derived from theories which involve the introduction of an absolute time-constant, such, for example, as a period of free molecular vibration, thus verifying the statement of the last paragraph. The somewhat exact correspondence between Verdet's experiments*, on the relation of magnetic rotation to dispersion, for a substance of simple chemical structure, like bisulphide of carbon, and the law deducible from the first of the three types of equations, is in accord with these considerations.

In the same way the *reversible* rotatory property of quartz, and sugar solutions, requires an additional term in the equation of motion of a wave parallel to z, of one of the types

$$\frac{d^3\theta}{dz\,dt^2} \quad\text{and}\quad \frac{d^3\theta}{dz^3};$$

where $\theta = u + v\sqrt{-1}$. The first of these represents a coefficient of inertia of the type $\rho + \sigma\cdot\frac{d}{dz}$, *i.e.*, one which, in the wave motion considered, has a varying term which is a harmonic function of z. The second connotes a law of elasticity involving differential coefficients of odd order of the displacement. Of these terms the latter is thus the only one that can be derived from an elastic or other statical theory based on a symmetry in the medium; while the former would require some sort of motional structure (*i.e.*, the existence of steady forces arising from theinertia opposing unrecognised steady motions) to justify its adoption.

On the Contacts of Systems of Circles. By ALEXANDER LARMOR, M.A., Fellow of Clare College, Cambridge. Communicated in abstract, November 12th, 1891. Received 29th February, 1892.

In his paper in the *Proceedings of the Royal Irish Academy*, Vol. IX., Part IV., Dr. Casey gave, in a very elegant form, as a relation among the lengths of their common tangents, the condition that four circles should touch the same circle. His discussion of this condition is by no

* Maxwell, *loc. cit.*

means complete, and, in the subsequent portion of the paper, after a proof that the condition is *necessary*, it is assumed to be *sufficient*.

The condition is now well known, and has been incorporated in text-books of geometry, and discussed by various writers; but I am not aware that anything substantial has been added to Casey's original treatment.

I have attempted to give a somewhat exhaustive discussion of this condition, whether we look upon it as a relation among the common tangents of the given circles, or as a relation among their angles of intersection, and have shown, by purely geometrical reasoning, that it is *sufficient* as well as *necessary*—that it is not satisfied by any relation between the circles except this contact relation.

The condition is then applied to the investigation of the contact relations of the eight circles which touch three given circles (due to Casey), and also of the eight circles each of which passes through three of the points of intersection of three given circles.* The latter case is also derived from the former by means of the polar transformation on the sphere.

From these results are derived the contact relations of the thirty-two right circular cones which touch three given covertical right circular cones, and of the thirty-two right circular cones which pass through three of the twelve lines of intersection of three given covertical right circular cones. Casey (*loc. cit.*) has shown that the former system of thirty-two right circular cones can be divided into a certain number of groups of four, each of which touches another right circular cone. By means of a transformation I have shown that a relation subsists for double this number of groups.

From these theorems are then derived, by the theory of projections, the contact relations of the thirty-two conics having double contact with a given conic, and touching three other conics having double contact with this given conic; and also of the thirty-two conics having double contact with a given conic and passing through three of the twelve points of intersection of three given conics having double contact with this one. The former system has been discussed analytically by Casey, but he has only detected half the full number of groups which possess the property under consideration.

Several important particular cases of these general theorems are also noticed.

* Communicated in abstract to Section A of the British Association, 1887.

We begin the discussion with the explanation of the nomenclature adopted, and the statement of some Lemmas which will be required in the course of the paper.

The small circle on a sphere, traced out by that pole of a tangent great circle, to a given small circle, which lies on the opposite side of the great circle, will be termed the polar of the given small circle, and the small circle traced out by the other pole will be termed the antipolar.

Two circles on a sphere will be said to have *internal* contact when they lie on the same side of their tangent great circle at their point of contact, and to have *external* contact when they lie on opposite sides of it.

It is important to remark that the geometry on a sphere is more symmetrical and admits of wider transformation than geometry on a plane. The process of inversion by reciprocal radii vectores on a plane corresponds to simple projection from a point, of the diagram on a sphere, and in this connexion its fundamental character, as regards similarity of small parts in a diagram and its inverse, appears in its true light. We have, in addition, a symmetrical method of polarizing on a sphere, wherein each point is replaced by its polar great circle. But this increased facility of transformation introduces its corresponding difficulties. A great circle has, in fact, two poles, so that any diagram has two polar diagrams, which must be discriminated. It is this circumstance which constitutes the chief obstacle to the general application of the method. It may be remarked that there is an analogous difficulty in the coordinate geometry of right circular cones. Thus, for example, the discussion given in Salmon's *Solid Geometry*, Art. 259 *seq.*, of Sir Andrew Hart's contact theorem relating to circles on a sphere, is not valid. The demonstrations apply to cones, not to circles; each cone meets the sphere in *two* opposite circles, and there is no means given of discriminating between them. The same difficulty occurs throughout in the treatment of small circles on a sphere by means of Cartesian equations.

It will be found that, for a transformation involving several contacts, the discrimination between the polar and antipolar diagrams, although at first sight it seems a matter of great complexity, may be effected by the use of the two following principles:—

(*a*) If two circles have internal contact with each other, their polars have internal contact with each other (and their antipolars have internal contact with each other).

(β) If two circles have external contact with each other, the polar of either has external contact with the antipolar of the other.

The Lemmas which we shall have occasion to use are the following:—

(i.) If any point and its polar plane with regard to a quadric be taken as centre and plane of projection, any plane section of the quadric projects into a conic having double contact with the section of the quadric by the plane of projection. If any other plane be taken as plane of projection, any plane section of the quadric projects into a conic having double contact with the section of the tangent cone by that plane.

When the plane section of the quadric intersects its section by the polar plane in real points, this proposition is evident; and therefore, by the principle of continuity, it holds generally.

(ii.) If two circles, on a sphere or *in plano*, are the inverses of two others with regard to a given point, they will all four touch each of a group of four circles which have a common radical centre situated at that point.

Consider the general case in which the circles are plane sections of a sphere. The two circles and their respective inverses lie on two cones having their common vertex at the given point. These cones have, in general, four common tangent planes, which pass through their common vertex and cut the sphere in four circles touching the two given circles and their inverses.

(iii.) Given three circles 1, 2, 3 and a point P, there is only one other point Q from which the tangents drawn to the three given circles are proportional to those from P, viz., the other point of concourse of the three circles passing through P and coaxal with 1, 2; 2, 3; 3, 1, respectively.

The straight line PQ passes through the radical centre of the circles 1, 2, 3, and it easily follows that P and Q are inverses with regard to the circle which cuts them orthogonally. The eight tangent circles of 1, 2, 3 consist of four inverse pairs with regard to their orthogonal circle.

Hence, if P is on neither circle of a given inverse pair, Q is not on either.

We now proceed, in effect, to discuss, and in some measure to systematize, the very remarkable complex of contacts which bind together the whole system of plane sections of a sphere, or quadric.

surface, that are constructed on the basis of any three plane sections (and their three opposites) by drawing in succession other plane sections themselves determined solely by conditions of contact.

1. If any four circles 1, 2, 3, 4 touch the same circle, a relation of the form

$$12 \,.\, 34 + 14 \,.\, 23 - 13 \,.\, 24 = 0$$

subsists among their common tangents, the common tangents involved being direct or transverse according as the contacts of the corresponding circles with the touched circle are of the same or opposite nature.

Where it is necessary, in the following analysis, to distinguish between direct and transverse common tangents, their lengths will be denoted by symbols of the types 12 and (12) respectively.

Let A, B (Fig. 1) be the centres of two circles, O that of a circle

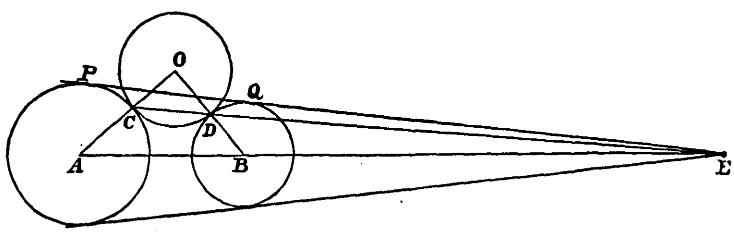

FIG. 1.

touching them both externally or both internally at C, D respectively; PQ a direct common tangent, and E the external centre of similitude.

Let ρ be the radius of the touching circle. Then, since ODB is a transversal drawn across the triangle ACE,

$$\frac{AO}{\rho} = \frac{AB}{BE} \cdot \frac{ED}{DC};$$

and, since OCA is a transversal drawn across the triangle EBD,

$$\frac{BO}{\rho} = \frac{AB}{AE} \cdot \frac{EC}{CD}.$$

Hence $$\frac{AO.BO}{\rho^2} = \frac{EC.ED}{AE.BE} \cdot \frac{AB^2}{CD^2} = \frac{EP.EQ}{AE.BE} \cdot \frac{AB^2}{CD^2} = \frac{t^2}{CD^2};$$

therefore $$\frac{t}{CD} = \frac{\sqrt{AO.BO}}{\rho},$$

where t is the length of the external common tangent PQ.

When the contacts are one external and one internal, if we make use of the corresponding Fig. 2, in which I the internal centre of similitude takes the place of E, we are led to this same relation; where t is now the length of the transverse common tangent.

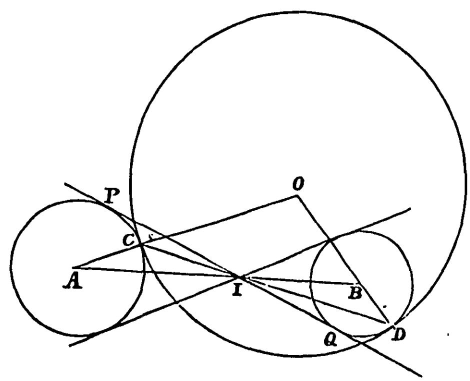

FIG. 2.

Let P_1, P_2, P_3, P_4 be the points of contact of the circles with their common tangent circle (Fig. 3); then, by Ptolemy's theorem,

$$P_1P_2 . P_3P_4 + P_1P_4 . P_2P_3 - P_1P_3 . P_2P_4 = 0.$$

Hence, denoting by 12 the common tangent of the circles 1 and 2,

$$12 . 34 + 14 . 23 - 13 . 24 = 0,$$

where the common tangents are direct or transverse according as the contacts with the touched circle are of the same or of opposite nature.

It is to be observed that the six common tangents involved must be either—

(i.) all direct,

(ii.) three direct and three transverse,

(iii.) two direct and four transverse;

and that in the two latter cases their distribution must be according to the types—

$$(12) \,.\, 34 + (14) \,.\, 23 - (13) \,.\, 24 = 0,$$

and

$$(12) \,.\, (34) + (14) \,.\, (23) - 13 \,.\, 24 = 0,$$

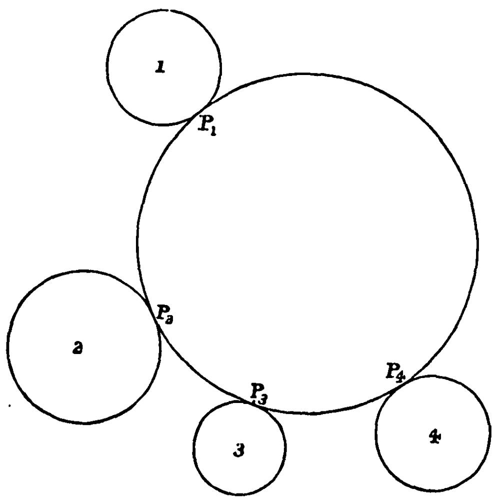

FIG. 3.

respectively, where, as defined above, (12) denotes the transverse common tangent of 1 and 2.

We also remark that the product affected with the negative sign is that of which the corresponding chords of contact intersect without being produced.

2. If the circle 4 reduce to a point, we see that

$$12 \,.\, 34 + 14 \,.\, 23 - 13 \,.\, 24 = 0,$$

$$-12 \,.\, 34 + 14 \,.\, 23 + 13 \,.\, 24 = 0,$$

or

$$12 \,.\, 34 - 14 \,.\, 23 + 13 \,.\, 24 = 0,$$

according as the point 4 is on the arc 13, 12, or 23, respectively, of either of the circles which touch 1, 2, 3 all internally or all externally.

If it be on either circle which has contacts of similar nature with 1 and 3, and of the opposite nature with 2, then (Fig. 4)

$$(12) \,.\, (34) + (14) \,.\, (23) - 13 \,.\, 24 = 0,$$

$$-(12) \,.\, (34) + (14) \,.\, (23) + 13 \,.\, 24 = 0,$$

or

$$(12) \,.\, (34) - (14) \,.\, (23) + 13 \,.\, 24 = 0.$$

If it be on either circle which has contacts of similar nature with 1 and 2, and of the opposite nature with 3, then (Fig. 5)

$$12\,.\,(34)+14\,.\,(23)-(13)\,.\,24=0,$$
$$-12\,.\,(34)-14\,.\,(23)+(13)\,.\,24=0,$$
or
$$12\,.\,(34)-14\,.\,(23)+(13)\,.\,24=0.$$

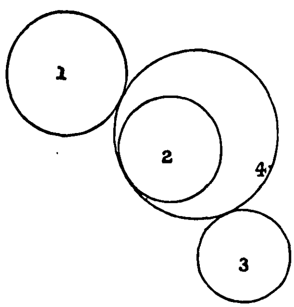

FIG. 4.

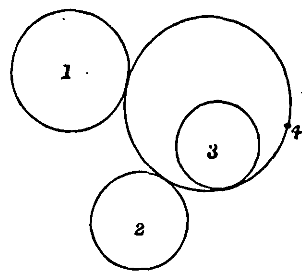

FIG. 5.

If it be on either circle which has contacts of similar nature with 2 and 3, and of the opposite nature with 1, then (Fig. 6)

$$(12)\,.\,34+(14)\,.\,23-(13)\,.\,24=0,$$
$$-(12)\,.\,34+(14)\,.\,23+(13)\,.\,24=0,$$
or
$$(12)\,.\,34-(14)\,.\,23+(13)\,.\,24=0.$$

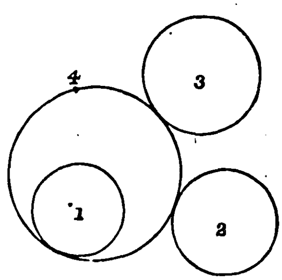

FIG. 6.

These alternatives hold in each case according as the point 4 is on the arc 13, 12, or 23, respectively, it being regarded as a point circle lying on the same side of the tangent circle as the circle 2.

3. Conversely, if any one of the relations which occur in the last

Art. subsist among the mutual common tangents of the circles 1, 2, 3, and the point 4, that point must lie on one or other of the two arcs of the pair of tangent circles of 1, 2, 3 for which that particular relation has here been proved to subsist.

Suppose (Fig. 7) that P is a position of the point 4, *on neither of the arcs* 13 of the pair of tangent circles for which the relation

$$12 . 34 + 14 . 23 - 13 . 24 = 0$$

is satisfied.

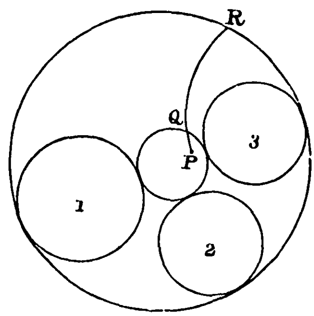

FIG. 7.

Through P describe a circle coaxal with 1 and 3, and let it cut either of these arcs in Q.

Then $\quad 12 . 3P + 23 . 1P - 13 . 2P = 0$, by hypothesis,

and $\quad 12 . 3Q + 23 . 1Q - 13 . 2Q = 0$, by Art. 2;

also $\quad 3P : 1P :: 3Q : 1Q$,

since P and Q are on a circle coaxal with 1 and 3.

Hence $\quad 3P : 2P : 1P :: 3Q : 2Q : 1Q$,

which is impossible, by Lemma iii.

4. If a relation of the form

$$12 \;.\; 34 \pm 14 \;.\; 23 \pm 13 \;.\; 24 = 0,$$

$$(12) \;.\; 34 \pm (14) \;.\; 23 \pm (13) \;.\; 24 = 0,$$

or

$$(12) \;.\; (34) \pm (14) \;.\; (23) \pm 13 \;.\; 24 = 0,$$

subsist among the common tangents of four circles 1, 2, 3, 4, they will have a common tangent circle.

Take that circle, say 4, whose radius is not greater than that of any of the three remaining circles. With the centre of each of the remaining circles as centre describe a circle, whose radius is equal to the sum or difference of its radius and that of the circle 4, according as the common tangent of it and 4 is transverse or direct.

These three new circles 1′, 2′, 3′, together with the centre of the circle 4 (a point circle) form a group of four circles having the same mutual common tangents as the four given circles, so that the given relation is satisfied for this system, and it follows by Art. (3) that the centre of the circle 4 lies on one or other of a pair of common tangent circles of 1′, 2′, 3′, and, hence, that 4 touches one or other of a pair of common tangent circles of 1, 2, 3.*

5. To express the common tangents of a pair of circles in terms of their radii and angle of intersection.

Let t, t' be the lengths of their direct and transverse common tangents,
δ the distance between their centres,
θ their angle of intersection (viz., the one which lies inside both).

Then, for circles *in plano* (Fig. 8),

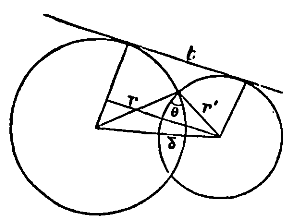

Fig. 8.

$$\delta^2 = r^2+r'^2+2rr'\cos\theta$$
$$= t^2+r^2+r'^2-2rr'$$
$$= t'^2+r^2+r'^2+2rr'.$$

Hence
$$\left.\begin{aligned} t^2 &= 2rr'(1+\cos\theta), \\ t'^2 &= -2rr'(1-\cos\theta), \end{aligned}\right\}$$

or
$$t = 2\sqrt{rr'}\,.\cos\frac{\theta}{2} \quad \text{..................(i.),}$$
$$t' = 2\sqrt{-1}\,.\,\sqrt{rr'}\,.\sin\frac{\theta}{2} \quad \text{..................(ii.).}$$

* It will touch both circles of the pair if, in addition to the given relation, the circles 1, 2, 3, 4 have a common orthogonal circle.

For circles on a sphere (Figs. 9, 10),

$$\cos\delta = \cos r\cos r' - \sin r\sin r'\cos\theta$$

$$= \sin r\sin r' + \cos r\cos r'\cos t$$

from triangle $OO'P$ (Fig. 9)

$$= -\sin r\sin r' + \cos r\cos r'\cos t'$$

from triangle $OO'P'$ (Fig. 10).

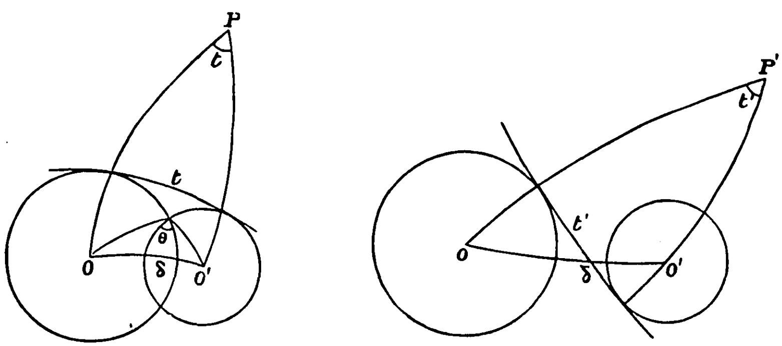

Fig. 9. Fig. 10.

Hence $$\left.\begin{aligned}\cos r\cos r'(1-\cos t) &= \sin r\sin r'(1+\cos\theta),\\ \cos r\cos r'(1-\cos t') &= -\sin r\sin r'(1-\cos\theta),\end{aligned}\right\}$$

or $$\left.\begin{aligned}\sin\frac{t}{2} &= \sqrt{\tan r\tan r'}\cos\frac{\theta}{2} \quad\text{..................(i.),}\\ \sin\frac{t'}{2} &= \sqrt{-1}\tan r\tan r'\sin\frac{\theta}{2} \quad\text{..................(ii.).}\end{aligned}\right\}$$

6. From Arts. (4) and (5), we see that the necessary and sufficient conditions that four circles *in plano* should have a common tangential circle is, when expressed in terms of their angles of intersection,

$$\cos\frac{12}{2}\cos\frac{34}{2} \pm \cos\frac{14}{2}\cos\frac{23}{2} \pm \cos\frac{13}{2}\cos\frac{24}{2} = 0,$$

$$\sin\frac{12}{2}\cos\frac{34}{2} \pm \sin\frac{14}{2}\cos\frac{23}{2} \pm \sin\frac{13}{2}\cos\frac{24}{2} = 0,$$

or $$\sin\frac{12}{2}\sin\frac{34}{2} \pm \sin\frac{14}{2}\sin\frac{23}{2} \pm \cos\frac{13}{2}\cos\frac{24}{2} = 0.$$

By inversion it is clear that this also holds for four circles on a sphere, and, in this latter case, the condition, when expressed in terms of the common tangents of the circles, is

$$\sin\frac{12}{2}\ \sin\frac{34}{2}\ \pm\ \sin\frac{14}{2}\ \sin\frac{23}{2}\ \pm\ \sin\frac{13}{2}\ \sin\frac{24}{2}=0,$$

$$\sin\frac{(12)}{2}\ \sin\frac{34}{2}\ \pm\sin\frac{(14)}{2}\ \sin\frac{23}{2}\ \pm\sin\frac{(13)}{2}\sin\frac{24}{2}=0,$$

or $$\sin\frac{(12)}{2}\sin\frac{(34)}{2}\pm\sin\frac{(14)}{2}\sin\frac{(23)}{2}\pm\sin\ \frac{13}{2}\ \sin\frac{24}{2}=0.$$

These conditions, in the case of four circles on a sphere, may also be established directly, by following a method precisely analogous to that adopted in the case of four circles *in plano*.

[I am indebted to Mr. Lachlan for the following alternative proof of the condition, directly applicable to the case in which the three circles intersect each other in real angles.

Let X_1, X_2, X_3, X_4 be four circles whose *external* angles of intersection are connected by the relation

$$\sin\tfrac{1}{2}\omega_{12}\,.\,\sin\tfrac{1}{2}\omega_{34}+\sin\tfrac{1}{2}\omega_{14}\,.\,\sin\tfrac{1}{2}\omega_{23}-\sin\tfrac{1}{2}\omega_{13}\,.\,\sin\tfrac{1}{2}\omega_{24}=0.$$

Let Y and Y' be two circles which touch X_1, X_2, X_3 in the same sense, *i.e.*, all internally or all externally; then either Y or Y' touches X_4 in the same sense as the others.

Now two circles can be drawn to cut X_2, X_3 at the same angles as X_4 cuts them, and at the same time to touch Y in the same sense as Y touches X_2, X_3. Let these circles be X_5, X_6. It is easy to see that, if P_5, P_6 be the points of contact of these with Y, P_5 and P_6 will lie on opposite arcs of Y which join the points of contact X_2, X_3. Hence, if P_5 be on the same arc as the point of contact of X_1,

$$\sin\tfrac{1}{2}\omega_{12}\,.\,\sin\tfrac{1}{2}\omega_{35}+\sin\tfrac{1}{2}\omega_{15}\,.\,\sin\tfrac{1}{2}\omega_{23}-\sin\tfrac{1}{2}\omega_{13}\,.\,\sin\tfrac{1}{2}\omega_{25}=0,$$

or $$-\sin\tfrac{1}{2}\omega_{12}\,.\,\sin\tfrac{1}{2}\omega_{35}+\sin\tfrac{1}{2}\omega_{15}\,.\,\sin\tfrac{1}{2}\omega_{23}+\sin\tfrac{1}{2}\omega_{13}\,.\,\sin\tfrac{1}{2}\omega_{25}=0,$$

according as P_5 is on the arc joining the points of contact of X_1, X_3, or on that joining the points of contact of X_1, X_2.

Hence, since

$$\sin\tfrac{1}{2}\omega_{25}=\sin\tfrac{1}{2}\omega_{24},\quad\text{and}\quad\sin\tfrac{1}{2}\omega_{35}=\sin\tfrac{1}{2}\omega_{34},$$

we must have $$\sin\tfrac{1}{2}\omega_{15}=\pm\sin\tfrac{1}{2}\omega_{14};$$

the lower sign is impossible, since all the angles are less than two right angles.

In the same manner there will be one circle, X_7 say, which touches Y' in the same sense as Y' touches X_2, X_3, such that

$$\sin\tfrac{1}{2}\omega_{27} = \sin\tfrac{1}{2}\omega_{24}, \quad \text{and} \quad \sin\tfrac{1}{2}\omega_{37} = \sin\tfrac{1}{2}\omega_{34},$$

and therefore $$\sin\tfrac{1}{2}\omega_{17} = \sin\tfrac{1}{2}\omega_{14}.$$

Hence $$\omega_{15} = \omega_{17} = \omega_{14} \quad \text{or} \quad 2\pi - \omega_{14};$$

the latter alternative is impossible, since $2\pi - \omega_{14} > \pi$.

Hence the circles X_5, X_7 cut X_1, X_2, X_3 at the same angles as X_4 cuts them.

Let X be the orthogonal circle of X_1, X_2, X_3, and let X_4' be the inverse of X_4 with respect to X; then it follows that X_5 and X_7 must coincide with X_4 and X_4'. That is to say, one of the pair X_5, X_7 must coincide with X_4.

Hence X_4 touches Y or Y'. That is to say X_1, X_2, X_3, X_4 touch another circle, each in the same sense.*]

7. The eight circles which can be drawn to touch three given circles on a sphere, or *in plano*, can be divided into fourteen groups of four, each of which touches another circle (Hart and Casey).

Let us assume that the three given circles intersect, as in (Fig. 11).

The three given circles divide the sphere into eight triangles. Consider any one of these triangles, and the four circles related to it, as the inscribed and escribed circles of a plane triangle.

Then, since these four circles, 1, 2, 3, 4, say, touch the three given circles as a, b, c, d touch α, β, γ in Fig. 11, the following relations

* It is interesting to observe that when X_4 cuts X_1, X_2, X_3 at equal angles, the condition becomes

$$\sin\tfrac{1}{2}\omega_{12} + \sin\tfrac{1}{2}\omega_{23} - \sin\tfrac{1}{2}\omega_{13} = 0,$$

the interpretation of which is that Y, Y' and X_4 touch each other at the same point. For, by inverting X_1, X_2, X_3 into three equal circles, this condition reduces to the relation

$$t_{12} + t_{23} - t_{13} = 0,$$

among the common tangents, which shows that these three circles touch two parallel straight lines, and it is clear that X_4 inverts into a straight line parallel to them. Hence, in the original figure, Y, Y' and X_4 touch each other at the origin of inversion.—A. L.

subsist among their common tangents, viz.,

$$\sin\frac{14}{2}\sin\frac{(23)}{2}+\sin\frac{24}{2}\sin\frac{(13)}{2}-\sin\frac{12}{2}\sin\frac{(34)}{2}=0,$$

$$\sin\frac{13}{2}\sin\frac{(24)}{2}+\sin\frac{(14)}{2}\sin\frac{23}{2}-\sin\frac{(34)}{2}\sin\frac{12}{2}=0,$$

$$\sin\frac{(12)}{2}\sin\frac{34}{2}+\sin\frac{14}{2}\sin\frac{(23)}{2}-\sin\frac{13}{2}\sin\frac{(24)}{2}=0.$$

Hence $\sin\frac{(14)}{2}\sin\frac{23}{2}+\sin\frac{(12)}{2}\sin\frac{34}{2}-\sin\frac{(13)}{2}\sin\frac{24}{2}=0,$

showing that the circles 1, 2, 3, 4 touch another circle, the contacts with 2, 3, 4 being of similar nature, and the contact with 1 of the opposite nature.

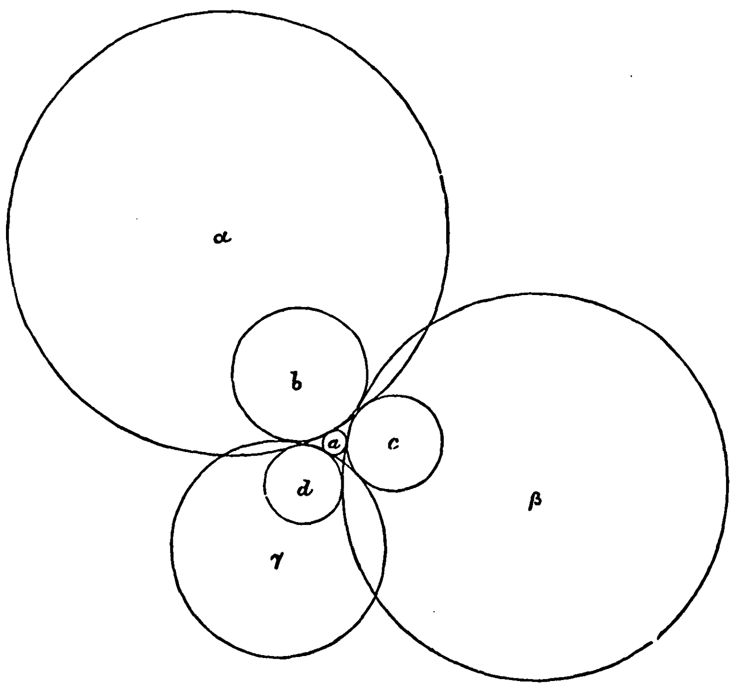

Fig. 11.

As there are eight triangles, there are eight groups of this species. Again, the eight circles which touch three given circles on a sphere clearly consist of 1, 2, 3, 4, and their inverses 1′, 2′, 3′, 4′ with regard to the point of intersection of the planes of the three given circles.

The six groups of four, viz.,

$$11'22',\ 11'33',\ 11'44',\ 22'33',\ 22'44',\ 33'44',$$

consist each of two circles and their inverses with regard to this point, and therefore (by Lemma ii.) are each tangential to four circles, *i.e.*, to another circle in addition to the three given circles.

8. The eight circles which can be drawn through the points of intersection of three given circles, taken three by three, can be divided into eight groups of four, possessing the property that each circle of any group touches *four* other circles.

Let us denote the points of intersection of the three given circles by A, B, C, A', B', C', respectively, and the point of intersection of their planes by O.

The eight circles in question are those circumscribing the triangles

$$\begin{array}{ll} ABC, & A'B'C', \\ ABC', & A'B'C, \\ AB'C, & A'BC', \\ A'BC, & AB'C'. \end{array}$$

Consider the six groups of four:—

$$\begin{array}{cccccc} ABC, & ABC', & AB'C, & AB'C, & AB'C', & ABC, \\ AB'C', & AB'C, & ABC, & AB'C', & ABC', & ABC', \\ A'B'C', & A'B'C, & A'BC', & A'BC', & A'BC, & AB'C', \\ A'BC, & A'BC', & A'B'C', & A'BC, & A'B'C, & A'B'C. \end{array}$$

Each of these groups consists of two circles and their inverses with regard to the point O.

Hence (Lemma ii.) the four constituents of each group touch each of a group of four circles.

Consider the two groups:—

$$\begin{array}{ll} ABC, & A'B'C', \\ AB'C', & A'BC, \\ A'BC', & AB'C, \\ A'B'C, & ABC'. \end{array}$$

The angles of intersection of the circles in the former group are given by the following scheme (as may easily be seen from the figure), in which α, β, γ denote the angles of intersection of the three given circles,

	ABC	*AB'C'*	*A'BC'*	*A'B'C*
ABC		$\beta\sim\gamma$	$\gamma\sim\alpha$	$\alpha\sim\beta$
AB'C'	$\beta\sim\gamma$		$\alpha+\beta-\pi$	$\alpha+\gamma-\pi$
A'BC'	$\gamma\sim\alpha$	$\alpha+\beta-\pi$		$\beta+\gamma-\pi$
A'B'C	$\alpha\sim\beta$	$\alpha+\gamma-\pi$	$\beta+\gamma-\pi$	

and, if we call the circles, respectively, 1, 2, 3, 4, and suppose $\alpha>\beta>\gamma$, it is easily verified that their angles of intersection satisfy the following four relations:—

$$\sin\frac{14}{2}\cos\frac{23}{2}+\sin\frac{12}{2}\cos\frac{34}{2}-\sin\frac{13}{2}\cos\frac{24}{2}=0,$$

$$\sin\frac{12}{2}\cos\frac{34}{2}-\sin\frac{23}{2}\cos\frac{14}{2}+\sin\frac{24}{2}\cos\frac{13}{2}=0,$$

$$-\sin\frac{24}{2}\cos\frac{13}{2}+\sin\frac{14}{2}\cos\frac{23}{2}+\sin\frac{34}{2}\cos\frac{12}{2}=0,$$

$$-\sin\frac{23}{2}\cos\frac{14}{2}+\sin\frac{13}{2}\cos\frac{24}{2}+\sin\frac{34}{2}\cos\frac{12}{2}=0,$$

whence (Art. 6) the circles 1, 2, 3, 4 touch four other circles.

In the same manner, or by inversion with regard to the centre of the sphere, it may be proved that the latter group

A'B'C',
A'BC,
AB'C,
ABC',

touch four other circles.

9. The contact relations of the system of eight circles which pass through the points of intersection of three given circles can also, in the two latter cases, be derived, by the following transformation, from those of the system of eight circles which touch three given circles.

Consider the figure consisting of three great circles on a sphere, and their eight tangential circles. Each great circle has two poles. Let A, B, C be the poles of the three given great circles, respectively, which do not lie on the same side as the inscribed circle of the spherical triangle formed by them, and let A', B', C' be their three remaining poles.

The polar figure corresponding to the three great circles, and their eight tangential circles, consists of three great circles intersecting in A, A', B, B', C, C', respectively, and the eight circles passing through these points three by three.

The inscribed and three escribed circles of the original spherical triangle touch a circle (Sir A. Hart's circle) the nature of the contacts being, in the case we have chosen (Fig. 11), the first internal and the last three external. Hence (Lemmas a, β) the polar of Hart's circle touches the polar of the first internally, and the antipolars of the other three externally, *i.e.*, the four circles passing through the points

$$ABC,$$
$$A'B'C,$$
$$A'BC',$$
$$AB'C'.$$

Now we observe that the original spherical triangle has three co-lunar triangles, and that each of these has a Hart circle of its own; hence, in the polar figure, we see that the four circles passing through the points

$$ABC',$$
$$A'BC,$$
$$AB'C,$$
$$A'B'C',$$

touch the polar of the Hart circle of a co-lunar triangle; and, inverting with regard to the centre of the sphere, that the four circles passing through the points

$$A'B'C,$$
$$AB'C',$$
$$A'BC',$$
$$ABC,$$

touch the antipolar of the Hart circle of a co-lunar triangle.

Thus this system of four circles touches the polar of the Hart circle of the original triangle, and the antipolars of the Hart circles of the three co-lunar triangles.

10. To find the radii of these four circles.

The radius r of the Hart circle of any spherical triangle is connected with the radius R of the circumscribing circle of that triangle by the relation

$$\tan R = 2 \tan r$$

(Salmon, *Solid Geometry*, Art. 261).

Hence the radius ρ of any of the four circles, to which the circles

$$ABC,$$
$$A'B'C,$$
$$A'BC',$$
$$AB'C',$$

are tangential, is given by the relation

$$2 \cot \rho = \cot r_1, \text{ or } \tan \rho = 2 \tan r_1,$$

r_1 being the radius of the inscribed circle of one of the four spherical triangles ABC, $A'B'C$, $A'BC'$, $AB'C'$.

11. The sixty-four circles which can be drawn to touch three given circles 1, 2, 3 on a sphere and their three antipodal circles 1′, 2′, 3′, in such a manner that each touches 1 or 1′, 2 or 2′, and 3 or 3′, can be divided into two hundred and twenty-four groups of four, each of which touches another circle.

Eight circles can be drawn to touch each of the groups

123, 1′2′3′,
12′3′, 1′23,
1′23′, 12′3,
1′2′3, 123′.

The theorem of Art. (7) shows that these sixty-four circles can be divided into $8 \times 14 = 112$ groups of four, each of which touches another circle.

Let us take, for considerations of symmetry in the distribution of the contacts, the case in which no two of the circles 1, 2, 3 intersect.

Let a, b, c, d, a', b', c', d' denote the tangent circles of 1, 2, 3, the nature of their respective contacts with the latter being defined by the scheme

	a	b	c	d	a'	b'	c'	d'
1	*ext.*	*ext.*	*int.*	*int.*	*int.*	*int.*	*ext.*	*ext.*
2	*ext.*	*int.*	*ext.*	*int.*	*int.*	*ext.*	*int.*	*ext.*
3	*ext.*	*int.*	*int.*	*ext.*	*int.*	*ext.*	*ext.*	*int.*

and let

i, ii, iii, iv, v, vi, vii, viii denote their respective polars,

i′, ii′, iii′, iv′, v′, vi′, vii′, viii′, their respective antipolars.

Then a, b, c, d touch a Hart circle, all internally, in the case we have chosen; therefore (Lemmas α, β) i, ii, iii, iv touch the polar of this Hart circle, all internally.

It is also to be observed that i, ii, iii, iv do not form a Hart group in the polar figure.

For, let α, β, γ, α', β', γ' be the respective polars and antipolars of 1, 2, 3; then, by the foregoing scheme, and Lemmas (α, β), the nature of the contacts of i, ii, iii, iv with $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, is according to the scheme

	α	β	γ	α'	β'	γ'
i				*ext.*	*ext.*	*ext.*
ii		*int.*	*int.*	*ext.*		
iii	*int.*		*int.*		*ext.*	
iv	*int.*	*int.*				*ext.*

Thus i, ii, iii, iv touch a circle, although they do not touch any three of the circles $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$.

Again, a, a', b, b' (a Hart group of the second species) touch a Hart circle in addition to 1, 2, 3. The nature of their contacts with 1, 2, 3 is defined by the first scheme; whence the nature of their contacts with their Hart circle is inferred according to the scheme

	a	b	a'	b'
1	*ext.*	*ext.*	*int.*	*int.*
2	*ext.*	*int.*	*int.*	*ext.*
3	*ext.*	*int.*	*int.*	*ext.*
Hart circle	*ext.*	*ext.*	*int.*	*int.*

Therefore the polar of this Hart circle touches i′, ii′ externally, and v, vi internally (Lemmas α, β).

It is further to be observed that i′, ii′, v, vi do not form a Hart group in the polar figure, for the nature of their contacts with $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ is according to the scheme

	α	β	γ	α'	β'	γ'
i′	*ext.*	*ext.*	*ext.*			
ii′	*ext.*				*int.*	*int.*
v	*int.*	*int.*	*int.*			
vi	*int.*				*ext.*	*ext.*

Thus i′, ii′, v, vi touch a circle, although they do not touch any three of the circles $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$.

Hence, corresponding to each of the fourteen Hart groups of the circles touching 1, 2, 3, we have a group of four circles touching α or α', β or β', and γ or γ', which touch a circle, although they are not tangential to the same three circles. Thus, the sixty-four circles which can be drawn to touch 1 or 1′, 2 or 2′, and 3 or 3′, can be divided into 8×14 Hart groups of four, each of which touches another circle; and also into 8×14 *other* groups of four, each of which touches another circle; *i.e.*, in all, into 224 groups of four, each of which touches another circle.

12. By joining the circles in the last Art. to the centre of the sphere by right circular cones, we observe that:—

The thirty-two right circular cones that can be drawn, with a given point as vertex, to touch three given right circular cones with that point as vertex, can be divided into one hundred and twelve groups of four, each of which is tangential to another right circular cone with that point as vertex.

13. In the same manner, from Art. (8), we deduce that:—

The thirty-two right circular cones that can be drawn, with a given point as vertex, to pass through the twelve lines of intersection, taken three by three, of three given right circular cones having that point as vertex, can be divided into thirty-two groups of four, each of which is tangential to four other right circular cones with that point as vertex.

This proposition may also be looked upon as a degenerate case of Art. (12), when the three given right circular cones become straight lines.

14. In the theorem of Art. (7), suppose the three given circles to be three great circles, and we deduce that:—

The four right circular cones that can be drawn, with a given point as vertex, to touch three given planes intersecting in that point, are tangential to four other right circular cones with that point as vertex.

This may likewise be looked upon as a degenerate case of the theorem of Art. (12).

15. In the theorem of Art. (8), suppose the three given circles to be three great circles, and we deduce that:—

The four right circular cones that can be drawn through three given concurrent straight lines touch four other right circular cones whose common vertex is the point of intersection of the three given straight lines.

This theorem may also be regarded as a degenerate case of the theorem of Art. (13).

16. The thirty-two conics that can be drawn having double contact with a given conic, and touching three given conics which have also double contact with this given conic, can be divided into one hundred and twelve groups of four, each of which touches another conic having double contact with the same conic.

In the theorem of Art. (12), project the sphere into a quadric surface; and the right circular cones become cones having their common vertex at the centre of the quadric, and having plane section with it.

Apply Lemma i., taking the centre of the quadric as origin, and any plane as plane of projection; the section of the figure by the plane of projection consists of three given conics, having double contact with the section of the asymptotic cone by the plane of projection, and the thirty-two conics that can be drawn touching them, and having double contact with the section of the asymptotic cone by the plane of projection; and the theorem enunciated follows at once from that of Art. (12).

17. The thirty-two conics which can be drawn having double contact with a given conic, and passing through three of the twelve points of intersection of three given conics which have double contact with this given conic, can be divided into thirty-two groups of four, each of which touches *four* conics having double contact with this given conic.

This follows from the theorem of Art. (13), by the method employed in the last Art.

18. The four conics that can be drawn touching the sides of a given triangle, and having double contact with a given conic, touch *four* conics which have double contact with this given conic.

This follows from the theorem of Art. (14), by the application of the same method.

19. The four conics that can be drawn through three given points, and having double contact with a given conic, touch *four* conics which have double contact with this given conic.

This follows from the theorem of Art. (15), by the application of the same method.

Thursday, January 14th, 1892.

Prof. GREENHILL, F.R.S., President, in the Chair.

Mr. Ralph Holmes, B.A., St. John's College, Cambridge, Mathematical Lecturer at King's College, London, and Mr. E. T. Dixon, Trinity College, Cambridge, were elected members.

On the motion of Mr. Walker, seconded by Mr. Brooksmith, the Treasurer's report was unanimously adopted.

The President drew the attention of the members present to the loss the Society had sustained by the death of Prof. Kronecker, who was elected an honorary member January 14th, 1875.

The following communications were made:—

The Harmonic Functions for the Elliptic Cone: Mr. Hobson.

Some Theorems relating to a System of Coaxal Circles: Mr. R. Lachlan.

Note on Dirichlet's Formula for the Number of Classes of Binary Quadratic Forms for a Complex Determinant: Prof. G. B. Mathews.

Researches in Calculus of Variations (third paper): Mr. E. P. Culverwell.

Impromptu Communications were made by Mr. Elliott and Major MacMahon.

The following presents were received :—

"Beiblätter zu den Annalen der Physik und Chemie," Vol. xv., No. 11; 1891.

"Atti del Reale Instituto Veneto di Scienze, Lettere, ed Arti," Vol. xxxviii., 7th series, Vol. i., Pt. 10; Vol. ii., Pts. 1–9.

"Physical Society of London—Proceedings," Vol. xi., Pt. 2, Dec., 1891.

"Nautical Almanac" for 1895.

"Proceedings of Royal Society," Vol. l., No. 303.

"Memoir on the Coefficients of Numbers," by B. Seal, M.A.; Calcutta, 1891.

"Bulletin de la Société Mathématique de France," Vol. xix., No. 7.

"Rendiconti del Circolo Matematico di Palermo," Vol. v., Pt. 6, 1891, Nov.–Dec.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Vol. xxv., Pts. 3, 4; Harlem, 1891.

"Atti della Reale Accademia dei Lincei," Vol. vii., 9–11.

"Educational Times," January, 1892.

"Annals of Mathematics," Vol. vi., No. 3.

"Journal of College of Science, Japan," Vol. iv., Pt. 2; Tokyo, 1891.

"Engineering Review," Vol. i., No. 8, Dec. 5th, 1891.

"Acta Mathematica," xv., 3, 4.

"Transactions of Royal Irish Academy," Vol. xxix., Pt. 17.

"Cayley, Mathematical Papers," Vol. iv. (2 copies).

"Atti della R. Accademia di Napoli," 2nd Series, Vol. iv.; Napoli, 1891.

"Washington Observations, made during the year 1886, at the U. S. Naval Observatory;" Washington, 1891.

"Die Willkürlichen Functionen in der Mathematischen Physik—Inaugural-Dissertation zur Erlangung der Doctorwürde, zu Königsberg i. Pr." (24 Okt., 1891), von Arnold Sommerfeld; Königsberg in Pr., 1891.

"The Mathematical Magazine: a Journal of Elementary and Higher Mathematics," edited by A. Martin, Ph.D., Vol. ii., No. 5, Oct., 1891; Washington, 1891.

"A Direct and General Method of Finding the Approximate Values of the Real Roots of Numerical Equations to any Degree of Accuracy," by J. W. Nicholson, A.M.; New Orleans, 1891.

"The Life-Romance of an Algebraist," by George Winslow Pierce; Boston, 1891. From the Author.