$$\frac{\sqrt{b}}{\sqrt{M}} = \frac{k}{m},$$

we get $$\frac{\sqrt{B}+\sqrt{b}}{\sqrt{M}} = \frac{l+k}{m}.$$

Since $$l+k = 2m,$$

$$\frac{\sqrt{B}+\sqrt{b}}{\sqrt{M}} = 2.$$

Therefore, $$\sqrt{M} = \tfrac{1}{2}(\sqrt{B}+\sqrt{b}).$$

NOTE: This theorem evidently holds true for a tetrahedron.

TO FIND APPROXIMATE SQUARE ROOTS.

BY NORMAN ANNING,
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This method cannot, like the rules recently published in SCHOOL SCIENCE AND MATHEMATICS for squaring numbers, lay claim to the word "mentally"; but for speed it will compare favorably with the method ordinarily taught. As regards accuracy an error in any step is corrected by those that follow.

Let $n = ab$ and suppose $a < b$.

It is not hard to prove that when a and b are nearly equal, more exactly when $a < b < 9a$, the arithmetic mean, $\frac{a+b}{2}$, is nearer to $\sqrt{n}$ than is either a or b.

To find $\sqrt{n}$ find a and b, two factors of n, which are as nearly equal as possible, then find arithmetic means successively of

$$a \text{ and } b,$$

$$\frac{a+b}{2} \text{ and } n\Big/\frac{a+b}{2} \text{ and so on.}$$

This process shuts $\sqrt{n}$ in a region which becomes narrower at each step. The nearer b/a is to unity the more rapid the approximation.

Examples.

(1) $\sqrt{20}$. $\quad 20 = 4\times5, \frac{4+5}{2} = 4\cdot5,$

$$= 4\cdot5\times4\cdot444, \ \frac{4\cdot5+4\cdot444}{2} = 4\cdot472,$$

$$= \sqrt{20} \text{ to three places.}$$

(2) $\sqrt{38}$. $\quad 38 = 6\times6\tfrac{1}{3},$

$$= 6\tfrac{1}{6}\times6\tfrac{6}{37}, \frac{6\cdot1667+6\cdot1622}{2} = 6\cdot1644,$$

$$= \sqrt{38} \text{ to four places.}$$

(3) $\sqrt{1+2x}$ (x small).

$$1+2x = 1\times(1+2x). \ \frac{1+1+2x}{2} = 1+x.$$

$$1+2x = (1+x)\times\frac{1+2x}{1+x}.$$

$$= (1+x)\times(1+x-x^2+\ldots), \ \tfrac{1}{2}(1+x+1+x-x^2)$$

$$= 1+x-\frac{x^2}{2}.$$

(4) $\sqrt{2}$. $2 = 1\times 2,$
$= 3/2 \times 4/3,$
$= 17/12 \times 24/17.$ $\frac{1}{2}(17/12 + 24/17) = 577/408 = 1\cdot 414215 \ldots$
$= \sqrt{2}$ to five places.

The fraction $577/408$ is the eighth convergent of the continued fraction $1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\ \ldots$.

A PROBLEM IN ELIMINATION.[1]

By Albert Babbitt.

The elimination of x from two equations (in elementary mathematics) consists in finding a relation between the coefficients of the given equations which must hold if the two equations are consistent, that is if they are both satisfied by a common value of x. In order that the elimination be possible it is necessary that the number of equations *exceed* the number of the unknowns entering into the equations, i. e., in order to eliminate one unknown there must be given at least two equations; for the elimination of two unknowns at least three equations must be given.

We have given the two trigonometric equations

$$\frac{\cos(a-3x)}{\cos^3 x} = \frac{\sin(a-3x)}{\sin^3 x} = b.$$

And let it be required to eliminate x.

The given equations may be written thus:

$$\cos(a-3x) = b\cos^3 x. \qquad (1)$$

$$\sin(a-3x) = b\sin^3 x. \qquad (2)$$

However, since $\cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$, and

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

(1) and (2) can be rewritten as follows:

$$\cos(a-3x) = \frac{b}{4}(\cos 3x + 3\cos x). \qquad (3)$$

$$\sin(a-3x) = \frac{b}{4}(3\sin x - \sin 3x). \qquad (4)$$

Squaring and adding (3) and (4) we have

$$\frac{b^2}{16}[1+9+6(\cos 3x\cos x - \sin 3x\sin x)] = 1, \text{ or}$$

$$b^2(10+6\cos 4x) = 16. \qquad (5)$$

Now, multiplying through (3) by $\cos 3x$, and (4) by $\sin 3x$, we obtain,

$$\cos 3x\cos(a-3x) = \frac{b}{4}(\cos^2 3x + 3\cos x\cos 3x) \qquad (6)$$

$$\sin 3x\sin(a-3x) = \frac{b}{4}(3\sin x\sin 3x - \sin^2 3x) \qquad (7)$$

Subtracting (7) from (6),

$$\cos 3x\cos(a-3x) - \sin 3x\sin(a-3x) = \frac{b}{4}[1+3(\cos x\cos 3x - \sin x\sin 3x)],$$

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[1] This problem has quite a history. It was offered at the St. Petersburg (now Petrograd) Institute for Railroad Engineers for several successive years to almost every student who took the entrance examinations in trigonometry, and not a single one of them was able to solve it. The question was raised by some whether the professors who offered this problem knew themselves how to solve it.