Yoke-Chains and Multipartite Compositions in connexion with the Analytical Forms called "Trees."

By Major P. A. MacMahon, R.A., F.R.S.

[*Read April 9th*, 1891.]

A yoke-chain, an expression adopted at the suggestion of Professor Cayley, is a geometrical configuration composed of line branches.

The simple line branch —— is indifferently a yoke or a chain.

The combinations  &c. are yokes;

whilst &c. are chains;

and generally we form chains by combining chain-wise any number of yokes, and also generally we form yokes by combining yoke-wise any number of chains.

E.g.,  is a chain,

 is a yoke.

Yoke-chains may be viewed as diagrammatic representations of the combinations of resistances of linear electrical conductors, or of the capacities of electrical condensers. The yoke and the chain represent parallel and series combinations of resistances, but series and parallel combinations of capacities.

The theory of yoke-chains does not include every combination of resistances, but only those which are made up of two or more combinations, either in parallel or in series. In the case of five resistances, for example, we find a combination such as the Wheatstone net, which is not decomposable into other combinations either in parallel or in series. Such networks, considered by Kirchhoff and Maxwell, do not come into view here.

Consider in the first place the different yoke-chains that can be formed from a given number of branches.

From a single branch we can merely form the yoke or chain.

From two branches we form the chain of Fig. 1 or the yoke of Fig. 2,

Fig. 1.

Fig. 2.

I define the yoke of Fig. 2 to be the yoke-chain conjugate to the chain of Fig. 1, and I further regard the yoke or chain as being self-conjugate.

Every chain is a chain of yokes, and every yoke is a yoke of chains.

I make the following definitions :—

Definition.—The conjugate of any chain is formed by placing the conjugates of the component yokes in a yoke.

Definition.—The conjugate of any yoke is formed by placing the conjugates of the component chains in a chain.

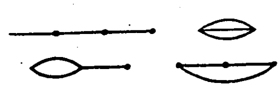
The process of conjugation is necessarily reversible, as is obvious from the definitions.

The second of the above definitions is derived from the first by interchanging the words chain and yoke.

We have thus the notion of conjugate yoke-chains, and it is clear that the yoke-chains of a given order (*i.e.*, of a given number of branches) may be arranged in conjugate pairs, each pair comprising a chain and a yoke.

Further, the whole of the yoke-chains may be arranged in a chain set and a yoke set, and either set is derivable by conjugation from the other.

Passing now to the case of three branches, we form the chain set by placing each of the forms of order 2 in chain with the form of order 1. We thus obtain the left-hand column below :

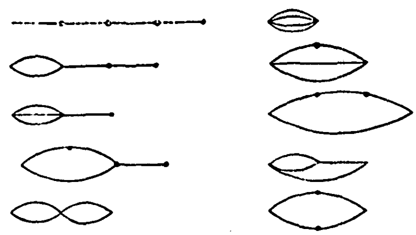


To form the right-hand column, we first place the three self-conjugate components of the form in yoke, and then place the conjugates of the components  of the form

viz., and , in yoke.

We thus obtain the left-hand chain set, and the right-hand yoke set.

In forming the forms of order 4, we place each of the four forms of order 3 in chain with the form of order 1, and further place the yoke form of order 2 in chain with itself. We have thus $5\,(=4+1)$ forms, constituting the chain-column of order 4. These are given below, and also the conjugate yoke-column.



To form the chain-column of order 5, each of the ten forms of order 4 is placed in chain with the form of order 1, and further each of the yoke forms of order 3 is placed in chain with the yoke form of order 2. There are thus, in all, $2\,(10+2) = 24$ forms. In general, the forms of order n are formed from the forms of lower orders. To form the chain-column, as many processes must be performed as there are non-unitary partitions of n, or as there are partitions of n composed of the integers

$$2,\ 3,\ 4,\ \ldots\ n.$$

We first place each form of order $n-1$ in chain with the form of order 1. This is the first process, and thereafter we have a process associated with every non-unitary partition of n with the exception of the partition consisting merely of the number n itself.

For the partition $(n-2,\ 2)$ we take every yoke form of order $n-2$ in chain with the yoke form of order 2. For the partition $(n-p,\ p)$, $p>1$, we take every yoke form of order $n-p$ in chain with every yoke form of order p.

Denote by B_p the number of chain or of yoke forms of order p. Then, if $p=1$, the complete number of forms is $B_1 = 1$; but if $p>1$ the complete number of forms is $2B_p$.

In forming synthetically the number B_n we have above found a portion $2B_{n-1}$, and corresponding to the partition $(n-p,\ p)$, $p>1$, a portion $B_{n-p}\,B_p$, for this represents the number of ways of combining one of $n-p$ things with one of p things of a different sort.

If, however, $n-p=p$, the corresponding portion is not B_p^2, but $\dfrac{B_p\,(B_p+1)}{2}$, viz., the number of homogeneous products of B_p things, two and two together. So also, corresponding to the partition $(l^\lambda m^\mu \ldots)$, we have the number

$$\frac{B_l(B_l+1)\ldots(B_l+\lambda-1)}{\lambda!}\cdot\frac{B_m(B_m+1)\ldots(B_m+\mu-1)}{\mu!}\ldots,$$

and finally we have the relation

$$B_n = 2B_{n-1}+B_{n-2}B_2+B_{n-3}B_3+\ldots$$
$$\ldots+\Sigma\,\frac{B_l(B_l+1)\ldots(B_l+\lambda-1)}{\lambda!}\cdot\frac{B_m(B_m+1)\ldots(B_m+\mu-1)}{\mu!}\ldots,$$

the numbers l, m, &c., ... being >1.

This relation leads at once to the expression of the law of the numbers B in the form

$$(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots$$
$$=1+(B_1-1)x+(2B_2-B_1)x^2+2(B_3-B_2)x^3+\ldots+2(B_p-B_{p-1})x^p+\ldots,$$

or, multiplying up by $1-x$,

$$(1-x)^{-B_1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots$$
$$=1+B_1x+2\,(B_2x^2+B_3x^3+B_4x^4+\ldots).$$

This formula, wherein $B_1 = B_2 = 1$, is convenient for calculating the numbers $2B_p$.

We find

B_1	$2B_2$	$2B_3$	$2B_4$	$2B_5$	$2B_6$	$2B_7$	$2B_8$	$2B_9$	$2B_{10}$ ...
1	2	4	10	24	66	180	522	1532	4984 ...

There is a paper by Cayley in the *Philosophical Magazine*, Vol. XIII. (1857), also *Collected Papers*, Vol. III., No. 203, "On the Theory of the Analytical Forms called Trees." The following passage occurs (*Collected Papers, loc. cit.*, pp. 245, 246)* :—

* Other references are—

"On the Analytical Forms called Trees," Cayley, *Phil. Mag.*, Vol. XX. (1860), pp. 337–341.

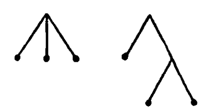
"On the Mathematical Theory of Isomers," Cayley, *Phil. Mag.*, Vol. XLVII. (1874), p. 444.

"On the Analytical Forms called Trees, with Application to the Theory of Chemical Combinations," Cayley, *British Association Report*, 1875, pp. 257–305.

"On the Analytical Forms called Trees," Cayley, *American Journal of Mathematics*, Vol. IV., pp. 266–268.

In the latter paper the notion of the centre or bicentre of number is stated to be due to M. Camille Jordan, but I cannot find the reference.

"I have had occasion for another purpose to consider the question of finding the number of trees with a given number of free branches, bifurcations at least. Thus, when the number of free branches is three, the trees of the form in question are those in the annexed figure, and the number is therefore two. It is not difficult to see that



we have in this case (B_r being the number of such trees with r free branches)

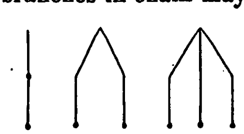
$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots$$
$$= 1+x+2B_2x^2+2B_3x^3+2B_4x^4+\&c."$$

In view of this interesting identity of enumeration, though in the absence of information in regard to the purpose for which Cayley investigated the subject, I propose to examine in detail the correspondence between yoke-chains and the trees with free branches.

A single tree represents either member of a certain conjugate pair of yoke-chains, according to the interpretation placed upon the combination of knots and branches which constitutes the tree.

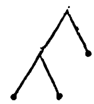
In the first place, we may restrict attention to the chain combinations of the several orders.

One, two, three, &c. branches in chain may be denoted by the trees



and this representation would be in some respects the most consistent with what follows. The idea is that a branch

denotes a chain of order 1, and the upper knots are then joined by branches to a single knot to denote that the several chains are to be joined in chain. The trees, however, become simplified if we agree to represent a single linear resistance, not by a branch but by a terminal knot. The three trees above then become

Consider next the tree 

which may be taken to denote the resistances

placed in series. In order that this tree may represent the combination

,

it is necessary to suppose that the tree

represents not the combination but rather its conjugate yoke form

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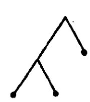
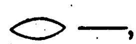
The ultimate step in interpreting a tree will be to take a number of forms in chain. The trees which represent these forms will be found pendent to the second row of knots in the tree. We must therefore agree always to interpret the trees which originate from the second row of knots as yoke combinations. On this convention the tree



denotes the chain combination of the yoke combinations denoted by the trees

viz., the yokes , placed in chain, or

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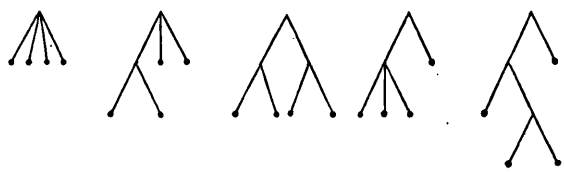
The above principle enables us to form in succession the free-branch trees of the various orders.

The five trees of order 4 are

I. II. III. IV. V.

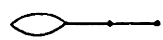


Each corresponds to some partition of four other than the number 4 itself.

The tree I. [partition (1^4)] clearly denotes

 ;

the tree II., partition (21^2), denotes

 ;

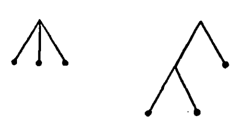
the tree III., partition (2^2), gives

 ;

trees IV. and V. both belong to the partition (31); this follows from the fact that there are two trees of the order 3, and either may be placed as a pendent to a knot of the tree

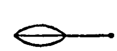
in order to form a tree of order 4.

According to the rule, we must give parallel interpretations to the trees



prior to taking them in series, in each case, with the tree denoted by a single knot.

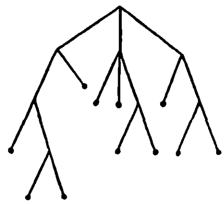
We have thus—

Tree IV. ≡ 

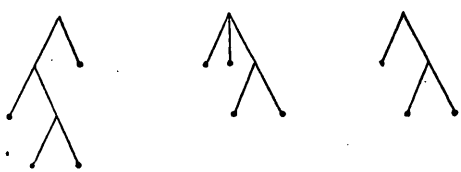
Tree V. ≡ 

and the five trees have been placed in correspondence with the five series combinations of order 4.

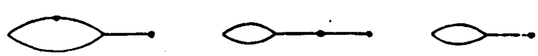
As another example I take the tree



The series equivalences of the trees

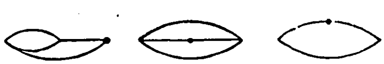


are



To form the complete combination we have merely to take the conjugates of these in series.

The conjugates are



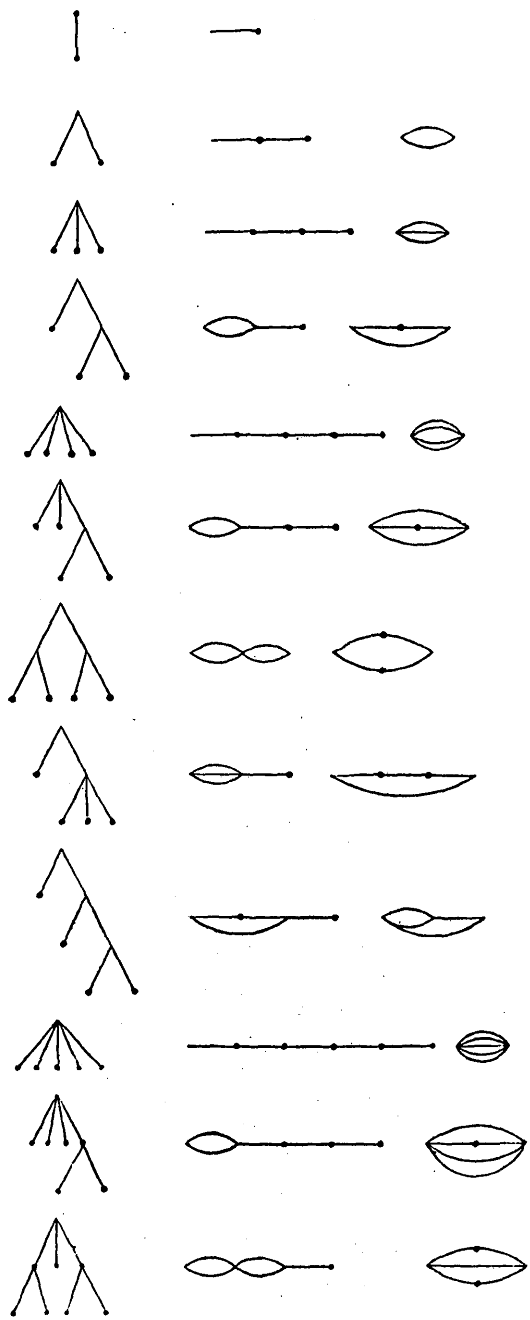
so that the tree denotes the combination

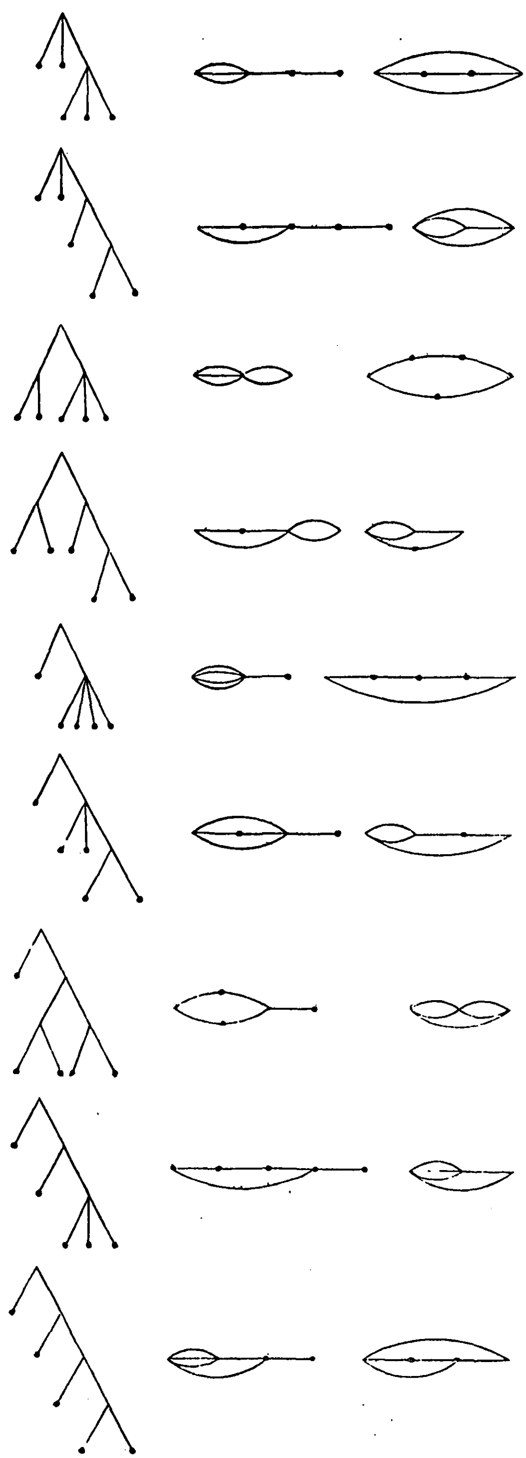
In this way any tree may be interpreted so as to denote a chain.

If in the foregoing rules the words chain and yoke be interchanged, the result will be the conjugate yoke combination.

A tree therefore may be taken to be a representation of either combination of a conjugate pair at pleasure, or if we please of both combinations of such a pair.

I give below a Table showing the correspondence of trees with yoke-chains of the first six orders.





The capacity of condensers placed in parallel is formed according to the same law as the resistance of linear conductors placed in series.

In general the capacity of any combination of condensers is formed according to the same law as the resistance of the conjugate combination of linear conductors.

There is a very simple result connected with the conjugate combinations of equal linear conductors which may possibly have escaped notice.

If any combination of linear conductors, each of r ohms resistance, be formed so that the combined resistance is

$$\frac{m}{n}\, r \text{ ohms};$$

then the resistance of the conjugate combination is

$$\frac{n}{m}\, r \text{ ohms}.$$

The inductive proof is easy, for suppose two series combinations to have resistances

$$\frac{a}{b}\, r \text{ and } \frac{a'}{b'}\, r \text{ ohms},$$

and to be such that the resistances of the conjugate combinations are

$$\frac{b}{a}\, r \text{ and } \frac{b'}{a'}\, r \text{ ohms}.$$

Placing the two combinations in parallel, the resistance is

$$\frac{\frac{aa'}{bb'}}{\frac{a}{b}+\frac{a'}{b'}}\, r \text{ ohms},$$

which is

$$\frac{1}{\frac{b}{a}+\frac{b'}{a'}}\, r \text{ ohms},$$

and the reciprocal of the multiplier of r is

$$\frac{b}{a}+\frac{b'}{a'},$$

which, multiplied into r, represents the resistance of the combination formed by placing the conjugates of the component combinations in series.

Hence, since the law evidently holds in the simplest cases, it must be true in general.

2. In the *Philosophical Magazine*, 1860, Professor Cayley has enumerated the trees with a given number of terminal knots. He remarks :—

" We have here

$$\phi m = 1.2.3\ldots(m-1) \text{ coefficient } x^{m-1} \text{ in } \frac{1}{2-\exp x},$$

giving the values

$$\phi m = 1, 1, 3, 13, 75, 541, 4683, 47293, \ldots,$$

for $$m = 1, 2, 3, 4, 5, 6, 7, 8, \ldots."$$

This enumeration is identical with that of the compositions of certain multipartite numbers.*

The correspondence is between the trees with m terminal knots, and the compositions of the multipartite number

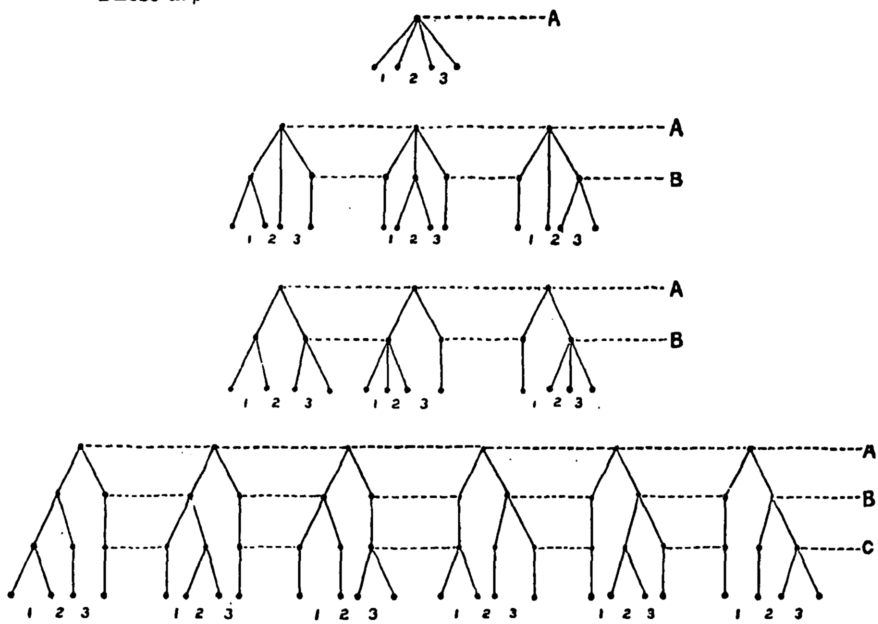
$$\overline{1^{m-1}},$$

where 1^{m-1} denotes 1 repeated $m-1$ times, and the bar — means that the partition does not denote the partition (1^{m-1}) of a single number $m-1$, but rather the multipartite number having the multipartite weight 1, 1, 1, 1, ..., $m-1$ times.

To identify each tree with a composition, consider, for example, the 13 trees with 4 terminal knots.

* H. J. S. Smith and J. W. L. Glaisher have termed partitions in which the order of the parts is essential " compositions." For multipartite numbers see the author, " Memoir on Symmetric Functions of the Roots of Systems of Equations," *Phil. Trans. R. S. of London*, Vol. CLXXXI. (1890), A., pp. 481–536; and for compositions of multipartite numbers, the author, the *Messenger of Mathematics*, Vol. XX. (1890).

These are



We have to identify each of these trees with a composition of the tripartite number $\overline{111}$. The number being tripartite, whilst the terminal knots are four in number, the idea is presented of considering not the terminal knots, but the spaces between them, which are but three in number. These spaces are numbered 1, 2, 3 from left to right in each tree. Consider, moreover, the knots in the first, second and third rows, omitting the terminal row of knots. These are marked A, B, C; in each row we have knots from which descend two or more branches. Of these branches any two that are adjacent bound an area which is in direct communication with one of the numbered spaces between the terminal knots. In any row of knots we must observe the connexion between each angle, formed by the

descent of two adjacent branches from a knot, and the terminal spaces.

In the second of the last line of trees, beginning with row A, we see an angle in communication with space 3, but not with spaces 1 or 2; this connexion between angles and spaces may be denoted by

$$\overline{001};$$

similarly for row B, there is one angle in connexion with space 1, so that we have

$$\overline{100};$$

in row C, one angle leads to space 2, so that we have

$$\overline{010};$$

the whole connexion between angles and spaces is represented by the composition

$$(\overline{001},\ \overline{100},\ \overline{010})$$

of the multipartite number $\overline{111}$.

Interpreting each tree in succession, we get the whole of the 13 compositions of $\overline{111}$, viz.:—in order

$$(\overline{111}),$$

$$(\overline{011},\ \overline{100}),\quad (\overline{101},\ \overline{010}),\quad (\overline{110},\ \overline{001}),$$

$$(\overline{010},\ \overline{101}),\quad (\overline{001},\ \overline{110}),\quad (\overline{100},\ \overline{011}),$$

$$(\overline{001},\ \overline{010},\ \overline{100}),\quad (\overline{001},\ \overline{100},\ \overline{010}),\quad (\overline{010},\ \overline{100},\ \overline{001}),$$

$$(\overline{010},\ \overline{001},\ \overline{100}),\quad (\overline{100},\ \overline{001},\ \overline{010}),\quad (\overline{100},\ \overline{010},\ \overline{001}).$$

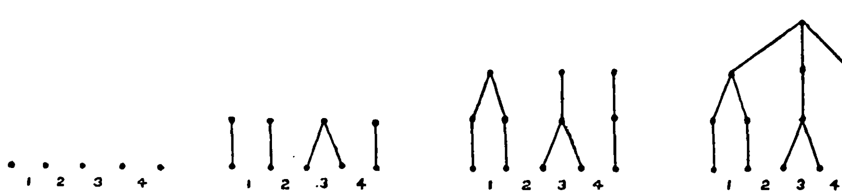
This principle of interpretation is perfectly general. The number of angles in a tree must be less by one than the number of terminal knots, and each row from the first (or top) to the penultimate must contain at least one angle.

If any composition of a multipartite number of the form which presents itself in this theory be given, it is extremely easy to form the corresponding tree. To form the tree, it is best to commence with the right-hand part of the composition, and to then proceed regularly towards the left.

Suppose given the composition

$$(\overline{0101},\ \overline{1000},\ \overline{0010}),$$

the successive operations are



This correspondence is valuable, as putting us in possession of a graphical method through which the compositions may be studied.

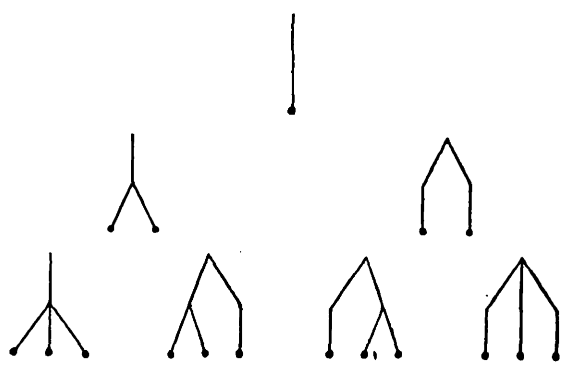
3. The compositions of the multipartite numbers

$$\overline{\text{I}},\ \overline{\text{II}},\ \overline{\text{III}},\ \overline{\text{IIII}},\ \&\text{c.},$$

into a definite number of parts, zero parts not excluded, may be graphically represented by trees with a definite number of terminal knots, and also a definite altitude (or number of rows of knots).

The trees of altitude 1 are trivial, as merely denoting the succession of integer numbers 1, 2, 3, &c.

Passing to those of altitude 2, we have, for 1, 2, 3, &c., terminal knots,



&c.;

and in the first place it is seen that these trees may be considered to be graphical representations of the compositions of the integer numbers 1, 2, 3, &c.; the last, for example, denoting the compositions of 3,

viz., 3, 21, 12, 111.

Consequently the number of trees of altitude 2 with m terminal knots is 2^{m-1}.

We may also interpret these trees as before by paying attention to the inter-terminal knot spaces.

We thus obtain the compositions

$$(01),\ (10),$$

$$(\overline{00}\ \overline{11}),\ (\overline{01}\ \overline{10}),\ (\overline{10}\ \overline{01}),\ (\overline{11}\ \overline{00}),$$

&c.,

in which we now have the compositions of the multipartites

$$1,\quad \overline{11},\quad \overline{111},\quad \&c.,$$

into two parts, zero not excluded.

In particular we have just obtained a graphical proof of the theorem:—

"The whole number of compositions of the number $m+1$ is equal to the number of compositions of the multipartite $\overline{1^m}$ into two parts, zero parts not excluded."

In general, the trees of altitude n and having m terminal knots are representations of the compositions of the multipartite

$$\overline{1^{m-1}},$$

into n parts, zeros not excluded.

It is to be shown that the trees of altitude n and having m terminal knots are

$$n^{m-1}$$

in number.

All the trees of a given altitude n are derivable from the trees of the next lower altitude.

We may start with a tree of altitude 1, having p terminal knots, $p \ngtr m$, and append any p trees of altitude $n-1$ of which the sum of the terminal knots is m. We must give p all integer values from 1 to m in succession, and append the appropriate trees.

Let then $T_{m,n}$ denote the number of trees in question.

It is easy to see that the above considerations lead to the relation

$$T_{m,n} = \Sigma \frac{(\pi_1+\pi_2+\pi_3+\ldots)!}{\pi_1!\ \pi_2!\ \pi_3!\ \ldots} T^{\pi_1}_{p_1,n-1}\ T^{\pi_2}_{p_2,n-1}\ T^{\pi_3}_{p_3,n-1}\ \ldots,$$

the summation being for all partitions

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$$

of the number m.

This relation is expressed algebraically by the formula

$$T_{1,n}x+T_{2,n}x^2+T_{3,n}x^3+\ldots = \frac{T_{1,n-1}x+T_{2,n-1}x^2+T_{3,n-1}x^3+\ldots}{1-(T_{1,n-1}x+T_{2,n-1}x^2+T_{3,n-1}x^3+\ldots)},$$

and since obviously $\qquad T_{m,1} = 1,$

we reach without difficulty the formula

$$T_{1,n}x+T_{2,n}x^2+T_{3,n}x^3+\ldots = \frac{x}{1-nx}.$$

Hence, as above stated, $T_{m,n} = n^{m-1}$.

We have established the theorem :—

"The number of compositions of the multipartite number

$$\overline{1^{m-1}}$$

into n parts, zeros not excluded, is

$$n^{m-1},$$

and there is a one-to-one correspondence between these compositions and the trees of altitude n which have m terminal knots."

On Functions determined from their Discontinuities and a certain form of Boundary Condition. By Mr. W. BURNSIDE.

[*Read May 14th,* 1891.]

The problem of uniform streaming in two dimensions (of incompressible fluid or electricity) may be stated in the following form: to determine a single-valued function of $x+iy$ which in a given region of the x, y plane shall have a single given infinity and whose imaginary part shall be constant over the boundary of the region.

It is the object of the following paper to prove certain general theorems with respect to functions of a complex variable which are determined by conditions similar to but more general than these, and to show how to construct the functions themselves in certain simple cases.

I suppose that m closed non-intersecting curves are given, $C_1, C_2 \ldots C_m$, no one of which separates any two others. The region bounded by these curves is called R; and I consider a function w in the region R which shall be everywhere one-valued and continuous, except at n given points $z_1, z_2 \ldots z_n$, at each of which it has a given simple infinity (so that $w-\frac{a_r}{z-z_r}$ is finite when $z=z_r$), and which at the boundaries shall be such that the imaginary part of $we^{i\theta_r}$ is constant over C_r, where $\theta_1, \theta_2 \dots \theta_m$ are given angles.

Assuming the function w to exist, I prove that it is completely determined, except as regards an additive constant, by the given conditions, and that in R it takes every value n times. It follows at once that if $w_1, w_2 \dots w_n$ are functions satisfying the same boundary conditions and each with a single infinity, then

$$w = w_1 + w_2 + \dots + w_n.$$

I then go on to show how to form the functions when the boundaries are circles.

First consider the nature of the condition at the boundaries. If

$$w = u + iv,$$

then, at C_r, $u \sin\theta_r + v\cos\theta_r$ is constant. Hence

$$\frac{\partial u}{\partial s}\sin\theta_r + \frac{\partial v}{\partial s}\cos\theta_r = 0,$$

if $\dfrac{\partial}{\partial s}$ denote differentiation along the arc of C_r. But $\dfrac{\partial}{\partial n}$ denoting differentiation along the normal to C_r,

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n} \text{ and } \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s};$$

and if at any point on C_r $\dfrac{\partial u}{\partial s}$ vanishes then also $\dfrac{\partial u}{\partial n}$ vanishes, and the point is a double point on the u-curve passing through it.

Now u and v cannot both be constant along C_r without w being everywhere constant; hence there must be an even number of points on C_r at which u and v have (as regards displacements on C_r) maximum or minimum values, and at which the curves u = constant and v = constant have double points. Suppose now that there are two different functions w' and w'' satisfying the given conditions in and at the boundaries of R; their difference $w'-w''$ $(= w)$ will be a function everywhere one-valued and continuous within R, and such that

$$u \sin\theta_r + v\cos\theta_r$$

is constant over C_r, where the suffix may have any value from 1 to m.

Suppose A and B are two points on C_1, at which

$$u = u_0.$$

The curve $$u = u_0,$$

which may, without loss of generality in the reasoning, be supposed not to extend to infinity, will consist of a branch starting from A and abutting on some other part of the boundary C_i, say at A_i; leaving C_i at another point B_i, and abutting on C_j; and so on, at last returning to B. Between A and B on C_1 is a point at which the u-curve, say

$$u = u_1,$$

has a double point. The curve

$$u = u_1$$

will either not meet C_i, or abut on it at two points between A_i and B_i, and a u-curve drawn for some value which u takes twice between A_i and B_i, will certainly abut on one bounding curve less than

$$u = u_0.$$

This process may be continued, and at last there must be a curve

$$u = u_r,$$

which abuts only on one bounding curve C_s. But between the two points on C_s where

$$u = u_r$$

meets it, there is one at which the u-curve has a double point. This u-curve then must either cut

$$u = u_r,$$

which is contrary to the supposition that w is one-valued, or it must be closed, which is contrary to the supposition that w is everywhere finite. Hence the supposition that $w'-w''$ is not constant leads to a contradiction. The function then, if it exists, is completely determined, except as regards an additive constant, by the given conditions.

For sufficiently great values of u_0 and v_0, the curves

$$u = u_0 \quad \text{and} \quad v = v_0$$

consist each of n small closed approximately circular curves passing through $z_1, z_2 \ldots z_n$ in pairs, and each pair intersecting in one other point.

Hence, when u_0 and v_0 are great enough, w takes the value u_0+iv_0 n times. Now following the reasoning that Prof. F. Klein applies to the similar problem for a one-valued function on a Riemann's surface ("Ueber Riemann's Theorie der Algebraischen Functionen," p. 46), the curves $u = u_0, \quad v = v_0,$

can by continuous variation of u_0 and v_0 only lose or gain an intersection at a double point on the u- and v- curves or at the boundary. But as u_0 and v_0 change continuously through values corresponding to a double-point, a pair of points of intersection approach, coincide, and then separate into two again, so that no point of intersection is gained or lost in this way. Also, if

$$u = u_0$$

abuts on a boundary at A and B, the values of v at A and B are the same (from the boundary condition); and, since between A and B there must be a double point, the values of v on the curve

$$u = u_0,$$

as it approaches A and leaves B, must be either continually increasing or continually decreasing. Hence, as regards intersections, there is no difference between a curve that meets the boundary and one that does not. It follows therefore that the number of points of intersection is always the same as when u_0 and v_0 are very great; or in other words, w takes every value n times in the region R.

If $w_1, w_2 \dots w_n$ are n functions, each with a single infinity in the region R, and all satisfying the same boundary conditions as before, then $w_1+w_2+\dots+w_n$ is a single-valued function in R with n given infinities and satisfying the same boundary conditions; but it has just been seen that there is only one such function, and therefore any function of the kind considered can be formed by combining linearly functions with a single infinity. The analogy between the functions w_1, w_2, &c., in the region R, and linear functions of z in the infinite plane is obvious.

If w_1 is the function with a single infinity at z_1, the equation

$$w_1 = f_1(z),$$

establishes the conformable representation of the region R on the infinite w-plane, the bounding curves S_1, S_2, &c., corresponding to the two sides of finite portions of the lines

$$u \sin\theta_1 + v\cos\theta_1 = c_1,$$

$$u \sin\theta_2 + v\cos\theta_2 = c_2,$$

&c.,

where c_1, c_2, &c. are constants depending on the bounding curves and on z_1; and similarly

$$w = f(z)$$

establishes the conformable representation of R on an n-sheeted Riemann's surface, in which m finite lengths of certain straight lines are to be regarded as boundaries.

Now if in R a closed curve be drawn that does not surround one of the curves C, it can be reduced to a point by continuous deformation without leaving R. Hence the same must be true of the above Riemann's surface; or, in other words, the surface itself, leaving the finite bounding lines out of account, must be simply connected. Since it has n sheets, there must therefore be $2(n-1)$ branch-points; and therefore within the region R (*i.e.*, excluding the boundary), there must be $2n-2$ double points on the u- and v-curves.

Now suppose that, on S_r, u, and therefore v, has p_r maxima and p_r minima. Then I have shown, in a paper in the *Messenger of Mathematics* (Vol. xx., p. 66), that the whole number of double-points of the u- or the v-curves in R, including those at the boundaries, is

$$2n+\Sigma p_r+m-2.$$

But it has been shown here that in this case there are $2n-2$ double points inside R and $2\Sigma p_r$ on the boundaries.

Therefore $$\Sigma p_r = m;$$

but p_r cannot be less than unity, and hence

$$p_1 = \ldots = p_r = \ldots = p_n = 1;$$

or u and v will each have one maximum and one minimum at each boundary.

I go on now to consider the actual formation of the functions when the boundaries are circles.

With a single circle the solution is simple and well-known; but it will be convenient to express the result directly in this case for the sake of explaining the notation to be used in the further cases.

The conjugate imaginary of z will always be represented by z', the modulus of z by $|z|$, and the radii and centres of the various circles by r_1, z_1; r_2, z_2, &c. For a single circle the function w must satisfy the following conditions:—

$$w-\frac{A}{z-a}$$

must be finite for all points on the same side of the circle

$$|z-z_1| = r_1$$

as a, and the imaginary part of $we^{i\theta_1}$ must be constant at the circle.

Now, it is easy to verify that the imaginary part of

$$\frac{Ae^{i\theta_1}}{z-a} - \frac{A'e^{-i\theta_1}}{a'-z_1'-\dfrac{r_1^2}{z-z_1}}$$

vanishes at the circle

$$|z-z_1| = r_1,$$

for, writing

$$z-z_1 = r_1 e^{i\theta},$$

the expression becomes

$$\frac{Ae^{i\theta_1}}{r_1e^{i\theta}+z_1-a} + \frac{A'e^{-i\theta_1}}{r_1e^{-i\theta}+z_1'-a'},$$

the sum of two conjugate imaginaries, which is real.

Hence
$$w = \frac{A}{z-a} - \frac{A'e^{-2i\theta_1}}{a'-z_1'-\dfrac{r_1^2}{z-z_1}} + \text{constant}.$$

But
$$\frac{1}{a'-z_1'-\dfrac{r_1^2}{z-z_1}} = \left(\frac{r_1}{a'-z_1'}\right)^2 \frac{1}{z-z_1-\dfrac{r_1^2}{a'-z_1'}} + \frac{1}{a'-z_1'},$$

and $z_1 + \dfrac{r_1^2}{a'-z_1'}$ is the inverse point of a in the circle $|z-z_1| = r_1$, which will always be represented by a_1; therefore

$$w = \frac{A}{z-a} - \left(\frac{r_1}{a'-z_1'}\right)^2 \frac{A'e^{-2i\theta_1}}{z-a_1} + \text{const}.$$

Starting from the finite form so obtained for the case of a single circle, a method which is essentially the same as the physical method of images may be applied to form a series, which, if convergent, will give the required function for the case of two circular boundaries.

The two circles might first, by a linear transformation, be replaced by two concentric circles, but the simplification of the algebraic work thereby introduced is not great; while the point at infinity being then a singular point for the functions involved, the result would lose something of its generality. Consider then the case in which the boundary consists of two non-intersecting circles

$$|z-z_1| = r_1 \quad \text{and} \quad |z-z_2| = r_2.$$

By applying continually the previous result a series may be obtained of the form

$$w = \frac{A}{z-a} \begin{matrix} + \dfrac{A_1}{z-a_1} + \dfrac{A_{12}}{z-a_{12}} + \dots \\ + \dfrac{A_2}{z-a_2} + \dfrac{A_{21}}{z-a_{21}} + \dots; \end{matrix}$$

where
$$A_1 = -\left(\frac{r_1}{a'-z_1'}\right)^2 A' e^{-2i\theta_1},$$

$$A_{12} = -\left(\frac{r_2}{a_1'-z_2'}\right)^2 A_1' e^{-2i\theta_2};$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

and a_{12} is the inverse point of a_1 in the circle $|z-z_2| = r_2$, &c. Each term of this series corrects over one circle the error introduced over it by the preceding term, and hence, if the series is convergent, it clearly represents the function required.

The form of the series may be simplified by the following considerations.

Two consecutive inversions are equivalent to a linear substitution, and hence the quantities a, a_1, a_{12}, a_{21}, a_{121}, &c., in the denominators can be derived from a and a_1 (or from a and a_2) by transforming these by means of the direct or inverse powers of a linear substitution.

Suppose that
$$a_{12} = \frac{aa+b}{ca+d};$$

any term $a_{121\dots}$ or $a_{212\dots}$ with an even number of suffixes can then be written in the form

$$\frac{a_n a + b_n}{c_n a + d_n},$$

where $\left(z, \dfrac{a_n z + b_n}{c_n z + d_n}\right)$ is a power of $\left(z, \dfrac{az+b}{cz+d}\right)$, and any term with an odd number of suffixes can be written in the form

$$\frac{a_n a_1 + b_n}{c_n a_1 + d_n}.$$

Also, since
$$a_{12} = z_2 + \frac{r_2^2}{a_1'-z_2'},$$

therefore
$$da_{12} = -\left(\frac{r_2}{a_1'-z_2'}\right)^2 da_1',$$

and
$$da_1 = -\left(\frac{r_1}{a'-z_1'}\right)^2 da'.$$

Therefore
$$A_{12} = -\left(\frac{r_2}{a_1'-z_2'}\right)^2 e^{-2i\theta_2} A_1'$$
$$= \left(\frac{r_2}{a_1'-z_2'}\right)^2 e^{-2i\theta_2}\left(\frac{r_1}{a-z_1}\right)^2 A e^{2i\theta_1}$$
$$= e^{2i(\theta_1-\theta_2)} A \frac{da_{12}}{da},$$

which result can be generalized at once.

The series therefore can be written in the form

$$w = A\sum_{-\infty}^{\infty} e^{2in(\theta_1-\theta_2)} \frac{1}{z-f_n(a)} \frac{df_n(a)}{da}$$
$$+A_1\sum_{-\infty}^{\infty} e^{2in(\theta_1-\theta_2)} \frac{1}{z-f_n(a_1)} \frac{df_n(a_1)}{da_1},$$

where f_n represents the result of repeating n times the linear substitution

$$\left(z, \frac{az+b}{cz+d}\right).$$

Now the substitution $\left(z, \dfrac{az+b}{cz+d}\right)$

has for its double points the limiting points of the coaxal system to which the two circles belong; while it may easily be verified that the multiplier is real and different from unity. It may therefore be written in the form

$$\frac{f(z)-g}{f(z)-h} = K\frac{z-g}{z-h},$$

and hence
$$f_n(a) = \frac{(hK^n-g)\,a-hg\,(K^n-1)}{(K^n-1)\,a+h-gK^n};$$

therefore
$$\frac{df_n(a)}{da} = \frac{(h-g)^2 K^n}{\left[(K^n-1)\,a+h-gK^n\right]^2}$$
$$= \frac{(h-g)^2}{\left[K^{\frac{1}{2}n}(a-g)-K^{-\frac{1}{2}n}(a-h)\right]^2}.$$

It follows that $\sum_{-\infty}^{\infty}\dfrac{df_n(a)}{da}$ is a uniformly convergent series, and therefore at once that the two infinite series in the expression for w are uniformly convergent and represent continuous functions of z.

If $\theta_1-\theta_2$ is commensurable with π, the expression for w can be put in a finite form by means of theta-functions, but not otherwise. Suppose, for instance, that

$$\theta_1-\theta_2=\frac{\pi}{p},$$

where p is an integer; then the terms in w which contain $Ae^{(2is\pi)/p}$ $(s<p)$ as a coefficient are

$$\sum_{-\infty}^{+\infty}\frac{1}{z-f_{s+np}(a)}\frac{df_{s+np}(a)}{da}.$$

Now
$$\frac{1}{z-f_{s+np}(a)}\frac{df_{s+np}(a)}{da}+\frac{1}{z-f_{s-np}(a)}\frac{df_{s-np}(a)}{da}$$

$$=-\frac{\partial}{\partial a}\log\left[z-\frac{(hK^{np}-g)f_s(a)-hg(K^{np}-1)}{(K^{np}-1)f_s(a)+h-gK^{np}}\right]$$
$$\times\left[z-\frac{(hK^{-np}-g)f_s(a)-hg(K^{-np}-1)}{(K^{-np}-1)f_s(a)+h-gK^{-np}}\right]$$

$$=-\frac{\partial}{\partial a}\log\frac{(z-h)\left[f_s(a)-g\right]-(z-g)\left[f_s(a)-h\right]K^{-np}}{f_s(a)-g-\left[f_s(a)-h\right]K^{-np}}$$
$$\times\frac{(z-g)\left[f_s(a)-h\right]-(z-h)\left[f_s(a)-g\right]K^{-np}}{f_s(a)-h-\left[f_s(a)-g\right]K^{-np}}$$

$$=-\frac{\partial}{\partial a}\log\frac{1-\dfrac{z-g}{z-h}\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}}{1-\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}}\;\frac{1-\dfrac{z-h}{z-g}\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}}{1-\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}}.$$

The terms in question can then be written

$$-\frac{\partial}{\partial a}\log\frac{\left(1-\dfrac{z-g}{z-h}\dfrac{f_s(a)-h}{f_s(a)-g}\right)\prod_1^\infty\left(1-\dfrac{z-g}{z-h}\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}\right)\times\left(1-\dfrac{z-h}{z-g}\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}\right)}{\left(1-\dfrac{f_s(a)-h}{f_s(a)-g}\right)\prod_1^\infty\left(1-\dfrac{f_s(a)-h}{f_s(a)-g}K^{-np}\right)\left(1-\dfrac{f_s(a)-g}{f_s(a)-h}K^{-np}\right)},$$

where finally the denominator can be omitted as a mere constant.

Omitting a further constant, this is the same as

$$-\frac{\partial}{\partial a}\log\theta_1(x_s, q),$$

where
$$q = K^{-ip},$$

and
$$x_s = \frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{f_s(a)-h}{f_s(a)-g} = \frac{1}{2i}\log\frac{z-g}{z-h}\,\frac{a-h}{a-g}+\frac{is}{p}\log q.$$

Now
$$\frac{\partial x_s}{\partial a} = \frac{1}{2i}\,\frac{h-g}{(a-h)(a-g)},$$

and the part of w which has A for a coefficient can therefore be written in the form

$$\frac{iA}{2}\,\frac{h-g}{(a-h)(a-g)}\sum_{s=0}^{s=p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(x_s)}{\theta_1(x_s)}.$$

The quantity $\dfrac{h-g}{(a_1-h)(a_1-g)}$, which occurs as a factor in the other half of w, may easily be put in the form

$$\left(\frac{a'-z_1'}{r_1}\right)^2\frac{h'-g'}{(a'-h')(a'-g')},$$

so that, if y_s is written for the result of putting a_1 for a in x_s, the second part of w becomes

$$-\frac{iA'}{2}\,\frac{h'-g'}{(a'-h')(a'-g')}\,e^{-2i\theta_1}\sum_{s=0}^{s=p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(y_s)}{\theta_1(y_s)}.$$

Finally, simplifying the constants by writing

$$\frac{iA}{2}\,\frac{h-g}{(a-h)(a-g)}\,e^{i\theta_1} = B,$$

$$e^{i\theta_1}w = B\sum_{0}^{p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(x_s)}{\theta_1(x_s)} + B'\sum_{0}^{p-1} e^{(2is\pi)/p}\,\frac{\theta_1'(y_s)}{\theta_1(y_s)}.$$

The function so obtained represents the space outside two non-intersecting circles conformably on an infinite plane in which two finite non-intersecting straight lines inclined at an angle π/p are to be regarded as boundaries.

If $\theta_1 = \theta_2 = 0$, so that $p = 1$,

$$w = B\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h}\frac{a-h}{a-g}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h}\frac{a-h}{a-g}\right)} + B'\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h}\frac{a_1-h}{a_1-g}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h}\frac{a_1-h}{a_1-g}\right)}$$

is a function having a single infinity at a in the space outside the two circles, and constant imaginary part at their circumferences.

If, further, $a = \infty$, and therefore

$$a_1 = z_1,$$

$$w = B\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h}\right)} + B'\frac{\theta_1'\left(\frac{1}{2i}\log\frac{z-g}{z-h} - \frac{1}{2i}\log\frac{z_1-g}{z_1-h}\right)}{\theta_1\left(\frac{1}{2i}\log\frac{z-g}{z-h} - \frac{1}{2i}\log\frac{z_1-g}{z_1-h}\right)},$$

where the modulus of the theta-functions is $K^{-\frac{1}{2}}$, gives the uniform streaming motion of fluid in any direction past two right circular cylinders. This particular case of functions of the kind considered has already been dealt with by Mr. Hicks and Mr. Greenhill in Vols. XIV. and XVI. of the *Quarterly Journal of Mathematics*.

In this case the original expression for w is

$$w = A\Sigma\frac{1}{z-f_n(a)}\frac{df_n(a)}{da} + A_1\Sigma\frac{1}{z-f_n(a_1)}\frac{df_n(a_1)}{da_1};$$

therefore

$$\frac{\partial w}{\partial z} = -A\Sigma\frac{1}{\left[z-f_n(a)\right]^2}\frac{df_n(a)}{da} - A_1\Sigma\frac{1}{\left[z-f_n(a_1)\right]^2}\frac{df_n(a_1)}{da_1}.$$

Now, if $$f_n(a) = \frac{a_n a + b_n}{c_n a + d_n}, \quad \left[a_n d_n - b_n c_n = 1\right],$$

$$\frac{1}{\left[z-f_n(a)\right]^2}\frac{df_n(a)}{da} = \frac{1}{\left[z(c_n a + d_n) - c_n a - b_n\right]^2}$$

$$= \frac{(-c_n z + a_n)^{-2}}{\left(a - \frac{d_n z - b_n}{-c_n z + a_n}\right)^2}$$

$$= \frac{1}{\left[a - f_{-n}(z)\right]^2}\frac{df_{-n}(z)}{dz},$$

or
$$\frac{\partial w}{\partial z} = -\Sigma\left[\frac{A}{[a-f_n(z)]^2}+\frac{A_1}{[a_1-f_n(z)]^2}\right]\frac{df_n(z)}{dz}$$
$$= -\Sigma\left[\frac{A}{[a-f_n(z)]^2}+\frac{A_1}{[a_1-f_n(z)]^2}\right](c_nz+d_n)^{-2}.$$

The form of this function is precisely that of those that M. Poincaré calls theta-fuchsian, except that the factor c_nz+d_n occurs in the power -2, while the general form of a theta-fuchsian function as considered by M. Poincaré, is

$$\sum_n \phi\left(\frac{a_nz+b_n}{c_nz+d_n}\right)(c_nz+d_n)^{-2m},$$

where m is not less than 2.

The group of substitutions to which the above $\frac{\partial w}{\partial z}$ belongs, being derived from a single fundamental substitution, is the simplest possible, and is only incidentally referred to by M. Poincaré in his memoir (*Acta Mathematica*, Vol. I.).

The previous investigation shows that for the discontinuous infinite group derived from a single fundamental substitution (which must be hyperbolic or loxodromic to give a discontinuous group),

$$\Sigma\,(c_nz+d_n)^{-2}$$

is a convergent series, and that therefore theta-fuchsian functions of the form

$$\sum_n \phi\left(\frac{a_nz+b_n}{c_nz+d_n}\right)(c_nz+d_n)^{-}$$

do really exist.

Now, if
$$\frac{\partial w}{\partial z} = f(z),$$

then, from the fundamental properties of these functions,

$$f\left(\frac{az+b}{cz+d}\right) = f(z)(cz+d)^2;$$

and therefore
$$f\left(\frac{az+b}{cz+d}\right)d\,\frac{az+b}{cz+d} = f(z)\,dz;$$

or, if
$$z' = \frac{az+b}{cz+d},$$

$$\int^{z'} f(z')\, dz' = \int^{z} f(z)\, dz,$$

so that the variable part of w must remain unchanged by the substitution
$$\left(z,\ \frac{az+b}{cz+d}\right),$$

or, in other words, by any substitution of the group.

The function w therefore has its characteristic properties, not only in the original region R, but in any of the regions into which this may be transformed by the substitutions of the group.

These latter results can be derived directly from the expression of w in theta-functions; but with a view to the consideration of the case when the boundary consists of three or more circles, in which the group of substitutions involved is derived from more than one fundamental substitution, and the functions can therefore not be expressed in terms of theta-functions of one variable, it appears desirable to show how in this simplest case the idea of a group of substitutions and of functions which are unchanged by it is involved.

Before going on to consider the functions for streaming motion and the analogous functions considered here for the case of three or more circular boundaries, it will be necessary to investigate the convergence of series analogous to the one just dealt with, but arising from a more extended group of substitutions. This I shall do in another paper, which I hope to have the honour of laying before the Society.