M'_2, M'_3, \ldots$, we have, from (29),

$$\left.\begin{aligned} \text{corrected value of } \mu_2 &= \mu'_2 - \cdot\dot{3} \\ \text{corrected value of } \mu_4 &= \mu'_4 - 2\mu'_2 + \cdot 4\dot{6} \\ \text{corrected value of } \mu_5 &= \mu'_5 - \tfrac{10}{3}\mu'_3 \end{aligned}\right\}. \qquad (31)$$

11. The following are some numerical examples* showing the extent of the inaccuracy due to determining "Σx", "Σx^2", ... by the approximate formulæ (28), instead of by the accurate formula (25).

In the first example the selection is "representative," the numbers being exactly proportional to the corresponding areas of the figure of frequency. The "most probable" values of $\Sigma x, \Sigma x^2, \ldots$, and of

* The first two of these illustrations are given in the paper referred to in note (*) on p. 362. A small error in one of the values has been corrected.

$\mu_2, \mu_3, \ldots$ are in this case obtained by a simple integration, and the differences between these values and the values given by (28) or (30) are due solely to the omitted term

$$\frac{n}{A}\left[h\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

which is equal to the second and third terms in (24). In the second example, taken from Kramp's table of values of $\int_t^\infty e^{-t^2}dt$, the selection is approximately representative, the integer nearest to n'_r being taken in each case. The most probable values are found by calculating each of the mean values $(x''')_r$ separately, and taking the sum $\Sigma n_r(x''')_r$; the differences between these and the values given by (28) or (30) are due principally to the term

$$\frac{n}{A}\left[h\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

but partly also to the fractional differences $n_r - n'_r$ occurring in the last term of (25). The third example is formed from the second by adding to or subtracting from the number in each class a number comparable with the "probable error" of the total number in the class. It may, therefore, be taken as a typical illustration of the amount of error that would occur in an actual case of random selection. The values of μ_2, μ_4, and μ_8 for the community selected being 2, 12, and 1680, the "probable errors" in μ_2 and μ_4 for a random selection of 88,622 individuals would be ·0064 and ·0888 respectively. The actual "errors" for the selected values of $n_1, n_2, \ldots, n_p$ are ·0156 and ·1815, so that the example is really an unfavourable one; but the formulæ in (30) give the values of μ_2 and μ_4 within ·0014 and ·0346 respectively, which is as close an approximation as we could expect to get.

In each case the "rough" values of $M_1, M_2, \ldots$ and the corresponding values of μ_2, μ_3, μ_4 are given for comparison, so as to show the degree of inaccuracy in the even moments μ_2, μ_4 which would be caused by omitting the corrections afforded by (28) or (30).

(i.) *Curve of Frequency* $z \propto (x+4\frac{1}{2})^5 (5\frac{1}{2}-x)^6$.

$n = 10^{12}$.

(Representative Selection.)

$x = -4\frac{1}{2}$ to $-3\frac{1}{2}$	541,231,822
$-3\frac{1}{2}$ to $-2\frac{1}{2}$	18,864,047,410
$-2\frac{1}{2}$ to $-1\frac{1}{2}$	98,443,459,630
$-1\frac{1}{2}$ to $-\frac{1}{2}$	216,942,703,570
$-\frac{1}{2}$ to $\frac{1}{2}$	278,001,526,318
$\frac{1}{2}$ to $1\frac{1}{2}$	228,994,738,642
$1\frac{1}{2}$ to $2\frac{1}{2}$	119,611,449,550
$2\frac{1}{2}$ to $3\frac{1}{2}$	34,697,711,410
$3\frac{1}{2}$ to $4\frac{1}{2}$	3,852,951,310
$4\frac{1}{2}$ to $5\frac{1}{2}$	50,180,338
Total	1,000,000,000,000

	$\Sigma x \div 10^6$	$\Sigma x^2 \div 10^6$	$\Sigma x^3 \div 10^6$	$\Sigma x^4 \div 10^6$	$\Sigma x^5 \div 10^6$
Rough value	115387	1871774	827127	9429592	8085022
Corrected value	115387	1788441	798281	8522872	7412577
Most probable value ...	115385	1788462	798077	8524038	7405402

	μ_2	μ_3	μ_4
Rough value	1·85847	·18227	9·19204
Corrected value.........	1·77513	·18227	8·29197
Most probable value...	1·77515	·18207	8·29803

(ii.) *Curve of Frequency* $z \propto e^{-\frac{1}{2}(x-\cdot 2)^2}$.

$n = 88{,}622$.

(Approximately Representative Selection.)

$x = -6\frac{1}{2}$ to $-5\frac{1}{2}$	2	$x = \frac{1}{2}$ to $1\frac{1}{2}$	21,005
$-5\frac{1}{2}$ to $-4\frac{1}{2}$	37	$1\frac{1}{2}$ to $2\frac{1}{2}$	11,259
$-4\frac{1}{2}$ to $-3\frac{1}{2}$	355	$2\frac{1}{2}$ to $3\frac{1}{2}$	3,733
$-3\frac{1}{2}$ to $-2\frac{1}{2}$	2,098	$3\frac{1}{2}$ to $4\frac{1}{2}$	765
$-2\frac{1}{2}$ to $-1\frac{1}{2}$	7,670	$4\frac{1}{2}$ to $5\frac{1}{2}$	97
$-1\frac{1}{2}$ to $-\frac{1}{2}$	17,338	$5\frac{1}{2}$ to $6\frac{1}{2}$	8
$-\frac{1}{2}$ to $\frac{1}{2}$	24,255	Total ...	88,622

	Σx	Σx^2	Σx^3	Σx^4	Σx^5
Rough value	17726	188168	111560	1196948	1169816
Corrected value............	17726	180783	$107128\frac{1}{2}$	1105449	1079434
Most probable value......	17726	180783	107123	1105535	1079202

	μ_2	μ_3	μ_4
Rough value	2·08325	·00074	13·00394
Corrected value.........	1·99992	·00074	11·99148
Most probable value...	1·99992	·00068	11·99250

(iii.) *Curve of Frequency* $z \propto e^{-\frac{1}{2}(x-\cdot 2)^2}$.

$n = 88{,}622.$

(Random Selection.)

$x = -6\frac{1}{2}$ to $-5\frac{1}{2}$	3	$x = \frac{1}{2}$ to $1\frac{1}{2}$	20,965
$-5\frac{1}{2}$ to $-4\frac{1}{2}$	31	$1\frac{1}{2}$ to $2\frac{1}{2}$	11,289
$-4\frac{1}{2}$ to $-3\frac{1}{2}$	374	$2\frac{1}{2}$ to $3\frac{1}{2}$	3,783
$-3\frac{1}{2}$ to $-2\frac{1}{2}$	2,153	$3\frac{1}{2}$ to $4\frac{1}{2}$	773
$-2\frac{1}{2}$ to $-1\frac{1}{2}$	7,657	$4\frac{1}{2}$ to $5\frac{1}{2}$	102
$-1\frac{1}{2}$ to $-\frac{1}{2}$	17,326	$5\frac{1}{2}$ to $6\frac{1}{2}$	12
$-\frac{1}{2}$ to $\frac{1}{2}$	24,154	Total ...	88,622

	Σx	Σx^2	Σx^3	Σx^4	Σx^5
Rough value	17798	189716	113060	1218440	1216388
Corrected value............	17798	182331	$108610\frac{1}{2}$	1126167	1124767
Most probable value......	17795	182202	108288	1122875	1116061

	μ_2	μ_3	μ_4
Rough value	2·10040	·00198	13·23706
Corrected value	2·01707	·00198	12·21603
Most probable value ...	2·01563	−·00038	12·18146

III. Extension to Two or More Variables.

12. The method of § 4 can be extended to any number of variables. The expression

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}\cdot\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}\cdots-\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}\cdot\frac{\sinh D_1'}{D_1'}\,\frac{D_2'}{\sinh D_2'}\cdots,$$

can be expanded so as to contain only positive integral powers (including the power 0) of D_1, D_2, D_1', D_2', ... ; and it vanishes when D_1+D_2, $D_1'+D_2'$, ... are simultaneously equal to zero. Thus we see

that
$$\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,\, y,\, \ldots)\, \phi(x,\, y,\, \ldots)\, dx\, dy \ldots \tag{32}$$
is equal to
$$\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,\, y,\, \ldots)\, dx\, dy \ldots \left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\frac{P_2}{4!}(\tfrac{1}{2}h)^4\frac{d^4}{d\xi^4}-\ldots\right\}$$
$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\frac{P_2}{4!}(\tfrac{1}{2}k)^4\frac{d^4}{d\eta^4}-\ldots\right\}\ldots\phi(\xi,\, \eta,\, \ldots)+R, \tag{33}$$

where R consists of terms involving the boundary values of $f(x, y, \ldots)$, $\phi(x, y, \ldots)$, and their partial differential coefficients with regard to $x, y, \ldots$. Hence, if the values of $x, y, \ldots$ range from $x = x_0$ to $x = x_0+ph$, from $y = y_0$ to $y = y_0+qk$, ..., and if $f(x, y, \ldots)$ and its first few differential coefficients are negligible for the extreme values of *each* of the variables $x, y, \ldots$, we have, approximately,

$$\Sigma\Sigma \ldots \int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} \ldots f(x,\, y,\, \ldots)\, \phi(x,\, y,\, \ldots)\, dx\, dy \ldots$$
$$= \Sigma\Sigma \ldots \int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} \ldots f(x,\, y,\, \ldots)\, dx\, dy \ldots$$
$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\ldots\right\}\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\ldots\right\}\ldots\phi(\xi,\eta,\, \ldots), \tag{34}$$

the summations being made for the $p\times q\times\ldots$ terms corresponding to the different values of $\xi, \eta, \ldots$.

It will be sufficient to consider the case of two variables. Let D_1 and D_2 denote $\frac{1}{2}h\frac{d}{dx}$ operating on $f(x, y)$ and on $\phi(x, y)$ respectively, and let D_1' and D_2' have a similar meaning for $\frac{1}{2}k\frac{d}{dy}$. Then

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}\cdot\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}$$
$$= \frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}\cdot\frac{\sinh D_1'}{D_1'}\,\frac{D_2'}{\sinh D_2'}$$
$$+\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}\left\{\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}-\frac{\sinh D_1'}{D_1'}\,\frac{D_2'}{\sinh D_2'}\right\}$$
$$+\frac{\sinh D_1'}{D_1'}\,\frac{D_2'}{\sinh D_2'}\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}\right\}$$
$$+\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}\right\}$$
$$\times\left\{\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}-\frac{\sinh D_1'}{D_1'}\,\frac{D_2'}{\sinh D_2'}\right\}; \tag{35}$$

and therefore $$\frac{1}{hk}\int_{x_0}^{x_p}\int_{y_0}^{y_q} f(x,y)\,\phi(x,y)\,dx\,dy$$

$$= \Sigma\Sigma\frac{1}{hk}\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dx\,dy\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}$$

$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(\xi,\eta)$$

$$+\Sigma\frac{1}{h}\left[\chi(D_1',D_2')\,.\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h} f(x,y)\,dx\,.\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}\phi(\xi,y)\right]_{y=y_0}^{y=y_q}$$

$$+\Sigma\frac{1}{k}\left[\chi(D_1,D_2)\,.\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dy\,.\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(x,\eta)\right]_{x=x_0}^{x=x_p}$$

$$+\Phi(x_p,y_q)-\Phi(x_0,y_q)-\Phi(x_p,y_0)+\Phi(x_0,y_0), \tag{36}$$

where

$$\chi(D_1,D_2)\equiv\tfrac{1}{2}\frac{D_2}{D_1}\left\{\left(\coth\overline{D_1+D_2}-\frac{1}{D_1+D_2}\right)-\left(\coth D_2-\frac{1}{D_2}\right)\right\}, \tag{37}$$

$$\Phi(x,y)\equiv\chi(D_1,D_2)\,\chi(D_1',D_2')\,f(x,y)\,\phi(x,y). \tag{38}$$

The signs of summation in the first term denote, of course, that the expression is to be summed for each value of ξ with each value of η; while in the second the summation is for all values of ξ, and in the third it is for all values of η. These terms, which denote values along the edges of the area of integration, with the four concluding terms, which denote values at the corners, can be expanded as in (19); or they can be expressed in terms of differences, as in (5A) and (24). They are negligible when $f(x,y)$ and its first few differential coefficients $\frac{df}{dx}$, $\frac{df}{dy}$, ... vanish for all values of x when $y=y_0$ or y_q, and for all values of y when $x=x_0$ or x_p; and in these cases we have, approximately,

$$\int_{x_0}^{x_p}\int_{y_0}^{y_p} f(x,y)\,\phi(x,y)\,dx\,dy$$

$$=\Sigma\Sigma\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dx\,dy\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}$$

$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(\xi,\eta). \tag{39}$$

13. Now let the measurements of x and of y for n individuals be given by a table of double entry, showing for values of r from 1 to p, and for values of s from 1 to q, the number of individuals for which x lies between $x_0+(r-1)(x_p-x_0)/p$ and $x_0+r(x_p-x_0)/p$, and y between $y_0+(s-1)(y_q-y_0)/q$ and $y_0+s(y_q-y_0)/q$. Let "$\Sigma x^m y^t$" denote the most probable value of $\Sigma x^m y^t$ for the n individuals (or n times the most probable mean value of $x^m y^t$ for the original community), and let S_{mt} denote the rough value of $\Sigma x^m y^t$ obtained by taking the values of x and of y for all the individuals in each sub-class to be the arithmetic means of the values of x and of y respectively which determine the sub-class. Then, if $M_m (= S_{m0})$ and $N_t (= S_{0t})$ denote the "rough" values of Σx^m and of Σy^t obtained in this way, we have, approximately,

$$\left.\begin{aligned}
\text{“}\Sigma xy\text{”} &= S_{11}{}^{*} \\
\text{“}\Sigma x^2y\text{”} &= S_{21}-\tfrac{1}{12}N_1h^2 \\
\text{“}\Sigma xy^2\text{”} &= S_{12}-\tfrac{1}{12}M_1k^2 \\
\text{“}\Sigma x^3y\text{”} &= S_{31}-\tfrac{1}{4}S_{11}h^2 \\
\text{“}\Sigma x^2y^2\text{”} &= S_{22}-\tfrac{1}{12}M_2k^2-\tfrac{1}{12}N_2h^2+\tfrac{1}{144}nh^2k^2 \\
\text{“}\Sigma xy^3\text{”} &= S_{13}-\tfrac{1}{4}S_{11}k^2 \\
\text{“}\Sigma x^4y\text{”} &= S_{41}-\tfrac{1}{2}S_{21}h^2+\tfrac{7}{240}N_1h^4 \\
\text{“}\Sigma x^3y^2\text{”} &= S_{32}-\tfrac{1}{4}S_{12}h^2-\tfrac{1}{12}M_3k^2+\tfrac{1}{48}M_1h^2k^2 \\
\text{“}\Sigma x^2y^3\text{”} &= S_{23}-\tfrac{1}{4}S_{21}k^2-\tfrac{1}{12}N_3h^2+\tfrac{1}{48}N_1h^2k^2 \\
\text{“}\Sigma xy^4\text{”} &= S_{14}-\tfrac{1}{2}S_{12}k^2+\tfrac{7}{240}M_1k^4 \\
&\text{&c.}
\end{aligned}\right\} \qquad (40)$$

IV. The Spurious Curve of Frequency.

14. Let $\frac{1}{h}\int_{x-\frac{1}{2}h}^{x+\frac{1}{2}h} f(x)\,dx$ be denoted by $f_1(x)$. If we call $z=f(x)$ the principal curve, then $z=f_1(x)$ may be called the derived curve.

* There is an apparent contradiction in this and similar formulæ when the value of y for each individual is equal or very nearly equal (or proportional) to that of x, *i.e.*, when the distributions of x and of y are very closely correlated; for the formula appears to give M_2 instead of $M_2-\frac{1}{12}h^2$. But, if y were always equal to x, there would be a discontinuity all along the line $y=x$; and therefore the formula, which is based on the hypothesis of continuity, would not apply at all. When y is only very nearly equal to x it will be seen that the sets of planes corresponding to consecutive values of x and y cut the solid of frequency in such a way that the volumes, which in forming the values of S_1 are concentrated at points outside the line $y=x$, are appreciable.

The curves are related in such a way that the ordinate to the latter curve at any point $x = \xi$ is proportional to the area of the former curve comprised between the ordinates at $x = \xi - \frac{1}{2}h$ and $x = \xi + \frac{1}{2}h$. If these ordinates are MP and $M'P'$, and if h is relatively small, the extremity of the ordinate to the derived curve will lie inside the area formed by the arc PP' and the chord PP'. Thus the derived curve lies inside the principal curve where the latter is convex, and outside it where it is concave; any point of intersection of the two curves being near a point of inflexion of the principal curve.

If $z = f(x)$ ranges from $x = x_0$ to $x = x_p$, then $z = f_1(x)$ only ranges from $x = x_0 + \frac{1}{2}h$ to $x = x_p - \frac{1}{2}h$. But, if $z = f(x)$ touches the base ($z = 0$) at its extremities, it may be considered to coincide with it beyond this range: and $z = f_1(x)$ may thus be regarded as extending from $x = x_0 - \frac{1}{2}h$ to $x = x_p + \frac{1}{2}h$; the curve, like the principal curve, touching the base at its extremities. We shall only consider this class of cases.

Let the areas of the principal curve between the successive ordinates corresponding to $x = x_0$, $x = x_0 + h$, $x = x_0 + 2h$, ..., $x = x_0 + ph$, be $A_1, A_2, \ldots, A_p$. Then the ordinates of the derived curve at the points $x = x_0 - \frac{1}{2}h$, $x = x_0 + \frac{1}{2}h$, ..., $x = x_p - \frac{1}{2}h$, $x = x_p + \frac{1}{2}h$ are $0, A_1/h, A_2/h, \ldots, A_p/h, 0$. Hence, from (2B), we have, approximately,

$$\text{area of derived curve} = \int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\,dx$$

$$= h\left\{\frac{A_1}{h} + \frac{A_2}{h} + \ldots + \frac{A_p}{h}\right\}$$

$$= \text{area of principal curve};$$

first moment of derived curve

$$= \int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\,x\,dx$$

$$= h\left\{\frac{A_1}{h}\xi_1 + \frac{A_2}{h}\xi_2 + \ldots + \frac{A_p}{h}\xi_p\right\}$$

$$= \text{first moment of principal curve, by (17).}$$

Thus the two curves have the same area and the same mean value of x. From (2B) we have as the general formula relating to the derived curve, when $A_1, A_2, \ldots, A_p$ are given,

$$\int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\,\phi(x)\,dx = \sum_{r=1}^{r=p} A_r \phi(x_r). \tag{41}$$

15. When the result of a random selection of n individuals is to give numbers $n_1, n_2, \ldots, n_p$ in the successive classes corresponding to equidistant divisions of the scale of measurement, these numbers are, except for the "errors" due to random selection, proportional to the corresponding areas of the figure of frequency in the original community. In order, therefore, to represent the data graphically, we should draw a curve such that the areas between the successive ordinates at the $p+1$ points of the scale are proportional to $n_1, n_2, \ldots, n_p$. The calculation of the ordinates of this curve, which may be regarded as the true curve of frequency for the n individuals, involves troublesome interpolations; and, to avoid this, statisticians often draw a curve whose ordinates at the middle points of the divisions of the scale are proportional to $n_1, n_2, \ldots, n_p$. This curve may be called the *spurious curve of frequency.* It bears the same relation to the true curve of frequency that the derived curve of § 14 bears to the principal curve. Thus it has the same area as the true curve, and its central ordinate starts from the same point on the base, but it is flattened down so as to be inside the true curve at or about the mean and outside it at the extremities; and therefore it has a greater mean square of deviation.

If we deal with the spurious curve instead of with the true curve, we must apply the formula (41) instead of the corresponding formula derived from (17). Hence, in order to find the values of μ_2, μ_3, &c., for the spurious curve, we must use the "rough" values $M_1, M_2, M_3, \ldots$, without the introduction of the corrections given by (28) or (30).

The divergence of the spurious curve from the true curve is greater or less according as h is greater or less. The same facts may therefore be represented by any number of different spurious curves, according to the unit of measurement we adopt. Thus example (A) of § 7 would give as ordinates of the spurious curve, when the unit of measurement is 10 lbs.:—

Weight.	Ordinate.
115 lbs.	133
145 lbs.	1,240
175 lbs.	492
205 lbs.	75
235 lbs.	10

but, if the unit had been 30 lbs., the ordinates (reduced in the ratio of 1 : 3) would have been

Weight.	Ordinate.
115 lbs.	166
145 lbs.	1,003
175 lbs.	559
205 lbs.	104
235 lbs.	17

which would have given a very different curve. Moreover, the results obtained by using the spurious curve are useless if we require an accurate test of any particular hypothesis. The arguments which lead us to expect to find a curve of a particular kind [*e.g.*, in Prof. Karl Pearson's recent researches, a curve whose equation is of the form $z \propto (x-x_0)^{\alpha} (x_p-x)^{\beta}$] relate to the curve of frequency itself, not to the spurious curve; and, by attempting to fit a curve of this kind to the spurious curve, we shall obtain incorrect results.

APPENDIX.

Moments of a Polygon.

The question of the proper formula to be used for Σx^m has been incidentally discussed by Prof. Karl Pearson in a recent memoir.* His method is to draw ordinates proportional to the numbers given by the observations, and to join the tops of these ordinates by straight lines, so as to form a polygon. It will be seen that this polygon lies inside the spurious curve where the latter is convex, and outside it where it is concave; and that, when $z = f(x)$ has close contact with the base at its extremities, the area of the polygon is approximately equal to that of the curve, and the first moments of the two areas are also approximately equal. The "mean square of deviation" of the polygon is, therefore, greater than that of the spurious curve, which, again, is greater than that of the true curve.

It is interesting, though the result is of no practical value, to obtain the moments of such a polygon in terms of the areas, instead of in terms of the ordinates. Let ordinates $z_0, z_1, \ldots, z_p$ be drawn at points $x = x_0,\ x = x_1,\ \ldots,\ x = x_p$, the common difference of abscissa being h, and let the tops of the ordinates be joined so as to form a polygon. This polygon consists of a series of trapezia of areas $A_1, A_2, \ldots, A_p$, where

$$A_r = \tfrac{1}{2}h\,(z_{r-1}+z_r).$$

* *Phil. Trans.*, Vol. CLXXXVI. (1895), A, p. 349.

The m^{th} moment of the polygon about $x = 0$ is

$$\sum_{r=1}^{r=p} \int_{x_{r-1}}^{x_r} zx^m \, dx.$$

Let this be denoted by P_m. Then, as Prof. Pearson shows,

$$\int_{x_{r-1}}^{x_r} zx^m \, dx$$

$$= -\frac{1}{(m+1)(m+2)} \frac{1}{h} (z_r - z_{r-1})(x_r^{m+2} - x_{r-1}^{m+2}) + \frac{1}{m+2} (z_r x_r^{m+1} - z_{r-1} x_{r-1}^{m+1}) \tag{42}$$

and this gives $$\sum_{r=1}^{r=p} \int_{x_{r-1}}^{x_r} zx^m \, dx$$

$$= 2 \sum_{r=0}^{r=p} \left\{ z_r \left(\frac{m!}{2!\, m!} x_r^m h + \frac{m!}{4!\,(m-2)!} x_r^{m-2} h^3 + \dots \right) \right\}$$
$$- \left[z \left(\frac{m!}{3!\,(m-1)!} x^{m-1} h^2 + \frac{m!}{5!\,(m-3)!} x^{m-3} h^4 + \dots \right) \right]_{x=x_0}^{x=x_p}, \tag{43}$$

the expression under the sign of summation being only taken once for the extreme values, instead of twice. The term given by the square brackets vanishes when z_0 and z_p are both zero.

To express P_m in terms of $A_1, A_2, \dots, A_p$, we have, from (42),

$$\int_{x_{r-1}}^{x_r} zx^m \, dx = -\frac{1}{(m+1)(m+2)} \frac{2}{h} (z_r x_r^{m+2} + z_{r-1} x_{r-1}^{m+2})$$
$$+ \frac{1}{(m+1)(m+2)} \frac{1}{h} (z_r + z_{r-1})(x_r^{m+2} + x_{r-1}^{m+2})$$
$$+ \frac{1}{m+2} (z_r x_r^{m+1} - z_{r-1} x_{r-1}^{m+1}). \tag{44}$$

To adapt the first term of the right-hand side, we use the formula*

$$(y+1)^{m+2} = y^{m+2} - E_1 \frac{(m+2)!}{2!\, m!} y^m + E_2 \frac{(m+2)!}{4!\,(m-2)!} y^{m-2} - \dots$$
$$+ T_1 \frac{(m+2)!}{1!\,(m+1)!} (y+1)^{m+1} - T_2 \frac{(m+2)!}{3!\,(m-1)!} (y+1)^{m-1} + \dots, \tag{45}$$

* "On the Relations between Bernoulli's and Euler's Numbers," *Quarterly Journal of Mathematics*, 1898, p. 18.

where $E_1, E_2, \ldots, T_1, T_2, \ldots$ are the coefficients in the expansions

$$\left.\begin{aligned} \operatorname{sech}\theta &= 1 - E_1\frac{\theta^2}{2!} + E_2\frac{\theta^4}{4!} - \ldots \\ \tanh\theta &= T_1\frac{\theta}{1!} - T_2\frac{\theta^3}{3!} + \ldots \end{aligned}\right\}. \tag{46}$$

If $$\xi_r = \tfrac{1}{2}(x_{r-1} + x_r),$$

this gives

$$x_r^{m+2} = \xi_r^{m+2} - E_1\frac{(m+2)!}{2!\,m!}(\tfrac{1}{2}h)^2\xi_r^m + E_2\frac{(m+2)!}{4!\,(m-2)!}(\tfrac{1}{2}h)^4\xi_r^{m-2} - \ldots$$
$$+ T_1\frac{(m+2)!}{1!\,(m+1)!}(\tfrac{1}{2}h)\,x_r^{m+1} - T_2\frac{(m+2)!}{3!\,(m-1)!}(\tfrac{1}{2}h)^3 x_r^{m-1} + \ldots,$$

with a similar equation obtained by changing x_r into x_{r-1} and $\frac{1}{2}h$ into $-\frac{1}{2}h$. Multiplying these equations by z_r and z_{r-1} respectively, and adding, substituting in (44), and adding the corresponding expressions for the other trapezia, we find, the summations being from $r = 1$ to $r = p$,

$$\Sigma\int_{x_{r-1}}^{x_r} zx^m\,dx$$
$$= \frac{E_1+1}{1.2}\Sigma A_r\xi_r^m - \frac{E_2-1}{3.4}\,\frac{m!}{2!\,(m-2)!}(\tfrac{1}{2}h)^2\,\Sigma A_r\xi_r^{m-2} + \ldots$$
$$+ \left[\frac{T_2}{2.3}\,\frac{m!}{1!\,(m-1)!}(\tfrac{1}{2}h)^2 zx^{m-1} - \frac{T_3}{4.5}\,\frac{m!}{3!\,(m-3)!}(\tfrac{1}{2}h)^4 zx^{m-3} + \ldots\right]_{x=x_0}^{x=x_p}, \tag{47}$$

so that, when $z_0 = 0$ and $z_p = 0$,

$$P_m = \Sigma A_r\xi_r^m - \tfrac{1}{12}\frac{m!}{2!\,(m-2)!}\Sigma A_r\xi_r^{m-2}h^2 + \tfrac{31}{240}\frac{m!}{4!\,(m-4)!}\Sigma A_r\xi_r^{m-4}h^4 - \ldots, \tag{48}$$

which agrees with (27) in the second term, but not in subsequent terms.