9. In the former figure (Vol. XXII., *l.c.*) we may mention that $\alpha\alpha'$, $\beta\beta'$ intersect on the symmedian line through C.

10. Let pq cut BC, CA, in D, E'; qr cut CA, AB in E, F'; and rp cut AB, BC in F, D'.

The points are given thus

$$D,\ 0,\ a^2b,\ c(a^2+b^2); \qquad D',\ 0,\ b(c^2+a^2),\ ca^2;$$
$$E,\ a(b^2+c^2),\ 0,\ b^2c; \qquad E',\ ab^2,\ 0,\ c(a^2+b^2);$$
$$F,\ c^2a,\ b(c^2+a^2),\ 0; \qquad F',\ a(b^2+c^2),\ bc^2,\ 0.$$

Hence we obtain $DE^2 = a^4b^2c^2\,(b^2+c^2+2bc\cos A)/\lambda^4$;

$$EE' = a^2bc^2/\lambda^2 = D'F,$$

i.e.,
$$EE'.AC = a^2b^2c^2/\lambda^2 = FF'.AB = DD'.BC.$$

11. The conic round $DEFD'E'F'$ is

$$b^2c^2\,(c^2+a^2)(a^2+b^2)\,\alpha^2+\ldots+\ldots = (2a^4+\lambda^2)\,bc\,(b^2+c^2)\,\beta\gamma+\ldots+\ldots.$$

12. We see that $AE = b^3c^2/\lambda^2$, $AF' = b^2c^3/\lambda^2$;

hence the hexagon

$$D'DE'EF'F = \Delta\,(1-\Sigma a^4b^4/\lambda^4) = 2\Delta a^2b^2c^2/\lambda^4.$$

Also the diagonals pass through the mid-points of $\Omega\Omega'$, which is therefore the centre of the conic in § 11.

Second Memoir on the Expansion of certain Infinite Products. By L. J. Rogers. Received April 2nd, 1894. Read April 12th, 1894.

1. If $A_r(\theta)$ denote the coefficient of $x^r/(1-q)(1-q^2)\ldots(1-q^r)$ in the expansion of

$$1 \div (1-2x\cos\theta+x^2)(1-2xq\cos\theta+x^2q^2)\ldots,$$

we have seen that the value of $A_r(\theta)$ is

$$2\cos r\theta+\frac{1-q^r}{1-q}\,2\cos(r-2)\,\theta+\frac{1-q^r}{1-q}\,\frac{1-q^{r-1}}{1-q^2}\,2\cos(r-4)\,\theta+\ldots \qquad \ldots\ldots\ldots(1),$$

and that certain series and infinite products have been expanded according to ascending orders of A's. Now suppose that any such series

$$a_0 + a_1 A_1(\theta) + a_2 A_2(\theta) + \dots$$

be equivalent to the Fourier series

$$b_0 + 2b_1 \cos\theta + 2b_2 \cos 2\theta + \dots .$$

We obtain, by equating coefficients of cosines of even multiples of θ,

$$\left.\begin{array}{ll}
a_0 + a_2 \dfrac{1-q^2}{1-q} + a_4 \dfrac{1-q^4}{1-q}\,\dfrac{1-q^3}{1-q^2} + a_6 \dfrac{1-q^6}{1-q}\,\dfrac{1-q^5}{1-q^2}\,\dfrac{1-q^4}{1-q^3} + \dots & = b_0 \\
a_2 + a_4 \dfrac{1-q^4}{1-q} + a_6 \dfrac{1-q^6}{1-q}\,\dfrac{1-q^5}{1-q^2} + a_8 \dfrac{1-q^8}{1-q}\,\dfrac{1-q^7}{1-q^2}\,\dfrac{1-q^6}{1-q^3} + \dots & = b_2 \\
a_4 + a_6 \dfrac{1-q^6}{1-q} + a_8 \dfrac{1-q^8}{1-q}\,\dfrac{1-q^7}{1-q^2} + \dots & = b_4 \\
a_6 + a_8 \dfrac{1-q^8}{1-q} + \dots & = b_6 \\
a_8 + \dots & = b_8 \\
\&\text{c.} &
\end{array}\right\} \qquad \dots\dots\dots\dots(2).$$

We may evidently, by multiplying these equations by suitable quantities, obtain a relation connecting the a's and b's in the form

$$a_0 + a_1\lambda + a_2\lambda^2 + \dots = b_0 + b_1\mu_1 + b_2\mu_2 + \dots \qquad \dots\dots\dots(3).$$

Now, since $\qquad a_0 + a_1 A_1(\theta) + \dots = b_0 + 2b_1 \cos\theta + \dots,$

and $\qquad 2\cos\theta \,.\, A_{r-1}(\theta) = A_r(\theta) + (1-q^{r-1})\, A_{r-2}(\theta)$

(see footnotes on p. 344, Vol. XXIV.), we get, after multiplying by $2\cos\theta$, which is $A_1(\theta)$,

$$a_0 A_1(\theta) + a_1 \{A_2(\theta) + (1-q)\} + a_2 \{A_3(\theta) + (1-q^2)\, A_1(\theta)\} + \dots$$
$$= 2b_1 + (b_0 + b_2)\cos\theta + (b_1 + b_3)\cos 2\theta + \dots \qquad \dots\dots\dots(4),$$

and hence, by (3),

$$a_1(1-q) + \{a_0 + a_2(1-q^2)\}\,\lambda + \{a_1 + a_3(1-q^3)\}\,\lambda^2 + \dots$$
$$= 2b_1 + (b_0 + b_2)\,\mu_1 + (b_1 + b_3)\,\mu_2 + \dots ;$$

i.e.,
$$\lambda\{a_0+a_1\lambda+a_2\lambda^2+\dots\}$$
$$+\frac{1}{\lambda}\{a_0+a_1\lambda+a_2\lambda^2+\dots\}-\frac{1}{\lambda}\{a_0+a_1\lambda q+a_2\lambda^2q^2+\dots\}$$
$$=b_0\mu_1+b_1(2+\mu_2)+b_2(\mu_1+\mu_3)+\dots.$$

If, however, as in Vol. XXIV., p. 337, δ_λ denote the operation which turns $f(\lambda)$ into $\frac{f(\lambda)-f(\lambda q)}{\lambda}$, we see that
$$(\lambda+\delta_\lambda)(b_0+b_1\mu_1+b_2\mu_2+\dots)$$
is identically the same series in the b's as
$$b_0\mu_1+b_1(2+\mu_2)+\dots.$$

Hence
$$\left.\begin{aligned}\mu_1&=\lambda\\ 2+\mu_2&=(\lambda+\delta_\lambda)\,\mu_1=\lambda^2+1-q\\ \mu_1+\mu_3&=(\lambda+\delta_\lambda)\,\mu_2\\ \dots\quad&\dots\quad\dots\quad\dots\end{aligned}\right\}\quad\dots\dots\dots\dots(5).$$

From these equations we may successively obtain the values of $\mu_1, \mu_2, \dots$, and by substituting in (3), and equating coefficients of powers of λ, we may obtain the values of $a_0, a_1, \dots$. It is, moreover, obvious that the terms containing a's with even suffixes may be equated to those containing b's with even suffixes.

The actual values of $\mu_1, \mu_2, \dots$, however, may be best determined by means of the identity
$$1+\frac{x\mu_1}{1-q}+\frac{x^2\mu_2}{1-q^2}+\frac{x^3\mu_3}{1-q^3}+\dots$$
$$=1+\frac{x}{1-q}(\lambda-x)+\frac{x^2}{1-q^2}(\lambda-x)(\lambda-xq)$$
$$+\frac{x^3}{1-q^3}(\lambda-x)(\lambda-xq)(\lambda-xq^2)+\dots\quad\dots\dots(6),$$
which we will now proceed to establish.

Calling the latter series F, we see that
$$\delta_\lambda F=x+x^2(\lambda-x)+x^3(\lambda-x)(\lambda-xq)+\dots\quad\dots\dots\dots(7),$$
since
$$\delta_\lambda(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-1})$$
$$=\frac{1}{\lambda}\{(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-1})$$
$$-q^{n-1}(\lambda q-x)(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-2})\}$$
$$=(1-q^n)(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-2}).$$

Again,

$$\delta_x F = \{\lambda - x(1+q)\} + x(\lambda - xq)\{\lambda - x(1+q^2)\}$$
$$+ x^2(\lambda - xq)(\lambda - xq^2)\{\lambda - x(1+q^3)\} + \ldots,$$

as is easily seen, and this

$$= (\lambda - xq) + x(\lambda - xq)(\lambda - xq^2) + x^2(\lambda - xq)(\lambda - xq^2)(\lambda - xq^3) + \ldots$$
$$- x - x^2(\lambda - xq) - x^3(\lambda - xq)(\lambda - xq^2) - \ldots \quad \ldots\ldots\ldots\ldots (8),$$

while, by (2),

$$\delta_x \delta_\lambda F = 1 + x(\lambda - x) + x^2(\lambda - x)(\lambda - xq) + \ldots$$
$$- q - xq(\lambda - xq) - x^2q^2(\lambda - xq)(\lambda - xq^2) - \ldots \quad \ldots (9).$$

Now $x(\lambda - x)\,\delta_x F = x(\lambda - x)(\lambda - xq) + x^2(\lambda - x)(\lambda - xq)(\lambda - xq^2) + \ldots$

$$- x^2(\lambda - x) - x^3(\lambda - x)(\lambda - xq) - \ldots,$$

and, since $\delta_x F$ may also be written in the form

$$(\lambda - x) + x(\lambda - x)(\lambda - xq) + x^2(\lambda - x)(\lambda - xq)(\lambda - xq^2) + \ldots,$$
$$- xq - x^2q^2(\lambda - xq) - x^3q^3(\lambda - xq)(\lambda - xq^2) - \ldots,$$

we see that

$$(1 - \lambda x + x^2)\,\delta_x F = (\lambda - x)$$
$$+ x^2(\lambda - x) + x^3(\lambda - x)(\lambda - xq) + \ldots$$
$$- xq - x^2q^2(\lambda - xq) - x^3q^3(\lambda - xq)(\lambda - xq^2) - \ldots,$$

which, by (9), $= \lambda - 2x + x\delta_x\delta_\lambda F$ ....(10).

If, then, F be arranged in powers of x in the form

$$1 + \frac{m_1}{1-q} + \frac{x^2 m_2}{1-q^2} + \ldots,$$

we see that $\qquad \delta_x F = m_1 + m_2 x + m_3 x^2 + \ldots,$

and (10) becomes

$$(1 + x^2)(m_1 + m_2 x + m_3 x^2 + \ldots)$$
$$= \lambda - 2x + x(\lambda + \delta_\lambda)(m_1 + m_2 x + m_3 x^2 + \ldots).$$

Equating coefficients of powers of x, we get

$$m_1 = \lambda,$$
$$2 + m_2 = (\lambda + \delta_\lambda)\,m_1,$$
$$m_1 + m_3 = (\lambda + \delta_\lambda)\,m_2,$$
$$\ldots \quad \ldots \quad \ldots$$

so that the m's are derived from one another in precisely the same way as the μ's in (5), and are therefore identical.

Hence the truth of (6) is established.

If we write q_n as an abbreviation for $1-q^n$, we may easily observe the formation of the coefficients.

In fact, since $(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-1})$

$$=\lambda^n-\frac{q_n}{q_1}\lambda^{n-1}x+q\frac{q_nq_{n-1}}{q_1q_2}\lambda^{n-2}x^2-q^3\frac{q_nq_{n-1}q_{n-2}}{q_1q_2q_3}\lambda^{n-3}x^3+\dots,$$

we get, from (6), $a_0+a_1\lambda+a_2\lambda^2+\dots$

$$=b_0+q_1b_1\left\{\frac{\lambda}{q_1}\right\}+q_2b_2\left\{\frac{\lambda^2}{q_2}-\frac{1}{q_1}\right\}$$
$$+q_3b_3\left\{\frac{\lambda^3}{q_3}-\frac{\lambda}{q_2}\frac{q_2}{q_1}\right\}$$
$$+q_4b_4\left\{\frac{\lambda^4}{q_4}-\frac{\lambda^2}{q_3}\frac{q_3}{q_1}+q\frac{1}{q_2}\right\}$$
$$+q_5b_5\left\{\frac{\lambda^5}{q_5}-\frac{\lambda^3}{q_4}\frac{q_4}{q_1}+q\frac{\lambda}{q_3}\frac{q_3}{q_1}\right\}$$
$$+q_6b_6\left\{\frac{\lambda^6}{q_6}-\frac{\lambda^4}{q_5}\frac{q_5}{q_1}+q\frac{\lambda^2}{q_4}\frac{q_4q_3}{q_1q_2}-q^3\frac{1}{q_3}\right\}\quad\dots\dots\dots(11),$$
$$+\dots,$$

where the coefficients in the bracketed series are formed on the analogy of those appearing in the development of $2\cos n\theta$ in powers of $2\cos\theta$, and q occurs in the r^{th} term of any series raised to the $\frac{1}{2}(r-1)(r-2)^{\text{th}}$ power.

Equating the several powers of λ, we see that

$$a_0=b_0-(1+q)\,b_2+q\,(1+q^2)\,b_4-q^3\,(1+q^3)\,b_6+\dots\quad\dots\dots(12),$$

the general term being

$$(-1)^r q^{\frac{1}{2}r(r-1)}\,(1+q^r)\,b_{2r},$$

while $(1-q)\,a_1=(1-q)\,b_1-(1-q^3)\,b_3+q\,(1-q^5)\,b_5-q^3\,(1-q^7)\,b_7-\dots,$

the general term being

$$(-1)^r q^{\frac{1}{2}r(r-1)}\,(1-q^{2r+1})\,b_{2r+1}\dots\dots\dots\dots\dots\dots\dots(13).$$

Similarly, series may be obtained for a_2, a_3, ... in terms of the b's, but for our present investigations it will not be necessary to quote them here.

Either of these relations gives us a method of expanding the square of the infinite product $(1-q)(1-q^2)(1-q^3)\ldots$ in powers of q.

For from the identity which gives $1 \div \Theta\left(\frac{2K\theta}{\pi}\right)$ as a series of partial fractions, which, on changing q^2, 2θ into q, θ, becomes

$$\frac{\Pi\,[1-q^n]^2}{\Pi\,[1-2q^{n+\frac{1}{2}}\cos\theta+q^{2n-1}]}$$

$$= \frac{1-q}{1-2q^{\frac{1}{2}}\cos\theta+q} - \frac{q\,(1-q^3)}{1-2q^{\frac{3}{2}}\cos\theta+q^3} + \frac{q^3\,(1-q^5)}{1-2q^{\frac{5}{2}}\cos\theta+q^5} - \ldots$$

$$\begin{aligned} = \ & (1-q+q^3-q^6+\ldots) \\ & +2q^{\frac{1}{2}}\cos\theta\,(1-q\cdot q+q^3\cdot q^2-q^6\cdot q^3+\ldots) \\ & +2q\cos 2\theta\,(1-q\cdot q^2+q^3\cdot q^4-q^6\cdot q^6+\ldots) \\ & +\ldots, \end{aligned}$$

we get, from (13),

$$\begin{aligned} \Pi\left[1-q^n\right]^2 = \ & (1-q)(1-q^2+q^5-q^9+\ldots) \\ & -q\,(1-q^3)(1-q^4+q^9-q^{15}+\ldots) \\ & +q^3\,(1-q^5)(1-q^6+q^{13}-q^{21}+\ldots) \\ & -q^6\,(1-q^7)(1-q^8+q^{17}-q^{27}+\ldots) \\ & +\ldots. \end{aligned}$$

Multiplying out the binomial factors on the right-hand side, and arranging the series in two blocks, it will be found that horizontal and vertical series are equal in pairs, starting from a series of terms running parallel to the diagonals of the blocks, so that

$$\begin{aligned} \Pi\left[1-q^n\right]^2 = \ & 1-2q+2q^3-2q^6+\ldots \\ & -q^2\,(1-2q^3+2q^7-2q^{12}+\ldots) \\ & +q^4\,(1-2q^4+2q^9-2q^{15}+\ldots) \\ & -q^{10}\,(1-2q^5+2q^{11}-2q^{18}+\ldots) \\ & +q^{14}\,(1-2q^6+2q^{13}+2q^{21}+\ldots) \\ & -\ldots, \end{aligned}$$

where the indices in the terms outside the brackets are of the form $n(3n\pm1)$, while those in the bracketed series form series whose differences are in arithmetic progression.

2. The series on the right-hand side of § 1, (6) can, in the cases where $\lambda = 1$ or where $\lambda = q^{\frac{1}{2}}$, be very easily arranged according to powers of x, by means of a functional equation which it satisfies.

Let $F(\mu, x) = 1 + \frac{\mu}{1-q}(\lambda - x) + \frac{\mu^2}{1-q^2}(\lambda - x)(\lambda - xq) + \dots$.

Then

$$F(\mu, x) - F(\mu q, x)$$
$$= \mu(\lambda - x) + \mu^2(\lambda - x)(\lambda - xq) + \dots$$
$$= \mu(\lambda - x) + \mu(\lambda - x)\{F(\mu, xq) - F(\mu q, xq)\} \quad \dots\dots\dots(1).$$

Moreover

$$\mu x\{F(\mu q, x) - F(\mu q^2, x)\}$$
$$= \mu^2 qx(\lambda - x) + \mu^3 q^2 x(\lambda - x)(\lambda - xq) + \dots$$
$$= -\mu^2(\lambda - qx)(\lambda - x) - \mu^3(\lambda - q^2 x)(\lambda - x)(\lambda - xq) - \dots$$
$$+ \mu^2\lambda(\lambda - x) + \mu^3\lambda(\lambda - x)(\lambda - xq) + \dots$$
$$= \mu(\lambda - x) - (1 - \mu\lambda)\{F(\mu, x) - F(\mu q, x)\} \quad \dots\dots\dots\dots(2).$$

Again,

$$F(\mu, xq) - F(\mu, x) = \mu x + \mu x\{F(\mu, xq) - F(\mu q, xq)\} \quad \dots\dots(3),$$

i.e.,

$$F(\mu, xq)(1 - \mu x) = F(\mu, x) - \mu x F(\mu q, xq) + \mu x.$$

By the help of (1), (2), and (3), we may eliminate all the functions except $F(\mu, x)$, $F(\mu q, xq)$, and $F(\mu q^2, xq^2)$, and obtain the equation

$$\frac{1-\mu\lambda}{1-\mu x}\{F(\mu, x) - F(\mu q, xq) + \mu x\}$$
$$+ \mu^2 xq^2 \frac{\lambda - xq}{1 - \mu xq^2}\{F(\mu q, xq) - F(\mu q^2, xq^2) + \mu xq^2\}$$
$$= \mu(1 - \mu xq^2)(\lambda - xq) \quad \dots\dots\dots\dots\dots\dots\dots\dots(4).$$

Let $\mu = x$, and write $F(x)$ for $F(x, x)$. Then

$$\frac{1-\lambda x}{1-x^2}\{F(x) - F(xq) + x^2\} + x^3q^2\frac{\lambda - xq}{1 - x^2q^2}\{F(xq) - F(xq^2) + x^2q^2\}$$
$$= x(1 - x^2q^2)(\lambda - xq).$$

Let $\lambda = 1$, and write $\psi(x)$ for $\frac{F(x) - F(xq) + x^2}{1+x}$, so that

$$\psi(x) + x^3q^2\psi(xq) = x - x^2q - x^3q^2 + x^4q^3.$$

From this relation we easily get

$$\psi(x) = x - x^2q - x^3q^2 + x^5q^5 + x^6q^7 - x^8q^{12} - x^9q^{15} + \dots \quad \dots\dots(5),$$

a series which is worthy of notice for its resemblance to the expansion of $\Pi_1^\infty\left[1-q^n\right]$ in powers of q.

Substituting for $\psi(x)$ and multiplying up, we get

$$F(x)-F(xq)=x-x^2q-x^3(q+q^2)-x^4q^2+x^5q^5+x^6(q^5+q^7)+\dots,$$

and, finally,

$$F(x)=1+\frac{x}{1-q}-\frac{qx^2}{1-q^2}-\frac{(q+q^2)\,x^3}{1-q^3}-\frac{q^2x^4}{1-q^4}+\frac{q^5x^5}{1-q^5}+\frac{(q^5+q^7)\,x^6}{-q^6}+\dots \qquad \text{............(6).}$$

Hence, by § 1, (6), $\qquad a_0+a_1+a_2+\dots$

$$=b_0+b_1-qb_2-(q+q^2)\,b_3-q^2b_4+q^5b_5+(q^5+q^7)\,b_6+q^7b_7-q^{10}b_8-\dots \qquad \text{............(7).}$$

The formation of these coefficients is sufficiently obvious. The coefficient of b_{3n} is $(-1)^n\left\{q^{\frac{1}{2}n(3n-1)}+q^{\frac{1}{2}n(3n+1)}\right\}$, while that of b_{3n-1} is $(-1)^n q^{\frac{1}{2}n(3n-1)}$, and that of b_{3n+1} is $(-1)^n q^{\frac{1}{2}n(3n+1)}$. Moreover we may, of course, equate the series of terms with even suffixes to each other, and those with odd.

Again, in (4), let $\lambda=q^{\frac{1}{2}}$, so that

$$\lambda-xq=q^{\frac{1}{2}}(1-\lambda x).$$

Then, in a manner similar to the above, we may show

$$F(x)=1+\frac{xq^{\frac{1}{2}}}{1-q}-\frac{x^2}{1-q^2}-\frac{q^{\frac{1}{2}}(1+q^2)\,x^3}{1-q^3}-\frac{q^4x^4}{1-q^4}+\frac{q^{\frac{5}{2}}x^5}{1-q^5}+\frac{q^4(1+q^4)\,x^6}{1-q^6}+\dots \qquad \text{............(8),}$$

so that, by (1), $\qquad a_0+q^{\frac{1}{2}}a_1+qa_2+\dots$

$$=b_0+q^{\frac{1}{2}}b_1-b_2-q^{\frac{1}{2}}(1+q^2)\,b_3-q^4b_4+q^{\frac{5}{2}}b_5+q^4(1+q^4)\,b_6+q^{\frac{15}{2}}b_7-q^8b_8-\dots$$

The series of alternate terms giving

$$a_0+qa_2+q^2a_4+\dots$$

$$=b_0-b_2-q^4b_4+q^4(1+q^4)\,b_6-q^8b_8-q^{20}b_{10}+q^{20}(1+q^8)\,b_{12}-\dots \qquad \text{...(9)}$$

is of special interest, as will be seen later.

The coefficient of b_{6n} is $q^{2n(3n-1)}+q^{2n(3n+1)}$, while that of b_{6n-2} and b_{6n+2} are respectively $-q^{2n(3n-1)}$ and $-q^{2n(3n+1)}$.

3. Closely analogous to the results obtained in the preceding sections are those derived from the coefficients of $x^r/(1-q^r)!$ in the expansion of

$$(1+2xq\cos\theta+x^2q^2)(1+2xq^3\cos\theta+x^2q^4)\dots,$$

which in Vol. XXIV., p. 352, was denoted by $B_r(\theta)$.

If the q series $\quad a_0 + a_1 B_1(\theta) + a_2 B_2(\theta) + \dots$

be equivalent to the same Fourier series

$$b_0 + 2b_1 \cos\theta + 2b_2 \cos 2\theta + \dots,$$

we have, as in § 1, (2),

$$\left.\begin{array}{ll} a_0 + a_2 q^{1+1} \dfrac{q_2}{q_1} + a_4 q^{3+3} \dfrac{q_4 q_3}{q_1 q_2} + a_6 q^{6+6} \dfrac{q_6 q_5 q_4}{q_1 q_2 q_3} + \dots & = b_0 \\ a_2 q^3 + a_4 q^{6+1} \dfrac{q_4}{q_1} + a_6 q^{10+3} \dfrac{q_6 q_5}{q_1 q_2} + \dots & = b_2 \\ a_4 q^{10} + a_6 q^{15+1} \dfrac{q_6}{q_1} + a_8 q^{21+3} \dfrac{q_8 q_7}{q_1 q_2} + \dots & = b_4 \\ a_6 q^{21} + a_8 q^{28+1} \dfrac{q_8}{q_1} + \dots & = b_6 \\ a_8 q^{36} + \dots & = \end{array}\right\} \quad \dots\dots(1),$$

whence a_0, a_2, ... may be expressed in terms of the b's. However, it is easy to see, from the formation of the indices of the powers of q which occur in the coefficient, that if by § 1, (2), we obtain a relation

$$m_0 a_0 + m_2 a_2 + m_4 a_4 + \dots = n_0 b_0 + n_2 b_2 + n_4 b_4 + \dots,$$

then, from (1) above, we get

$$m_0 a_0 + m_2 q^2 a_2 + m_4 q^6 a_4 + m_6 q^{12} a_6 + \dots$$
$$= n_0 b_0 + n_2 q^{-1} b_2 + n_4 q^{-4} b_4 + n_6 q^{-9} b_6 + \dots \quad \dots\dots(2),$$

while, in a similar manner, if

$$m_1 a_1 + m_3 a_3 + \dots = n_1 b_1 + n_3 b_3 + \dots,$$

then

$$m_1 q a_1 + m_3 q^4 a_3 + m_5 q^9 a_5 + \dots = n_1 b_1 + n_3 q^{-2} b_3 + n_5 q^{-6} b_5 + n_7 q^{-12} b_7 + \dots .$$

4. By the results obtained in § 1 it is obvious that, if conditions for convergency are satisfied, any cosine-series in θ may be expanded uniquely in the form

$$a_0 + a_1 A_1(\theta) + a_2 A_2(\theta) + \dots .$$

Let us therefore expand the product

$$(1 + 2\lambda q \cos\theta + \lambda^2 q^2)(1 + 2\lambda q^3 \cos\theta + \lambda^2 q^4) \dots$$

in this form.

Now it has been seen in Vol. xxiv., p. 345, that, if

$$f(\theta) = C_0 + C_1 A_1(\theta) + C_2 A_2(\theta) + \dots,$$

and $$\frac{f(\theta)}{P(\lambda)} = K_0 + K_1 A_1(\theta) + K_2 A_2(\theta) + \dots,$$

where $$P(\lambda) = (1-2\lambda\cos\theta+\lambda^2)(1-2\lambda q\cos\theta+\lambda^2q^2)\dots,$$

then $$K_0 + K_1 x + K_2 x^2 + \dots = \frac{1}{(x\lambda)}\,\frac{1}{(x\delta)}\,(C_0 + C_1\lambda + \dots),$$

where δ operates only on the λ's not contained in the C's; or, since

$$\frac{1}{(x\delta_\lambda)}\phi(\lambda) = \frac{1}{(\lambda\delta_x)}\phi(x),$$

we may write this relation in the form

$$K_0 + K_1 x + K_2 x^2 + \dots = \frac{1}{(\lambda x)}\,\frac{1}{(\lambda\delta_x)}\,(C_0 + C_1 x + C_2 x^2 + \dots),$$

where the C's are independent of x.

Hence $C_0 + C_1 x + C_2 x^2 + \dots = (\lambda\delta_x)(\lambda x)(K_0 + K_1 x + \dots)$.

If, then, $$P(\lambda) = C_0 + C_1 A_1(\theta) + \dots,$$

we have $$C_0 + C_1 x + C_2 x^2 + \dots = (\lambda\delta_x)(\lambda x),$$

because $$K_0 + K_1 A_1(\theta) + \dots = 1.$$

Now $$(\lambda\delta_x)(\lambda x)$$

$$= \left\{1 - \frac{\lambda\delta_x}{1-q} + \frac{q\lambda^2\delta_x^2}{(1-q)(1-q^2)} - \dots\right\}\left\{1 - \frac{\lambda x}{1-q} + \frac{q\lambda^2x^2}{(1-q)(1-q^2)} - \dots\right\},$$

and since $$\delta_x x^r = (1-q^r)\,x^{r-1}$$

by definition, we see that the coefficient of x^r is the series

$$(-1)^r\frac{q^{\frac{1}{2}r(r-1)}\lambda^r}{(1-q^r)!}\,\Sigma\,\frac{\lambda^{2s}q^{rs+s(s-1)}}{(1-q^s)!} \quad\dots\dots\dots\dots(1),$$

where $$s = 0, 1, 2, \dots.$$

Changing λ in $-\lambda q$, we have

$$(1+2\lambda q\cos\theta+\lambda^2q^2)(1+2\lambda q^2\cos\theta+\lambda^2q^4)\dots$$

$$= 1 + \frac{\lambda^2q^2}{1-q} + \frac{\lambda^4q^6}{(1-q)(1-q^2)} + \frac{\lambda^6q^{12}}{(1-q)(1-q^2)(1-q^3)} + \dots$$

$$+ \frac{q\lambda}{1-q}A_1(\theta)\left\{1 + \frac{\lambda^2q^3}{1-q} + \frac{\lambda^4q^8}{(1-q)(1-q^2)} + \dots\right\}$$

$$+ \frac{q^3\lambda^2}{(1-q)(1-q^2)}A_2(\theta)\left\{1 + \frac{\lambda^2q^4}{1-q} + \dots\right\}$$

$$+ \dots \quad\dots\dots\dots\dots\dots\dots(2).$$

Moreover, if we write $\chi(\lambda^2)$ for the series

$$1+\frac{\lambda^2q^2}{1-q}+\frac{\lambda^4q^6}{(1-q)(1-q^2)}+\dots,$$

which is the coefficient of $A_0(\theta)$, we can write this expansion in the form

$$\chi(\lambda^2)+\frac{q\lambda A_1(\theta)}{1-q}\chi(\lambda^2q)+\frac{q^3\lambda^2A_2(\theta)}{(1-q)(1-q^2)}\chi(\lambda^2q^2)$$
$$+\frac{q^6\lambda^3A_3(\theta)}{(1-q)(1-q^2)(1-q^3)}\chi(\lambda^2q^3)+\dots \qquad \dots\dots(3).$$

The function $\chi(\lambda^2)$ also satisfies the relation

$$\chi(\lambda^2)-\chi(\lambda^2q)=\lambda^2q^2\chi(\lambda^2q^2),$$

so that

$$\frac{\chi(\lambda^2)}{\chi(\lambda^2q)}=\frac{1}{1+}\,\frac{\lambda^2q^2}{1+}\,\frac{\lambda^2q^3}{1+}\,\frac{\lambda^2q^4}{1+}\dots \qquad \dots\dots\dots\dots\dots(4).$$

5. When $\lambda=q^{-\frac{1}{2}}$, we get some very interesting results. For then the equation § 4, (2), gives

$$(1+2q^{\frac{1}{2}}\cos\theta+q)(1+2q^{\frac{3}{2}}\cos\theta+q^3)\dots$$
$$=1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\dots$$
$$+\frac{q^{\frac{1}{2}}A_1(\theta)}{1-q}\left\{1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\dots\right\}$$
$$+\dots$$
$$=\phi(q)+\frac{q^{\frac{1}{2}}A_1(\theta)}{1-q}\psi(q)+\dots,\ \text{say.}$$

But, by the theory of theta functions, the above product

$$=\frac{1}{\Pi[1-q^n]}(1+2q^{\frac{1}{2}}\cos\theta+2q^2\cos2\theta+2q^{\frac{9}{2}}\cos3\theta+2q^8\cos4\theta+\dots).$$

Hence, by § 1, (12),

$$\Pi[1-q^n]\,\phi(q)=1-q^2(1+q)+q^9(1+q^3)-q^{21}(1+q^5)-\dots$$
$$=1-q^{\frac{5}{2}}(q^{-\frac{1}{2}}+q^{\frac{1}{2}})+q^{10}(q^{-1}+q^1)-q^{\frac{45}{2}}(q^{-\frac{3}{2}}+q^{\frac{3}{2}})+\dots$$
$$=\Pi[1-q^{5n}]\,\Pi[1-q^{5n\pm2}];$$

therefore

$$\phi(q)=1\div(1-q)(1-q^4)(1-q^6)(1-q^9)\dots$$
$$=1\div\Pi[1-q^{5n\pm1}]\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

Similarly, by § 1, (13),

$$\Pi\,[1-q^n]\,\psi(q) = 1-q-q^4(1-q^3)+q^{13}(q-q^6)-\dots$$
$$= 1-q^{\frac{5}{2}}(q^{-\frac{1}{2}}+q^{\frac{1}{2}})+\dots$$
$$= \Pi\,[1-q^{5n}]\,\Pi\,[1-q^{5n\pm 1}]\,;$$

therefore $$\psi(q) = 1\div\Pi\,[1-q^{5n\pm 2}] \quad\dots\dots(2).$$

Combining these results, we see that

$$\phi(q)\,\psi(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\dots}{(1-q)(1-q^2)(1-q^3)\dots},$$

and that

$$\frac{\phi(q)}{\psi(q)} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\dots = \frac{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots}$$
$$= (1-q)(1+q^2)(1+q^3)(1+q^7)(1+q^8)(1-q^9)(1-q^{11})(1+q^{12})\dots,$$

where the indices in the binomial factors include all numbers whose final digits are 1, 2, 3, 7, 8, or 9, the first and last being combined with minus signs, and the rest with plus signs.

Similarly,

$$\frac{\psi(q)}{\phi(q)} = (1+q)(1-q^3)(1+q^4)(1+q^6)(1-q^7)(1+q^9)(1+q^{11})(1-q^{13})\dots.$$

6. The series

$$\chi(\lambda) = 1+\frac{\lambda q^2}{1-q}+\frac{\lambda^2 q^6}{(1-q)(1-q^2)}+\dots$$

may be expressed in another form by means of Lemma IV. on p. 340, Vol. XXIV. For, since this lemma gives

$$(\lambda\mu\eta_1)(\lambda_1\mu)\,\frac{1}{(\lambda\delta_1)}\,\frac{\psi(\lambda_1)}{(\lambda_1\mu)} = \frac{1}{(\lambda\delta_1)}\,\psi(\lambda_1),$$

we get, by putting $\psi(\lambda_1)\equiv(-\lambda_1 q)$, and $\mu=-q$,

$$\frac{1}{(\lambda\delta_1)}\,(-\lambda_1 q) = (-\lambda q\eta_1)(-\lambda_1 q).$$

Expanding each of these products and performing the operations involved, we get

$$1+qH_1(\lambda,\lambda_1)+q^3H_2(\lambda,\lambda_1)+q^6H_3(\lambda,\lambda_1)+\dots$$
$$=(-\lambda_1 q)\left\{1+\frac{\lambda q}{(1-q)(1+\lambda_1 q)}+\frac{\lambda^2 q^3}{(1-q)(1-q^2)(1+\lambda_1 q)(1+\lambda_1 q^2)}+\dots\right\},$$

which, by symmetry,

$$=(-\lambda q)\left\{1+\frac{\lambda_1 q}{(1-q)(1+\lambda q)}+\frac{\lambda_1^2 q^3}{(1-q)(1-q^2)(1+\lambda q)(1+\lambda q^2)}+\ldots\right\} \quad \ldots\ldots\ldots\ldots(1).$$

Let $\lambda_1 = \lambda q^{\frac{1}{2}}$, so that

$$1+\Sigma x^r H_r(\lambda, \lambda_1) = 1 \div (1-\lambda x)(1-\lambda x q^{\frac{1}{2}})(1-\lambda x q)\ldots,$$

and $$H_r(\lambda, \lambda_1) = \lambda^r \div (1-q^{\frac{1}{2}})(1-q)\ldots(1-q^{\frac{1}{2}r});$$

then (1) becomes, after changing q into q^2,

$$1+\frac{\lambda q^2}{1-q}+\frac{\lambda^2 q^6}{(1-q)(1-q^2)}+\ldots,$$

i.e., $$\chi(\lambda) = \Pi[1+q^{2n+1}\lambda]\times$$

$$\left\{1+\frac{\lambda q^3}{(1-q^2)(1+\lambda q^3)}+\frac{\lambda^2 q^6}{(1-q^2)(1-q^4)(1+\lambda q^3)(1+\lambda q^5)}+\ldots\right\} \ldots(2),$$

$$= \Pi[1+q^{2n}\lambda]\times$$

$$\left\{1+\frac{\lambda q^3}{(1-q^2)(1+\lambda q^2)}+\frac{\lambda^2 q^8}{(1-q^2)(1-q^4)(1+\lambda q^2)(1+\lambda q^4)}+\ldots\right\} \ldots(3).$$

Again, in (1), let $\lambda_1 = -\lambda$, so that

$$1+\Sigma x^r H_r(\lambda, \lambda_1) = 1 \div (1-\lambda^2 x^2)(1-\lambda^2 q^2 x^2)\ldots,$$

and $$H_{2r}(\lambda, \lambda_1) = \lambda^{2r} \div (1-q^2)(1-q^4)\ldots(1-q^{2r}),$$

and $$H_{2r+1}(\lambda, \lambda_1) = 0;$$

then (1) becomes

$$1+\frac{q^3\lambda^2}{1-q^2}+\frac{q^{10}\lambda^4}{(1-q^2)(1-q^4)}+\ldots = (-\lambda q)\left\{1-\frac{\lambda q}{(1-q)(1+\lambda q)}+\ldots\right\}.$$

Changing q into q^2, and λ into λq, we get

$$1+\frac{\lambda^2 q^8}{1-q^4}+\frac{\lambda^4 q^{24}}{(1-q^4)(1-q^8)}+\ldots = \Pi[1+\lambda q^{2n+1}]\times$$

$$\left\{1-\frac{\lambda q^3}{(1-q^2)(1+\lambda q^3)}+\frac{\lambda^2 q^8}{(1-q^2)(1-q^4)(1+\lambda q^3)(1+\lambda q^5)}-\ldots\right\} \ldots(4),$$

the left side of which is $\chi(\lambda^2)$ in which q has been changed into q^4.

If, now, in (2), we put $\lambda = q^{-1}$, we get

$$\phi(q) = \Pi[1+q^{2n}]\left\{1+\frac{q}{1-q^4}+\frac{q^4}{(1-q^4)(1-q^8)}+\ldots\right\};$$

therefore $$\phi(q)+\phi(-q) = 2\Pi[1+q^{2n}]\times$$

$$\left\{1+\frac{q^4}{(1-q^4)(1-q^8)}+\frac{q^{16}}{(1-q^4)(1-q^8)(1-q^{12})(1-q^{16})}+\dots\right\}.$$

But in (4), if $\lambda = -q^{-2}$, we get

$$\phi(q^4) = \Pi[1-q^{2n-1}]\times$$

$$\left\{1+\frac{q}{(1-q)(1-q^2)}+\frac{q^4}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}.$$

Comparing these results, we see that

$$\phi(q)+\phi(-q) = 2\frac{\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}\phi(q^{16}) \quad\dots\dots\dots\dots(5).$$

Similarly,

$$\phi(q)-\phi(-q) = 2\Pi[1+q^{2n}]\left\{\frac{q}{1-q^4}+\frac{q^9}{(1-q^4)(1-q^8)(1-q^{12})}+\dots\right\},$$

which, by putting $l=+1$, and $-q^4$ for q in (4),

$$= \frac{2\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}q\psi(-q^4)\quad\dots\dots\dots\dots(6).$$

Again, by (3), we shall get

$$\psi(q) = \Pi[1+q^{2n}]\left\{1+\frac{q^8}{1-q^4}+\frac{q^8}{(1-q^4)(1-q^8)}+\dots\right\},$$

whence, by (4),

$$\psi(q)-\psi(-q) = 2\frac{\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}q^8\psi(q^{16})\quad\dots\dots\dots\dots(7),$$

and similarly, by (3), used twice,

$$\psi(q)+\psi(1-q) = 2\frac{\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}\phi(-q^4)\quad\dots\dots\dots\dots(8).$$

These four identities (5), (6), (7), (8) are sufficiently remarkable in themselves to call for mention at this point, although they may all be derived from the Θ-function values of the series $\phi(q)$, $\psi(q)$ obtained in the last section.

For instance, $$\phi(q)+\phi(-q)$$

$$= \frac{(1-q)(1-q^9)(1-q^{11})(1-q^{19})\dots+(1+q)(1+q^9)(1+q^{11})(1+q^{19})\dots}{(1-q^2)(1-q^{18})(1-q^{22})\dots\times(1-q^4)(1-q^6)(1-q^{14})(1-q^{16})},$$

by § 5, (1).

This numerator multiplied by

$$(1-q^{10})(1-q^{20})(1-q^{30})\dots$$
$$= 2\left\{1+q^{20}(q^{-8}+q^{8})+q^{80}(q^{-16}+q^{16})+\dots\right\}$$
$$= 2\Pi(1-q^{40n})(1+q^{12})(1+q^{28})(1+q^{52})(1+q^{68})\dots.$$

In this manner we reduce the left hand of (5) to an infinite product, which is easily seen to include identically all the factors in the right-hand side, after substituting for $\phi(q^{16})$ by § 5, (1).

It is not, however, in these identities that the special interest in the series $\phi(q)$ and $\psi(q)$ lies. These relations may be considerably simplified by substituting the functions

$$\phi(q)\times(1-q^{2})(1-q^{4})(1-q^{6})\dots \text{ and } \psi(q)\times(1-q^{2})(1-q^{4})(1-q^{6})\dots.$$

Let $u_{\pm r}$ denote $\phi(\pm q^{r})\,\Pi\left\{1-q^{2rn}\right\}$,

and $v_{\pm r}$ " $\psi(\pm q^{r})\,\Pi\left\{1-q^{2rn}\right\}$.

Then (5), (6), (7), and (8) become

$$u_1+u_{-1} = 2\Pi\left[1-q^{8n}\right]\phi(q^{16}) = 2\frac{\Pi\left[1-q^{8n}\right]}{\Pi\left[1-q^{32n}\right]}u_{16} \quad \text{.........(9)},$$

$$u_1-u_{-1} = 2qv_{-4} \quad \text{.........(10)},$$

$$v_1-v_{-1} = 2q^{3}\frac{\Pi\left[1-q^{8n}\right]}{\Pi\left[1-q^{32n}\right]}v_{16} \quad \text{.........(11)},$$

$$v_1+v_{-1} = 2u_{-4} \quad \text{.........(12)}.$$

Now, if we put $\lambda = q^{-1}$ in (3), we get

$$\phi(q) = (1+q)(1+q^{3})(1+q^{5})\dots$$
$$\times\left\{1+\frac{q^{2}}{(1+q)(1-q^{2})}+\frac{q^{6}}{(1+q)(1-q^{2})(1+q^{3})(1-q^{4})}+\dots\right\},$$

so that u_{-1}

$$= \Pi\left[1-\lambda(-q)^{n}\right]\left\{1+\frac{q^{2}}{(1-q)(1-q^{2})}+\frac{q^{6}}{(1-q)(1-q^{2})(1-q^{3})(1-q^{4})}+\dots\right\} \quad \text{............(13)}.$$

By § 2, (9), and § 3, (2), we see that, if

$$a_0+a_1B_1(\theta)+a_2B_2(\theta)+\dots \equiv b_0+2b_1\cos\theta+2b_2\cos 2\theta+\dots,$$

then

$$a_0+a_2q^{3}+a_4q^{8}+a_6q^{15}+\dots$$
$$= b_0-\frac{b_2}{q}-q^{4}\frac{b_4}{q^{4}}+q^{4}(1+q^{4})\frac{b_6}{q^{9}}-q^{8}\frac{b_8}{q^{16}}-\dots \quad \text{......(14)}.$$

Now, by the definition of $B_r(\theta)$,

$$(1+2q^{\frac{1}{2}}\cos\theta+q)(1+2q^{\frac{3}{2}}\cos\theta+q^3)\dots$$

$$=1+\frac{B_1(\theta)\,q^{-\frac{1}{2}}}{1-q}+\frac{B_2(\theta)\,q^{-1}}{(1-q)(1-q^2)}+\frac{B_3(\theta)\,q^{-\frac{3}{2}}}{(1-q)(1-q^2)(1-q^3)}+\dots$$

$$=\frac{1}{\Pi\,[1-q^n]}(1+2q^{\frac{1}{2}}\cos\theta+2q^2\cos 2\theta+2q^{\frac{9}{2}}\cos 3\theta+2q^8\cos 4\theta+\dots).$$

Hence, by (13), u_{-1} is the right-hand side of (14), where $b_{2r}=q^{2r^2}$.

Thus $$u_{-1}=1-q-q^8+q^{18}(1+q^4)-q^{24}-q^{45}+q^{56}(1+q^8)-\quad\dots(15).$$

This series may be systematically arranged, if we notice that by taking every fourth term we get powers of q whose indices are in hyper-arithmetic progression.

We then see that we may write the series in the form

$$1+q^{15.1^2}(q^{-2}+q^2)+q^{15.2^2}(q^{-4}+q^4)+\dots$$
$$-q\,\{1+q^{15.1^2}(q^{-8}+q^8)+q^{15.2^2}(q^{-16}+q^{16})+\dots\},$$

consisting of two Θ-series of the 15th order.

Changing q into $-q$, we arrive finally at the remarkable identity

$$\Pi\,[1-q^{2n}]\left\{1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\dots\right\}$$
$$=1-q^{15.1^2}(q^{-2}+q^2)+q^{15.2^2}(q^{-4}+q^4)-\dots$$
$$+q\,\{1-q^{15.1^2}(q^{-8}+q^8)+q^{15.2^2}(q^{-16}+q^{16})-\dots\}$$
$$=\Pi\,[1-q^{30n}]\left\{\begin{array}{l}(1-q^{13})(1-q^{17})(1-q^{43})(1-q^{47})\dots\\ \qquad +q\,(1-q^7)(1-q^{23})(1-q^{37})(1-q^{53})\dots\end{array}\right\},$$

and, remembering the product value of $\phi(q)$ obtained in § 7, we see that

$$\Pi\,[1-q^{2n}]\div\Pi\,[1-q^{30n}]\,\Pi\,[1-q^{5n\pm1}]$$
$$=\quad(1-q^{13})(1-q^{17})(1-q^{43})(1-q^{47})(1-q^{73})(1-q^{77})\dots$$
$$+q\,(1-q^7)(1-q^{23})(1-q^{37})(1-q^{53})(1-q^{67})(1-q^{83})\dots\quad\dots(16).$$

We may, moreover, obtain from (10) a similar expression for v_1, after changing $-q^4$ into q,

$$v_1=1-q^{15.1^2}(q^{-4}+q^4)+q^{15.2^2}(q^{-8}+q^8)-\dots$$
$$+q^3\,\{1-q^{15.1^2}(q^{-14}+q^{14})+q^{15.2^2}(q^{-28}+q^{28})-\dots\},$$

whence $\Pi\,[1-q^{2n}] \div \Pi\,[1-q^{30n}]\,\Pi\,[1-q^{6n\pm 2}]$

$$= (1-q^{11})(1-q^{19})(1-q^{41})(1-q^{49})(1-q^{71})(1-q^{79})\ldots$$
$$+q^{8}\,(1-q)(1-q^{29})(1-q^{31})(1-q^{59})(1-q^{61})(1-q^{89})\ldots .$$

Again, since, by (9),

$$u+u_{-1} = \Pi\,[1-q^{8n}]\,\phi\,(q^{16}),$$

we have $\Pi\,[1-q^{8n}]\,\phi\,(q^{16})$

$$= 1+q^{60.1^2}(q^{-4}+q^{4})+q^{60.2^2}(q^{-8}+q^{8})$$
$$-q\,\{q^{15.1^2}(q^{-8}+q^{8})+q^{15.3^2}(q^{-24}+q^{24})$$

$$= \Pi\,[1-q^{120n}]\left\{\begin{array}{l}(1+q^{56})(1+q^{64})(1+q^{176})(1+q^{184})\ldots \\ \quad -q^{16}\,(q^{-8}+q^{8})(1+q^{104})(1+q^{136})(1+q^{224})(1+q^{256})\ldots\end{array}\right\};$$

therefore, changing q^8 into q,

$$\Pi\,[1-q^{n}]\,\phi(q^{2}) \div \Pi\,[1-q^{15n}] = (1-q)(1-q^{3})(1-q^{4})(1-q^{5})\ldots(1-q^{N})\ldots,$$

where N is any integer which is not a multiple of 15, or whose last digit is not 2 or 8,

$$= (1+q^{7})(1+q^{8})(1+q^{22})(1+q^{23})\ldots$$
$$-q\,(1+q^{2})(1+q^{13})(1+q^{17})(1+q^{28})(1+q^{32})\ldots .$$

Similarly, from (11),

$$\Pi\,[1-q^{n}]\,\psi\,(q^{2}) \div \Pi\,(1-q^{15n}] = \Pi\,[1-q^{N}],$$

where N is not a multiple of 15, and does not end with a 4 or 6,

$$= (1+q^{4})(1+q^{11})(1+q^{19})(1+q^{26})\ldots$$
$$-q\,(1+q)(1+q^{14})(1+q^{16})(1+q^{29})\ldots .$$

7. We have seen that, if

$$a_0 \pm a_1 A_1(\theta)+a_2 A_2(\theta) \pm \ldots = b_0 \pm 2b_1 \cos\theta + 2b_2 \cos 2\theta \pm \ldots \quad \ldots(1),$$

then a_0, a_1, ... can separately be expanded in § 1 in series containing b's with simple coefficients. Moreover the series

$$a_0+a_1+a_2+\ldots \quad \text{and} \quad a_0+a_1 q^{\frac{1}{2}}+a_2 q+\ldots$$

have been similarly expanded in § 2.

These identities are only a few of a very large number of relations connecting simple series in the a's with simple series in the b's, which will be established in the subsequent sections of this memoir.

These can all be treated in the manner of § 6, (14), where by § 3 (2), we see that, if

$$a_0 + m_1 a_1 + m_2 a_2 + \ldots = b_0 + n_1 b_1 + n_2 b_2 + \ldots \qquad \ldots\ldots\ldots(2),$$

then $$a_0 + m_2 q^2 a_2 + m_4 q^6 a_4 + \ldots = b_0 + n_2 q^{-1} b_2 + n_4 q^{-4} b_4 + \ldots$$

and $$m_1 q^{\frac{3}{4}} a_1 + m_3 q^{\frac{15}{4}} a_3 + \ldots = n_1 q^{-\frac{1}{4}} b_1 + n_3 q^{-\frac{9}{4}} b_3 + \ldots,$$

which, applied to the Θ-function identity

$$\Pi[1-q^n]\left\{1 + \frac{B_1(\theta)\, q^{-\frac{1}{4}}}{1-q} + \frac{B_2(\theta)\, q^{-1}}{(1-q)(1-q^2)} + \ldots\right\}$$
$$= 1 + 2q^{\frac{1}{4}} \cos\theta + 2q \cos 2\theta + \ldots,$$

gives

$$\Pi[1-q^n]\left\{1 \pm \frac{q^{\frac{1}{4}} m_1}{1-q} + \frac{q m_2}{(1-q)(1-q^2)} \pm \frac{q^{\frac{9}{4}} m_3}{(1-q)(1-q^2)(1-q^3)} + \ldots\right\}$$
$$= 1 \pm q^{\frac{1}{4}} n_1 + q n_2 \pm q^{\frac{9}{4}} n_3 + q^4 n_4 \pm \ldots \qquad \ldots\ldots\ldots\ldots\ldots(3).$$

Many of the relations obtained will only lead to well-known identities, and in such cases the application of this section will not be quoted.

8. We have seen in § 4, (2), that, if

$$1 + \frac{B_1(\theta)\,\lambda}{1-q} + \frac{B_2(\theta)\,\lambda^2}{(1-q)(1-q^2)} + \ldots,$$

be expanded according to $A(\theta)$'s, the coefficient of $A_r(\theta)$ is

$$\Sigma \frac{\lambda^{r+2s}\, q^{\frac{1}{2}r(r+1)+rs+s(s+1)}}{(1-q^r)!\,(1-q^s)!},$$

when s has all integral values from 0 to ∞.

Now $\frac{1}{2}r(r+1) + rs + s(s+1) = \frac{1}{4}r^2 + \frac{1}{4}(r+2s)(r+2s+2)$,

so that, if any power λ^m of λ be changed into $\lambda q^{-\frac{1}{4}m(m+2)}$, the corresponding coefficient of $A_r(\theta)$ would be

$$\Sigma \frac{\lambda^{r+2s}\, q^{\frac{1}{4}r^2}}{(1-q^r)!\,(1-q^s)!}.$$

This is precisely the same thing as saying that

$$1 + \frac{B_1(\theta)\, q^{-\frac{3}{4}}\lambda}{1-q} + \frac{B_2(\theta)\, q^{-2}\lambda^2}{(1-q)(1-q^2)} + \ldots$$
$$= \left\{1 + \frac{A_1(\theta)\, q^{\frac{1}{4}}\lambda}{1-q} + \frac{A_2(\theta)\, q\lambda^2}{(1-q)(1-q^2)} + \ldots\right\}$$
$$\times \left\{1 + \frac{\lambda^2}{1-q} + \frac{\lambda^4}{(1-q)(1-q^2)} + \ldots\right\} \qquad \ldots(1).$$

Now suppose that, in consequence of a relation

$$a_0+a_1A_1(\theta)+\dots = \alpha_0+\alpha_1B_1(\theta)+\dots \qquad \text{............(2)},$$

we can establish some relation of the form

$$a_0+a_1\lambda_1+a_2\lambda_2+\dots = \alpha_0+\alpha_1 n_1 q^{\frac34}+\alpha_2 n_2 q^2+\dots \qquad \text{.........(3)};$$

then (1) gives relations connecting the A's and B's by equating coefficients of powers of λ; (2) establishes connexions between a_0, a_1, ... and α_0, α_1, ... by equating coefficients of the A's; on substituting for the α's in (3), we get relations connecting λ_1, λ_2, ... with m_1, m_2.

It will be easy to see, however, that these are simply expressed by substituting λ_r for $A_r(\theta)$, and m_r for $B_r(\theta)\,q^{-\frac14 r(r+2)}$, so that

$$1+\frac{m_1\lambda}{1-q}+\frac{m_2\lambda^2}{(1-q)(1-q^2)}+\dots$$

$$=\left\{1+\frac{\lambda_1 q^{\frac14}\lambda}{1-q}+\frac{\lambda_2 q\lambda^2}{(1-q)(1-q^2)}+\dots\right\}\left\{1+\frac{\lambda^2}{1-q}+\dots\right\} \quad \text{...(4)},$$

Now, in § 3, we have seen that, if

$$a_0+m_1a_1+m_2a_2+\dots = b_0+n_1b_1+n_2b_2+\dots \qquad \text{............(5)},$$

then $\quad a_0+m_1q^{\frac34}a_1+m_2q^2a_2+\dots = b_0+n_1q^{-\frac14}b_1+n_2q^{-1}b_2+\dots,$

i.e., $\quad a_0+a_1\lambda_1+a_2\lambda_2+\dots = b_0+n_1q^{-\frac14}b_1+n_2q^{-1}b_2+\dots \qquad \text{...(6)}.$

If, then, we know a relation of the form (5), we can by (4) obtain the coefficients λ_1, λ_2, ... which establish the equation (6), and *vice versâ*.

Example 1.—Let $\qquad \lambda_1 = \lambda_2 = \dots = 0,$

so that, by § 1, (12), we know the values of $n_1q^{-\frac14}$, n_2q^{-1}, &c. Then substituting in (5) the values of the m's given by (4), we get

$$a_0+a_2(1-q^3)+a_4(1-q^3)(1-q^4)+a_6(1-q^4)(1-q^5)(1-q^6)+\dots$$
$$= b_0-q(1+q)\,b_2+q^5(1+q^2)\,b_4-q^{12}(1+q^3)\,b_6+\dots .$$

Example 2.—In § 7, (1), by putting $\theta=\frac{\pi}{2}$, we get

$$a_0-a_2(1-q)+a_4(1-q)(1-q^3)-\dots = b_0-2b_2+2b_4-\dots .$$

The present section gives

$$a_0+a_2(1-q)+a_4(1-q)(1-q^3)+\dots = b_0-2qb_2+2q^4b_4-\dots .$$

Example 3.—By putting $\pm 2\cos\theta = q^{-\frac{1}{2}}+q^{\frac{1}{2}}$ in § 7, (1), we get

$$a_0 \pm a_1 q^{-\frac{1}{2}}(1+q^{\frac{1}{2}}) + a_2 q^{-1}(1+q^{\frac{1}{2}})(1+q) \pm \ldots$$
$$= b_0 \pm b_1 q^{-\frac{1}{2}}(1+q^{\frac{1}{2}}) + b_2 q^{-1}(1+q) \pm \ldots,$$

and the derived form is

$$a_0 \pm a_1 (1+q^{\frac{1}{2}}) + a_2 (1+q^{\frac{1}{2}})(1+q) \pm a_3 (1+q^{\frac{1}{2}})(1+q)(1+q^{\frac{3}{2}}) + \ldots$$
$$= b_0 \pm b_1 (1+q^{\frac{1}{2}}) + b_2 q^{\frac{1}{2}} (1+q) \pm b_3 q^{\frac{3}{2}} (1+q^{\frac{3}{2}}) + \ldots .$$

Example 4.—If $\qquad -2\cos 2\theta = q^{-1}+q,$

$$a_0 - a_2 q^{-1}(1-q) + a_4 q^{-2}(1-q)(1-q^3) - \ldots$$
$$= b_0 - q^{-1}(1+q^2)\, b_2 + q^{-2}(1+q^4)\, b_4 - \ldots,$$

and the derived form is

$$a_0 + a_2 q\,(1-q) + a_4 q^2 (1-q)(1-q^3) + \ldots$$
$$= b_0 - (1+q^2)\, b_2 + q^2 (1+q^4)\, b_4 - q^6 (1+q^6)\, b_6 + \ldots .$$

9. Quadratic transformation of q.

We have already seen, on pp. 175 and 343 of Vol. XXIV., that

$$\frac{(\mu\nu)}{P(\mu)\,P(\nu)} = 1 + H_1(\mu, \nu)\, A_1(\theta) + \ldots,$$

from which relation, by putting $\nu = \mu q^{\frac{1}{2}}$, and afterwards changing q into q^2, we have

$$\left\{1 - \frac{\mu^2 q}{1-q^2} + \frac{\mu^4 q^4}{(1-q^2)(1-q^4)} - \ldots\right\}\left\{1 + \frac{A_1(\theta)\,\mu}{1-q} + \frac{A_2(\theta)\,\mu^2}{(1-q)(1-q^2)} + \ldots\right\}$$
$$= 1 + \frac{A_1(\theta, q^2)}{1-q}\,\mu + \frac{A_2(\theta, q^2)}{(1-q)(1-q^2)}\,\mu^2 + \ldots \qquad \ldots\ldots\ldots\ldots(1),$$

where $A_r(\theta, q^2)$ is what $A_r(\theta)$ becomes when q is changed into q^2.

By equating coefficients of powers of μ, we get $A_r(\theta, q^2)$ in terms of $A_r(\theta)$, $A_{r-2}(\theta)$ … .

Now suppose that we have some function expanded according to both kinds of A, *i.e.*,

$$a_0 + a_1 A_1(\theta) + \ldots = \gamma_0 + \gamma_1 A_1(\theta, q^2) + \ldots \qquad \ldots\ldots\ldots\ldots(2).$$

Substituting from (1) in the right-hand side of (2), and equating coefficients of like A's, we get relations connecting the a's and γ's.

If this is written in the form

$$a_0+m_1a_1+m_2a_2+\ldots=\gamma_0+c_1\gamma_1+c_2\gamma_2+\ldots \qquad \ldots\ldots\ldots(3),$$

it will not be difficult to see, just as in the last section, that the m's and c's will be connected by the relation

$$1+\frac{c_1\mu}{1-q}+\frac{c_2\mu^2}{(1-q)(1-q^2)}+\ldots$$

$$=\left\{1+\frac{m_1\mu}{1-q}+\frac{m_2\mu^2}{(1-q)(1-q^2)}+\ldots\right\}\left\{1-\frac{\mu^2q}{1-q^2}+\frac{\mu^4q^4}{(1-q^2)(1-q^4)}-\ldots\right\}$$

$$\ldots\ldots\ldots\ldots(4).$$

It is evident, moreover, that if we know a relation connecting a series of γ's with a series of b's, we may change q to $q^{\frac12}$ throughout, provided we change γ_r into a_r.

In this manner we may extend very considerably the number of relations connecting series of a's with series of b's. It will not be necessary to work them all out in detail, since the method of deduction is the same for all.

Example 1.—Let $\qquad m_r=q^{\frac12 r},$

so that the right-hand side of (4) becomes

$$\frac{(1-\mu^2q)(1-\mu^2q^3)(1-\mu^2q^5)\ldots}{(1-\mu q^{\frac12})(1-\mu q^{\frac32})(1-\mu q^{\frac52})},$$

which

$$=(1+\mu q^{\frac12})(1+\mu q^{\frac32})\ldots$$

$$=1+\frac{\mu q^{\frac12}}{1-q}+\frac{\mu^2q^2}{(1-q)(1-q^2)}+\ldots.$$

Hence $\quad a_0+a_1q^{\frac12}+a_2q+a_3q^{\frac32}+a_4q^2+\ldots$

$$=\gamma_0+\gamma_1q^{\frac12}+\gamma_2q^2+\gamma_3q^{\frac92}+\gamma_4q^8+\ldots.$$

But since the left-hand side has been given in § 2, (8), and changing q into $q^{\frac12}$, and γ into a, we have

$$a_0+a_1q^{\frac12}+a_2q+a_3q^{\frac32}+a_4q^4+\ldots$$

$$=b_0+b_1q^{\frac12}-b_2-q^{\frac12}(1+q)\,b_3-q^2b_4+q^{\frac32}b_5+\ldots,$$

where, of course, the series of terms with even suffixes are equal, and those with odd.

Example 2.—Let $\qquad m_r = 0;$

then $a_0 = \gamma_0 - q(1-q)\gamma_2 + q^4(1-q)(1-q^3)\gamma_4 - \dots$

$$= b_0 - (1+q)\,b_2 + q\,(1+q^2)\,b_4 - q^3(1+q^3)\,b_6 - \dots, \text{ by § 1, (12)};$$

therefore $a_0 - q^{\frac{1}{2}}(1-q^{\frac{1}{2}})\,a_2 + q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})\,a_4 - \dots$

$$= b_0 - (1+q^{\frac{1}{2}})\,b_2 + q^{\frac{1}{2}}(1+q)\,b_4 - q^{\frac{3}{2}}(1+q^{\frac{3}{2}})\,b_6 + \dots.$$

By § 7, (3), we get the identity

$$\Pi[1-q^n]\left\{1 - \frac{q^{\frac{3}{2}}(1-q^{\frac{1}{2}})}{(1-q)(1-q^2)} + \frac{q^6(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{(1-q)(1-q^2)(1-q^3)(1-q^4)} - \dots\right\}$$

$$= \text{the } \Theta\text{-function } 1 - q\,(1+q^{\frac{1}{2}}) + q^{\frac{5}{2}}(1+q) - \dots.$$

Example 3. $\qquad a_0 + a_2(1-q) + a_4(1-q)(1-q^3) + \dots$

$$= \gamma_0 + \gamma_2(1-q)^2 + \gamma_4(1-q)^2(1-q^3)^2 + \dots$$

$$= b_0 - 2qb_2 + 2q^4b_4 - \dots, \text{ by the preceding section};$$

therefore $a_0 + a_2(1-q^{\frac{1}{2}})^2 + a_4(1-q^{\frac{1}{2}})^2(1-q^{\frac{3}{2}})^2 + \dots$

$$= b_0 - 2q^{\frac{1}{2}}b_2 + 2q^2b_4 - \dots,$$

whence, by § 7,

$$\Pi[1-q^n]\left\{1 + \frac{(1-q^{\frac{1}{2}})^2 q}{(1-q)(1-q^2)} + \frac{(1-q^{\frac{1}{2}})^2(1-q^{\frac{3}{2}})^2 q^4}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \dots\right\}$$

$$= 1 - 2q^{\frac{3}{2}} + 2q^6 - \dots.$$

Example 4. $\qquad a_0 + a_2(1-q^2) + a_4(1-q^3)(1-q^4) + \dots$

$$= \gamma_0 + \gamma_2(1-q) + \gamma_4(1-q)(1-q^3) + \dots$$

$$= b_0 - q\,(1+q)\,b_2 + q^5(1+q^2)\,b_4 - \dots, \text{ by § 9, Ex. 1};$$

therefore $\qquad a_0 + a_2(1-q^{\frac{1}{2}}) + a_4(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}}) + \dots$

$$= b_0 - q^{\frac{1}{2}}(1+q^{\frac{1}{2}})\,b_2 + q^{\frac{5}{2}}(1+q)\,b_4 - q^6(1+q^{\frac{3}{2}})\,b_6 + \dots.$$

By § 7, this gives

$$\Pi[1-q^n]\left\{1 + \frac{q\,(1-q^{\frac{1}{2}})}{(1-q)(1-q^2)} + \frac{q^4(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \dots\right\}$$

$$= 1 - q^{\frac{3}{2}}(1+q^{\frac{1}{2}}) + q^{\frac{13}{2}}(1+q) - q^{15}(1+q^{\frac{3}{2}}) + \dots$$

$$= \Pi[1-q^{\frac{7}{2}n}]\,\Pi[1-q^{\frac{1}{2}(7n\pm3)}].$$

Hitherto we have deduced results by changing q^2 into q. We may derive other identities by assuming values for the c's in § 4.

Example 5.—Let $c_r = 0$;

then
$$1+\frac{m_1\mu}{1-q}+\ldots = 1 \div (1-\mu^2 q)(1-\mu^2 q^3)\ldots$$
$$= 1+\frac{q\mu^2}{1-q^2}+\frac{q^2\mu^4}{(1-q^2)(1-q^4)}+\ldots;$$

therefore
$$\gamma_0 = a_0 + a_2 q\,(1-q) + a_4 q^2\,(1-q)(1-q^3)+\ldots$$
$$= b_0 - b_2\,(1+q^2) + b_4 q^2\,(1+q^4) - b_6 q^6\,(1+q^6)+\ldots,$$

by changing q into q^2, and a_0 into γ_0 in § 1, (12).

This result agrees with § 8, Ex. 4.

10. Second quadratic transformation of q.

By putting $r = -\mu$ in the identity quoted at the beginning of the last section, we get a relation

$$\left\{1+\frac{A_2(\theta)}{1-q^2}\mu^2+\frac{A_4(\theta)}{(1-q^2)(1-q^4)}\mu^4+\ldots\right\}$$
$$\times\left\{1-\frac{\mu^2}{1-q}+\frac{\mu^4}{(1-q)(1-q^2)}-\ldots\right\}$$
$$= 1+\frac{A_2(2\theta,\,q^2)}{1-q^2}\mu^2+\frac{A_4(2\theta,\,q^2)}{(1-q^2)(1-q^4)}\mu^4+\ldots.$$

As in the preceding sections, we see that, if in consequence of a known equation

$$a_0 + a_2 A_2(\theta) + a_4 A_4(\theta) + \ldots = e_0 + e_1 A_1(2\theta,\,q^2) + e_2 A_2(2\theta,\,q^2)+\ldots,$$

we can derive a relation

$$a_0 + a_2 m_2 + a_4 m_4 + \ldots = e_0 + e_1 k_1 + e_2 k_2 + \ldots,$$

then
$$1+\frac{k_1\mu^2}{1-q^2}+\frac{k_2\mu^4}{(1-q^2)(1-q^4)}+\ldots$$
$$=\left\{1+\frac{m_2\mu^2}{1-q^2}+\frac{m_4\mu^4}{(1-q^2)(1-q^4)}+\ldots\right\}$$
$$\times\left\{1-\frac{\mu^2}{1-q}+\frac{\mu^4}{(1-q)(1-q^2)}-\ldots\right\} \quad \ldots(1).$$

Example 1.—Let $k_r = 0$;

then
$$e_0 = a_0 + a_2\,(1+q) + a_4 q\,(1+q)(1+q^3)+\ldots.$$

But the expansion of e_0 in terms of the b's is evidently obtained by changing q into q^2, and b_r into b_{2r} in the expansion of a_0; therefore

$$a_0+a_2(1+q)+a_4q(1+q)(1+q^2)+a_6q^3(1+q)(1+q^2)(1+q^3)+\dots$$
$$=b_0-b_4(1+q^2)+b_8q^2(1+q^4)-b_{12}q^6(1+q^6)-\dots.$$

By § 7, we get

$$\Pi[1-q^n]\left\{1+\frac{q(1+q)}{(1-q)(1-q^2)}+\frac{q^5(1+q)(1+q^2)}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}$$
$$=\text{the }\Theta\text{-function }1-q^4(1+q^2)+q^{18}(1+q^4)-\dots.$$

Example 2.—From Ex. 1, § 8, we get

$$a_0+a_2(1+q)+a_4(1+q)(1+q^2)+\dots$$
$$=b_0-b_4q^2(1+q^2)+b_8q^{10}(1+q^4)-\dots;$$

whence

$$\Pi[1-q^n]\left\{1+\frac{q(1+q)}{(1-q)(1-q^2)}+\frac{q^4(1+q)(1+q^2)}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}$$
$$=\text{the }\Theta\text{-function }1-q^7(q^{-1}+q)+q^{23}(q^{-2}+q^2)-\dots.$$

Example 3.—Let $k_1=-q,\quad k_2=q^3,\quad k_3=-q^6\dots;$

then
$$a_0+a_2+a_4q^2+a_6q^6+a_8q^{12}+\dots$$
$$=e_0-e_1q+e_2q^3-e_3q^6+\dots$$
$$=b_0+qb_2-b_4-q(1+q^4)b_6-q^8b_8+\dots,\text{ by § 2, (8).}$$

Similarly,
$$a_0+a_2q+a_4q^4+\dots$$
$$=e_0-e_1+e_2-e_3+\dots$$
$$=b_0-b_2-q^2b_4+q^2(1+q^2)b_6-\dots,\text{ as in § 9, Ex. 1.}$$

Example 4.—Let $m_2=1+q,\ m_4=(1+q)(1+q^3),\ \dots;$

then, since

$$1+\frac{1+q}{1+q^2}\mu^2+\frac{(1+q)(1+q^3)}{(1-q^2)(1-q^4)}\mu^4+\dots=\frac{(1+\mu^2q)(1+\mu^2q^3)\dots}{(1-\mu^2)(1-\mu^2q^2)\dots},$$

we have $\quad 1+\dfrac{k_1\mu^2}{1-q^2}+\dots=1\div(1-\mu^4)(1-\mu^4q^4)\dots;$

therefore $a_0+a_2(1+q)+a_4(1+q)(1+q^3)+\dots$
$$=e_0+e_2(1-q^2)+e_4(1-q^2)(1-q^6)+\dots$$
$$=b_0-2q^2b_4+2q^8b_8-\dots,\text{ by § 8, Ex. 2.}$$

Example 5.—Let $$m_r = 0,$$

$$e_0-(1+q)\,e_1+(1+q)(1+q^2)\,e_2-\dots$$
$$= a_0 = b_0-(1+q)\,b_2+q\,(1+q^2)\,b_4-\dots,\ \text{by § 1, (12)};$$

therefore $$a_0-(1+q^{\frac{1}{2}})\,a_1+(1+q^{\frac{1}{2}})(1+q)\,a_2-\dots$$
$$= b_0-(1+q^{\frac{1}{2}})\,b_1+q^{\frac{1}{2}}\,(1+q)\,b_2-q^{\frac{3}{2}}\,(1+q^{\frac{3}{2}})\,b_3-\dots .$$

11. Cubic transformation of q.

The identity

$$\frac{(\lambda\mu)(\mu\nu)(\nu\lambda)}{P(\lambda)\,P(\mu)\,P(\nu)} = 1+\Sigma H_r\,(\lambda\mu\nu/\lambda,\mu,\nu)\,A_r(\theta),$$

given in Vol. XXIV., p. 346, when

$$\mu = \lambda q^{\frac{1}{3}},\quad \nu = \lambda q^{\frac{2}{3}},$$

and q is changed into q^3, becomes

$$1+\frac{A_1(\theta)}{1-q}\lambda+\frac{A_2(\theta)}{(1-q)(1-q^2)}\lambda^2+\dots$$
$$=\left\{1+\frac{\lambda^2 q}{1-q}+\frac{\lambda^4 q^2}{(1-q)(1-q^2)}+\dots\right\}\{1+\Sigma H_r\,.\,A_r(\theta,q^3)\},$$

where $$1+\Sigma H_r x^r \equiv \frac{(1-\lambda^3q^3x)(1-\lambda^3q^6x)\dots}{(1-\lambda x)(1-\lambda qx)(1-\lambda q^2x)\dots}.$$

If, then, $$a_0+a_1A_1(\theta)+\dots = f_0+f_1A_1(\theta,q^3)+\dots,$$
we see that $$a_0+a_2q\,(1-q^2)+a_4q^2\,(1-q^3)(1-q^4)+\dots$$
$$= b_0-(1+q^3)\,b_2+q^3\,(1+q^6)\,b_4-q^9\,(1+q^9)\,b_6+\dots .$$

Again, by § 9, we get

$$a_0+a_2q\,(1-q^2)+a_4q^2\,(1-q^3)(1-q^4)+\dots$$
$$= \gamma_0+\gamma_2q^2\,(1-q)+\gamma_4q^4\,(1-q)(1-q^3)+\dots,$$

so that $$a_0+a_2q\,(1-q^{\frac{1}{3}})+a_4q^2\,(1-q^{\frac{1}{3}})(1-q^{\frac{2}{3}})+\dots$$
$$= b_0-(1+q^{\frac{2}{3}})\,b_2+q^{\frac{5}{3}}\,(1+q^3)\,b_4-q^{\frac{4}{3}}\,(1+q^{\frac{4}{3}})\,b_6+\dots,$$

and, by § 7,

$$\Pi[1-q^n]\left\{1+\frac{q^2\,(1-q^{\frac{1}{3}})}{(1-q)(1-q^2)}+\frac{q^6\,(1-q^{\frac{1}{3}})(1-q^{\frac{2}{3}})}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}$$
$$= 1-q-q^{\frac{5}{3}}+q^{\frac{14}{3}}+q^{\frac{17}{3}}-\dots$$
$$= \Pi\,[1-q^{\frac{2}{3}n}]\,\Pi\,[1-q^{\frac{1}{3}(7n\pm 2)}]\ \ (cf.\ \text{Ex. 4, § 9}).$$

The foregoing examples will illustrate the great fertility of the method employed for deducing identities which are difficult to prove by other means. It may be noticed that, when all the b's are equated to unity, the expression for a_0 vanishes identically. The equation § 1, (4) would lead us to infer that a_1 would also vanish identically on the same supposition, as indeed is obvious from § 1, (13). Similarly, it may be shown that all the a's vanish identically when the b's are equated to unity. Consistently with this fact, it will be then seen that, if in any relation connecting an a-series with a b-series the coefficients of the a's form a convergent series, then the b-series vanishes identically, as in § 2, (9), § 8, Ex. 4, &c.; but, if the b-series does not vanish identically, then the coefficients of the a's form a divergent series, as in § 2, (7), § 8, Ex. 1, 2, 3, &c.

On Regular Difference Terms. By A. B. Kempe, M.A., F.R.S.

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1. Let $\alpha, \beta, \gamma, \ldots$ be a system S_n of n quantities, which may be termed *roots;* and let w differences $\alpha-\beta$, $\alpha-\beta$; $\beta-\gamma$, $\alpha-\gamma$; &c., be formed with these, each root entering into v of the differences. Then the product of these w differences will be called a *regular difference term* of the system S_n, and will be said to be of *degree n, order v,* and *weight w.*

2. The expression

$$(\alpha-\beta)^2(\beta-\gamma)(\gamma-\delta)^2(\delta-\alpha)$$

affords an example of a regular difference term of degree 4, order 3, and weight 6.

3. We may have difference terms into which the different roots do not all enter the same number of times; such difference terms are, however, *irregular.* A difference term will be irregular although each of the roots which enters into it enters the same number of times as the others, provided that there are other roots of the system under consideration which do not enter at all. Such a difference term will,