# A Possible Proof Of The Riemann Hypothesis

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## Abstract

The Zeta Function and one of its analytic continuations are defined as follows:

$$\forall s \in \mathbb{C} \mid Re(s) > 1,\ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\forall s \in \mathbb{C} \setminus \left\{1 + \frac{2\pi ik}{ln(2)} \mid k \in \mathbb{Z}\right\},\ \zeta(s) = \frac{\eta(s)}{\left(1 - 2^{1-s}\right)},\ where\ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

The Riemann Hypothesis states the following, for all the nontrivial zeros:

$$\zeta(s) = 0 \implies Re(s) = \frac{1}{2}$$

It has already been proved that $Re(s) \in ]0, 1[$ for all the nontrivial zeros.

**Firstly**, for $a = Re(s)$ and $b = Im(s)$, we'll prove that:

$$\zeta(s) = 0 \Rightarrow \eta(s) = 0 \Leftrightarrow \sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k / j))}{(kj)^a} \to 0\ as\ n \to +\infty$$

And since $\forall x \in \mathbb{R},\ -1 \leqslant \cos(x) \leqslant 1$, this implies that there exists a map $r_n$ satisfying $-1 \leqslant r_n \leqslant 1$ for all $n$ sufficiently large, and for which:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{(kj)^a} \to 0\ as\ n \to +\infty$$

**Secondly**, by reformulating it as a problem of quadratic equations, we will figure out that this holds true only if $\forall n \in \mathbb{N} \setminus [\![0, 3]\!],\ r_n \in \left[-\frac{1}{n-1}, -\frac{1}{n-3}\right]$ where $[\![0, 3]\!] = \{0, 1, 2, 3\}$, and therefore, that $r_n \sim -\frac{1}{n}\ as\ n \to +\infty$

And through various asymptotic equivalences, we will get:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \to 0\ as\ n \to +\infty$$

**Finally**, from there, <u>we'll consider $a = Re(s)$ as a map</u> $a_n = Re(s_n)$ converging to a real number $a_{+\infty} \in ]0, 1[$, rather than considering it as a fixed value (since we're dealing with infinity).
It is for convenience that we denote $\lim\limits_{n \to +\infty} a_n = a_{+\infty} \in ]0, 1[$.

Then we'll approximate each side's sum with integrals depending on $a_{+\infty}$,

and we shall distinguish three different cases:

- $a_{+\infty} \in ]0, \frac{1}{2}[$
- $a_{+\infty} \in ]\frac{1}{2}, 1[$
- $a_{+\infty} = \frac{1}{2}$

And conclude that the only case that is logically consistent is when $a_{+\infty} = \frac{1}{2}$.

## 1 Simplifying the expression

First of all, for the sake of simplification, let's write $s = a + ib$ where $a = Re(s)$ and $b = Im(s)$,
We can write the Eta function as follows:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{-ib}}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-ib\ln(n)}}{n^a}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(-b\ln(n))}{n^a} + i \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(-b\ln(n))}{n^a}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(b\ln(n))}{n^a} - i \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(b\ln(n))}{n^a}$$

If we assume $\zeta(s) = 0$, then by the expression of its analytic continuation $\zeta(s) = \frac{\eta(s)}{\left(1 - 2^{1-s}\right)}$, we also have $\eta(s) = 0$ and then $|\eta(s)|^2$ is null too:

$$|\eta(s)|^2 = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(b\ln(n))}{n^a}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(b\ln(n))}{n^a}\right)^2 = 0$$

$$thus\ as\ n \to +\infty,\ \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}\cos(b\ln(k))}{k^a}\right)^2 + \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}\sin(b\ln(k))}{k^a}\right)^2 \to 0$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k))\cos(b\ln(j))}{(kj)^a} + \frac{(-1)^{k+j-2}\sin(b\ln(k))\sin(b\ln(j))}{(kj)^a} \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} (-1)^{k+j-2}\left(\frac{\cos(b\ln(k))\cos(b\ln(j))}{(kj)^a} + \frac{\sin(b\ln(k))\sin(b\ln(j))}{(kj)^a}\right) \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} (-1)^{k+j-2}\left(\frac{\cos(b\ln(k) - b\ ln(j))}{(kj)^a}\right) \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k/j))}{(kj)^a} \to 0\ as\ n \to +\infty$$

$$\iff \sum_{k=1}^{n} \frac{(-1)^{2k-2}}{k^{2a}} + \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \; as \; n \to +\infty$$

$$\iff \sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \; as \; n \to +\infty$$

$$\forall k, j \in [\![1, n]\!], \forall b \in \mathbb{R}, \; -1 \leqslant \cos(b \ln(k/j)) \leqslant 1 \tag{1}$$

Thus there exists a map $r_n$ satisfying $-1 \leqslant r_n \leqslant 1$ for all $n$ sufficiently large, and for which:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{(kj)^a} \to 0 \; as \; n \to +\infty$$

And we end up with what curiously resembles a quadratic equation.

## 2 The "Russian Doll" Quadratic Equations

Now let's assume there is $x_1,..., x_n \in \mathbb{R}$ with $x_1 = 1$ so that:

$$\sum_{k=1}^{n} x_k^2 + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} x_k x_j = 0$$

And let's try and figure out which kind of map $r_n$ is.

But first, let's define $\forall n \in \mathbb{N}^*, \; u_n = \sum_{k=1}^{n} x_k^2$ , $v_n = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} x_k x_j$ and $p_n = \sum_{k=1}^{n} x_k$

Our previous equation becomes:

$$u_n \; + \; 2r_n v_n = x_n^2 + 2r_n p_{n-1} x_n + u_{n-1} + 2r_n v_{n-1} = 0$$

And now let's define $(f_n)_{n \in \mathbb{N}\setminus\{0,1\}}$ and $(g_n)_{n \in \mathbb{N}\setminus\{0,1\}}$ so that $\forall n \in \mathbb{N} \setminus \{0, 1\}$:

$$f_n u_n \; + \; g_n v_n = f_n x_n^2 + g_n p_{n-1} x_n + f_n u_{n-1} + g_n v_{n-1} = 0$$

Let's now express the delta $\Delta_n$ of this equation and find the expressions of $f_{n-1}$ and $g_{n-1}$ so that $\Delta_n = f_{n-1} u_{n-1} + g_{n-1} v_{n-1} \geqslant 0$:

$$\Delta_n = (g_n p_{n-1})^2 - 4 f_n (f_n u_{n-1} + g_n v_{n-1}),$$

$$p_{n-1}^2 = \left( \sum_{k=1}^{n-1} x_k \right)^2 = u_{n-1} + 2v_{n-1},$$

$$thus \; \Delta_n = (g_n p_{n-1})^2 - 4 f_n (f_n u_{n-1} + g_n v_{n-1}) = g_n^2 (u_{n-1} + 2v_{n-1}) - 4 f_n (f_n u_{n-1} + g_n v_{n-1})$$

$$\Delta_n = \left( g_n^2 - 4 f_n^2 \right) u_{n-1} + \left( 2g_n^2 - 4 f_n g_n \right) v_{n-1}$$

We conclude that $f_{n-1} = g_n^2 - 4f_n^2$ and $g_{n-1} = 2g_n^2 - 4f_n g_n$, and we see $\Delta_n$ is in turn a new quadratic equation:

$$\Delta_n = f_{n-1}x_{n-1}^2 + g_{n-1}p_{n-2}x_{n-1} + f_{n-1}u_{n-2} + g_{n-1}v_{n-2}$$

with a new $\Delta_{n-1}$ for which we must determine the conditions to ensure $\Delta_{n-1} \geqslant 0$, and so on until $\Delta_2$ (hence the comparison with a Russian doll).

But also, $\dfrac{g_{n-1}}{f_{n-1}} = \dfrac{2g_n^2 - 4f_n g_n}{g_n^2 - 4f_n^2} = \dfrac{2g_n(g_n - 2f_n)}{(g_n - 2f_n)(g_n + 2f_n)} = \dfrac{2g_n}{(g_n + 2f_n)} = \dfrac{2\frac{g_n}{f_n}}{\frac{g_n}{f_n} + 2}$

We observe that each time we calculate a $\Delta_{n-k}$, we actually apply $h : x \mapsto \dfrac{2x}{x+2}$ to the ratio $\dfrac{g_{n-k}}{f_{n-k}}$ to obtain $\dfrac{g_{n-k-1}}{f_{n-k-1}}$: $\forall k \in [\![1, n-3]\!]$, $\dfrac{g_{n-k-1}}{f_{n-k-1}} = \dfrac{2\frac{g_{n-k}}{f_{n-k}}}{\frac{g_{n-k}}{f_{n-k}} + 2}$.

In our precise case, $f_n = 1$ and $g_n = 2r_n$, so $\dfrac{g_n}{f_n} = 2r_n$; our $f_{n-1}$ and $g_{n-1}$ thus become:

$$f_{n-1} = \left(4r_n^2 - 4\right)f_n^2 = 4\left(r_n^2 - 1\right)f_n^2 = 4(r_n - 1)(r_n + 1)f_n^2$$
$$g_{n-1} = \left(2 \times 4r_n^2 - 4 \times 2r_n\right)f_n^2 = 8\left(r_n^2 - r_n\right)f_n^2 = 8r_n(r_n - 1)f_n^2$$

Thus, $\dfrac{g_{n-1}}{f_{n-1}} = \dfrac{8r_n(r_n - 1)f_n^2}{4(r_n - 1)(r_n + 1)f_n^2} = \dfrac{2r_n}{r_n + 1}$.

Now, let's prove by induction that $\forall k \in [\![1, n-2]\!]$, $\dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$:

Let's assume $\exists k \in [\![1, n-3]\!]$, $\dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$,

Then we have:

$$\frac{g_{n-k-1}}{f_{n-k-1}} = h\left(\frac{g_{n-k}}{f_{n-k}}\right) = 2 \times \frac{2r_n}{k \times r_n + 1} \times \frac{1}{\frac{2r_n}{k \times r_n + 1} + 2} = 2 \times 2r_n \times \frac{1}{2r_n + 2(k \times r_n + 1)}$$

$$\Leftrightarrow \frac{g_{n-k-1}}{f_{n-k-1}} = h\left(\frac{g_{n-k}}{f_{n-k}}\right) = \frac{4r_n}{2(r_n + k \times r_n + 1)} = \frac{2r_n}{(k+1) \times r_n + 1}$$

Which proves that $\forall k \in [\![1, n-2]\!]$, $\dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$.

Now, $\forall n \in \mathbb{N} \setminus [\![0, 3]\!]$, $\forall k \in [\![1, n-2]\!]$ we can express all the $\Delta_{n-k}$, and above all the following:

$\Delta_3 = f_2u_2 + g_2v_2 = f_2x_2^2 + f_2x_1^2 + g_2x_2x_1 = f_2x_2^2 + g_2x_2 + f_2$ (because $x_1 = 1$)

$\Delta_2 = g_2^2 - 4f_2^2 = 4 \times \left(\dfrac{r_n^2}{[(n-2) \times r_n + 1]^2} - 1\right) \times f_2^2$

To determine the positivity of $\Delta_2$ we only focus on the positivity of $\dfrac{r_n^2}{[(n-2)\times r_n+1]^2}-1$,

for we know $f_2^2$ and 4 are always positive.

$$\frac{r_n^2}{[(n-2)\times r_n+1]^2}-1\geqslant 0 \;\Leftrightarrow\; r_n^2\geqslant[(n-2)\times r_n+1]^2 \Leftrightarrow \left[1-(n-2)^2\right]r_n^2-2(n-2)r_n-1\geqslant 0$$

$$\Delta=4(n-2)^2-4\times(-1)\left[1-(n-2)^2\right]=4\left[(n-2)^2+1-(n-2)^2\right]=4>0$$

So solutions for all of our previous $\Delta_k$ exist;

$\forall n\in\mathbb{N}\setminus[\![0,3]\!]$, the quadratic coefficient $\left[1-(n-2)^2\right]$ is strictly negative, so:

$$r_n\in\left[\frac{2(n-2)-\sqrt{4}}{2\left[1-(n-2)^2\right]},\frac{2(n-2)+\sqrt{4}}{2\left[1-(n-2)^2\right]}\right]$$

which means:

$$r_n\in\left[\frac{(n-2)-1}{1-(n-2)^2},\frac{(n-2)+1}{1-(n-2)^2}\right]\Leftrightarrow r_n\in\left[\frac{(n-2)-1}{(1-n+2)(1+n-2)},\frac{(n-2)+1}{(1-n+2)(1+n-2)}\right]$$

$$\Leftrightarrow r_n\in\left[-\frac{1}{n-1},-\frac{1}{n-3}\right],\ \forall n\in\mathbb{N}\setminus[\![0,3]\!]$$

Therefore, as $n\to+\infty,\ r_n\sim-\dfrac{1}{n}$

In conclusion, for the following to be true, as $n\to+\infty$:

$$\sum_{k=1}^{n}x_k^2+2r_n\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}x_kx_j\to 0$$

We must have it in the following form:

$$\sum_{k=1}^{n}x_k^2-\frac{2}{n}\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}x_kx_j\to 0\ as\ n\to+\infty$$

Now we could simplify this:

$$\sum_{k=1}^{n}x_k^2-\frac{2}{n}\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}x_kx_j=\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}x_kx_j$$

$$=\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\left(\sum_{k=1}^{n}\sum_{j=1}^{n}x_kx_j-\sum_{k=1}^{n}x_k^2\right)=0$$

$$\Leftrightarrow\left(1+\frac{1}{n}\right)\times\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\sum_{k=1}^{n}\sum_{j=1}^{n}x_kx_j=0$$

And as $n\to+\infty$ the asymptotic equivalences give us the following:

$$\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\sum_{k=1}^{n}\sum_{j=1}^{n}x_kx_j\to 0\ as\ n\to+\infty$$

$$\Leftrightarrow \sum_{k=1}^{n} x_k^2 - \frac{1}{n} \times \left( \sum_{k=1}^{n} x_k \right)^2 \to 0 \; as \; n \to +\infty$$

Now to get back to our problem, if we assume that $\forall k \in [\![1, n]\!],\ x_k = \frac{1}{k^a}$, then we get, as $n \to +\infty$:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left( \sum_{k=1}^{n} \frac{1}{k^a} \right)^2 \to 0$$

Which is therefore - thanks to all we've seen up to now - the new formula on which we'll work from now on, and which is way more easy-to-handle and less obscure than:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \; as \; n \to +\infty$$

## 3 Comparison Of Asymptotic Behaviours

Now, We got this expression from the previous part:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left( \sum_{k=1}^{n} \frac{1}{k^a} \right)^2 \to 0, \; as \; n \to +\infty$$

Since we're dealing with infinity, instead of distinguishing the cases for different fixed values for $a \in \,]0,1[$, **I will speak of a map $(a_n)_{n \in \mathbb{N}^*}$ converging to a real number in $]0,1[$:** $\lim_{n \to +\infty} a_n \to a_{+\infty} \in \,]0,1[$ with a rate of convergence $\epsilon_n = a_n - a_{+\infty}$.

The sums with their corrections (obtained via Taylor expansions) become, as $n \to +\infty$:

$$\sum_{k=1}^{n} \frac{1}{k^{2a_{+\infty}}} - 2\epsilon_n \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} - \frac{1}{n} \times \left( \sum_{k=1}^{n} \frac{1}{k^{a_{+\infty}}} - \epsilon_n \sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} \right)^2 \to 0$$

The correction terms can be ignored for a fast convergence of $a_n$;
**We'll deal with fast and slow convergences**.

It has already been well-established in the literature [1, 2] that $a_{+\infty} \in \,]0,1[$ for all the nontrivial zeros, so $1 - a_{+\infty} > 0$ and then the **squared sum** can be approximated with the **following squared integral as follows if $a_n$ converges fastly to its limit**:

$$\left( \int_1^n \frac{1}{t^a} dt \right)^2 = \frac{\left(n^{1-a} - 1\right)^2}{(1-a)^2} \sim \frac{n^{2-2a}}{(1-a)^2} \; as \; n \to +\infty$$

to obtain the following (I omit the $n$ index of $a_n$ for convenience in these calculations):

$$\forall a \in \,]0,1[ \; and \; as \; n \to +\infty, \; \frac{1}{n} \times \left( \sum_{k=1}^{n} \frac{1}{k^a} \right)^2 \sim \frac{1}{n} \times \frac{n^{2-2a}}{(1-a)^2} = \frac{n^{1-2a}}{(1-a)^2}$$

**And for a slow convergence, the sum of the correction term added in the squared sum**:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{\ln(t)}{t^{a_{+\infty}}} dt \leqslant \sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} \leqslant 1 + \int_2^{n+1} \frac{\ln(t-1)}{(t-1)^{a_{+\infty}}} dt$$

$$with \int_1^{n} \frac{\ln(t)}{t^{a_{+\infty}}} dt = \frac{\ln(n) n^{1-a_{+\infty}}}{1-a_{+\infty}} - \frac{n^{1-a_{+\infty}} - 1}{(1-a)^2}$$

Therefore, since $1 - a_{+\infty} > 0$ we get the following asymptotic equivalence:

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} = \int_1^{n} \frac{\ln(t)}{t^{a_{+\infty}}} dt \sim \frac{\ln(n) n^{1-a_{+\infty}}}{1-a_{+\infty}} - \frac{n^{1-a_{+\infty}}}{(1-a)^2} as\ n \to +\infty$$

**As to the <u>sum of squares</u>, for a fast convergence**:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{1}{t^{2a}} dt \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \int_2^{n+1} \frac{1}{(t-1)^{2a}} dt$$

**And the sum of the correction term added for a slow convergence**:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{\ln(t)}{t^{2a_{+\infty}}} dt \leqslant \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \leqslant 1 + \int_2^{n+1} \frac{\ln(t-1)}{(t-1)^{2a_{+\infty}}} dt$$

So $\exists \mu \in [0,1] \mid \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \mu + \int_1^{n} \frac{\ln(t)}{t^{2a_{+\infty}}} dt\ as\ n \to +\infty$

We get to distinguish $a_{+\infty} \neq \frac{1}{2}$ and $a_{+\infty} = \frac{1}{2}$ for the sum of squares.

**If** $a_{+\infty} \neq \frac{1}{2}$:

**Fast convergence:**

$$\frac{(n+1)^{1-2a} - 1}{1-2a} \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \frac{(n+1-1)^{1-2a} - (2-1)^{1-2a}}{1-2a}$$

(I skipped the details of variable substitution on the right side)

We then obtain the following asymptotic equivalences, as $n \to +\infty$:

$$\frac{n^{1-2a} - 1}{1-2a} \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \frac{n^{1-2a} - 1}{1-2a}$$

Which means that as $n \to +\infty, \exists \lambda \in [0,1], \sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \lambda + \frac{n^{1-2a} - 1}{1-2a}$

**Sum of the correction term added for a slow convergence** (asymptotic equivalent as $n \to +\infty$):

$$\exists \mu \in [0,1] \mid \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t^{2a_{+\infty}}} dt + \mu = \frac{\ln(n) n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{n^{1-2a_{+\infty}} - 1}{(1-2a_{+\infty})^2} + \mu$$

**If** $a_{+\infty} = \frac{1}{2}$:

**Fast convergence:**

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \ln(n) \ as \ n \to +\infty$$

**Sum of the correction term for a slow convergence** (asymptotic equivalent as $n \to +\infty$):

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t} dt \ as \ n \to +\infty$$

$$\int_1^n \frac{\ln(t)}{t} dt = \ln(n)^2 - \int_1^n \frac{\ln(t)}{t}$$

$$\Leftrightarrow \int_1^n \frac{\ln(t)}{t} dt = \frac{\ln(n)^2}{2}$$

## We therefore have three different cases:

- $a_{+\infty} \in ]0, \frac{1}{2}[$
- $a_{+\infty} \in ]\frac{1}{2}, 1[$
- $a_{+\infty} = \frac{1}{2}$

## Case $a_{+\infty} \in ]0, \frac{1}{2}[$:

**If $a_n$ converges fastly enough to its limit, we can take the following for granted**:
$1 - 2a_{+\infty} > 0$ so $n^{1-2a}$ grows unboundedly as $n \to +\infty$, so:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \frac{n^{1-2a}}{1-2a} \ as \ n \to +\infty$$

Thus our expression:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{n^{1-2a}}{(1-a)^2} \to 0 \ as \ n \to +\infty$$

becomes, as $n \to +\infty$:

$$\frac{n^{1-2a}}{1-2a} - \frac{n^{1-2a}}{(1-a)^2} \to 0 \Leftrightarrow (1-a_{+\infty})^2 = 1 - 2a_{+\infty} \Leftrightarrow 1 - 2a_{+\infty} + a_{+\infty}^2 = 1 - 2a_{+\infty}$$

$\Leftrightarrow a_{+\infty}^2 = 0 \Leftrightarrow a_{+\infty} = 0$, **which contradicts** $a_{+\infty} \in ]0, \frac{1}{2}[$.

**If the convergence is slow, the expression with the correction terms is as follows:**

As $n \to +\infty$:

$$\frac{n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{2\epsilon_n \ln(n) n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} + \frac{2\epsilon_n \ln(n) n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} - \frac{\epsilon_n^2 \ln(n)^2 n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} \to 0$$

$$\Leftrightarrow n^{1-2a_{+\infty}} \left( \frac{1}{1-2a_{+\infty}} - \frac{2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1}{(1-a_{+\infty})^2} + \frac{2\epsilon_n \ln(n)}{(1-a_{+\infty})^2} - \frac{\epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2} \right) \to 0$$

$$\textit{which necessitates}:$$

$$\frac{1-2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1-2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2} \to 0$$

$$\Leftrightarrow \frac{(1-a_{+\infty})^2}{1-2a_{+\infty}} - \frac{1-2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2}{1-2\epsilon_n \ln(n)} \to 0$$

$$\Leftrightarrow \frac{a_{+\infty}^2}{1-2a_{+\infty}} - \frac{\epsilon_n^2 \ln(n)^2}{1-2\epsilon_n \ln(n)} \to 0$$

$$\textit{for } \beta = \epsilon_n \ln(n),\ \textit{this means}:$$

$$a_{+\infty}^2 - 2a_{+\infty}^2 \beta - (1-2a_{+\infty})\beta^2 \to 0 \textit{ as } n \to +\infty$$

$$\Delta = 4a_{+\infty}^4 + 4a_{+\infty}^2\left(1-2a_{+\infty}^2\right) = 4a_{+\infty}^2\left(1-a_{+\infty}^2\right) > 0$$

This equation admits two real solutions, let's not delve into the details but just call them $c_1$ and $c_2$, and just keep in mind that epsilon can then be expressed as:

$$\beta \in \{c_1, c_2\} \Leftrightarrow \epsilon_n \in \left\{ \frac{c_1}{\ln(n)}, \frac{c_2}{\ln(n)} \right\}$$

Since multiples of $\epsilon_n n^{1-2a_{+\infty}} \ln(n)$ and $\epsilon_n^2 n^{1-2a_{+\infty}} \ln(n)^2$ dominate the multiples of $\epsilon_n n^{1-2a_{+\infty}}$ and $\epsilon_n^2 n^{1-2a_{+\infty}}$, we neglected multiples of $\epsilon_n n^{1-2a_{+\infty}}$ and $\epsilon_n^2 n^{1-2a_{+\infty}}$ in the final parenthesis, and actually these latter **stay secretly** in the parenthesis while $\frac{1-2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1-2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2}$ **vanishes for** $\epsilon_n \in \left\{ \frac{c_1}{\ln(n)}, \frac{c_2}{\ln(n)} \right\}$.

We can express the sum of these "hidden" terms as $n^{1-2a_{+\infty}}\left(d_1\epsilon_n + d_2\epsilon_n^2\right),\ (d_1, d_2) \in \mathbb{R}^2$, and by the expression of $\epsilon_n$, they become $n^{1-2a_{+\infty}}\left(\frac{c_1 d_1}{\ln(n)} + \frac{c_1^2 d_2}{\ln(n)^2}\right)$ or $n^{1-2a_{+\infty}}\left(\frac{c_2 d_1}{\ln(n)} + \frac{c_2^2 d_2}{\ln(n)^2}\right)$, and therefore:

**Instead of this**:

$$n^{1-2a_{+\infty}} \left( \frac{1}{1-2a_{+\infty}} - \frac{2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1}{(1-a_{+\infty})^2} + \frac{2\epsilon_n \ln(n)}{(1-a_{+\infty})^2} - \frac{\epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2} \right) \to 0 \textit{ as } n \to +\infty$$

**We are left with this:**

$$n^{1-2a_{+\infty}}\left(\frac{c_1 d_1}{\ln(n)}+\frac{c_1^2 d_2}{\ln(n)^2}\right)\to 0 \text{ as } n\to+\infty, \text{ or } n^{1-2a_{+\infty}}\left(\frac{c_2 d_1}{\ln(n)}+\frac{c_2^2 d_2}{\ln(n)^2}\right)\to 0 \text{ as } n\to+\infty$$

And this is **impossible** because, firstly, if $1-2a_{+\infty}>0$, $\ln(n)=o\left(n^{1-2a_{+\infty}}\right)$, $\ln(n)^2=o\left(n^{1-2a_{+\infty}}\right)$, and secondly $\frac{1}{\ln(n)}$ and $\frac{1}{\ln(n)^2}$ have different decay rates, so the sum in the parenthesis can't even become a plain 0.

(I shall make $d_1$ and $d_2$ explicit in the following versions of this paper)

## Case $a_{+\infty}\in\left]\frac{1}{2},1\right[$:

**If $a_n$ converges fastly enough to its limit, we can take the following for granted**:

As $n\to+\infty,\ \exists\lambda\in[0,1],\ \sum_{k=1}^{n}\frac{1}{k^{2a}}\sim\lambda+\frac{n^{1-2a}-1}{1-2a}$

In this case, $1-2a_{+\infty}<0$,

Therefore as $n\to+\infty,\ \sum_{k=1}^{n}\frac{1}{k^{2a}}\to\lambda+\frac{1}{2a-1}$ and then $\sum_{k=1}^{n}\frac{1}{k^{2a}}-\frac{n^{1-2a}}{(1-a)^2}\xrightarrow[n\to+\infty]{}0$ becomes:

$\lambda+\frac{1}{2a-1}-\frac{n^{1-2a}}{(1-a)^2}\xrightarrow[n\to+\infty]{}0$, and since $1-2a_{+\infty}<0$ this means $\lambda+\frac{1}{2a-1}=0$

$\Leftrightarrow(2a-1)\lambda+1=0\Leftrightarrow 2a\lambda=\lambda-1\Leftrightarrow a_{+\infty}=\frac{\lambda-1}{2\lambda}\leqslant 0$ because $\lambda-1\leqslant 0$ while $2\lambda\geqslant 0$ for $\lambda\in[0,1]$, **which also contradicts** $a_{+\infty}\in\left]\frac{1}{2},1\right[$.

**If the convergence is slow, the expression with the correction terms is as follows:**

$$\lambda+\frac{1}{2a_{+\infty}-1}-2\epsilon_n\left(\frac{1}{1-2a_{+\infty}}+\mu\right)-\frac{1}{n}\left(\frac{n^{1-a_{+\infty}}}{1-a_{+\infty}}-\frac{\epsilon_n\ln(n)n^{1-a_{+\infty}}}{1-a}\right)^2\to 0,\ \text{as } n\to+\infty$$

where $\mu\in[0,1]$, thus as $n\to+\infty$;

$$\lambda+\frac{1}{2a_{+\infty}-1}-\frac{n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2}+\frac{2\epsilon_n\ln(n)n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2}-\frac{\epsilon_n^2\ln(n)^2\ n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2}\to 0$$

As $n\to+\infty,\epsilon_n\to 0 \text{ and } n^{1-2a_{+\infty}}\to 0$, and then:

$\lambda+\frac{1}{2a_{+\infty}-1}\to 0 \text{ as } n\to+\infty$ which means $a_{+\infty}=\frac{\lambda-1}{2\lambda}\leqslant 0$ **which contradicts** $a_{+\infty}\in\left]\frac{1}{2},1\right[$ once again.

## Case $a_{+\infty}=\frac{1}{2}$:

**If $a_n$ converges fastly enough to its limit, we can take the following for granted**:

As $n\to+\infty,\ \sum_{k=1}^{n}\frac{1}{k^{2a}}\sim\ln(n)$, so $\sum_{k=1}^{n}\frac{1}{k^{2a}}-\frac{n^{1-2a}}{(1-a)^2}\to 0 \text{ as } n\to+\infty$ becomes:

$$\ln(n) - \frac{n^{1-2a_n}}{(1-a_n)^2} \to 0 \; as \; n \to +\infty$$

**And now, let's reflect upon the conditions for this asymptotic equivalence to hold:**

- As said earlier, we deal with a map $(a_n)_{n \in \mathbb{N}^*}$ converging to a real number in $]0, 1[$ as $n \to +\infty$, $\frac{1}{2}$ in this case, rather than a fixed value $a = \frac{1}{2}$, otherwise it would mean that $\lim_{n \to +\infty} \ln(n) \to 4$ which is absurd,
- $\frac{1}{(1-a_n)^2} \to 4 \; as \; n \to +\infty$, so it doesn't affect the asymptotic behaviour of $n^{1-2a_n}$,
- $\ln(n)$ grows unboundedly as $n \to +\infty$, so we must have $1 - 2a_n > 0$ for all $n$ sufficiently large, for $n^{1-2a_n}$ to grow unboundedly as $n \to +\infty$ as well,
- Had we assumed that $\exists l > 0 \mid \lim_{n \to +\infty} 1 - 2a_n = l$, we would get $\ln(n) - \frac{n^l}{\left(\frac{l+1}{2}\right)^2} \to 0$ as $n \to +\infty$, which is impossible because $\forall l > 0, \; \ln(n) = o\left(n^l\right)$,
- So it is necessary that $1 - 2a_n$ be **strictly positive** for all $n$ sufficiently large while **converging to** $0^+$ as $n \to +\infty$, in order to adequately "bend" $n^{1-2a_n}$,

but even in this case, the expression of $\epsilon_n$ becomes something like $\epsilon_n = -\frac{\ln\left(\frac{\ln(n)}{4}\right)}{2\ln(n)}$ which is **too slow a rate of convergence to neglect the correction terms**; which we ironically did here.

## So we're left with one case, our last chance:

**If the convergence to $\frac{1}{2}$ is slow, the equivalence with the correction terms is as follows:**

$$\ln(n) - 2\epsilon_n\left(\frac{\ln(n)^2}{2}\right) - \frac{n^{1-2\frac{1}{2}}\left(1 - 2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2\right)}{\left(1 - \frac{1}{2}\right)^2} \to 0, \; as \; n \to +\infty$$

$$\Leftrightarrow \ln(n)(1 - \epsilon_n \ln(n)) - 4\left(1 - 2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2\right) \to 0, \; as \; n \to +\infty$$

$$\Leftrightarrow \ln(n) - 4 + \left(8\ln(n) - \ln(n)^2\right)\epsilon_n - 4\epsilon_n^2 \ln(n)^2 \to 0, \; as \; n \to +\infty$$

$\epsilon_n = \frac{1}{\ln(n)}$ is an ideal choice:

$$\ln(n) - 4 + \left(8\ln(n) - \ln(n)^2\right)\frac{1}{\ln(n)} - 4\left(\frac{1}{\ln(n)}\right)^2 \ln(n)^2$$

$$= \ln(n) - 4 + \frac{8\ln(n)}{\ln(n)} - \frac{\ln(n)^2}{\ln(n)} - 4$$

$$= \ln(n) - 4 + 8 - \ln(n) - 4 = 0$$

We have a good $\epsilon_n = \dfrac{1}{\ln(n)} \to 0 \; as \; n \to +\infty$.

So if $a_n$ tends to $\dfrac{1}{2}$ slowly, this adequate $\epsilon_n$ **exists, and voilà, we get the right result**.

$\dfrac{1}{2}$ is the only limit the map $a_n$ can reach as $n \to +\infty$, if it hopes to satisfy:

$$\sum_{k=1}^{n} \frac{1}{k^{2a_n}} - \frac{1}{n} \times \left( \sum_{k=1}^{n} \frac{1}{k^{a_n}} \right)^2 \to 0 \; as \; n \to +\infty$$

And we could ideally write $a_n$ as $a_n = \dfrac{1}{2} + \dfrac{1}{\ln(n)}$

# Conclusion:

For any nontrivial zero $s \in \mathbb{C} \setminus \left\{ 1 + \dfrac{2\pi i k}{ln(2)} \mid k \in \mathbb{Z} \right\}$, $\lim\limits_{n \to +\infty} \sum\limits_{k=1}^{n} \dfrac{(-1)^{k-1}}{k^s} = 0$

implies that $Re(s)$ be a map of $n$, $a_n = Re(s_n)$, the limit of which **necessarily** is: $\lim\limits_{n \to +\infty} a_n = \dfrac{1}{2}$.

Therefore, since $\zeta(s) = 0 \; \Rightarrow \eta(s) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^s} = 0$:

For any nontrivial zero $s \in \mathbb{C} \setminus \left\{ 1 + \dfrac{2\pi i k}{ln(2)} \mid k \in \mathbb{Z} \right\}$, $\zeta(s) = 0 \implies Re(s) = \dfrac{1}{2}$.

This proves the Riemann Hypothesis.

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function* (1986).

[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory* (2004)