We see by Art. (39) that, unless every first minor has the root $\alpha-1$ times at least, a solution can be deduced from the first minors which has some power of t greater than zero in the coefficient. Again, unless every second minor has the root $\alpha-2$ times at least, a solution can be deduced from the second minors with some power of t in the coefficient. On the whole, we infer that when α equal roots occur in the determinant, and the terms in the solution with t as a factor are to be absent, it is necessary as well as sufficient that all the first, second, &c. minors up to the $(\alpha-1)$th should be zero.

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*Investigation of the Character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density.** By Lord Rayleigh.

[*Read April 12th,* 1883.]

The well-known condition of equilibrium requires that the fluid be arranged in horizontal strata, so that its density σ is a function of the vertical coordinate z only. If this state of things be slightly departed from, we may regard the actual density at any point x, y, z as equal to $\sigma+\rho$, where ρ is a function of x, y, z, and the time t, which always remains small during the period contemplated. The component velocities u, v, w are equally to be regarded as small; they are connected by the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad \text{.......................(1).}$$

The equilibrium pressure p is a function of z only. If the actual pressure be called $p+\delta p$, the dynamical equations become, with omission of the squares of small quantities,

$$\frac{d\delta p}{dx}=-\sigma\frac{du}{dt},\quad \frac{d\delta p}{dy}=-\sigma\frac{dv}{dt},\quad \frac{d\delta p}{dz}=-g\rho-\sigma\frac{dw}{dt} \quad \text{......(2).}$$

One further equation is supplied by the condition that the density of every particle remains unchanged.

Thus
$$\frac{d\rho}{dt}+w\frac{d\sigma}{dz}=0 \quad \text{...........................(3).}$$

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* These calculations were written out in 1880, in order to illustrate the theory of cirrous clouds propounded by the late Prof. Jevons (*Phil. Mag.*, xiv., p. 22, 1857). Pressure of other work has prevented me hitherto from pursuing the subject.

By Fourier's theorem, and the general theory of disturbed equilibrium, we know that the complete solution of the present problem may be decomposed into partial solutions, for any one of which the variable quantities considered as functions of x vary as $e^{i\kappa x}$, as functions of y vary as $e^{i\kappa' y}$, and as functions of t vary as e^{int}. The *wave-lengths* of the disturbances parallel to x and y are λ, λ'; where $\lambda = 2\pi/\kappa,\ \lambda' = 2\pi/\kappa'$.

The introduction of these suppositions into (1), (2), and (3) leads to

$$i\kappa u + i\kappa' v + \frac{dw}{dz} = 0 \quad \text{.........................(4)},$$

$$\kappa\delta p = -n\sigma u, \quad \kappa'\delta p = -n\sigma v, \quad \frac{d\delta p}{dz} = -g\rho - in\sigma w \quad \text{...........(5)},$$

$$in\rho + w\frac{d\sigma}{dz} = 0 \quad \text{.............................(6)}.$$

Eliminating u and v between (4) and the two first of equations (5), we get

$$i(\kappa^2+\kappa'^2)\,\delta p - n\sigma\frac{dw}{dz} = 0 \quad \text{.......................(7)}.$$

Next eliminating δp between (7) and the last of equations (5), we find

$$i(\kappa^2+\kappa'^2)(g\rho + in\sigma w) + n\frac{d}{dz}\left(\sigma\frac{dw}{dz}\right) = 0 \quad \text{..............(8)}.$$

Finally between (6) and (8) we eliminate ρ, and thus obtain

$$\frac{d}{dz}\left(\sigma\frac{dw}{dz}\right) - (\kappa^2+\kappa'^2)\left(\frac{g}{n^2}\frac{d\sigma}{dz} + \sigma\right)w = 0 \quad \text{.........(9)};$$

or, as it may be also written,

$$\frac{d^2w}{dz^2} - (\kappa^2+\kappa'^2)\,w + \frac{d\sigma}{\sigma dz}\left\{\frac{dw}{dz} - (\kappa^2+\kappa'^2)\frac{g}{n^2}w\right\} = 0 \quad \text{......(10)}.$$

We will first apply this equation to the well-known case of two uniform fluids of densities σ_1, σ_2, separated by a horizontal boundary ($z = 0$), and for brevity we will omit to write κ'. For both regions of fluid, the general equation (10) reduces to

$$\frac{d^2w}{dz^2} - \kappa^2 w = 0 \quad \text{.............................(11)},$$

of which the solution is

$$w = Ae^{\kappa z} + Be^{-\kappa z} \quad \text{.............................(12)}.$$

By the condition at infinity, we are to take for the upper fluid $A = 0$, and for the lower $B = 0$. Moreover by continuity the value of w must be the same for both fluids at the separating surface. Thus we may write for the upper fluid $w = Be^{-\kappa z}$, and for the lower $w = Be^{\kappa z}$. The second boundary condition is obtained by integrating equation

(9) across the surface of transition. Thus

$$\left[\sigma\frac{dw}{dz}\right]_2-\left[\sigma\frac{dw}{dz}\right]_1-\frac{g\kappa^2}{n^2}(\sigma_2-\sigma_1)=0;$$

whence
$$n^2=g\kappa\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\quad\text{..........................(13)},$$
the known solution.

If the upper fluid be the lighter, $\sigma_2<\sigma_1$, and n^2 is positive. This indicates stability with harmonic oscillations, whose frequency increases without limit with κ; that is, as the wave-length diminishes. If, on the other hand, $\sigma_2>\sigma_1$, the equilibrium is unstable, and the instability (measured by the rate at which a small disturbance is multiplied in a given time) is greater the smaller the wave-length. If the disturbance be not limited to two dimensions, we have simply to replace κ by $\sqrt{(\kappa^2+\kappa'^2)}$.

We know from the general theory that only real values of n^2 are admissible in (9), and that if $\frac{d\sigma}{dz}$ be negative throughout, all the values of n^2 are positive, but if $\frac{d\sigma}{dz}$ be positive throughout, all the values of n^2 are negative. In order to prove this from the equation, suppose that w and w' are two solutions corresponding to different values of n^2, say n^2 and n'^2. Then

$$\int w'\frac{d}{dz}\left(\sigma\frac{dw}{dz}\right)dz=\kappa^2\int\left(\frac{g}{n^2}\frac{d\sigma}{dy}+\sigma\right)ww'\,dz,$$

or, on integration by parts between two finite or infinite limits for which w, w' vanish,

$$\int\sigma\frac{dw}{dz}\frac{dw'}{dz}dz+\kappa^2\int\sigma ww'\,dz+\kappa^2\frac{g}{n^2}\int\frac{d\sigma}{dz}ww'\,dz=0\quad\text{.......(14)}.$$

In this equation w and w' may be interchanged if n'^2 be written for n^2. Hence

$$\int\sigma\frac{dw}{dz}\frac{dw'}{dz}dz+\kappa^2\int\sigma ww'\,dz=0\quad\text{.....................(15)},$$

$$\int\frac{d\sigma}{dz}ww'\,dz=0\quad\text{..........................(16)}.$$

If now n^2 could be complex, there would be two solutions of the form $w=\alpha+i\beta$, $w'=\alpha-i\beta$, and equation (15) would become

$$\int\sigma\left\{\left(\frac{d\alpha}{dz}\right)^2+\left(\frac{d\beta}{dz}\right)^2\right\}dz+\kappa^2\int\sigma(\alpha^2+\beta^2)\,dz=0,$$

which cannot be true if, as we suppose, σ is everywhere positive.

Again, suppose in (14) that $w'=w$. Thus

$$\int\sigma\left\{\left(\frac{dw}{dz}\right)^2+\kappa^2w^2\right\}dz+\frac{\kappa^2g}{n^2}\int\frac{d\sigma}{dz}w^2dz=0\quad\text{...............(17)},$$

from which it is evident that, if $\frac{d\sigma}{dz}$ be of one sign throughout, n^2 can only be of the opposite sign.

These conclusions are limited to the cases for which every mode of disturbance is stable, or every mode unstable; but we know that if $\frac{d\sigma}{dz}$ be *anywhere* positive, instability must ensue. To see this from equation (9), we may regard it as the condition (according to the methods of the Calculus of Variations) that $\int \frac{d\sigma}{dz} w^2 dz$ is a maximum or minimum, while $\int \sigma \left\{ \left(\frac{dw}{dz}\right)^2 + \kappa^2 w^2 \right\} dz$ is given, $-\frac{n^2}{g\kappa^2}$ being the then value of the ratio of the integrals. If $d\sigma / dz$ be anywhere positive, the first integral admits of a positive value, and therefore of a positive maximum, so that one value at least of n^2 is negative, and one mode of disturbance is unstable.

The simplest case of a variable density which we can consider is that obtained by supposing $\frac{d\sigma}{\sigma dz}$ to be constant, equal say to β, or, on integration,

$$\sigma = \sigma_0 e^{\beta z} \quad \text{.......................(18)};$$

so that all strata of equal thickness are similarly constituted, differing only in absolute density. In this case, with omission of κ' as before (10) becomes

$$\frac{d^2w}{dz^2} + \beta \frac{dw}{dz} - \kappa^2 \left(1 + \frac{g\beta}{n^2}\right) w = 0 \quad \text{..............(19)}.$$

If m_1, m_2 be the roots of

$$m^2 + \beta m - \kappa^2 (1 + g\beta n^{-2}) = 0 \quad \text{....................(20)},$$

the general solution of (19) is

$$w = Ae^{m_1 z} + Be^{m_2 z} \quad \text{............................(21)},$$

A and B being arbitrary constants.

Let us now suppose that the fluid is bounded by impenetrable horizontal planes at $z = 0$ and at $z = l$. Since w vanishes with z, $B = -A$, so that (21) becomes

$$w = A (e^{m_1 z} - e^{m_2 z}) \quad \text{............................(22)}.$$

Again, since w vanishes when $z = l$, $e^{m_1 l} - e^{m_2 l} = 0$, or $e^{(m_1 - m_2) l} = 1$, whence

$$(m_1 - m_2)\, l = 2a i\pi \quad \text{.........................(23)},$$

a being an integer. Thus (22) may be written

$$w = Ae^{\frac{1}{2}(m_1+m_2)z} \left\{ e^{\frac{1}{2}(m_1-m_2)z} - e^{-\frac{1}{2}(m_1-m_2)z} \right\} = A'e^{-\frac{1}{2}\beta z} \sin a \frac{\pi z}{l} \quad \text{......(24)},$$

by (20), (23), A' being a new arbitrary constant. The values of n corresponding to the various values of a are obtained by comparison of (20) and (23). From the former

$$(m_1-m_2)^2 l^2 = \beta^2 l^2 + 4\kappa^2 l^2 (1+g\beta n^{-2}) \quad\text{.................(25)},$$

so that

$$\kappa^2 l^2 (1+g\beta n^{-2}) = -\tfrac{1}{4}\beta^2 l^2 - a^2\pi^2 \quad\text{.....................(26)},$$

or

$$\frac{g\beta}{n^2} = -\frac{\frac{1}{4}\beta^2 l^2 + \kappa^2 l^2 + a^2\pi^2}{\kappa^2 l^2} \quad\text{......................(27)}.$$

From (27) we see that the disturbances are all stable if β is negative, that is, if the density diminishes upwards, and that in the contrary case they are all unstable. The smallest admissible value of a is unity, and this corresponds to the greatest numerical value of n^2. Contrary to what is met with in most vibrating systems, there is (in the case of stability) a limit on the side of rapidity of vibration, but none on the side of slowness. In the case of instability we are principally interested in the mode for which the instability is greatest, and this also corresponds to the unit value of a. When a is greater than unity, there are internal nodal planes, as appears from (24).

If l, κ, and a are given, n^2 is numerically greatest when β is such that

$$\tfrac{1}{4}\beta^2 l^2 = \kappa^2 l^2 + a^2\pi^2.$$

If l, a, β be regarded as given, n^2 increases numerically from zero when κ is zero, up to a finite limit when κ is infinite; or, in the case of stability, as the wave-length diminishes from ∞ to 0, the frequency of vibration rises from 0 to a finite value, given by

$$n^2 = -g\beta \quad\text{..............................(28)},$$

which is independent both of a and of l. These vibrations are isochronous with the vibrations of a pendulum whose length is equal to the distance between two strata, of which the densities are as $e : 1$.

If the disturbance be not limited to two dimensions, we must write $\sqrt{(\kappa^2+\kappa'^2)}$ for κ^2. The completely expressed value of w, corresponding to one normal mode of disturbance, is then

$$w = Ae^{-\frac{1}{2}\beta z} \sin\frac{a\pi z}{l} \cos\frac{2\pi(x-x_0)}{\lambda} \cos\frac{2\pi(y-y_0)}{\lambda'} \cos n(t-t_0) \quad\text{...(29)}.$$

We will now apply the solution to the investigation of the case in which the density for all values of z less than 0 is σ_1, and for all values of z greater than l is σ_2, the transition from the one density to the other being in accordance with the law $\sigma = \sigma_1 e^{\beta z}$, so that

$$\sigma_2 = \sigma_1 e^{\beta l} \quad\text{..............................(30)}.$$

When $z > l$, $w \propto e^{-\kappa z}$, so that for $z=l$, $dw/w\,dz = -\kappa$; similarly for the lower fluid, when $z=0$, $dw/w\,dz = +\kappa$. Thus, by (21), the boundary conditions are $m_1 A e^{m_1 l} + m_2 B e^{m_2 l} = -\kappa (A e^{m_1 l} + B e^{m_2 l})$,

$$m_1 A + m_2 B = +\kappa (A + B),$$

whence, by elimination of $A : B$,

$$e^{m_1 l} (m_1 + \kappa)(m_2 - \kappa) = e^{m_2 l} (m_2 + \kappa)(m_1 - \kappa).$$

This, in connection with (20), determines the admissible values of n. It may be written

$$\frac{m_1 m_2 + \kappa (m_2 - m_1) - \kappa^2}{m_1 m_2 - \kappa (m_2 - m_1) - \kappa^2} = e^{(m_2 - m_1) l},$$

or
$$\frac{\kappa (m_2 - m_1)}{m_1 m_2 - \kappa^2} = \tanh \tfrac{1}{2} (m_2 - m_1)\, l.$$

By (20) this may be put into the form

$$\frac{\kappa (m_2 - m_1)}{\frac{1}{4}\beta^2 - \kappa^2 - \frac{1}{4} (m_2 - m_1)^2} = \tanh \tfrac{1}{2} (m_2 - m_1)\, l,$$

or, if for brevity we write θ for $(m_2 - m_1)\, l$,

$$\frac{\kappa l \,.\, \theta}{\frac{1}{4}\beta^2 l^2 - \kappa^2 l^2 - \frac{1}{4}\theta^2} = \tanh \tfrac{1}{2}\theta \quad\text{.......................(31).}$$

This equation determines θ; and then, by (20),

$$\theta^2 = l^2 (m_2 - m_1)^2 = l^2 \{(m_2 + m_1)^2 - 4 m_1 m_2\}$$
$$= \beta^2 l^2 + 4\kappa^2 l^2 (1 + g\beta n^{-2}) \quad\text{.............................(32),}$$

giving n in terms of θ.

Before going farther, we may verify these results by applying them to the case of a sudden transition, for which l vanishes, while βl remains finite. The principal solution of (31) gives $\theta^2 = \beta^2 l^2$ approximately, so that

$$\tfrac{1}{4}\beta^2 l^2 - \tfrac{1}{4}\theta^2 = \kappa l \,.\, \beta l \,.\, \coth \tfrac{1}{2}\beta l.$$

Using this in (32), we get

$$g\beta n^{-2} = -\kappa^{-1}\beta \coth \tfrac{1}{2}\beta l,$$

whence
$$n^2 = -g\kappa \tanh \tfrac{1}{2}\beta l = -g\kappa \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1},$$

as before.

Other solutions of (31) are obtained by supposing $\theta^{-1} \tanh \tfrac{1}{2}\theta$ to vanish, whence $\theta = i \,.\, a \,.\, 2\pi$, a being an integer other than zero, These are of no importance, as the corresponding values of n vanish.

When the layer of transition is of finite thickness, the general solution expressed by (31), (32) is rather complicated. A simplification, which does not involve much loss of interest, may be effected by supposing that the whole change of density is small, so that (31), (32) become $$\frac{\kappa l \,.\, \theta}{\kappa^2 l^2 + \frac{1}{4}\theta^2} = -\tanh \tfrac{1}{2}\theta \quad\text{.......................}(33),$$

$$g\beta n^{-2} = \frac{\theta^2}{4\kappa^2 l^2} - 1 \quad\text{..........................}(34).$$

From (33), $$\pm\frac{\kappa l - \frac{1}{2}\theta}{\kappa l + \frac{1}{2}\theta} = e^{\frac{1}{2}\theta},$$

whence $$-\tanh \tfrac{1}{4}\theta = \frac{\theta}{2\kappa l}, \text{ or } \frac{2\kappa l}{\theta}\text{.......................}(35).$$

Equation (35) cannot be satisfied by any real value of θ. If we write $\theta = i\phi$, we get in place of it,

$$\tan \tfrac{1}{4}\phi = -\frac{\phi}{2\kappa l}, \text{ or } \frac{2\kappa l}{\phi}\text{.......................}(36);$$

and in place of (34), $$g\beta n^{-2} = -\frac{\phi^2}{4\kappa^2 l^2} - 1 \quad\text{..........................}(37).$$

The series of admissible values of ϕ, given by (36), extends to infinity, but the higher roots correspond to small values of n^2, which are of little interest. Whether the equilibrium be stable or unstable, the most important root is the smallest. It lies in the first quadrant, and is given by the second alternative of (36). The progress of n^2 as a function of κl is easily traced. When κl is small, $\phi^2 = 8\kappa l$, and $g\beta n^{-2} = -2\,(\kappa l)^{-1}$, which leads to $n^2 = -g\kappa\frac{\sigma_2 - \sigma_1}{2\sigma}$, the known result for a rapid transition. As κl increases, $\frac{1}{4}\phi$ ranges from 0 to $\frac{1}{2}\pi$, and $\frac{\phi^2}{4\kappa^2 l^2}$ or $\cot^2 \frac{1}{4}\phi$ ranges from infinity to zero. Thus the numerical value of n^2 continually increases, until for an infinitely small wave-length it approaches the finite limit $-g\beta$, beyond which it cannot pass. The principal result of the substitution of a gradual for an abrupt transition is to arrest the further increase of n^2, after the wave-length has diminished so far as to become comparable in magnitude with the thickness of the layer of transition. In the case of the limiting value of n^2, the length of the equivalent pendulum is

$$l \div (\log \sigma_2 - \log \sigma_1).$$

If, for example, the extreme difference of densities amounted to one per cent., the length of the equivalent pendulum would be 100 times the thickness of the layer of transition.

For actual calculation (36), (37) may advantageously be written

$$\tfrac{1}{2}\kappa l = \tfrac{1}{4}\phi \times \tan \tfrac{1}{4}\phi\text{..............................}(38),$$

$$-\tfrac{1}{2}\kappa g \,.\, \beta l \,.\, n^{-2} = \tfrac{1}{2}\kappa l \div \sin^2 \tfrac{1}{4}\phi \quad\text{....................}(39),$$

the right-hand member of (39) being equal to unity, when κl is small.

Ascribing arbitrary values to $\frac{1}{4}\phi$, we can readily calculate corresponding values of κl and $\frac{1}{2}\kappa l/\sin^2\frac{1}{4}\phi$, and thus exhibit the effect upon the equilibrium of a gradual increase in the thickness of the layer of transition, the extreme densities (determined by βl) and the wave-length being given.

$\frac{1}{4}\phi$	κl	$\frac{1}{2}\kappa l/\sin^2\frac{1}{4}\phi$
0	κl	1·000
10°	·06155	1·021
20°	·2541	1·086
30°	·6046	1·209
40°	1·172	1·418
50°	2·080	1·772
60°	3·628	2·418
70°	6·713	3·801
80°	15·838	8·165
90°	κl	$\frac{1}{2}\kappa l$

Equations of the Loci of the Intersections of three Tangent Lines and of three Tangent Planes to any Quadric $u=0$. By Professor WOLSTENHOLME.

[*Read April 12th*, 1883.]

If $u=0$ be the rational equation of the second degree of any quadric, the equation of the tangent cone whose vertex is (xyz) is (using XYZ for current coordinates), when referred to its vertex as origin,

$$4uU=\left(X\frac{du}{dx}+Y\frac{du}{dy}+Z\frac{du}{dz}\right)^2;$$

and the coefficient of X^2 in this is

$$2u\frac{d^2u}{dx^2}-\left(\frac{du}{dx}\right)^2;$$

that of $2YZ$ is

$$2u\frac{d^2u}{dy\,dz}-\frac{du}{dy}\frac{du}{dz},$$

Hence if we write A, B, C, F, G, H for

$$\frac{d^2u^{\frac{1}{2}}}{dx^2},\ \frac{d^2u^{\frac{1}{2}}}{dy^2},\ \frac{d^2u^{\frac{1}{2}}}{dz^2},\ \frac{d^2u^{\frac{1}{2}}}{dy\,dz},\ \frac{d^2u^{\frac{1}{2}}}{dz\,dx},\ \frac{d^2u^{\frac{1}{2}}}{dx\,dy},$$