A CHILD WANTS NICE AND NO MEAN NUMBERS

A child wants nice and no mean numbers

Mathematics in primary education

2^{nd} edition

Major parts were written at the occasion of M’s sixth birthday in 2012

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Prologue

Mathematics education is a mess. Earlier books *Elegance with Substance* (EWS) (2009, 2015) and *Conquest of the Plane* (COTP) (2011) [1] present a diagnosis:

Mathematicians are trained for abstraction while education is an empirical issue.

When abstract thinking mathematicians enter a classroom they meet with real live pupils. They must experience cognitive dissonance, and they tend to resolve this by relying on tradition. They will tend to regard themselves as evidence that this works. However, mathematical formats have grown historically. Those aren't necessarily designed for didactics. EWS and COTP re-engineer mathematics education for didactic purpose. Each nation is advised to have a parliamentarian enquiry into mathematics education, in order to identify the proper policy for improvement, and to make funds available for change.

This book looks at mathematics in primary education. Its contents can be included in the list of examples where tradition is not as friendly to pupils as can be re-engineered.

I am professionally involved in mathematics education at the level of highschool and the first year of higher education, and thus these thoughts on elementary school are prospective only. Perhaps the proper word is *amateurish.* My very plea is for professional standards, and thus I am sorry to say that I cannot provide this myself for elementary school. For example, Domahs et al. (eds) (2012) discuss finger counting and numerical cognition, with theory and empirical research: which I haven't read or studied, and thus it is quite silly of me to discuss the topic. This qualifier holds for this whole book.

My only defence for this book – or the articles that it collects and re-edits – is that I want to organise my thoughts on this. If parliaments will already need to investigate the issue, with much more funds than I can muster, then it seems acceptable that I organise my marginal comments on primary education too. There is also a good reason why I must collect my thoughts on this. Thinking about education in highschool and the first year of higher education caused questions about more elementary mathematics. It seems rather natural to wonder whether some issues cannot be dealt with in elementary school.

To be sure: it is not at all clear whether the world is served by this book. However, I am still under the impression that these articles support the general diagnosis in EWS and COTP. It may also be that my intuition is wrong and that the questions posed here have good answers, which I only missed because I did not study the issues fully. The book however achieves its goal when it provides some new ideas and perspectives for the *true* researchers of elementary education, and when it indeed provides some additional support for the general diagnosis of EWS and COTP that parliaments must take steps.

This book has a Dutch counterpart in Colignatus (2012a) that was written at the occasion of my son M.'s sixth birthday. These books only partly overlap. Various Dutch texts on local conditions are not interesting for an English translation. The present book includes some new articles since 2012. I thank Yvonne Killian for her permission to use some of her ideas on presenting the Pythagorean Theorem in elementary school.

A shocking discovery in 2014 w.r.t. Holland was that abstract thinking Hans Freudenthal (1905-1990) sabotaged the empirical theory by Pierre van Hiele (1909-2010); [2] see also the discussion in Colignatus (2014, 2015). Readers interested in primary education will not quickly read §15.2 of COTP on the right approach by Van Hiele and the erroneous approach by Freudenthal. For that reason page 135+ below copies that text.

[1] Reviews by Gamboa (2011) and Gill (2012).

[2] https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/

Contents in brief

Contents

Introduction

The West reads and writes **text** from the left to the right, while Indian-Arabic **numbers** are from the right to the left. English pronounces 14 as *fourteen* instead of *ten & four,* and switches order from 21, to *twenty-one.* This order is already better, yet there still is an issue, for structurally the latter is *two·ten & one*. Pronunciation as *ten & four* and *two·ten & one* gives so much more clarity that pupils could learn arithmetic much faster. [3]

Seen from the perspective of the pupil, the traditional pronunciation can be called *mean* and the mathematically proper way is *nice*.

We can express this more diplomatically by referring to the place value system, a.k.a. the positional system. The numbers themselves already fully use the place value system. The traditional pronunciation only *partly* uses the place value system. The present suggestion is that the pronunciation of the numbers fully uses the place value system too.

In *ten* + *ten* = *two·ten,* the result is immediately available in the positional system itself. Thus it would also be advantageous for pupils to grow aware, as a learning goal by itself, not only of the positional system itself, since this is already a learning goal, but also of its relation to language.

Thus teaching arithmetic does not only deal with number but also language. Education errs in regarding English as perfect. English as a language appears to be a crummy dialect of mathematics. A new learning goal will be to recognise the dialect *for what it is*.

The key notion thus is to regard traditional English as a dialect indeed, and extend lessons on arithmetic with clarification of the dialect. This book develops the proposal (i) to teach in a nice language (ii) to clarify the translation of *nice* to *mean* so that pupils grow aware of the pitfalls in the dialect. The translation of mathematical pronunciation to standard English would be like handling any dialect. Given that children learn other languages with ease, while this concerns only a small set of words and concepts, this translation cannot be much of a burden. Perhaps the reluctance in the USA and the UK to learn other languages and accept dialects is a larger bottleneck than possible doubts about the didactic advantages of using mathematics.

The chapter *Marcus learns counting and arithmetic with ten* contains a stylized lesson for six-year olds. This is not intended for actual use in class but provides an example to start thinking about this for research and development. Six-year olds can still be orienting on left and right, for example (perhaps also because of this), and it could take more refined material to handle the issues (like particular feedback on progress and error).

Sadly, though, all fingers are already in use for the numbers 1 – 10, and there are no fingers available to practice on the decimal system itself. Perhaps lower arms help out.

The second type of issues below are more directly on arithmetic (algebra) and (analytic) geometry. The texts relate to the ideas of Pierre van Hiele and Dina Van Hiele - Geldof about levels of insight (understanding, abstraction). These didactic ideas directly transfer from my experience with highschool. Education in highschool requires algebraic insight, and this is based upon arithmetic mastered in elementary school. Van Hiele thought that algebra could be started in elementary school already, and would even be the best subject to start in elementary school with formal deduction and proof.

[3] The middle dots are unpronounced, and are better than hyphens in numbers, to prevent possible confusion with the negative sign.

Pierre van Hiele also proposed to have vectors in elementary school. He was hesitant about formal proofs with geometry there. Killian (2006) (2012) designed a proof of the Pythagorean Theorem that however feels very natural for this environment, and I have seen it work wonderfully for pupils in an enrichment course in elementary school.

This 2nd edition has a major update.

(1) There is a key role for the ampersand in the pronunciation of numbers that is now recognised – while this use of the ampersand was rejected in the first edition.
(2) I proposed Xur and Yur in 2008 [4], then $\Theta = 2\,\pi = 6.28...$ in 2011, and the name Archi for Θ in 2012. [5] Obviously π is handy in some formulas for which $\Theta\ 2^H$ might seem arcane (when unfamiliar). Thus I kept an eye open for new insights. It was a surprise to realise that pupils in elementary school may rather begin with disks and area before they proceed with circles and circumference. This became p131.
(3) Colignatus (2018a) is included here on p21. Research on number sense and competence in arithmetic tends to be invalid because of the issue on pronunciation. This research requires a standard on pronunciation in order to attain validity, even when education itself is slow in change. (An important step is that the German association Zwanzigeins, with chairman Peter Morfeld, takes an interest in this discussion. [6])
(4) Colignatus (2018b) further develops the relation between pronunciation and the education on the place value system. Different levels in the curriculum are recognised. This paper uses *Mathematica* and this software allows that one can actually hear how the numbers would be pronounced, and how things would be for kids who still live in a world of sounds (before reading and writing). This paper is not included here, and the reader is referred to the separate paper, also available in the Wolfram cloud.
(5) Colignatus (2018cd) on the negative numbers are mentioned in the section on arithmetic below, p89. The abstract of (2018c) is included on p102.
(6) Colignatus (2018e), here included on p145, is a letter to the makers of the US *Common Core State Standards* (CCSS) and *Trends in International Mathematics and Science Study* (TIMSS). This letter reports about the issue of this book, using this recent finding on the negative numbers as a hoped-for eye-opener.
(7) Within mathematics education research (MER) we must distinguish between traditional mathematics education (TME) and “reform” or “realistic” mathematics education (RME) and my own proposal of re-engineering mathematics education. [7] Unfortunately, there is a math war between TME and RME. [8] [9] A key factor is that mathematicians are trained on abstraction and not trained on empirical research. Testing on competence in mathematics is often delegated to psychometrics. However, psychometricians may lack understanding of didactics of mathematics. There is a letter to the integrity of research committee of Leiden University, p169, and a supporting analysis using institutional economics, p175. Remarkably this issue adds some 50 pages to this new edition, increasing the weight of the meta-argument.

Our order of discussion thus is: numbers, arithmetic, geometry, meta-commentary.

[4] http://www.wiskundebrief.nl/456.htm#2
[5] https://boycottholland.wordpress.com/2012/02/18/mathematical-constant-archimedes/
[6] https://zwanzigeins.jetzt/
[7] https://zenodo.org/communities/re-engineering-math-ed/about/
[8] https://boycottholland.wordpress.com/2016/01/24/graphical-displays-about-the-math-war/
[9] https://www.theglobeandmail.com/opinion/article-in-the-ongoing-math-wars-both-sides-have-a-point/

Medical School as a model for education

2014-07-18 [10]

In Medical School, doctors are trained while doing both research and treating patients. Theory and practice go hand in hand. We should have the same for education. Teachers should get their training while doing theory and learning to teach, without having to leave the building. When graduated, teachers might teach at plain schools, but keep in contact with their alma mater, and return on occasion for refresher updates.

Some speak about a new education crisis (e.g. in the USA). The above seems the best solution approach. It is also a model to reach all existing teachers who need retraining. Let us now look at the example of mathematics education.

Professor Hung-Hsi Wu [11] of UC at Berkeley is involved in improving K12 math education since the early 1990's. He explains how hard this is, see two enlightening short articles, one in the *AMS Notices* 2011 [12] and one interview in the *Mathematical Medley* 2012. [13] These articles are in fact remarkably short for what he has to tell. Wu started out rather naively, he confesses, but his education on education makes for a good read. It is amazing that one can be so busy for 30 years with so little success while around you Apple and Google develop into multi-billion dollar companies.

Always follow the money, in math education too. A key lesson is that much is determined by textbook publishers. Math teachers are held on a leash by the answers books that the publishers provide, as an episode of *The Simpsons* shows when Bart hijacks his teacher's answers book. [14] As a math teacher myself I tend to team up with my colleagues since some questions are such that you need the answers book to fathom what the question actually might be (and then rephrase it properly).

At one point, the publishers apparently even ask Wu whether he has an example textbook that they might use as a reference or standard that he wants to support. The situation in US math education appears to have become so bad that Wu discovers that he cannot point to any such book. Apparently he doesn't think about looking for a UK book or translating some from Germany or France or even Holland or Russia. In the interview, Wu explains that he only writes a teacher's education book now, and leaves it to the publishers to develop the derived books for students, with the different grade levels, teacher guides and answers books. One can imagine that this is a wise choice for what a single person can manage. It doesn't look like an encouraging situation for a nation of 317 million people. One can only hope that the publishers would indeed use quality judgement and would not be tempted to dumb things down to become acceptable to both teachers and students. In a world of free competition perhaps an English publisher would be willing to replace "rigour" by "rigor" and impose the A-levels also in the US of A.

In my book *Elegance with Substance* (2009, 2015) I advise the parliaments of democratic nations to investigate their national systems of education in mathematics. Reading the experience by Wu suggests that this still is a good advice, certainly for the US.

[10] https://boycottholland.wordpress.com/2014/07/18/the-medical-school-as-a-model-for-education/
[11] http://math.berkeley.edu/~wu/
[12] http://www.ams.org/notices/201103/rtx110300372p.pdf
[13] http://math.berkeley.edu/~wu/Interview-MM.pdf
[14] http://www.wired.com/2013/11/simpsons-math/

About the subject of logic, professor Wu in the interview p14 suggests that training math teachers in mathematical logic would not be so useful. He thinks that they better experience logic in a hands-on manner, doing actual proofs. I disagree. My book *A Logic of Exceptions* (1981 unpublished, 2007, 2011) would be quite accessible for math teachers, shows how important a grasp of formal logic is, and supports the teaching of math in fundamental manner. The distinction between necessary and sufficient conditions, for example, can be understood from doing proofs in geometry or algebra, but is grasped even better when the formal reasons for that distinction are seen. I can imagine that you want to skip some parts of ALOE but it depends upon the reader what parts those are. Some might be less interested in history and philosophy and others might be less interested in proof theory. Overall I feel that I can defend ALOE as a good composition, with some new critical results too.

Thus, apart from what parliaments do, I move that the world can use more logic, even in elementary school.

Update 2015:

Editing the 2nd edition of *Elegance with Substance* (2015), now available, I was struck again by the empirical observation on the diversity of students and pupils. Evidence based education (EBE) may never attain the sample sizes that are required for statistical testing of theories that allow for such diversity. This fits the Medical School model: there is an important role for individual observation and personal hands-on experience to deal with empirical variety. Methodology and statistics remain important, of course, but in balanced application.

It appears that professor Wu is updating some files. There is a rationale that such updates cause new file names and hence new links. A consequence is that old links break. My suggestion is to keep the old file names and links, and only insert the updated text. I have done so one my website and it works fine. Major changes can always be discussed in an appendix. Only fundamental new texts require new links.

One such update concerns professor Wu's text on fractions. [15] The text follows from professor Wu's objective to neatly develop the tradtional approach. Reading it again, I am struck again by the cumbersomeness of that approach. Much more elegant is the suggestion by Pierre van Hiele to abolish fractions, and use the multiplicative form. See this short introduction [16] and the longer discussion in *A child wants nice and no mean numbers* (2015).

Update 2018: On the latter, see p89 below.

[15] https://math.berkeley.edu/~wu/CCSS-Fractions_1.pdf (new link, as long as it lasts)
[16] https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/

English as a dialect for a didactic number system

The problem

The issue came to my attention by Gladwell (2008:228):

> "Ask an English-speaking seven-year-old to add *thirty-seven* plus *twenty-two* in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add *three·tens-seven* and *two·tens-two,* and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It's *five·tens-nine*." (Hyphen edited.)

My alternative suggestion is to use *five·ten & nine,* thus (i) no 'tens' and (ii) the use of a middle dot and ampersand (smaller font). The hyphen is unattractive since it is too similar to subtraction. The dot is not pronounced, like the hyphen or comma. Thus there is not only the notation of 59 and the pronunciation, but also the notation of the pronunciation. [17]

Gladwell (2008:228) also emphasizes the importance of mental working space:

> "(…) we store digits in a memory loop that runs for about two seconds."

English numbers are cumbersome to store. He quotes Stanislas Dehaene:

> "(…) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits."

The problem has an internationally quick fix: Use the Cantonese system and sounds for numbers. It would be good evidence based education (EBE) to determine whether this would be feasible for an English speaking environment (e.g. start in Hong Kong).

Decimal system

There is more to it. The decimal number system has, for digits *a, b, c, d, ….*:

$$...dbca = a \times 10^0 + b \times 10^1 + c \times 10^2 + d \times 10^3 + \ldots$$

The West reads and writes text from the left to the right while Indian-Arabic numbers are from the right to the left. Thus 19 is *nineteen* instead of *ten & nine.* Human psychology apparently focuses on the lowest digits that have been learned first. The order switches in English at *twenty-one*, when attention shifts to the most important weight. While English switches order at 21, Dutch continues in the wrong order till 99 (*negen-en-negentig*). Thus instead of saying *...dcba* (most important weight first) we have reversed pronunciation ...*dcab* for the numbers below 20 (English) or 100 (Dutch). See Ejersbo & Misfeldt (2011) for the Danish convolutions.

Can we do something about these linguistic pecularities ? A key observation is that for higher numbers like 125 the Indian-Arabic writing order happily co-incides with our attention for the most important weights of the digits. Let this order be the guide. Let us agree that 21 is *two·ten & one.* The article *Marcus learns counting and arithmetic with ten* (page 35) explains how this works. The idea is that the most important weight is pronounced first, and that *ten* is the weight (and not *tens*).

Conclusion: We can apparently handle the pecularities of the natural languages. But also at an appalling cost of teaching in primary education. Instead, there is a number system with didactic clarity so that pupils could learn arithmetic more easily: the decimal

[17] 2015: A relevant reference to Barrow (1993) is discussed on page 208 below.

positional system. The translation to English would be a mere matter of learning another dialect, which cannot be a burden given the ease by which children learn other languages, and also given the small set of words and concepts. Perhaps the English and American reluctance to learn other languages and accept dialects is a larger bottleneck than possible doubts about the didactic advantages. The key notion is to regard English as a dialect indeed, and extend lessons on arithmetic with clarification of the dialect.

Language pecularities

For English it may be easier to switch from *nineteen* to *ten & nine* and from *twenty* to *two·ten.* Other languages may have to make a greater adjustment. Consider Dutch as an example for handling such pecularities.

English distinguishes *ten* and *teen* in *nineteen* while Dutch uses *tien* everywhere, such as *negentien* for 19. A possible switch in Dutch to *tien & negen* runs against the problem that the new pronunciation of 90 would be *negen·tien* (English *nine·ten*). It would wreak havoc that the new pronunciation of 90 would be the old 19.

An option in Dutch is to use a new plural: *tienen* (rather than *tientallen* for the numbers of ten). However, the plural *tens* is not needed, and may cause later problems for higher powers such as *ten·ten·ten* for *thousand.* Thus *tens* and *tienen* can be used in discussion but not official pronunciation.

The solution in Dutch is to introduce a new label *tig* which can be done since 20 = *twintig* and 30 = *dertig* and so on. This is presented in Colignatus (2012a).

The equivalent for English would be to use *ty* [18] so that we would get *two·ty* and *three·ty*. The latter is not necessary since we can already use *ten.* Perhaps *two·ty* is better than *two·ten* but *ten* does fine. Better to have *hundred* = *ten·ten* than *ty·ty*.

English tends to use a hyphen: *twenty-two*. Dutch tends to concatenate words and has *tweeëntwintig* with the sudden umlaut to prevent "confusion" over vowels. (Thus an original confusion is solved by introducing another one.) For pupils learning the structure of the number system it is useful to avoid complexity. The middle dot then is better than a hyphen since the subject area is arithmetic and there might be a confusion with the minus sign. Thus Dutch *twee·tig & twee* is fine. Or switch to English or Cantonese.

Positional system and multiplication

It is a question at what age pupils can understand and actually learn multiplication. It is an option to see whether they already can multiply for the numbers up to 5 before progressing with the numbers above 20. When multiplication is known then it is easier to highlight the numerical structure. We can write (using 'times' and 'to time' rather than 'multi-plus' and 'multiplication'):

$$\ldots\ c \times \text{hundred} + b \times \text{ten} + a = \ldots cba$$

and then explain to the pupils, at least at some stage, that the number on the right is pronounced like on the left but without pronouncing "times" and pronouncing "plus" as "and". This is how the positional system supports understanding of arithmetic. At some points this may conflict with the assumed abstraction level of the pupils and perhaps the need to first learn to pronounce numbers before understanding the structure in the pronunciation. But when pupils are learning arithmetic, then we should also discuss how the positional system supports this.

[18] Ty has an origin like Gothic tigjus = tens, decades. https://www.etymonline.com/word/-ty

Notes on *Marcus learns counting and arithmetic with ten*

This discussion quickly becomes more complicated than needed. It is better to proceed with *Marcus learns counting and arithmetic with ten,* since this clarifies what the ideas entail. This is not spelling reform but targeted bilingualism. Please keep in mind:

(1) This text contains a stylized presentation for six-year olds. This is not intended for actual use in class but contains the framework for starting to think about that.
(2) The idea is to write *five·ten & nine,* where the dot is not pronounced and the order helps to decode the position.
(3) Much of arithmetic can be already done by proper pronunciation. Having this creates room for the operators plus and minus.
(4) Numbers are called *low* and *high* instead of *small* and *large,* since the latter would refer to absolute sizes, and a wrong convention might become a block in the later introduction of negative numbers.
(5) Putting the tables of addition together in a big table gives the opportunity to discuss patterns.
(6) Addition of many numbers uses the separation of numbers of ten (or higher) as an intermediate step. After some experience the pupils will use the direct method.

Multiplication is a long word

Before we can proceed with *Marcus* there is the issue that the word *multiplication* itself is long and rather awkward. In Dutch it is *vermenigvuldigen.* Apparently multiplication does not belong to the Indo-European core words like *mom* or *water.* Pupils in elementary school seem to lack easy words to express what they are doing.

Surprisingly, David Tall mentions that *of* is used for multiplication, see page 91 below. Thus *five of two makes ten* would be unambiguous.

I would explain that as *grouping five groups of two makes ten,* and then erase the *group* words. [19] We could call × the *of-sign*, and say *John ofs five and two to get ten,* rather than *John multiplies ....* The surface of a rectangle as *five by two makes ten* might perhaps also be used: *Mary bies two and five to get ten*. But verbs *to of* and *to by* are absent from online dictionaries and even Mathworld. [20]

My guess is that historically the development of *five of two is ten* into a verb *to of* was blocked by prim mathematicians who stuck to Latin *multiplex*. The Italian *volta* generates the English *times* with a reference to Father Time [21] – like in French *fois* and Dutch *keer, maal.* [22] Even when emphasis is put upon the notion of repetition, it is actually distractive. Multiplication is not only repetition of same sizes, but rather the *grouping* of those: creating a set of sets.

Times actually is a prefix *((five times) two) gives ten.* Five times two hamburgers need not be the same event as two times five hamburgers, if we allow for different days. The *times* prefix forces a demonstration that *((two times) five) gives ten too.* Once symmetry has been established we can create a *new* infix *five times two gives ten.* This is needlessly complex, and only gives an infix, i.e. without a rich and easy vocabulary with verb,

[19] Set theory has *joining five sets of two gives a set of ten.*
[20] http://mathworld.wolfram.com/Multiplication.html
[21] Amusing is http://math.stackexchange.com/questions/1150438/the-word-times-for-multiplication. But informative is Mauro Allegranza: "This latin plicō, like the ancient Greek : πλεκτός - "plaited, twisted", comes from Indo-European pleḱ- : “to plait, to weave”." Apparently related to *fold.* A weaving loom indeed reminds of a rectangle for multiplication. Folding a piece of paper however is an example for exponential growth.
[22] https://translate.google.com/#nl/en/keer

adjective and so on. It makes more sense to directly use an infix that actually has a rich and easy vocabulary that expresses symmetry directly.

The question becomes what synonyms for *times* there are. The Webster thesaurus on *times* is absent, with *time* only as a noun, and it is disappointing on *multiply.* [23] The idea of *double*, *triple*, *quadruple*, ... invites to think about a *multiple*, or *multi-plus,* indeed. But *run* is not *multi-walk.* When you are running then you don't want to be reminded continuously that you are actually walking but only faster.

It appears to be a relevant research objective to establish easy words for arithmetic so that pupils can discuss what they are doing without stumbling over the syllables and losing places in working memory. It is fine that mathematicians have developed the words *multiply* and *multiplication* so that adults may know what they are speaking about, but these words are overly complex for First or even Second Grade.

It is not clear how the verb *to group* is used for other applications, but if it is not used much then *group five of two makes ten* would be clearer than *times* on what multiplication is. My proposal in 2012 for Dutch was to use the verb *malen* (English *to mill*)*,* given the already conventional Dutch *vijf maal twee geeft tien.* This was my first reaction to get rid of *vermenigvuldigen.* But *groepeer vijf van twee geeft tien* looks better.

For now, the paper *Marcus learns counting and arithmetic with ten* uses the verb *to time*. Hopefully there is scope for *to group*, or *to of* or *to by* eventually, with *tables of to of.*

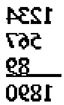
Appendices

Some issues have been put in appendices.

Appendix: A novel way to look at numbers

An option is to mirror the numerals. Thus 19 becomes ϘƖ. It does not take much time to get used to, and when working from left to right then the handling of the overflow in addition feels rather natural, see Table 1.

Table 1. Addition, also in the mirror



1234
567
89
1890

However, the number system is well established, and given the psychological preference to know the size (the digit with the most weight) the present graphical order might be alright. A discussion on the four combinations of Indian / mirror and writing / pronouncing is put in **Appendix B**: *Number sense and sensical numbers.*

Appendix: Fingers and hand

Embodiment or gestures are important for the development of number sense. The decimal positional system can be supported by using fingers and ells (lower arms), see p83. See **Appendix C** on p215 on using fingers and hands. The particular system that is presented there will not be quickly used in first grade itself. It may be of use for students who are training for teachers in elementary school, and who want to re-experience what it is to learn a positional system.

[23] http://www.merriam-webster.com/thesaurus/multiply

The need for a standard for the mathematical pronunciation of the natural numbers. Suggested principles of design. Implementation for English, German, French, Dutch and Danish

September 2-9, 2015 & May 14 2018 (amendment on ampersand) & Sept 14 2018 [24]

Abstract

Current English for 14 is *fourteen* but mathematically it is *ten & four*. Research on number sense, counting, arithmetic and the predictive value for later mathematical abilities tends to be methodologically invalid when it doesn't measure true number sense that can develop when the numbers are pronounced in mathematical proper fashion. Researchers can correct by including proper names in the research design, but this involves some choices, and when each research design adopts a different scheme, also differently across languages, then results become incomparable. A standard would be useful, both ISO for general principles and national implementations. Research may not have the time to wait for such (inter-) national consensus. This article suggests principles of design and implementations for said languages. This can support the awareness about the need for a process towards ISO and national consensus, and in the mean time provides a baseline for research.

Keywords number sense, counting, arithmetic, mathematical ability, invalidity, design, standards, language, pronunciation, metastudy, number processing, numerical development, inversion effects, language-moderated effects, Google Translate

MeSH Terms Child, Child Development, Educational Measurement, Humans, Intelligence, Longitudinal Studies, Mathematics/education, Mathematics/methods, Mental Processes, Students

American Mathematical Society: MSC2010
- 00A35 Methodology and didactics
- 97F02 Arithmetic, number theory ; Research exposition

Journal of Economic Literature: JEL
- P16 Political economy
- I20 General education

Introduction

There is the distinction between (1) a mathematical pronunciation of the natural numbers (0, 1, 2, 3, ...) and (2) the pronunciation of those numbers in the natural languages (English, German, ..., French). While we will use the term "natural language" those languages clearly have been subjected to changes by influential authors and often even committees. Thus the present discussion on a standard on mathematical pronunciation is no breach upon nature itself.

Subsequently we observe that the distinction between (1) and (2) hinders research on number sense, counting and arithmetic, and their predictive value for later mathematical competence. Research methods may suffer from methodological invalidity when they mistake "number sense in natural language" for "true number sense with mathematical

[24] https://doi.org/10.5281/zenodo.774866

pronunciation". Researchers can try to correct by providing pupils with mathematical names, as Ejersbo & Misfeldt (2015) do. There is a risk that researchers implement their own interpretation of what mathematical names are, so that comparison of results becomes more and more difficult or impossible. Hence, a standard for such mathematical pronunciation will be useful, for achieving both validity and comparability.

For such a standard, we first establish the need, then propose principles of design, and then implement those principles to generate proposals for English, German, French, Dutch and Danish. It must be hoped that there will be a process towards consensus on such standards, both in ISO manner and national implementation. This article hopes to generate interest for such a process. In the mean time, researchers who are already in need of a baseline might be helped by the present suggestions.

The present issue differs principally from spelling reform. The spelling of a number ("29"), remains the same. Only its pronunciation changes. The new pronunciation will be spelled in common fashion too. This issue is not about spelling but about bilingualism and mathematical ability. A discussion in the media is by Shellenbarger (2014) in the WSJ.

The need for a standard

Professor Fred Schuh of TU Delft in 1943 observed that the Dutch pronunciation of the numbers was awkward. While English has *twenty-seven* in the order of written 27, Dutch has *zeven-en-twintig*. He again discussed this in Schuh (1949) and formulated a proposal for change, focussing on the numbers above 20. The proposal reached the Dutch minister of education, see Stoffels (1952), but it was not adopted.

Researchers in Norway had observed the same problem, and the Norse parliament (Storting) adopted a change in 1950, which we see reflected in the pronunciation after 1951. [25] I am not aware of an evaluation report. [26] Pixner et al. (2011) observe that the Czech language allows both kinds of pronunciation, and they show that the mathematical order causes less errors than the inverted order.

Various authors look into number sense, counting and arithmetic, in which there is an interplay of language, embodiment (fingers), nonsymbolic forms (e.g. dots), symbols (Indian - Arabic numbers), and working memory. Dowker & Roberts (2015) and Mark & Dowker (2015) compare English, Welsh and Cantonese. Zuber et al. (2009), Moeller et al (2011), Klein et al. (2013) indicate that inversion in German slows down the learning progress w.r.t. mathematics proper. In Holland, Friso - Van den Bos (2014), Xenidou-Dervou (2015) and Xenidou-Dervou et al. (2015) indicate the same for Dutch.

Hopefully this research generates interest amongst policy makers to adopt changes like in Norway 1950/51. However, such changes may still be limited w.r.t. a full mathematical pronunciation. Also English isn't perfect. It would be better to have ten & one for 11 and two·ten & one for 21. Thus the challenge is larger, also for English and Norse.

Recent studies that compare the performances in languages suffer from the problem that they may study the obvious. Schuh (1949) didn't need modern statistics to arrive at the logical conclusion that number-names are better pronounced as they are written. The real problem lies in the policy making process, see Colignatus (2015ab).

The research on the development of number sense tends to suffer from methodological invalidity. In truth, number sense is defined with the use of mathematical pronunciation. The reason for this is that numbers themselves are defined as such. A natural language tends to be a dialect of the mathematical pronunciation. One should not take a dialect as the norm. Studies that do not allow children to develop number sense by using the

[25] http://blogs.transparent.com/norwegian/learning-norwegian-numbers/
[26] I have asked this question at http://www.matematikksenteret.no/

mathematical names, will not observe true number sense, but "number sense in natural language". It may be admitted that one can develop statistical measures on such observations, but such a result is an awkward construct of both true number sense and confusion in language, in unclear mixture, without scientific relevance. [27]

The research on the development of number sense will also benefit from when researchers have deeper roots in mathematics education research (MER). The research quoted above derives mainly from the realm of (neuro-) psychology, and the problems on relevance, validity and comparability might have been observed at an earlier stage when there had been more awareness about what it actually is that pupils must learn. For a mathematician as Fred Schuh the pronunciation *zeven-en-twintig* is obviously illogical, while a neuro-psychologist may record it statistically as an "inversion", and actually think that this is how numbers are pronounced also mathematically, given that mathematicians also use such names. When (neuro-) psychologists would look deeper into MER, they must be warned that this field is not without problems of its own, however. See Colignatus (2015ab) for a longer discussion.

Relevant for research is the question whether pupils can deal with the difference between mathematical names and natural language dialect names. We see that many children can manage, see the examples of Czech, bilingual Chinese, bilingual English & X (e.g. in Holland), and in Ejersbo & Misfeldt (2015). The problem is not with children but in the policy making process, see Colignatus (2015ab).

Thus, researchers interested in number sense, validity and relevance, will tend to follow the example by Ejersbo & Misfeldt (2015) and include in the research design an instruction for pupils for using mathematical names. Perhaps researchers can find schools that are willing to participate in experiments with dual names, given that these aren't really much of experiments since we know that most children can deal with it. When parents are properly informed and first receive a training in the mathematical names, they might readily sign consent forms.

Colignatus (2015b) contains a chapter *Marcus learns counting and arithmetic with ten.* This text contains a stylized presentation for six-year olds. This is not intended for actual use in class but contains the framework for starting to think about that. There are translations for German, French, Danish and Dutch, that is: at this moment of writing the text still is in English but the numbers have been replaced by those in **Appendix D** below. This can also be used to instruct parents.

The real bottleneck then becomes comparability of research results. There are still questions of design. Different researchers might use different rules, and thus we would lose comparability. This establishes the need for a standard.

Principles of design

It is easy to suggest a "mathematical pronunciation of numbers in German", but what would that be ? When we use current *zehn* for 10, then there arises a problem, since the present pronunciation of 19 could be the mathematical pronunciation of 90. This will generate great confusion, and Germans would have to check continuously whether others are using current or mathematical names. However, German might replace *zehn* by *zig* or adopt English *ten* or scientific *deca* (though two syllables).

Number	*Math in English*	*English*	*Math in German ?*	*German*	*Math in German !*
19	ten & nine	nineteen	zehn & neun	neunzehn	**zig & neun**
90	nine·ten	ninety	neun·zehn	neunzig	**neun·zig**

[27] See also my weblog text https://boycottholland.wordpress.com/2015/08/29/research-on-number-sense-tends-to-be-invalid/

The proposed principles of design are:

(7) Pronunciation fully follows the place value system … *c* × hundred + *b* × ten + *a* = …*cba.* The current convention to start with the digit with the highest place value is fine. (See Colignatus (2015b) for lesser alternatives in pronunciation and order.) Much of arithmetic can be done by proper pronunciation (e.g. 2 × 10 + 4 = 24).

(8) In writing out the pronunciation, also in educational texts, the connectives middle-dot (unpronounced) and ampersand (pronounced) are used. We thus say *five·ten* & *nine* for 59, where the dot is not pronounced and the order helps to decode the position. The middle dot is preferred over the hyphen since the latter may be confused with the minus-sign. [28]

(9) **Insert August 20 2018:** (3a) For everyday use (in school) there is simplification in the pronunciation of 1 and 0. The proposed standard has simplified 11 = "ten & one" and not the nonsimplified "one·ten & one·one". (3b) On occasion the nonsimplified form can be used. A teaching objective is that pupils should understand the positional system, and the nonsimplified pronunciation indeed is more informative on this than the simplified pronunciation. However, while the nonsimplified form must be shown for such purpose, the everyday use is served by the simplified form. See Colignatus (2018b) for software that can show both forms, with default simplification. See below for more discussion of this aspect in education.

(10) There is awareness of the distinction between the process of calculation and the result given by the number. The process would be *two times ten plus four* and the result would be *two·ten* & *four.* On occasion *two of ten and four* might have the double role of both process and result. Operators might be bracketed or coloured it indicate that they are not pronounced, as in *two (times) ten plus four.* It must be tested whether young children would be served by a phase in which those operators are still pronounced also for the number result. Also elder pupils might at occasion be reminded of it. Also other names than times must be researched (e.g. the verb *to of*). Plus and minus however would be universal (given that "and" might not be commutative, as in *he missed the train and arrived late at work*).

(11) There are no exceptions in pronunciation of the digits in different place value positions. For example, German currently uses *sieben* in 7 and 27 and *sieb* in 70. A choice must be made for one name only. As a rule the shortest name is selected. For English some authors use *tens* as in *two·tens* & *one,* but *ten* is the value of the place, and must be used consistently. Multiplication can be scalar multiples (2 km) or consists of making groups, and can be expressed by the word *times,* or find another word that expresses this better, such as *grouping.*

(12) A key point for the standard is that it is identified where languages can make choices. Thus, a proposal for German identifies such a choice between *zig* and *ten.* It is up to German what it selects, but the standard helps German identify the choice.

(13) If the name of 10 cannot be used as a base (e.g. German *zehn* and Dutch *tien*) then it is tried to find a close substitute already in use (e.g. *zig* in German and *tig* in Dutch), while often a clear option is to use English *ten* or scientific *deca.*

(14) The above only gives the cardinals. There are also the ordinals (first, second, third, ...) and the fractions (that abuse the ordinals, e.g. "a fifth"). The fractions are solved by using $y\ x^H = y / x$ = "*y* per *x*" ($H = -1$). The ordinals are solved by adopting a single extension, e.g. English "th" (one-th, two-th, three-th, ...) or Dutch "de" (een-de, twee-de, drie-de, ...). There is no linguistic morphing (Dutch *tig-de* doesn't become *tig-ste*).

[28] See the use of the minus-sign in the place value system (a chapter in Colignatus (2015a)): https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/

[29] Colloquial words like English *first* and French *premier* will gradually adopt a meaning of *"to begin with*" rather than an ordinal number.

(15) The rule is that mathematical names are used *in calculation*. The national natural language is explained as a dialect of mathematics. It is an explicit educational goal to identify the national language as such a dialect.

(16) It will be useful to denote mathematical pronunciation with a label, say *English-M* and *Deutsch-M*. This now holds for numbers but this may apply to more phenomena later on, notably for the vocabulary. This suits translations too, e.g. Google Translate.

(17) These principles are targeted at becoming a consensus ISO standard. Countries define their own mathematical pronunciation based upon such a standard, and include own national improvements. For example, 7 in Dutch is consistently *zeven* in 7, 27 and 70, but when Dutch changes, it might opt for a single syllable *zeef* anyway. English might prefer *thir* over *three,* with *thirteen, thirty* and *third* then becoming *ten* & *thir, thir·ten* and *thir-th.* (This choice though is not likely, because of potential confusion between *thir·ten* and *thirteen.*)

A suggestion is to have an expert meeting on this. In the mean time it still seems wise to provide this paper that identifies the issue. While the proposals in this paper may already be used in research to enhance comparability, ISO & national standards would be needed for further use such as in official education requirements (US Common Core) and eventually national adoption also in courts of justice.

Amendment May 14 2018

Colignatus (2018b) (update today or later) provides software in *Mathematica* to show how it all would hear and look, taking advantage of the modern facilities for sounds and translation. Revisiting the issue causes the following amendments.

(1) The symbol *Đ* (capital eth) can be used as symbolic 10, and be pronounced as “deka”. The number 10 is universal already, but when each language pronounces it differently, then the universal pronunciation of *Đ* = 10 = deka may help at times. For example, $Đ^0$, $Đ^1$, $Đ^2$, $Đ^3$, ... indicates the place values and does not invite to do an actual calculation.

(2) It is better to use the (smaller) ampersand (&) to separate the place value positions. This is used above but is a major revision of the earlier text of 2015 and deserves clarification. Thus also for higher positions as e.g. 657 = six·hundred & five·ten & seven.

The connectives "&" and "·" have an important role in the pronunciation and writing of the words of the numbers. They differ from the mathematical operators "plus" and "group" (multi-plus), since + and × have commutation, association and distribution.

- The ampersand (&) is the ghost of addition, but simply "and", and not as the operator "plus" with all its properties. The ampersand should be pronounced to separate the place value positions. It is already (often) pronounced in German, Dutch and Danish, and other languages better adopt this practice too. It may take some time to get used to this but afterwards you will wonder why you never did before.
- The center dot (not pronounced) is the ghost of multiplication of the weight and the place value. It is not pure multiplication, like 5 days 2 hamburgers is not quite the same as 2 days 5 hamburgers.

Kids in kindergarten and Grade 1 live in a world of sounds. Thus it is important to also provide them with the &-separator of the place value positions, so that they have this anchor to distinguish which from what. For adults and native speakers of English it may

[29] See the importance of the ordinals for developing number sense (a chapter in Colignatus (2015a)): https://boycottholland.wordpress.com/2014/08/01/is-zero-an-ordinal-or-cardinal-number-q/

seem superfluous. Indeed, I myself in (2015a, footnote 10, and also the former version of this proporal for a standard) found the use of “&” "distractive", and proposed to use the center dot for “&” too: thus as 25 = two·ten·five, without the distinction and merely as an unpronounced connective. However, after much consideration, the empirical observation is that the &-separator really is there. Its existence must be acknowledged instead of hidden from sight.

Namely, in natural language, putting two terms alongside, like in 2 km, means a scalar multiplication. In multiplication as grouping, kids learn to use the times-symbol, but you do not use it for 2 km, like 2 × 1 km. Later students will learn that 2 *a* is multiplication in general, also dropping the times-symbol. If they would have been trained by the pronunciation of the very numbers (and this *a* would be a number, in this scenario like in *a* = 25 = two·ten·five, thus without the “&”) then we create a conundrum: (1) within "*a* = 25 = two·ten·five" the lack of an interfix means addition and (ii) outside of this, in 2 *a*, the lack of an interfix means multiplication ? We should not create conundrums. Thus 25 = two·ten & five.

Indeed, in kindergarten and Grade 1 kids will tend to focus on the & as an important new symbol in their universe, but this is not "distractive" but only fortunate, because it will form a stepping stone for the later learning on addition, i.e. using plus. Eventually they would tend to focus on the figures in the numbers and not the connectives.

Addendum June 28 2018. (i) New findings Van der Ven et al. (2017) and Bussi et al. (eds) (2018) have not been included here but may be mentioned. (ii) I discovered that there is the use of “tigus” (proto-Germanic) and “tigjus” (Gothic) for 10s (more sources). (iii) This suggestion to achieve a standard finds support at https://zwanzigeins.jetzt

Addendum September 14 2018: Hyphens used in common terms like *twenty-one.*

Implementation

The implementation of these principles of design to English, German, French, Dutch and Danish results in the proposals in **Appendix D.** (They are also used in *Marcus learns counting and arithmetic with ten* and its online translations.)

For English, German, Dutch and Danish we skip the elaboration of the numbers 50-100 since these follow the system from 20-50.

For French, the numbers for 70-99 are fully written out however. This again shows the difficulty of international comparisons.

Addendum August 20 2018. The pronunciation in the natural language is called “partial” with respect to the place value system. Education and research are better served with the full pronunciation. Colignatus (2018b) shows (also with software) how the full pronunciation has a basic nonsimplified form while everyday use is better served with simplification.

See above point (3) on teaching of the place value system and simplification. There are (3a) the standard for everyday use (in school) with simplification, and (3b) the question how to teach the positional system and the proposed standard with simplification. This teaching might need its own standard too. However, didactics would require more research. Thus the following considerations are preliminary:

(1) The proposed standard for everyday use has simplification. It is more natural to pronounce 9 as “nine” instead of “nine·one”.
(2) It is most sensible to start from day 1 in kindergarten with using the proposed standard with the simplified pronunciation. This is what the pupils must learn. It would be problematic to first learn the nonsimplified form and later unlearn it again.

(3) The nonsimplified pronunciation must occur at least sometimes during education, to clarify to pupils how the positional system works, and to clarify the role of zero.
(4) Researchers who have wondered about the basic or the simplified form as a standard, better see this as an issue in didactics. There is no need for uncertainty about what the standard for everyday use should be. There is only the empirical question about the didactics of (4a) the place value system and (4b) its simplification in pronunciation. (I thank Peter Morfeld of Zwanzigeins for a discussion on this.)

A suggestion for the didactics is as follows. Suppose that a bike has an odometer (distance meter) and that the display changes as in below table (imagine the digit-wheels turning). The pure pronunciation clearly shows the positional values and their weights. This allows pupils to get to understand how this system works and why zero is so important. A principle is that leading zero's are not pronounced, so that 9 is nine·one without simplification but 109 would give one.hundred & zero·ten & nine·one.

Last two digits in an odometer	*Nonsimplified pronunciation*	*Simplified*
...09	(zero·ten &) nine·one	nine
...10	one·ten & zero·one	ten
...11	one·ten & one·one	ten & one

Conclusions

The mathematical pronunciation of numbers is straightforward. The only bottleneck is consensus, as language tends to be social phenomenon. (It remains amazing that two people who haven't met before appear able to speak the same language.)

The principles of design are based upon the place value system, full adherence, minimal distance from current natural language, and a preference for short words. The principles allow the identification of choices to be made.

A prospective implementation is useful, firstly as an example of what it all might mean, secondly to provide researchers, who cannot wait for (inter-) national consensus to continue with their research goals, with a baseline suggestion. Both aspects would support the process towards such ISO & national results.

See **Appendix D** for the proposed implementation for some languages.

Research on number sense tends to be invalid

2015-08-30 [30]

The preceding weblog text considered the pronunciation of numbers in English, German, French, Dutch and Danish. [31]

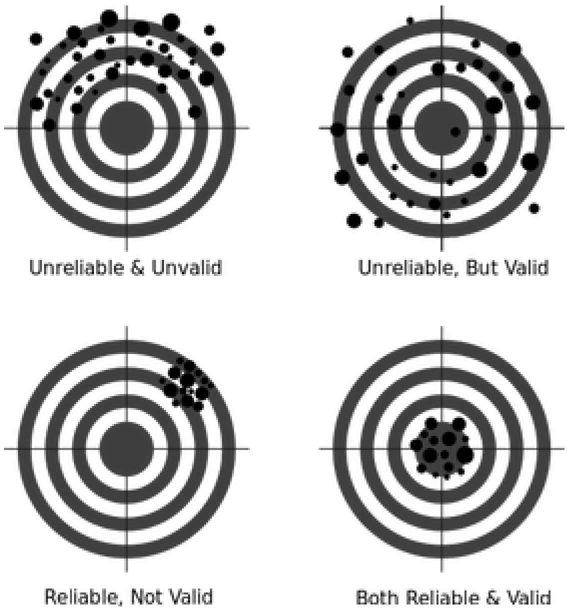
There better be a general warning about invalidity of current research on number sense.

Warning 1. The object of study concerns a chaotic situation

> Research on how children learn numbers, counting and arithmetic, is mostly done in the context of the current confusing pronunciations. This is like studying people walking a tightrope while saying the alphabet in reverse order. This will not allow conclusions on the separate abilities: (a) dealing with arithmetic, (b) dealing with a confusing dialect.

In methodological terms: common studies suffer from invalidity. (Wikipedia. [32]) They aren't targeted at their research objective: number sense. Perhaps they intend to, but they are shooting into a fog, and they cannot be on target.

A positive exception is this article by Lisser Rye Ejersbo and Morten Misfeldt (2015), *"The relationship between number names and number concepts".* [33] They provide pupils with the mathematical names of numbers and study how this improves their competence. This reduces the chaos that other studies leave intact.



Validity and reliability (source: wikimedia commons)

[30] https://boycottholland.wordpress.com/2015/08/29/research-on-number-sense-tends-to-be-invalid/
[31] https://boycottholland.wordpress.com/2015/08/29/mathematical-pronunciation-of-numbers/
[32] https://en.wikipedia.org/wiki/Validity_%28statistics%29
[33] http://pure.au.dk/portal/en/persons/lisser-rye-ejersbo%28b66e2df6-0692-4c95-9d99-10fb812a4cf5%29/publications/the-relationship-between-number-names-and-number-concepts%28029859a5-697b-4d92-8750-62435f58075a%29.html

It is insufficient to state that you want to study *"number sense in the current situation".* When you grow aware that the current situation seriously hinders number sense, then you ought to see that your research objective is invalid, since the current situation confuses number sense. If you still want to study number sense in the current situation, hit yourself with a hammer, since apparently this is the only thing that will still stop you.

Warning 2. Results will be useless

> Results of studies within the current chaos will tend to be useless: (a) They cannot be used w.r.t. mathematical pronunciation, since they don't study this. (b) Once the mathematical pronunciation is implemented, results on number sense within the current chaotic situation are irrelevant.

Warning Sub 2. Don't be confused by a possible exception

> There seems to be one exception to warning 2: the comparison of English, which has low chaos in pronunciation, to other situations with higher chaos (Dutch, German, French, Danish). This presumes similar setup of studies, and would only be able to show that mathematical pronunciation indeed is better. Which we already know. It is like establishing over and over again that drinking affects driving. The usefulness of this kind of study thus must be doubted too. One should not be confused in thinking that it would be useful.

Indeed, we might imagine a diagram with a horizontal axis giving skill in addition with outcomes in the range 10-20 and a vertical axis giving skill in addition with outcomes in the range 20-50, both giving the ages when satisfactory skills have been attained, and then plot the results for English, German, French, Dutch and Danish. We would see that English has lower ages, and French might actually do better than German, since the strange French number names are for 70-99. It might make for a nice diagram, but the specific locations don't really matter since we already know the main message. [34]

For example, Xenidou-Dervou (2015:14) states:

> "Increasingly more studies are suggesting that this inconsistency between spoken and written numbers can have negative effects on school-aged children's symbolic processing (e.g., Helmreich et al., 2011)."

Compare this with our earlier observation [35] that professor Fred Schuh of TU Delft already proposed on these grounds a reform of pronunciation in Dutch in 1943, 1949 and 1952 … Parliament in Norway (their "Storting") decided in July 1950 to rename the numbers above 20 in English fashion.

It is not only problematic that Xenidou-Dervou isn't aware of this, but also that she doesn't see that the current chaotic situation invalidates her own research setup.

She remarks (2015:14) that the logical clarity (Schuh's insight) has not been subjected to statistical testing. This may be true. When you don't understand that drinking affects driving, then you might require statistics. Doing such tests is as relevant as statistical research on verifying that drinking affects driving. She states (my emphasis):

> "To the best of our knowledge, the effect that the language of numbers can have in the development of a core system of numerical cognition such as children's symbolic approximation skills [using Arabic numbers], *controlling for their*

[34] Ann Dowker et al. recently compared English and Welsh, http://www.psy.ox.ac.uk/publications/535741, and English and Cantonese, http://www.psy.ox.ac.uk/publications/514955, which other languages have number names that better conform to the mathematical pronunciation. I read the summaries with interest but also observed that they mainly confirm what logic already requires. A quick fix for education is to use Cantonese.

[35] https://boycottholland.wordpress.com/2015/08/29/mathematical-pronunciation-of-numbers/

> *nonsymbolic approximation skills* [using representations like dots but apparently not fingers] has not been previously addressed."

Thus, the statistics on drunk driving are corrected for the performance on drunk riding a bicycle. It might be suggested that nonsymbolic number sense would be independent from language, and we might readily accept this for numbers smaller than 10, but to properly test this for 11-99 we need a large sample of *Kaspar Hausers* [36] who are unaffected by language. Xenidou-Dervou's correction does not remove the contamination by language.

Statistical tests may indeed be used to establish that large males tend to have a higher tolerance for drinking than small females, and to test legal standards. But questions like these are not at issue in the topic of number sense.

The relevant points are:

- It is already logically obvious that a change to mathematical pronunciation will be beneficial. There is no need for statistical confirmation, e.g. by comparing English with other language situations. To suggest that such research would be necessary is distractive w.r.t. the real scientific question (see next).
- The study of number sense can only be done validly in a situation with mathematical pronunciation, without the noise of the current chaotic situation of the national language dialects.

(PM. This is inverse of the case that there was statistical information that smoking was highly correlated with lung cancer, but that the tabacco industry insisted upon biological evidence. This analogy might arise when researchers would have stacks of statistical results proving that weird pronunciation is highly correlated with slow acquisition of mathematical understanding and skill, while there would be a strong lobby for maintaining national pronunciation who insist upon biological evidence. Thus do not confuse these statistical situations.)

Curiously, the press-release [37] on Xenidou-Dervou's promotion event and publication of the thesis of January 7 2015 states that she 'discovered' something which was already well known to Fred Schuh in 1943, 1949, 1952, if not some present-day teachers and children themselves:

> "From age 5 the influence of teaching is larger than of natural abilities. What hinders Dutch children is the way how numbers are pronounced in Dutch. These relations have been found by Iro Xenidou-Dervou (…)"
>
> "One of the teachers in the researched schools could confirm this with an anecdote from practice. She had heard one pupil telling another pupil doing a calculation: "Do it in English, that is easier.""
>
> "Xenidou-Dervou thus suggests to start in Holland with education in symbolic calculation [with Arabic numbers] already before First Grade [age 6]."

Perhaps we might already start with Arabic numbers before First Grade indeed. Some children already watch Sesame Street. It would be more advisable to do something about pronunciation however. It is perhaps difficult to maintain common sense when you are in a straight-jacket of thesis research.

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[36] https://en.wikipedia.org/wiki/Kaspar_Hauser
[37] http://www.vu.nl/nl/nieuws-agenda/nieuws/2015/jan-mrt/rekenonderwijs-meest-bepalend-voor-rekenprestaties.asp

Warning 3. Such studies will not discover the true cause for the current chaotic situation

> The barrier against the use of mathematical pronunciation doesn't lie with the competences of children but with the national decision making structure. Thus, most current studies on education and number sense will never discover, let alone resolve, the true problem.

That the mathematical pronunciation will be advantageous is crystal clear. Of course it helps when you are allowed to first walk the tightrope and only then say the alphabet in reverse. Thus we have to look at the national decision making structure to see why this isn't done.

Of key importance are misconceptions about mathematicians. Policy makers and education researchers often think that mathematicians know what they are doing while they don't. Education researchers may be psychologists with limited interest in mathematics per se. Few are critical of what children actually must learn.

We may accept that psychology is something else than mathematics education, but when a psychologist researches the education of mathematics then we ought to presume that they know about mathematics education. When they don't understand mathematics education then they should not try to force it into their psychological mold, and go study something else.

Two relevant books of mine on this issue are:

- *A child wants nice and not mean numbers*, [38] for primary education
- *Elegance with Substance*, [39] for education in general but targeted at highschool and first year of tertiary education.

Warning 4. Mathematics education research has breaches of scientific integrity

> Current research on education and number sense assumes that there is an environment with integrity of science. However, there is a serious breach by Hans Freudenthal (1905-1990) w.r.t. the results of his Ph. D. student Pierre van Hiele (1909-2010). [40] Van Hiele discovered the key educational relevance of the distinction between concrete versus abstract, with levels of insight, while Freudenthal interpreted that as the distinction between applied and pure mathematics, and henceforth used his elbows to get Van Hiele out of the way. Freudenthal was an abstract thinking mathematician who invented his own reality. There now exists a Freudenthal "Head in the Clouds Realistic Mathematics" Institute in Utrecht. Its employees behave as a sect, reject criticism, will not look into Freudenthal's breach of integrity of science, and will not undo the damage. See my letter to IMU / ICMI. [41] Other researchers tend not to know about this, and tend to accept "findings" from Utrecht assuming that it has a "good reputation".

This warning holds in general

Just to be sure: this warning on invalidity of research on number sense is general. We might for example think of issues discussed in the *Oxford Handbook of Numerical Cognition* (2015), edited by Ann Dowker. Or think about issues discussed by Korbinian

[38] http://thomascool.eu/Papers/NiceNumbers/Index.html
[39] http://thomascool.eu/Papers/Math/Index.html
[40] https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/
[41] https://boycottholland.wordpress.com/2014/09/02/for-imu-icmi-integrity-of-science-in-dutch-research-in-didactics-of-mathematics/

Moeller et al. (2011), [42] or Elise Klein et al. (2013). [43] But, this weblog is about a major problem in Holland, and thus it might help to make some remarks concerning the anatomy of Holland.

Comment w.r.t. the Dutch MathChild project

The Dutch MathChild project can be found here, [44] with contacts in Belgium, UK and Canada. Its background is in psychology and not in mathematics education.

The Amsterdam thesis by Iro Xenidou-Dervou (2015) isn't fully online and it should be.[45]

There is the full thesis by Ilona Friso-Van den Bos (2014). [46] She did the thesis at the dept. of education & pedagogy in Utrecht, but now she is at the Freudenthal “Head in the Clouds Realistic Mathematics” Institute (FHCRMI). I looked at this thesis only diagonally. Issues quickly become technical and this is secondary to the first question about validity. At first glance the thesis does not show sect behaviour (allowing for contagion from FHCRMI to other places at Utrecht University). The names of Freudenthal and Van Hiele are not in the thesis. The thesis has a neuro-psychological setup with a focus on working memory, which suggests some distance from mathematics education.The scheme of the thesis is that you define a test for number sense, a test for working memory, and a test for mathematical proficiency (try to imagine this without number sense and working memory), and then use children to see what model parameters can be estimated. Criticism 1 is that “mathematics achievement” is in the title and used frequently (see also the picture on p282), and taken for Holland as the CITO score (p160), which has a high FHCRMI content (so we find contagion indeed). Criticism 2 is that working memory belongs to the current fashion in neuro-psychology but is less relevant for mathematics education. For ME it is important to get rid of Freudenthal’s misconceptions and to look at Van Hiele levels of insight. Thus, get proper use of working memory, rather than train it to become a bit larger to do crummy FHCRMI math.

Criticism 3 concerns our present issue: the handling of the pronunciation of numbers. The thesis gives:

> “(…) a difference between participants from linguistic backgrounds in which number words are inverted (e.g., saying six-and-twenty instead of twenty-six), because these inversions have been suggested to be a source of difficulty in number processing (Klein et al., 2013), and that errors related to inversion can be associated with central executive performance (Zuber, Pixner, Moeller, & Nuerk, 2009).” (p82)
>
> “Publication year and inversion of number words did not play a role in the prediction of effect sizes.” (p97)

On p197-198 we find, my emphasis:

> “An alternative explanation for the deviation in findings between previous studies (e.g., Barth & Paladino, 2011) and the current study is that in all previous studies, children were taught in English, in which the number system is more uniform than the Dutch number system. Dutch number words include the ones before the tens, instead of tens before ones (e.g., instead of saying thirty-five, one would say five-and-thirty), which is inconsistent with the order of written numerals. **This may make it more difficult for young children to gain insight into the number system**, and might explain the large number of children being

[42] http://www.ncbi.nlm.nih.gov/pubmed/21498043
[43] http://www.ncbi.nlm.nih.gov/pmc/articles/PMC3733006/
[44] http://vu.mathchild.nl/en/home/
[45] http://dare.ubvu.vu.nl/handle/1871/52176
[46] http://www.narcis.nl/publication/RecordID/oai%3Adspace.library.uu.nl%3A1874%2F297856/id/1/Language/NL/uquery/Making sense of numbers/coll/publication

> placed in the random group during kindergarten, leading children to prevail in using less mature placement strategies and skipping the strategy with three reference points to inform number line placements in favour of the most advanced strategy, which is making linear placements. This hypothesis, however, rests under the assumption that children make placements through interpretation of verbal number words, either by transcoding the written number or by listening closely to the experimenter reading the numbers out loud. A study by Helmreich et al. (2011) indeed suggested that inversion errors may be of influence on number line placements in primary school children, although an important difference with the current study was that no numbers were read out loud by the experimenter, making the chance of inversion errors larger. **More experimental studies are needed** to investigate similar differences in findings and manipulate strategy use through variations in instruction in various groups."

Criticism 3 thus generates the sub-criticisms:

(1) It is not only problematic that Friso-Van den Bos doesn't give the earlier reference to professor Fred Schuh of TU Delft in 1943, 1949 and 1952, but also that she doesn't see that the current chaotic situation invalidates her own research setup. Yes, we do see that she makes a correction at times, but the point is that the proper correction is that the thesis as a whole is shelved, since the situation that she studies cannot render the data that she needs.

(2) It is curious that she states that "more experimental studies are needed". Compare this with a study of drunken driving in London, Paris, Oslo, Athens, … to test whether there are differences … I cannot understand how an educator can observe the crooked pronunciation of numbers, and not see immediately how important it is to *remove* the bottleneck rather than further *research* it. This is like finding a cancer and not remove it but argue that it needs more study. One might say that it is "only a Ph. D. study", but the idea of a dissertation is that it shows that one can do scientific research by oneself individually. A researcher should be able to spot issues on validity. (Perhaps most Ph. D. students are too young or perhaps standards are too low given current academic culture.)

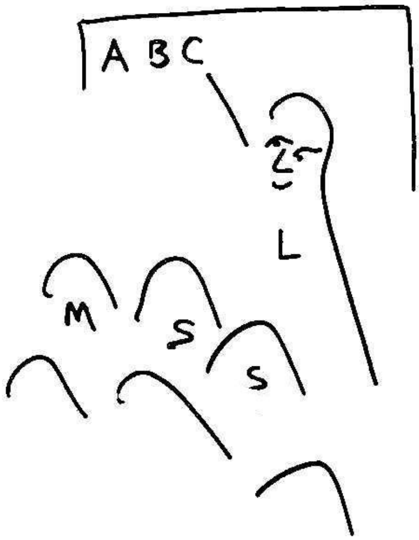
Concluding on the responsibility of educators of mathematics

As in the earlier weblog text, the main responsibility lies with Parliament: to investigate the issue.

It will still be the educators of mathematics who have the responsibility to re-engineer the mathematical pronunciation of numbers, to be used in education, and subsequently also in society and courts of justice. As a teacher of mathematics, I have presented my suggestions in the earlier weblog text, see here. [47]

[47] https://boycottholland.wordpress.com/2015/08/29/mathematical-pronunciation-of-numbers/

Marcus learns counting and arithmetic with ten


A B C
L
M
S
S

1. Marcus and his friends at school

Marcus is now at school.

His friends Sam and Susan are in his class too.

They have reading, writing and arithmetic.

The teacher is called Linda.

Miss Linda shows how to do it.

2. Marcus knows ten digits

Marcus knows the letters of the alphabet.

He uses the letters to make words.

Marcus also knows the ten digits.

We use these to make the first numbers.

zero	0
one	1
two	2
three	3
four	4
five	5
six	6
seven	7
eight	8
nine	9
ten	10

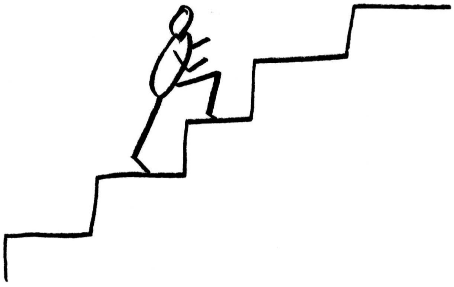
Do you see the difference between a digit and a number ?
A number is recorded with the digits.

A hand has 5 fingers.

Two hands have 10 fingers.

When you calculate with zero then you better use candy. (It must be able to disappear.)

It is Marcus's birthday and he brought cookies !



3. Count and add

Numbers can be used for counting.

You count when you say: 0, 1, 2, 3, 4, 5, … and so on.

Numbers can be used for addition.

You add when you say plus and then what it adds up to.

Or when you write numbers with + and then =.

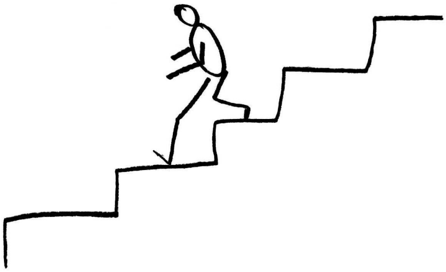
This adds 1.

zero plus one is one	0 + 1 = 1
one plus one is two	1 + 1 = 2
two plus one is three	2 + 1 = 3
three plus one is four	3 + 1 = 4
four plus one is five	4 + 1 = 5
five plus one is six	5 + 1 = 6
six plus one is seven	6 + 1 = 7
seven plus one is eight	7 + 1 = 8
eight plus one is nine	8 + 1 = 9
nine plus one is ten	9 + 1 = 10

You can add also in a column.

number	1	2	3	4	5	6	7	8	9
plus	1	1	1	1	1	1	1	1	1
is	2	3	4	5	6	7	8	9	10

You may switch the first and second row, with the same outcome.



4. Count down and subtract

Numbers can be used to count down.

This is when you say: 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0

Numbers can be used for subtraction.

You subtract when you say minus and then what is the difference.

Or when you write numbers with – and then =.

This subtracts 1.

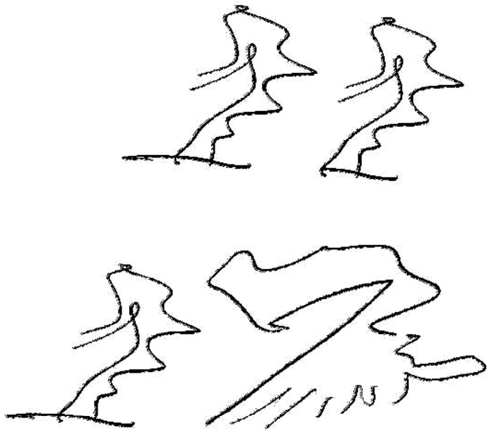
one minus one is zero	1 – 1 = 0
two minus one is one	2 – 1 = 1
three minus one is two	3 – 1 = 2
four minus one is three	4 – 1 = 3
five minus one is four	5 – 1 = 4
six minus one is five	6 – 1 = 5
seven minus one is six	7 – 1 = 6
eight minus one is seven	8 – 1 = 7
nine minus one is eight	9 – 1 = 8
ten minus one is nine	10 – 1 = 9

Check: 9 – 2 = 7 because 7 + 2 = 9.

You can subtract also in a column.

number	1	2	3	4	5	6	7	8	9
minus	1	1	1	1	1	1	1	1	1
is	0	1	2	3	4	5	6	7	8

You may not switch the first and second rows, because the outcome is different. (You will learn this later on.)



5. From ten to two·ten

Sam says: ten is the highest number.

Not true, Marcus says, eleven is higher.

Eleven is a weird number, Susan says.

It is the same as ten & one but people also say eleven.

Yes, Marcus says, for ten & two they say twelve.

That is easy for telling the hour.

number	10	10	10	10	10	10	10	10	10	10
plus	1	2	3	4	5	6	7	8	9	10
is	11	12	13	14	15	16	17	18	19	20

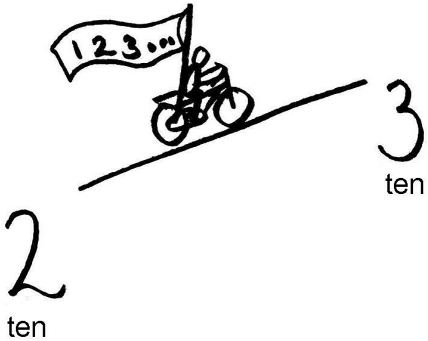
Ten plus ten is two·ten. You write a dot but don't say it.

Miss Linda explains that people say the numbers in different orders. Then the names sound differently. It is useful to know this. But words like eleven and twelve will not be used in calculation.

Marcus, Sam and Susan learn the numbers to two·ten.

They also learn that they can say twenty. But not in calculation.

		Reverse order but not in calculation
ten	10	ten
ten & one	11	eleven
ten & two	12	twelve
ten & three	13	thirteen
ten & four	14	fourteen
ten & five	15	fifteen
ten & six	16	sixteen
ten & seven	17	seventeen
ten & eight	18	eighteen
ten & nine	19	nineteen
two·ten	20	twenty


1 2 3 ...
3
ten
2
ten

6. From two·ten to three·ten

Sam says: two·ten is the highest number.

Not true, Marcus says.

Two·ten plus one gives two·ten & one.

This is higher.

And so on, Marcus says.

Miss Linda explains that people say also *twenty-one*.
But not in calculation.

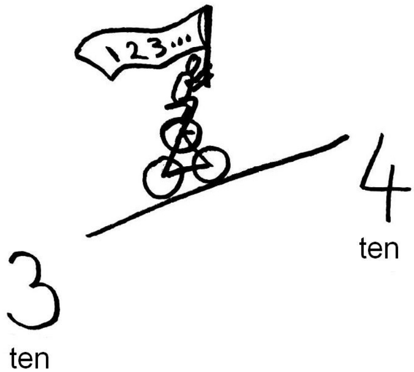
number	20	20	20	20	20	20	20	20	20	20
plus	1	2	3	4	5	6	7	8	9	10
is	21	22	23	24	25	26	27	28	29	30

Two·ten plus ten gives three·ten.

They learn that people also can say thirty. But not in calculation.

Marcus, Sam and Susan now learn the numbers to three·ten.

		Also used but not in calculation
two·ten	20	twenty
two·ten & one	21	twenty-one
two·ten & two	22	twenty-two
two·ten & three	23	twenty-three
two·ten & four	24	twenty-four
two·ten & five	25	twenty-five
two·ten & six	26	twenty-six
two·ten & seven	27	twenty-seven
two·ten & eight	28	twenty-eight
two·ten & nine	29	twenty-nine
three·ten	30	thirty


123...
4
ten
3
ten

7. From three·ten to four·ten

Sam says: three·ten is the highest number.

Not true, Marcus says.

Three·ten plus one gives three·ten & one.

This is higher.

And so on, Marcus says.

Sam and Susan don't believe it.

Marcus says: if you don't believe it, then calculate it yourselves.

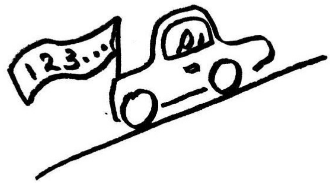
number	30	30	30	30	30	30	30	30	30	30
plus	1	2	3	4	5	6	7	8	9	10
is	31	32	33	34	35	36	37	38	39	40

Three·ten plus ten is four·ten.

They learn that they also can say forty. But not in calculation.

Marcus, Sam and Susan now learn the numbers to four·ten.

		Also used but not in calculation
three·ten	30	thirty
three·ten & one	31	thirty-one
three·ten & two	32	thirty-two
three·ten & three	33	thirty-three
three·ten & four	34	thirty-four
three·ten & five	35	thirty-five
three·ten & six	36	thirty-six
three·ten & seven	37	thirty-seven
three·ten & eight	38	thirty-eight
three·ten & nine	39	thirty-nine
four·ten	40	forty



5
ten

4
ten

8. From four·ten to five·ten

Sam says: four·ten is the highest number.

Not true, Marcus says.

Four·ten plus one gives four·ten & one.

And so on, Marcus says.

Sam and Susan now agree with him.

number	40	40	40	40	40	40	40	40	40	40
plus	1	2	3	4	5	6	7	8	9	10
is	41	42	43	44	45	46	47	48	49	50

Four·ten plus ten gives five·ten.

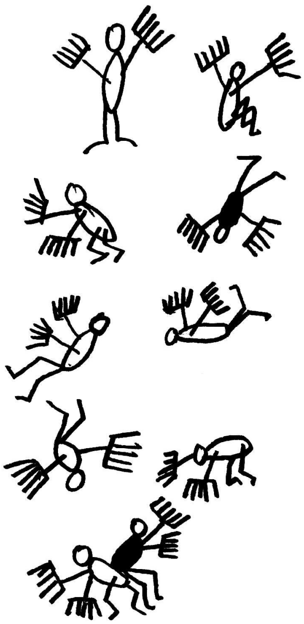
Five children with each ten fingers have five·ten fingers in total.

They learn to count to five·ten.

		Also used but not in calculation
four·ten	40	forty
four·ten & one	41	forty-one
four·ten & two	42	forty-two
four·ten & three	43	forty-three
four·ten & four	44	forty-four
four·ten & five	45	forty-five
four·ten & six	46	forty-six
four·ten & seven	47	forty-seven
four·ten & eight	48	forty-eight
four·ten & nine	49	forty-nine
five·ten	50	fifty

Miss Linda applauds.

They are such smart kids !



9. Ten·ten is hundred

Miss Linda says:

Shall I show you the numbers to a hundred ?

Hundred, Susan asks, what is that ?

Hundred, Miss Linda explains, that is ten·ten.

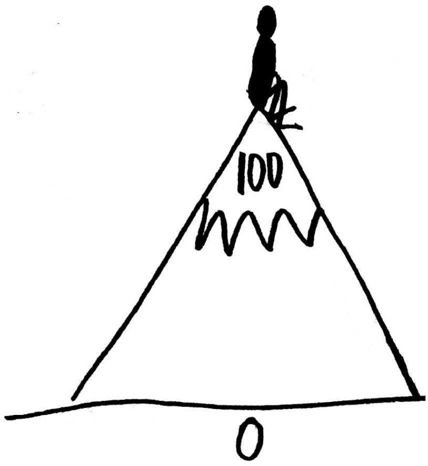
Ten children with ten fingers have ten·ten fingers jointly.

Hundred is a word that we use in calculation too.

And so on, Marcus says, raising his hand with one finger.

Miss Linda laughs.

Yes, she says, that is a hundred and one.


100
0

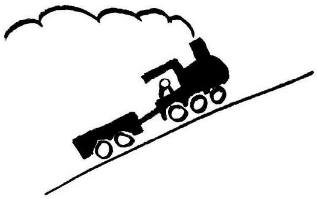
10. Hundred and one numbers

Miss Linda shows the numbers to hundred.

0	10	20	30	40	50	60	70	80	90
1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

These are the numbers of ten.

		Also used but not in calculation
ten	10	ten
two·ten	20	twenty
three·ten	30	thirty
four·ten	40	forty
five·ten	50	fifty
six·ten	60	sixty
seven·ten	70	seventy
eight·ten	80	eighty
nine·ten	90	ninety
ten·ten, hundred	100	



11

ten

10

ten

11. Above hundred

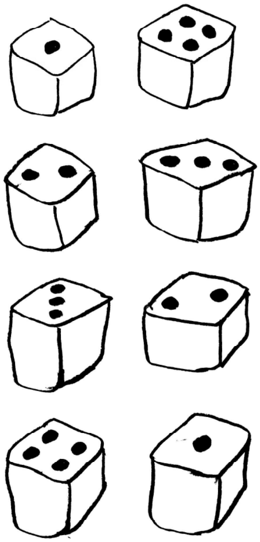
Sam says: hundred is the highest.
Not true, Marcus says.
Hundred plus one is hundred & one.
And so on, Marcus says.

Didn’t you pay attention, Sam ?
Miss Linda already said this, didn’t she ?
Sam and Susan now agree with him.

Miss Linda nods. Hundred & one is 101.

number	100	100	100	100	100	100	100	100	100	100
plus	1	2	3	4	5	6	7	8	9	10
is	101	102	103	104	105	106	107	108	109	110

Miss Linda says: let us look at the numbers less than hundred.



12. The tables of addition to ten

Miss Linda says: let us look at the table of addition.

When we add 1, 2 and 3 with themselves and each other, then we get this table.

+	1	2	3
1	1 + 1 = 2	1 + 2 = 3	1 + 3 = 4
2	2 + 1 = 3	2 + 2 = 4	2 + 3 = 5
3	3 + 1 = 4	3 + 2 = 5	3 + 3 = 6

And so on, Marcus says.

Miss Linda nods.

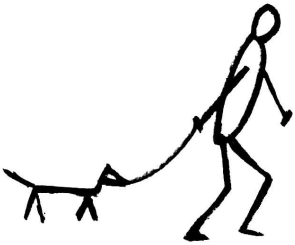
When we add the numbers to ten then we get this table.

+	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16
7	8	9	10	11	12	13	14	15	16	17
8	9	10	11	12	13	14	15	16	17	18
9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	20

Do you see that five fingers plus five fingers is ten fingers ?

And four fingers plus six fingers is ten fingers too.

Do you see that ten plus ten is two·ten ?



13. Mental addition in steps

Susan may pick a number. She says 4.

Sam may pick a number. He says 8.

Miss Linda ask Marcus to add these up. What is 4 + 8 ?

Marcus counts down from 4 to 3.

For the second number he counts up from 8 to 9.

number	4	3
plus	8	9
is		

Marcus counts down from 3 to 2, and up from 9 to 10.

number	4	3	2
plus	8	9	10
is			12

Marcus looks in the table. Yes, 4 + 8 = 12.

Miss Linda explains what is easy to do:

- If the first number is less than 5 you count down, and for the second number you count up.
- If the first number is 5 or more you count up, and for the second number you count down.

14. Mental addition with jumps

When you learn the table of addition by heart then it goes faster.

Then you don't make steps but jumps.

How do you do these sums ?

Does everyone in class have the same outcome ?

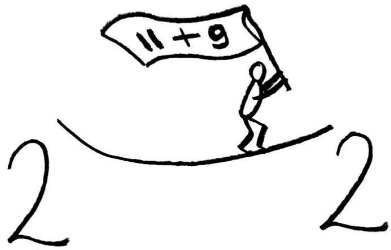
5 + 6 =

7 + 8 =

9 + 3 =

2 + 6 =

4 + 7 =



ten ten

15. The table of addition of two·ten

Miss Linda says: When I use small writing then I can make the table of addition for 1 to 20.

Two·ten plus two·ten is four·ten.

+	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

Susan may pick a number. She says 9.

Sam may pick a number. He says 14.

Miss Linda asks Marcus to add these. What is 9 + 14 ?

Marcus counts from 9 to 10, and down from 14 to 13.

number	9	10
plus	14	13
is		23

Marcus checks the table. Yes, 9 + 14 = 23.

16. Adding more numbers

Sam may pick a number. He says 7.

Susan may pick a number. She says 11.

Marcus may pick a number. He says 6. It is his sixth birthday.

What is 7 + 11 + 6 ?

The friends start adding the three numbers.

When they find 0 or 10 then they stop changing them.

number	7	8	9	10
plus	11	10	10	10
plus	6	6	5	4
is				24

You can also add numbers one by one:

7 + 11 + 6 =

18 + 6 = 24

Another sum: 27 + 36 = ... ?

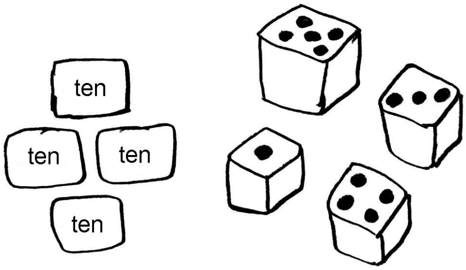
two·ten & seven plus three·ten & six =

step: seven plus six = ten & three

step: the latter ten plus two·ten plus three·ten = six·ten

six·ten & three = 63

You can do it differently but this method works always.


ten
ten
ten
ten

17. Adding many numbers

They may pick one or two numbers each.

Sam says 5 and 11. Susan says 20 and 3. Marcus says 14.

What is 5 + 11 + 20 + 3 + 14 ?

The class wants to find out what these numbers add up to.

Miss Linda shows a fast way.

She takes the numbers of ten apart.

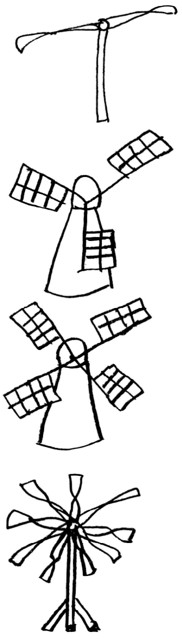
number	5		5	5
plus	11	10	1	11
plus	20	20	0	20
plus	3		3	3
plus	14	10	4	14
is		40	13	53

Five·ten & three. That is a high number !

Marcus shows another way to do it.

$$
\begin{aligned}
&5 + 11 + 20 + 3 + 14 = \\
&16 + 20 + 3 + 14 = \\
&36 + 3 + 14 = \\
&39 + 14 = \\
&40 + 13 = \\
&50 + 3 = 53
\end{aligned}
$$

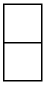
He thinks that the way by Miss Linda is faster.



18. Group, of, by, times

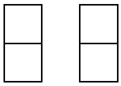
The class counts how many tiles a stoop has.
How many groups are there ? How many are there in a group ?

Here is a group of two tiles. How many tiles are there ?
One group of two = one of two = one by two = one times two = ?



one times two tiles (one by two)
1 × 2 = 2 tiles together (one of two)

Two groups of two tiles. How many tiles are there ?

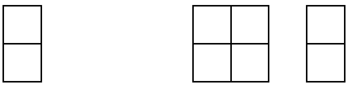


two times two tiles (two by two)
2 × 2 = 4 tiles together (two of two)

Three groups of two tiles. How many tiles are there ?

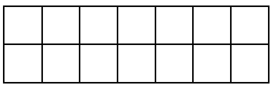
three times two tiles (three by two)
3 × 2 = 6 tiles together (three of two)

Four groups of two tiles. How many tiles are there ?



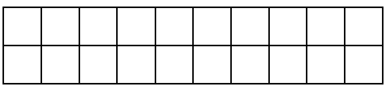
four times two tiles (four by two)
4 × 2 = 8 tiles together (four of two)

Seven groups of two tiles. How many tiles are there ?

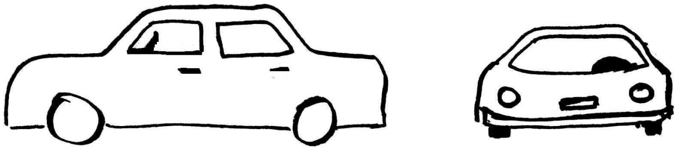


seven times two tiles (seven by two)
7 × 2 = 14 tiles together (seven of two)

Ten groups of two tiles. How many tiles are there ?



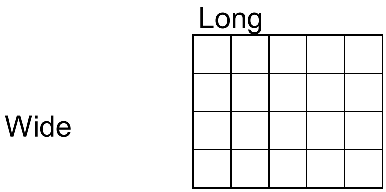
ten times two tiles (ten by two)
10 × 2 = 20 tiles together (ten of two)



19. Length by width

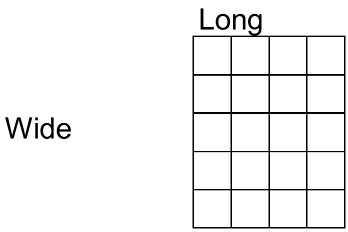
A stoop has length and width.
We take length horizontal (*laying*) and width vertical (*standing*).

This stoop is 5 tiles long and 4 tiles wide.
How many tiles are there ?



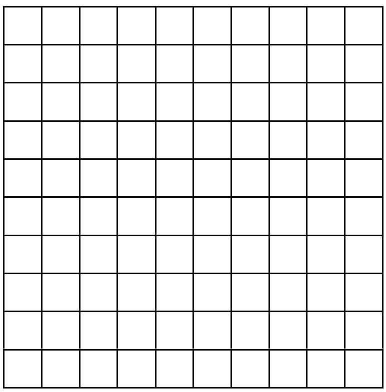
length times width is all
5 times 4 tiles (5 by 4) (5 of 4)
5 × 4 = 20 tiles all together
5 groups of 4 gives 20

This stoop is 4 tiles long (horizontally) and 5 tiles wide (vertically).
How many tiles are there ?



length times width is all
4 times 5 tiles (4 by 5) (4 of 5)
4 × 5 = 20 tiles all together
4 groups of 5 gives 20

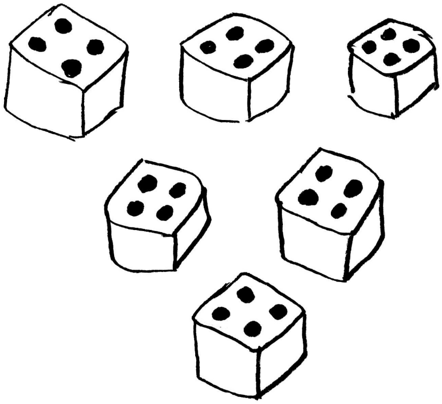
This stoop is 10 tiles long and 10 tiles wide.
How many tiles are there ?



ten times ten tiles
10 × 10 = 100 tiles all together

PM. What is the difference with
§10. Hundred and one numbers (p55) ?

Give an example when you cannot do times ?



20. The table of group, of, by, times

Miss Linda says: now we look at the table of *group, of, by, times.*

When we time 1, 2 and 3 with themselves and each other, then we get the following table.

×	**1**	**2**	**3**
1	1 × 1 = 1	1 × 2 = 2	1 × 3 = 3
2	2 × 1 = 2	2 × 2 = 4	2 × 3 = 6
3	3 × 1 = 3	3 × 2 = 6	3 × 3 = 9

And so on, Marcus says.

Miss Linda nods.

When we time 1 to 10 with themselves and each other, then we get the following table.

×	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

5 Children with each 10 fingers is 5 × 10 = 50 fingers.

And 4 children with each 6 marbles is 4 × 6 = 24 marbles.

21. Speech is silver, silence is golden

Miss Linda shows these sums:

2 × 10 + 7 = 20 + 7 = 27 = two·ten & seven

two ten plus seven speech

times silence

6 × 10 + 3 = 60 + 3 = 63 = six·ten & three

six ten plus three speech

times silence

The name of a number is how it is calculated with ten.

You can understand how numbers are spoken now that you have learned what *group, of, by, times* is.

Idea: write × with red, and + with green.

22. A present for Marcus

Miss Linda says:

Marcus has his birthday and I have a present for him.

Marcus, here are the very high numbers.

			Short - also in calculation
10^1	ten	10	ten
10^2	ten·ten	100	hundred
10^3	ten·ten·ten	1,000	thousand
10^4	ten·ten·ten·ten	10,000	ten·thousand
10^5	ten·ten·ten·ten·ten	100,000	hundred·thousand
10^6	ten·ten·ten·ten·ten·ten	1,000,000	million

In this way you make a high number:

number	5000	five·thousand
plus	300	three·hundred
plus	80	eight·ten
plus	7	seven
is	5387	five·thousand & three·hundred & eight·ten & seven

Miss Linda explains:

There are three·hundred·million people in the USA.

Sam says: that is the highest number that I know.

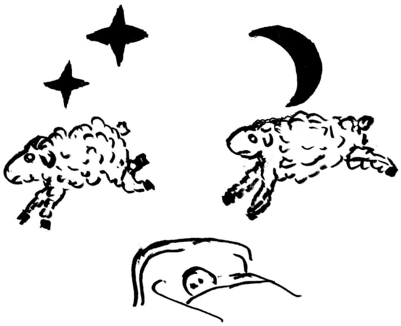
Not true, Marcus says.

300·million plus one gives 300·million & one.

And so on, Marcus says.

Miss Linda laughs.

She says: Today your name is *Marcus And so on*.



23. Marcus counts sheep

It is evening and Marcus is in bed.

His head is full of numbers.

He cannot sleep.

He counts sheep.

One, two, three, four, five, …

Thousand, thousand and one, thousand and two, …

Million, million and one, million and two, million and three, ….

Million·million, million·million and one, ….

Marcus says: and so on.

He falls asleep happily.

Decimal positions using fingers and ells

Introduction

The idea is to allow pupils to grow aware of the positional system much earlier. This will allow them to achieve faster insight in the structure of numbers and arithmetic, including multiplication. Whether this is so, of course needs to be tested in research.

I did not quickly see a developed system on the internet that satisfies some basic conditions: (i) use of fingers and (ii) use of the positional system for those finger signs. My intention here is to clarify what those conditions are.

For example, the American Sign Language drops out, since it doesn't use the positional system. [48] There is some conflicting information on the German DGS (base 5 ?). [49]

The following develops a method to use the lower arms (ells) to signify the numbers of ten: 10, 20, ..., 100. This leaves the hands free to fill in the intermediate digits. The method is a proposal for further research, not a proposal for implementation.

Number sense, process and result

A *sense of number* is natural to many mammals and at least humans, see Piazza & Dehaene (2004). We teach children to use their fingers to count to ten. Milikowski (2010):

> "Kaufmann concludes: a brain doing arithmetic needs the fingers for a long while for support. They apparently help to build a bridge from the concrete to the abstract. In other words: the use of the fingers helps the brain to learn the meaning of the digits."

There is a problem for the numbers higher than 10, since there are no more fingers. Pupils find it difficult to master the positional shift.

In Holland, First Grade is limited to addition and subtraction with the numbers to 20 – a bit comparable to the US *Common Core*. This will be related to the positional shift, the illogical pronunciation of the numbers (*nineteen* instead of *ten & nine* and *twenty* instead of *two·ten*), and the fact that multiplication may quickly give such awkward numbers. When we take a fresh look at the issue then we may agree that learning the numbers to 20 does not have a priority in itself. Unless research would show that First Grade can only grasp number size but not multiplication.

A calculation like 2 × 10 + 4 is not much of a calculation since it is precisely the definition for the number 24 within the positional system (two·ten & four). There is a distinction between calculation as the process and number as the result. EWS:29 has:

> Gray & Tall developed this distinction into the idea of a 'procept'. Tall (2002) seems to embed the 'procept' into the 2nd Van Hiele level:
>
> > "The Symbolic-Proceptual World of symbols in arithmetic, algebra and calculus that act both as PROcesses to do (eg 4+3 as a process of addition) and conCEPTs to think about (eg 4+3 as the concept of sum.)"

[48] https://www.youtube.com/watch?v=teK9oqqOo6g
[49] http://journal.frontiersin.org/article/10.3389/fpsyg.2011.00256/full
and http://www.sign-lang.uni-hamburg.de/alex/lemmata/oberbegr/zahlen.htm

> I have a small problem with this use of vocabulary, in that a concept is not necessarily static and may well be a process too. It is not necessary to limit the distinction between verb and noun to symbols only. It is not entirely clear whether it is really useful to use a new word “procept” to indicate that verbs and nouns are connected, and that processes hopefully give a result and that results tend to be created by processes. That said, the Gray & Tall idea remains important. It points to the phenomemon that mathematics can use *deliberate vagueness* in order to make efficient use of the same symbols. See Colignatus (2014, 2015) for Van Hiele and Tall.

For process, result and their connection, it seems advantageous for pupils to be aware how the positional system works. With that awareness pupils not only count but know that the number system supports both counting and arithmetic. This awareness may already start when they begin using fingers to learn to count.

Caveat

The Prologue of this book has indicated my lack of knowledge about this topic. Libraries have been written about the education on numbers, counting and arithmetic. Domahs, Kaufmann and Fischer (eds) (2012) show a mature field of research which I have not looked into. I have not read the latter reference and lack time to do so. The following is only prospective. These comments have been triggered by the apparent lack of a system that satisfies the mentioned two conditions, but perhaps I did not look well enough.

Base 10 versus base 6

Originally in 2012, I wrote *Numbers in base six in First Grade ?,* here put in the Appendix. This article wonders about a training on the positional system itself, by using fingers and hand in base 6, before using base 10. The fingers on the right hand count the single digits, and the fingers on the left hand count the number of (completed) right hands. The idea is rather radical and will not be quickly adopted. Few parents will offer their children to experiment with. (The pupils might become confused between the senary and decimal systems, for example.) [50]

The only reason to include that article here in the Appendix is that it was a useful stepping stone to think more generally about gestures to indicate number position:

(a) Pupils use the fingers because of their great educational value.
(b) Cognition about the positional system better is an explicit learning goal so that pupils can achieve insight in the structure of numbers and arithmetic, including multiplication.
(c) A question for empirical research is: can pupils in First Grade already multiply ?

The objective becomes: can we think of a *positional sign system* ? The following develops a suggestion how the lower arms (ells) might be used to identify the numbers of ten (10, 20, ..., 100). The fingers are used for the intermediate numbers.

---

[50] In 2012 I wrote seductively: "We might agree on this: Counting the fingers on the back of the hand (with the thumbs in the middle) we use the decimal system, and, counting the fingers on the palm of the hand (with the thumbs sticking out) we use base six, i.e. the senary system. In a senary system with two hands, the right hand for the units 0 to 5 and the left hand for the number of right hands, in the order of the Indian-Arabian positional system. When we have this foundation in First Grade and below then the later change to the decimal system seems a repetition of moves, relatively simple and enlightening." Of course I advised to get evidence, but now in 2015 it seems better to develop a system of gestures for the decimal system anyway, to use from the beginning.
PM. This discussion and the Appendix cover the same subject except for base 10 or 6. It is useful that both discussions can stand by themselves. Some texts thus are copies.

Design principles

A system of signs is in Table 2. Design principles have been:

(1) For arithmetic it is easier to look at your palm and check how the thumb holds down other fingers.

(2) Zero is given by the neutral position of two fists, palms up.

(3) Only the ell (lower arm) is used (since stretching the full arm causes problems in class).

(4) The numbers are assigned in clockwise rotation.

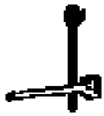
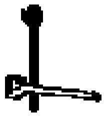
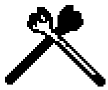
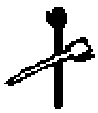
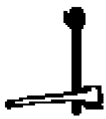
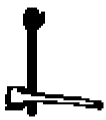
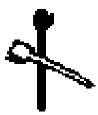
(5) To support the positional shift: *All ten fingers out* is equivalent to the next position with *fists.* For example, ten can also be presented by two fists, crossed at the wrists, left over right, see Table 2. This allows a stepwise transition from fingers to fists. Eventually one of these phases may be skipped. (Thinking continually in terms of equality and replacement will slow down the process of counting.)

(6) The numbers up to and including 50 use the distinction between a fist at the wrist versus a fist at the inside of the elbow. This suits younger pupils. For 60-100 we must use the middle positions of the ells too.

(7) At 50 the hands turn over (from palms up to palms down). At 50 the right ell over left ell (for 40) also switches to left ell over right ell, to allow a new clockwise round.

(8) The table needs only mention the tens (10, 20, ..., 100). Numbers in-between have some fingers out. There is no need for a scheme on fingers.

Table 2. Number, gesture (sign), description

Number	*Gesture*	*Description (ell = lower arm)*
0		Two fists parallel, palms up
10		Two hands parallel, palms up, all fingers stretched
10		Two fists crossing at the wrists, palms up (thumbs together), left ell over right ell
20		Two fists, palms up (thumbs facing each other), left wrist over right elbow (inside of the elbow)
30		Two fists, palms up (thumbs facing each other), right wrist over left elbow (inside of the elbow)
40		Two fists crossing at the wrists, palms up (thumbs together), right ell over left ell
50		Two fists crossing at the wrists, palms down (thumbs not facing each other), left ell over right ell
60		Two fists, palms down (thumbs not facing each other), left wrist over middle of right ell
70		Two fists, palms down (thumbs not facing each other), left wrist over right elbow (inside of the elbow)
80		Two fists, palms down (thumbs not facing each other), right wrist over left elbow (inside of the elbow)
90		Two fists, palms down (thumbs not facing each other), right wrist over middle of left ell
100		Two fists crossing at the wrists, palms down (thumbs not facing each other), right ell over left ell

Related research questions

The advantage of having above signs is that we can consider the introduction of multiplication. For these pupils it seems better to speak about ‘times’ and ‘to time’ (or to repeat) rather than the long terms ‘multiplication’ and ‘multiply’ (multi-plus).

The curious point is:

> When pupils in First Grade can master above sign system (say to 50) then this itself shows that they can master elementary multiplication. Counting groups of ten namely is multiplication by ten. Can they multiply different numbers ?

The discussion of a rectangle and its surface shows that *times* is commutative. Thus, the order of *times* does not matter. When there are five cats with each two eyes then there are 5 × 2 = 10 eyes in total. With five cats you have five left eyes and five right eyes, thus 5 + 5 = 2 × 5 = 10.

A calculation like 2 × 12 = 24 contains operations that seem doable at this level, using the property that 2 × 10 + 4 is the formula for the number 24. Table 3 uses those higher numbers to make the issue nontrivial. How high can the numbers be for First Grade ?

Table 3. Example multiplications

12		10 + 2		19		10 + 9	
2	×	2	×	4	×	4	×
4				36			
20	+			40	+		
24				76			

Many pupils of age six could learn this. Would there be a sufficient number of them to introduce the approach in the general curriculum ?

Counting groups of ten is a higher level of abstraction (the levels identified by Pierre van Hiele). Counting is the ticking-off of the elements of a set. It is a higher level of abstraction to see a set as a new unit of account, and then tick off the sets.

The following is an important insight with respect to *times*:

> A result like 5 × 2 = 10 is trivial for us but only since we learned this by heart.

Some authors argue that pupils need not learn the *table of times* by heart but must first feel their way. This runs against logic. If you don’t learn the *table of times* by heart then you remain caught in the world of addition. This is very slow and does not contribute to understanding. Remember what *times* is:

(1) Taking a set of sets
(2) To know how you can count single elements but that it is faster to only count the border totals
(3) To know which table to use to look it up (namely × instead of +)
(4) And get your result faster because you know the table by heart
(5) To know all of this.

This discussion shows the advantage of knowing what *times* means. Who knows what it is can understand how the numbers are contructed, and can also understands what arithmetic is (the collection of the weights of the powers of the base number). For this reason it is didactically advantageous to have *times* available as quickly as possible.

It becomes a serious research question whether more should be done with set theory in primary education. Apparently pupils are willing and able to memorize long lists of data but it might be enlightening for them to discuss what a set is, and that multiplication concerns the determination of the cardinal number of a set of sets.

Conclusions

Current problems in teaching arithmetic may have to do less with the number system itself, see for comparison the 1950s. In Holland since then there has been a curious move towards *not* learning the tables by heart, see Milikowski (2004). We may already see a big improvement when misunderstandings like these are resolved. That said, it still is a separate issue to think about the number system and its relation to arithmetic.

Libraries have been filled on number and arithmetic but the present discussion seems to includes these useful points:

(1) The sign system with ells seems doable.
(2) This book gives another perspective as well, with the proposal to revise the names of the decimal numbers (with 11 = ten & one and so on).
(3) Research in both didactics and brains could look with priority whether First Grade can multiply. When pupils can learn about the positional system then this already shows their elementary grasp of *times*. Can pupils also multiply with other numbers ? Five cats with two eyes each gives ten eyes. Seems doable as well. When a range of numbers can be found then this can be exploited to develop arithmetic.
(4) Above discussion may also help to beter target learning aims for Second Grade. Problems like $2 \times 10 + 4 = 24$ highlight the structure of number as well.

Re-engineering arithmetic

Introduction

The following three weblog articles from 2014 are highly useful for an overview.

The analysis has been developed further into (2018cd) in *Mathematica* with software, not included here.

Summaries of (2018cd) are:.

(1) Elementary education discusses negative numbers only in Grade 6 or higher. There appears to exist a remarkable misconception. Let us first accept as proper that the lowest Van Hiele level is in direct experience, *and* that this level is the starting point for didactics in elementary school. However, we meet with the argument “there are no negative apples” with the subsequent judgement that negative numbers would be unfit for elementary education. It may also be that educators around 1900 had less understanding of negative numbers, whence they prohibited negative numbers, after which this prohibition continues as dogma. Whatever this be, Colignatus (2018c) clarifies that kids can *turn around*, and that this fits the negative numbers. Thus there is a direct experience at the lowest Van Hiele level that provides a sound base for didactics. This relates to Van Hiele’s idea that pupils already can work with vectors. Having the negative numbers available in the early phase of elementary school makes for a much clearer curriculum.

(2) Colignatus (2018d) extends with tables on addition and subtraction. When pupils can use negative numbers, then subtraction can be done in place value manner, which is more transparant than the current methods.

On rational numbers, ratios, fractions, percentages, decimal numbers and the number line, but also the relation to algebra in secondary education, Dutch readers may compare the following approach that uses H = -1 with Treffers, Streefland & De Moor (1994) who provide an outline for the Dutch curriculum. Their pages 205-216 give an overview discussion of the same subject, much driven by the properties of the fraction bar, and labeling algebra as “empty formalism” (p212) that must be reduced.

Confusing math in elementary school

2014-08-25 [51]

The problems in Russia-Ukraine, Irak-Syria and Israel-Gaza are so large since the combatants are hardly aware of the concept of fair division and sharing. Something must have gone wrong in elementary school with division and fractions. Let us see whether we can improve education, not only for future dictators but for kids in general.

English as a dialect

In 2012 I suggested that English can best be seen as a dialect of mathematics. [52] The case back then was the pronunciation of the integers, e.g. 14 as *fourteen* (English) instead of *ten & four* (math & Chinese). The decimal positional system isn't merely a system of recording but it contains switches in the *unit of account*. In this system the step from 9 to 10 means that *ten* becomes a new unit of account, and the step from 99 to 100 means that *hundred* (ten·ten) becomes a new unit of account. This relies on the ability to grasp a whole and the notion of cardinality. Having a new unit of account means that it is valid to introduce the new words *ten* and *hundred*, so that 1456 as a number differs from a pin-code with merely mentioning of the digits. When the numbers are pronounced properly then pupils will show greater awareness of these elements and become better in arithmetic – and arithmetic is crucial for division and fractions.

When education is seen as trying to plug mathematics into the mold of English as a natural language, then this is an invitation to trouble. It is better to free mathematics from this mold and teach it in its own structural language. It is a task for the teaching of English to show that it is a somewhat curious dialect.

Rank numbers

After the recent discussion of ordinal or cardinal 0, [53] it can be mentioned that the ordinals are curiously abused in the naming of fractions. Check the pronunciation of 1/2, 1/3, 1/4, 1/5, … With number 4 = four and the rank 4th = fourth, the fraction 3/4 is pronounced as *three·fourths*. What is rank *fourth* doing in the pronunciation of 3/4 ? School kids are excused to grow confused.

Supposedly, when cutting up a cake in four parts, one can rank the pieces into the first, second, third and fourth piece. Assuming equal pieces, or fair division, then one might borrow the name of the last rank number *fourth* to say that all pieces are *a fourth*. This is inverse cardinality. Presumably, this is how natural language developed in tandem with budding mathematics. Such borrowing of terms is conceivable but not so smart to do. It is confusing.

The creation of *a fourth*, as a separate concept in the mind, also takes up attention and energy, but it doesn't produce anything particularly useful. Malcolm Gladwell alerted us to that the Chinese language pronounces 3/4 as "*out of four parts, take three*". [54] Shorter would be "3 out of 4″. This directly mentions the parts, and there is no distracting step in-between.

For a reason discussed below we better avoid the "of" in "out of". Thus it might be even shorter to use "3 from 4″, but a critical reader alerted me to that his might be seen as subtraction. Thus "3 out 4″ seems shortest. However, there is also the issue of ratio

[51] https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/ with ampersand 2018
[52] https://boycottholland.wordpress.com/2012/04/01/english-as-a-dialect-of-mathematics/
[53] https://boycottholland.wordpress.com/2014/08/01/is-zero-an-ordinal-or-cardinal-number-q/
[54] http://gladwell.com/outliers/rice-paddies-and-math-tests/

versus rate. In a ratio the numerator and the denominator have the same dimension (say apples) while in a rate they are different (say meter per second).

Thus the overall best shortest pronunciation would be “3 **per** 4″, which is neutral on dimensions, and actually can be used in most European languages that are used to “percent”.

This pronunciation facilitates direct calculation, like “*one per four* plus *three per four* gives *four per four,* which gives *one”.*

Dividing and sharing

The Dutch word for “divide” ("delen") also means “share”. Sharing a cake tends to generate a new unit of account, namely the *part.* In fair division each participant gets a part of the same size, which becomes: the same part. This process focuses on the denominator and generates a larger number and not a smaller number. [55] It actually relies on multiplication: the denominator times the new unit of account (the part) gives the original cake again. The process of sharing is rather opposite to the notion of division that gives a fraction, that maintains the old unit of account and generates a smaller number on the number line.

A fraction 3/4 or *three per four*, when three cakes must be shared by four future dictators, requires the pupil to establish the proportional ratio with *three cakes per four cakes* (virtually giving each a cake even though there are no four cakes but only three), and then rescale from the four hypothetical cakes down to one cake.

PM. The pupil must have a good control of *active* versus *passive* voice. [56] The relation is that “4 kids share 3 cakes” (active) and “3 cakes are being shared by 4 kids” (passive). Thus *3 per 4* or *3 out of 4* is shorthand for “3 units taken out of 4 units”. But also “4 (kids) take out of 3 (cakes)”). It is not “3 (kids) take out of 4 (cakes)” which would give 1 + 1/3 per kid, and would require a discussion of mixed numbers. “Out of” is ambiguous.

Hence it is unfortunate that the Dutch language uses the same word for both *sharing* and *dividing*. Fraction 3/4 reads in Dutch as “3 shared by 4 gives three·fourths” (“3 gedeeld door 4 geeft drie·vierde”), which thus combines the two major stumbling blocks: (a) the sharing/dividing switch in the unit of account, (b) the curious use of rank words. When 3/4 = *three per four* would be used, then the stumbling blocks disappear, and teaching could focus on the difference between the *process* of dividing and the *result* of the fractional number on the number line.

David Tall (2013) points to a related issue in the language on sharing and dividing:

> “The notion of a fraction is often introduced as an *object,* say ‘half an apple’. This works well with addition. (…) What does ‘half an apple *multiplied by* half an apple’ mean? (…) However, if a fraction is seen flexibly as a *process,* then we can speak of the process ‘*half* [halve] an apple’ and then take ‘a *third* of half an apple’ (…) the idea is often simply introduced as a rule, ‘of means multiply’, which can be totally opaque to a learner meeting the idea for the first time.” (p97)

Note that Tall’s book is rather confused [57] so that you better wait for a revised edition. He indeed does not mention above issues (a) and (b). But this latter observation on the *process* and *result* of division is correct.

The rank words thus are abused not only as nouns but also as verbs (“take a *third* of half of an apple”). We better translate into “(*one per three*) of (*one per two*)”, which gives “(one times one) per (three times two)”. The mathematical procedure quickly generates the

[55] Indeed in absolute sizes: not only greater but also larger, not only lesser but also smaller.
[56] http://en.wikipedia.org/wiki/English_passive_voice
[57] http://arxiv.org/abs/1408.1930

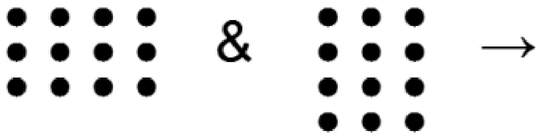
result. The didactic challenge becomes to help kids understand what is involved rather than to master confused language.

Multiplication

Speaking about Tall and multiplication: Apparently the English pronunciation of the tables of multiplication can be wrong too. E.g. *'two fours are eight'* refers to two groups of four, and thus implies an order, while merely *'two times four is eight'* gives the symmetric relation in arithmetic. Tall's book p94 contains a table with 3 rows and 4 columns – see Figure 1 – and Tall argues:

> "the idea of three cats with four legs is clearly different from that of four cats with three legs. The consequence is that some educators make a distinction between 4 x 3 and 3 x 4. (…) I question whether it is a good policy to *teach* the difference. (…) [reference to Piaget] (…) So a child who has the concept of number should be able to see that 3 x 4 is the same as 4 x 3."

Figure 1. An exercise in marbles



0	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	

Tall doesn't provide this explanation: Pierre van Hiele focuses on the distinction *'concrete versus abstract',* and would focus on the table, so that children would master the insight that the order does not matter for arithmetic. Once they have mastered arithmetic, they might consider *'reality versus model'* cases like on the cats and their legs without becoming confused by arithmetical issues hidden in those cases. Instead, Hans Freudenthal with his 'realistic education of mathematics' (RME) would present kids with the 'reality versus model' cases (e.g. also five cups with saucers and five cups without saucers, a 3D table), and argue that this would inspire kids to re-invent arithmetic, though with some guidance ("guided re-invention"). Earlier, I wondered why Freudenthal blocked empirical research in what method works best (and my bet is on Van Hiele). [58] See p 135.

Conclusion

Overall, the scope for improvement is huge. It is advisable that the Parliaments of the world investigate failing math education and its research. When kids have improved skills in arithmetic and language, they would have more time and interest to participate in and understand issues of fair division. Hurray for World Peace !

PM 1. *Conquest of the Plane* pages 77-79 & 207-210 discuss proportions and fractions.

PM 2. See also COTP for the distinction between standard static division y / x and dynamic division y // x.

PM 3. Some say "3 over 4″ for 3/4, hinting at the notation with a horizontal bar. I wonder about that. The "3 per 4″ is actually shorter for "3 taken out of 4″, and this puts emphasis on what is happening rather than on the shape of the notation. An alternative is "3 out of 4″ but my inclination was to avoid the "of" as this is already used for multiplication. Also,

[58] https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/

my original training has been to reserve “*n* over *k*” for the binomial coefficient [59] (that can be taught in elementary school too). However, a reader alerted me to Knuth’s suggestion [60] to use “*n* choose *k*” for the binomial coefficient, and that is better indeed. In any case I would still tend to avoid the “over”.

It was also commented that “3 from 4″ sounded like subtraction: but my proposal is to adhere to “3 minus 4″ for “3 – 4″ as opposed to “3 plus 4″ for addition. It is just a matter to introduce plus and minus into general usage, so that it is always clear what they are. Note that we are speaking about mathematics as a language and not about English as a natural language. Also -1 would be “negative-1″ or “min-1″, with the sign “min” differing from the operator “minus”.

[59] http://en.wikipedia.org/wiki/Binomial_coefficient
[60] http://arxiv.org/abs/math/9205211

Taking a loss

2014-08-30 [61]

Sharp readers will have observed that Vladimir Putin of Russia closely follows the suggestions in this weblog. After the last discussion of "*To invade or not to invade ?"* [62] we now see the *"Alea iacta est"* with Russian tanks crossing the Ukrainean border.

Putin's dilemma reminded of Shakespeare and the Danish prince Hamlet: *"To be or not to be ?"* We shouldn't be surprised that we got a response from Peter Harremoës from Denmark as well. [63]

On the issue of taking a loss, be it the Crimea or now larger parts of the Ukraine, or children losing their fingers in Iraq-Syria or Israel-Gaza, but rather mathematically more general in the form of the subtraction of numbers in arithmetic, and thus the creation of negative numbers, Harremoës has developed a creative new approach that might stop the combatants in amazement. His 2000 article [64] might stop you too, since it still is in Danish, and Google Translate still isn't perfect. Harremoës mentions that he considers an extension in English at some time, so let us keep our fingers crossed till then – while we still have those.

In the mean time I would like to take advantage of some minor points on subtraction, partly relying on Peter's article and thanking him for some additional explanation too.

Notation of negative numbers

Namely, in the last weblog discussion on confusing math in elementary school [65] I stated that it is important to distinguish the *operator* **minus** from the *sign* **min**. Peter referred to $a - (-b)$ and commented that problems of subtraction better be transformed into addition, and that subtraction can be seen on an abstract level as much more complex (or mathematically simple) than commonly thought.

One of his proposals is to create a separate symbol for -1 without the explicit showing of the min-sign. He took an example from history in which *1-with-a-dot-on-top* already stood for -1. I have wondered about this, and would suggest to take a symbol that is available on the keyboard without much ado, where we e.g. already have i = Sqrt[-1].

A-*ha* ! Doesn't the reader hear the penny drop ? Let us take i = quarter turn, H = half turn = i^2 = -1, then $i^3 = H\, i = -i$ = 3 / 4 turn = three per four turn, and $H\, H$ = full turn = 1.

It would appear that H best be pronounced as 'eta', [66] both for international exchange, and in sympathy for German teachers who would otherwise have to pronounce $H\, H$ as 'haha', which would form a challenge for the German sense of humour. I considered suggesting small η or h but the nice thing about H is that it has a shade of -1 in it. In elementary school we can use just the Harremoës-operator H = -1 without the complex numbers. Later in highschool when complex numbers would arise we can usefully refer to H as something that would already be known (or forgotten).

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[61] https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/ with ampersand 2018
[62] https://boycottholland.wordpress.com/2014/08/17/putins-ultimatum-to-himself-to-invade-or-not-to-invade/
[63] http://www.harremoes.dk/Peter/
[64] http://www.harremoes.dk/Peter/talnot.pdf
[65] https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/
[66] http://en.wikipedia.org/wiki/H

Properties of H

Kids can understand that a debt is an opposite from a credit, or that losing the Ukraine is opposite to winning it. Thus if *a* is an asset then *H a* is a liability of the same absolute size.

- Calculation of gains and losses could be done with $a + H\ b$ for counting down, or $H\ b + a$ for counting up.
- If you lose a debt, then you gain. Losing a debt $H\ b$ then would be introduced as $a + H\ (H\ b) = a + b$.

  (2015: This might be misunderstood. Having a debt might be written as $a + H\ b$. Losing that debt then is $a + H\ b + H\ H\ b = a$. The above takes $a' = a + H\ b$.)

Actually, I suppose that it would be even better to start with the absolute difference between two numbers, $\Delta[a, b]$. A sum would be to determine that $\Delta[a, H\ b] = a + b$, presuming that *a* and *b* are nonnegative integers.

Thus *H* would be used in the creation of the negative numbers and the introduction of subtraction, and for later remedial teaching for who didn't get it or lost it. Peter Harremoës seems to be of the opinion that there would be no need, in principle, to introduce minus and min, but agrees that people would currently want to stick to common notions. Once the basics of *H* are grasped, it is no use to grind them in, since it is better to switch to minus and min that must be ground in because of that commonality.

First the **min** sign and the negative integers are introduced by extending the number line: $-1 = H\ 1$, $-2 = H\ 2$, … $-100 = H\ 100$ and so on. The teacher can show that applying *H* means making half turns, or moving from the right to the left, or back.

Subsequently the **minus** operator is introduced as $a - b = a + H\ b$.

Hence there arises the exercise $a - (-b) = a - H\ b = a + H\ (H\ b) = a + 1\ b = a + b$.

Or the relation between minus and min: $-b = 0 - b = 0 + H\ b = H\ b$.

A pupil who has mastered arithmetic will do $a - (-b) = a + b$ directly. Otherwise return to remedial teaching and practice with *H* again.

Positional system

Arithmetic seems simplest in a positional system. Earlier, we already discussed that English better is regarded as a dialect of mathematics. [67] A number like 15 is better pronounced as *ten & five* than as *fifteen*. A sum 15 + 36 then fluently (yes!) translates into "*ten & five* plus *three·ten & six* equals (*one* plus *three*)·*ten*·(*five* plus *six*), equals *four·ten* plus *ten & one*, equals *five·ten & one*" which is 51. Let me introduce the suggestion that pupils can use balloons in handwriting or brackets in typing to indicate not only the digits but also the values in the positional system. See Figure 2.

Figure 2. Adding 15 and 36 using the positional system with balloons or brackets

15 + 36 = (1+3)(5+6) = (4)(11) = 40 + 11 = 51

15 + 36 = [1 + 3][5 + 6] = [4][11] = 40 + 11 = 51

In the same manner, the positional system allows us to state –1234 = [-1][-2][-3][-4], where we might rely on *H* if needed.

[67] https://boycottholland.wordpress.com/2012/04/01/english-as-a-dialect-of-mathematics/

Subtraction in the positional system

For subtraction, the algorithm for $a - b$ is to keep that order if $a \geq b$, or otherwise reverse and calculate $-(b - a)$. But, it is useful to show pupils the following method if they forget about reversing the order. For example, 16 – 34 = 16 + [-3][-4] and the rest follows by itself, see Table 4. [68]

Table 4. Do 16 minus 34 when forgetting to reverse the order

Introduction / Remedial	With typewriter using [...]	Mastered
16 -34 ---- plus 1 - 3 6 - 4 -2 2 -20 + 2 -18	16 -34 ---- plus [1 - 3][6 - 4] [-2][2] -20 + 2 -18 or 16-34 = [1-3][6-4] = [-2][2] = -20 + 2 = -18	16 -34 ---- plus 2 -20 ---- plus -18

Comparing to subtraction in American or Austrian ways

One might compare the above with other expositions on subtraction. An obvious portal is the wikipedia article on subtraction, [69] while google gives some pages e.g. from the UK [70] or the USA. [71] Some texts seem somewhat overly complex.

Originally I thought that the subtraction $a - b$ for $a \geq b$ would be harmless, but on close consideration there is a snake in the grass.

A point is that corrections are made *above* the plus-bar, so that the original question is altered.

- In the Wikipedia example of the Austrian method the final sum doesn't add up anymore.
- The Wikipedia example of the American method is okay, provided that indeed 7 is replaced by 6, and 5 is replaced by 15.

But this is not a proper positional notation anymore. The method also assumes that you use pen and paper, which is infeasible in a keyboard world.

A new method for subtraction

In Table 5 below there are two examples on the right that keep the original sum intact, and that only use the working area *below* that original sum.

One approach is to rewrite 753 = [6][15][3] and the other approach is to do the borrowing a bit later, which is faster.

[68] 2015: The original weblog article has a minus-bar instead of a plus-bar, but it is better to consistenly use only plus-bars.
[69] http://en.wikipedia.org/wiki/Subtraction
[70] http://www.cimt.plymouth.ac.uk/projects/mepres/book7/bk7i15/bk7_15i1.htm
[71] http://www.themathpage.com/arith/subtract-whole-numbers-subtract-decimals.htm

These methods rely on the trick of using balloons or brackets to put values and sub-calculations into a positional place. If we allow for adaptation above the plus-bar, then the use of *H* = -1 and *T* = 10 would work as well, without the need to dash out digits. The second column combines the American & Austrian methods with the Harremoës operator *H,* indeed treated as a digit, and using [H][T][0] = HT0 = 0. See Table 5. [72]

Table 5. Do 753 minus 491, in the American manner (source Wikipedia), with comments and an alternative new way on the right

Wikipedia (American)	*H* = -1, *T* = 10	American [...]	Direct
6 15 ~~7~~~~5~~3 -491 262	HT0 753 -491 ------- plus 262	753 -491 ------- plus REWRITE 6[15]3 -(4 9 1) ----------- plus 262	753 -491 ------- plus 3[-4]2 2[10-4]2 262

Evaluation

Evaluating these methods, my preference is for the last column:

- It follows the work flow, in which the negative value is discovered by doing the steps.
- The method accepts negative numbers instead of creating some fear for them.
- A pupil with experience would not need the 2[10-4]2 line and directly jump to the answer, so that the number of lines is the same as in the first and second column.

The American method (also used in Holland) with HT0 = 0 inserted as a help line creates the suggestion as if borrowing is required before one can do the subtraction, which goes against the earlier training to be able to do a subtraction that results into a negative value. The borrowing is only required to finalize into a final number in standard notation.

Overall, my conclusion is that the emphasis in teaching should be on the positional system. The understanding of this makes arithmetic much easier.

Secondly, the Harremoës operator *H* indeed is useful to first understand the handling of credit and debt, before introducing the number line and the notation *a* – *b*.

Thirdly, in a combination of the two earlier points, this operator also appears useful into decomposing -1234 = [-1][-2][-3][-4].

I want to thank Peter again for starting all this (apart from the more advanced ideas in his article). For completeness, let me refer to the 2012 paper *A child wants nice and not mean numbers*, [73] with a discussion of the pronunciation of the numbers and some more exercise on the positional system.

But these mathematical operations don't explain that Ukraineans first lose the Ukraine but subsequently gain it once they have turned into Russians.

Addendum 2018: Colignatus (2018d) also with software develops the tables further.

[72] 2015: The original weblog article uses minus-bars, but this causes a problem in the second column when there are three rows. The consistent use of plus-bars is better. 2018: *Ð* = 10.
[73] No longer available: since it has become this present book.

With your undivided attention

April 9 2014 [74]

Both President Obama of the USA and President Putin of the Russian Federation have somewhat illogical positions. Obama repeats the ritual article 5 *"An attack on one is an attack on all"* but the Ukraine is not a fraction of NATO. So what is the USA going to do about the Ukraine ? Putin holds that Russia defends all Russians everywhere but claims that Russia is not involved with the combatants in the Ukraine. His proposed 7 point plan contains a buffer zone so that he creates fractions in a country on the other side of the border. Overall, we see the fractional division of the Ukraine starting, as already predicted in an earlier entry in this weblog.

What is it with fractions, that Presidents find so hard, and what they apparently didn't master in elementary school, like so many other pupils ? There are two positions on this. The first position is that mathematics teachers are right and that kids must learn fractions, with candy or torture, whatever works best. The second position is that kids are right and that fractions may as well be abolished as both useless and an infringement of the *Universal Declaration of Human Rights* (article 1). [75] Let us see who is right.

An abolition of fractions

Could we get rid of fractions ? We can replace 1 / *a* or *one·per-a* by using the exponent of -1, giving a^{-1} that can be pronounced as *per-a.* In the earlier weblog entry on subtraction [76] we found the Harremoës operator H = -1. [77] Clearest is $a^H = 1 / a$. Before we introduce the negative numbers we might consider to introduce the new notation for fractions. The trick is that we do not say that. We just introduce kids to the operator with the following algebraic properties:

$$0^H = \textit{undefined}$$

$$a\ a^H = a^H\ a = 1$$

$$(a^H)^H = a$$

Getting rid of fractions in this manner is not my idea, but it was considered by Pierre van Hiele (1909-2010), [78] a teacher of mathematics and a great analyst on didactics, in his book *Begrip en Inzicht* (1973:196-204), thus more than 40 years ago. His discussion may perhaps also be found in English in *Structure and Insight* (1986). Note that $a^H = 1 / a$ already had been considered before, certainly in axiomatics, but the Van Hiele step was to consider it for didactics at elementary school.

From the above we can deduce some other properties.

Theorem 1

$$(a\ b)^H = a^H\ b^H$$

[74] In FMNAI and https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/
[75] http://www.un.org/en/documents/udhr/
[76] https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/
[77] It is one single symbol but still reminds of "-1". Pronounce the operator as "eta".
[78] http://en.wikipedia.org/wiki/Van_Hiele_model

Proof. Take $x = a\ b$. From $x^H\ x = 1$ we get $(a\ b)^H\ (a\ b) = 1$. Multiply both sides with $a^H\ b^H$, giving $(a\ b)^H\ (a\ b)\ a^H\ b^H = a^H\ b^H$, giving the desired. Q.E.D.

Theorem 2

$$H^H = H$$

Proof. From addition and subtraction we already know that $H\ H = 1$. Take $a\ a^{\ H} = 1$, substitute $a = H$, get $H\ H^H = 1$, multiply both sides with H, get $H\ H\ H^H = H$, and thus $H^H = H$. Q.E.D.

It remains to be tested empirically whether kids can follow such proofs. But they ought to be able to do the following.

Simplification

The expression $10 * 5^H$ or *ten per five* can be simplified into $10 * 5^H = 2 * 5\ 5^H = 2$ or *two each*.

Equivalent fractions

Observing that 6 / 12 is actually 1 / 2 becomes $6 * 12^H = 6 * (2 * 6)^H = 6 * 2^H * 6^H = 2^H$. Alternatively all integers are factorised into the primes first. Note that equivalent fractions are part of the methods of simplification.

Multiplication

$$a\ b^H * c\ d^{\ H} = (a\ c)\ (b\ d\ )^{\ H}$$

Comparing fractions

Determining whether $a\ b^H > c\ d^H$ or conversely: this reduces by multiplication by $b\ d$, giving the equivalent question whether $a\ d > c\ b$ or conversely.

Rebasing

That $(a / b = c) \Leftrightarrow (a / c = b)$ may be shown in this manner:

$$a\ b^H\ = c$$

$$a\ b^H\ \ (b\ c^H) = c\ \ (b\ c^H)$$

$$a\ c^H\ = b$$

Addition

Van Hiele's main worry was that we can calculate 2 / 7 + 3 / 5 = 31 / 35 but without much clarity what we have achieved. Okay, the sum remains smaller than 1, but what else ? Translating to percentages 2 / 7 ≈ 28.5714% and 3 / 5 = 60%, so the sum ≈ 88.5714%, is more informative, certainly for pupils at elementary school. This however requires a new convention that says that 0.6 is an exact number and not an approximate decimal, see *Conquest of the Plane* (2011c). The argument would be that working with decimals causes approximation error, and that first calculating 31 / 35 and then transferring to decimals would give greater accuracy for the end result. On the other hand it is also informative to see the decimal constituants, e.g. observe where the greatest contribution comes from.

Another argument is that 2 / 7 + 3 / 5 = 31 / 35 would provide practice for algebra. But why practice a particular format if it is unhandy ? The weighted sum can also be written in terms of multiplication. Compare these formats, and check what is less cluttered:

$$a / b + c / d = (a / b + c / d)\ (b\ d) / (b\ d) = (a\ d + c\ b) / (b\ d)$$

$$a\ b^H + c\ d^{\ H}\ = (a\ b^H + c\ d^{\ H})\ (b\ d)\ (b\ d)^{\ H} = (a\ d + c\ b)\ (b\ d)^{\ H}$$

Subtraction

In this case kids would have to see that H can occur at two levels, like any other symbol.

$$a\ b^H + H\ c\ d^{\,H} = (a\ b^H + H\ c\ d^{\,H})\ (b\ d)\ (b\ d)^{\,H} = (a\ d + H\ c\ b)\ (b\ d)^{\,H}$$

Mixed numbers

A number like *two-and-a-half* should not be written as *two-times-a-half* or 2½ . *Elegance with Substance* (2009) already considers to leave it at 2 + ½. Now we get $2 + 2^H$.

Division

Part of division we already saw in simplification. The major stumbling block is division by another fraction. Compare:

$$a\ /\ b\ /\ \mathbf{\{}c\ /\ d\mathbf{\}} = (a\ /\ b)(d\ /\ c)\ /\ \mathbf{\{}\ (c\ /\ d)\ (d\ /\ c)\ \mathbf{\}} = (a\ /\ b)(d\ /\ c)\ \ /\ \mathbf{\{}\ 1\ \mathbf{\}} = (a\ d)\ /\ (b\ c)$$

$$a\ b^H * (c\ d^{\,H})^H = \ a\ b^H\ \ * c^{\,H}\ d = (a\ d)\ (b\ c\ )^{\,H}$$

Supposedly, kids get to understand this by e.g. dividing 1/2 by 1/10 so that they can observe that there are 5 pieces of 10^H = 1/10 that go into 2^H = 1/2. Once the inversion has been established as a rule, it becomes a mere algorithm that can also be applied to arbitrary numbers like $34^H\ (127^H)^H$ = 1/34 / (1/127). The statement “divide *per-two* by *per-10*″ becomes more general:

(divide by per-*a*) = (multiply by *a*)

Dynamic division

A crucial contribution of *Elegance with Substance* (2009:27) and *Conquest of the Plane* (2011c:57) is the notion of dynamic division, that allows an algebraic redefinition of calculus. [79]

With $y\ x^H = y\ /\ x$ as normal static division then dynamic division $(y\ x^D) = y\ //\ x$ becomes:

$(y\ x^D) \equiv \{y\ x^H$, unless x is a variable and then: assume $x \neq 0$, simplify the expression $y\ x^H$, declare the result valid also for the domain extension $x = 0$ }.

A trick might be to redefine $y\ /\ x$ as dynamic division. It would be somewhat inconsistent however to train on x^H and then switch back to the $y\ /\ x$ format that has not been trained upon. On the other hand, some training on the division slash and bar is useful since it are formats that occur.

Van Hiele 1973

Van Hiele in 1973 includes a discussion of an axiomatic development of addition and subtraction and an axiomatic development of multiplication and division. This means that kids would be introduced to group theory. This axiomatic development for arithmetic is much easier to do than for geometry. Since mathematics is targeted at 'definition, theorem, proof' it makes sense to have kids grow aware of the logical structure. He suggested this for junior highschool rather than elementary school, however. It is indeed likely that many kids at that age are already open to such an insight in the structure of arithmetic. This does not mean a training in axiomatics but merely a discussion to kindle the awareness, which would already be a great step forwards.

His 1973 conclusions are:

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[79] https://boycottholland.wordpress.com/video/

Advantages

1. In the abolition of fractions $1/a$ a part of mathematics is abolished that contains a technique that stands on its own.
2. One will express theorems more often in the form of multiplication rather than in the form of division, which will increase exactness. (See the problem of division by zero.)
3. Group theory becomes a more central notion.
4. In determining derivatives and integrals, it no longer becomes necessary to transform fractions by means of powers with negative exponents. (They are already there.)

Disadvantages

1. Teachers will have to break with a tradition.
2. It will take a while before people in practice write $3\ 4^H$ instead of $3 / 4$.
3. Proponents will have to face up to people who don't like change.
4. We haven't studied yet the consequences for the whole of mathematics (education).

His closing statement: "We do not need to adopt the new notation overnight. It seems to me very useful however to consider the abolition of the algorithms involving fractions."

Conclusion

Given the widespread use of $1 / a$, we cannot avoid explaining that $a^H = 1 / a$. The fraction bar is obviously a good tool for simplication too, check $6 * (2 * 6)^H$.

Similarly, issues of continuity and limits $x \rightarrow 1$ for expressions like $(1 + x)\ (1 - x^2)^H$ could benefit from a bar format too. This would also hold alternatively for $(1 + x)\ (1 - x^2)^D$.

But, awareness of this, and the ability to transform, is something else than training in the same format. If training is done in algorithms in terms of a^H then this becomes the engine, and the fraction slash and bar merely become input and output formats that are of no significance for the actual algebraic competence.

Hence it indeed seems that fractions as we know them can be abolished without the loss of mathematical insight and competence.

Addendum 2015: Page 79 mentions the powers 10^*n* so that pupils can get used to them at an early stage. Killian observed that they may become confused between $10\ n$ and 10^n. Discussing this with her we agreed that 10^*n* is best till pupils are used to the concept and might become relaxed with 10^n. There is reason to do this before division. The confusion on powers namely could also hold for $10\ H$ and 10^H. Testing this innovation thus requires attention. The formulas above look less appealing with x^*H* everywhere. In that case we might perhaps as well write $1 / x$ but retain the idea of multiplication, i.e. that the meaning of $1 / x$ is that $x \times (1 / x) = 1$, and maintain this consistently. For example $1 / 4 = (1 / 2) \times (1 / 2)$ because $2 \times 2 \times (1 / 4) = 1$; and don't use $1 / (2 \times 2)$ with its unnecessary concepts. Overall, it seems better to make sure that pupils do not grow confused between $10\ H$ and 10^H.

A comparable design issue is that it is better to first introduced the system of co-ordinates before introducing fractions. Because, once co-ordinates are available, then we can introduce Proportion Space (see COTP) too, and explain more about x^H.

Addendum 2018: Colignatus (2018c) further develops the fundamental role for $H = -1$.

Arithmetic with H = -1: subtraction, negative numbers, division, rationals and mixed numbers

Abstract of Colignatus (2018c)

April 2 and May 23 2018 [80] [81]

Abstract

$H = -1$ is an universal constant. H represents a half turn along a circle, like i represents a quarter turn. Kids know what it is to turn around and walk back along the same path. H creates the additive inverse with $x + H\,x = 0$ and the multiplicative inverse with $x\,x^H = 1$ for $x \neq 0$. Pronounce H as "ehta" or "symbolic negative one". The choice of H is well-considered: its shape reminds of -1 and even more (-1). Pierre van Hiele (1909-2010) already proposed to use $y\,x^{-1}$ and drop the fraction bar y / x with its needless complexity. Students must learn exponents anyway. The negative exponent might confuse pupils to think that they must subtract something, but the use of an algebraic symbol clinches the proposal. Also 5/2 can be written as $2 + 2^H$, so that it is clearer where it is on the number line. This approach also causes a re-evaluation of the didactics of the negative numbers. The US Common Core has them only in Grade 6 which is remarkably late. The negative numbers arise from the positive axis x by rotating or alternatively mirroring into $H\,x$. Algebraic thinking starts with the rules that $a + H\,a$ can be replaced by 0 and that $H\,H$ can be replaced by 1. Subtraction $a - b \geq 0$ may be extended into $a - b < 0$ with its present didactics, e.g. $2 - 5 = 2 - (2 + 3) = 2 - 2 - 3 = 0 - 3 = -3$, but there is an intermediate stage with familiar addition $2 + 5\,H = 2 + (2 + 3)\,H = 2 + 2\,H + 3\,H = 0 + 3\,H = 3\,H$, that does not require (i) the switch at the brackets from plus to minus and (ii) the transformation of binary 0 - 3 to number -3. The expression $a - (-b)$ involves (scalar) multiplication which indicates why pupils find this hard, and $a + H\,H\,b$ is clearer. The use of H would affect the whole curriculum. There appears to be a remarkable incoherence in mathematics education and its research w.r.t. the negative numbers, which reminds of the problems that the world itself had since the discovery of direction by Albert Girard in 1629 and the introduction of the number line by John Wallis in 1673. This notebook provides a package to support the use of H in *Mathematica.* The notebook and package are intended for researchers, teachers and (Common Core) educators in mathematics education. Pupils in elementary school would work with pencil and paper of course.

[80] PDF https://doi.org/10.5281/zenodo.1251686
[81] Mathematica notebook with packages https://doi.org/10.5281/zenodo.1241382

Teaching Simpson's paradox at elementary school – with *H*

2017-03-05 [82]

[This is the same text as the former weblog (here [83]), but now we follow Van Hiele's argument for the abolition of fractions. The key property is that there are numbers x^H such that $x\, x^H = 1$ when $x \neq 0$, and the rest follows from there. Thus we replace (y / x) with $y\, x^H$ with $H = -1$.]

Robert Siegler participates in the "*Center for Improved Learning of Fractions*" (CILF) [84] and was chair of the IES 2010 research group "*Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*" (report [85]) (video [86]).

IES 2010 key advice number 3 is:

> "Help students understand why procedures for computations with fractions make sense."

The first example of this helping to understand is:

> "A common mistake students make when faced with fractions that have unlike denominators is to add both numerators and denominators. [ref 88] Certain representations can provide visual cues to help students see the need for common denominators." (Siegler et al. 2010:32, [87] refering to Cramer, K., & Wyberg, T. 2009 [88])

For $a\, b^H$ "and" $c\, d^H$ kids are supposed to find $(a\, d + b\, c)\, (b\, d)^H$ instead of $(a + c)\, (b + d)^H$.

Obviously this is a matter of definition.

For "plus" we define: $a\, b^H + c\, d^H = (a\, d + b\, c)\, (b\, d)^H$.

But we can also define "superplus": $a\, b^H \oplus c\, \mathrm{d}^\mathrm{H} = (a + c)\, (b + d)^H$.

The crux lies in "and" that might not always be "plus".

When $(a + c)\, (b + d)^H$ makes sense

There are cases where $(a + c)\, (b + d)^H$ makes eminent sense. For example, when $a\, b^H$ is the batting average in the Fall-Winter season and $c\, d^H$ the batting average in the Spring-Summer season, then the annual (weighted) batting average is exactly $(a + c)\, (b + d)^H$. Kids would calculate correctly, and Siegler et al. 2010 are suggesting that the kids would make a wrong calculation ?

The "superplus" outcome is called the "mediant". [89] See a Wolfram Demonstrations project case [90] with batting scores.

[82] https://boycottholland.wordpress.com/2017/03/05/teaching-simpsons-paradox-at-elementary-school-with-h/
[83] https://boycottholland.wordpress.com/2017/01/30/teaching-simpsons-paradox-at-elementary-school/
[84] https://sites.google.com/a/udel.edu/fractions/
[85] https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/fractions_pg_093010.pdf
[86] https://www.youtube.com/watch?v=ngGCmSk7hRY
[87] https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/fractions_pg_093010.pdf
[88] https://www.researchgate.net/publication/250890089_Efficacy_of_Different_Concrete_Models_for_Teaching_the_Part-Whole_Construct_for_Fractions
[89] https://en.wikipedia.org/wiki/Mediant_(mathematics)

Adding up fractions of the same pizza thus differs from averaging over more pizzas.

We thus observe:

- Kids live in a world in which ($a + c$) ($b + d$)^H makes eminent sense.
- Telling them that this is "a mistaken calculation" is actually quite confusing for them.
- Thus it is better teaching practice to explain to them when it makes sense.

There is no alternative but to explain Simpson's paradox also in elementary school. See the discussion about the paradox in the former weblog entry. [91] The issue for today is how to translate this to elementary school.

[Some readers may not be at home in statistics. Let the weight of b be $w = b$ ($b + d$)^H. Then the weight of d is 1 – w. The weighted average is (a b^H) w + (c d^H) (1 – w) = ($a + c$) ($b + d$)^H.]

Cats and Dogs

Many examples of Simpson's paradox have larger numbers, but the Kleinbaum et al. (2003:277) "ActivEpi" example has small numbers (see also here [92]). I add one more to make the case less symmetrical. Kady Schneiter rightly remarked [93] that an example with cats and dogs will be more appealing to students. She uses animal size (small or large pets) as a factor, but let me stick to the idea of gender as a confounder. Thus the kids in class can be presented with the following case.

- There are 17 cats and 16 dogs.
- There are 17 pets kept in the house and 16 kept outside.
- There are 17 female pets and 16 male pets (perhaps "helped").

There is the phenomenon – though kids might be oblivious why this might be "paradoxical":

1. For the female pets, the proportion of cats in the house is larger than the proportion for dogs.
2. For the male pets, the proportion of cats in the house is larger than the proportion for dogs.
3. For all pets combined, the proportion of cats in the house is smaller than the proportion for dogs.

The paradoxical data

The paradoxical data are given as follows. Observe that kids must calculate:

- For the cats: 6 7^H = 0.86, 2 10^H = 0.20 and (6 + 2) (7 + 10)^H = 0.47.
- For the dogs: 8 10^H = 0.80, 1 6^H = 0.17 and (8 + 1) (10 + 6)^H = 0.56.

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[90] http://demonstrations.wolfram.com/BattingAveragesWeightedAveragesAndSimpsonsParadox/
[91] https://boycottholland.wordpress.com/2017/01/28/simpsons-paradox-and-how-to-correct-it/
[92] https://arxiv.org/abs/0708.1401
[93] http://ww2.amstat.org/publications/jse/v21n1/schneiter.pdf

	Female	Cat	Dog	Total
(1)	House	6	8	14
(2)	Outside	1	2	3
(3)	Total	7	10	17
(4)	(1) / (3)	**0.86**	**0.80**	
	Male	Cat	Dog	Total
(5)	House	2	1	3
(6)	Outside	8	5	13
(7)	Total	10	6	16
(8)	(5) / (7)	**0.20**	**0.17**	
	Joint	Cat	Dog	Total
(9)	House	8	9	17
(10)	Outside	9	7	16
(11)	Total	17	16	33
(12)	(9) / (11)	**0.47**	**0.56**	

A discussion about what this means

Perhaps the major didactic challenge is to explain to kids that the outcome must be seen as “paradoxical”. When kids might not have developed “quantitative intuitions” then those might not be challenged. It might be wise to keep it that way. When data are seen as statistics only, then there might be less scope for false interpretations.

Obviously, though, one would discuss the various views that kids generate, so that they are actively engaged in trying to understand the situation.

The next step is to call attention to the sum totals that haven’t been shown above.

	Totals	Cat	Dog	Total
(3)	Female	7	10	17
(7)	Male	10	6	16
(11)	Total all	17	16	33
(13)	(3) / (11)	**0.41**	**0.63**	0.52
(14)	(7) / (11)	**0.59**	**0.38**	0.48

It is straightforward to observe that the *F* and *M* are distributed in unbalanced manner.

The correction

It can be an argument that there should be equal numbers of *F* and *M*. This causes the following calculations about what pets would be kept at the house. We keep the observed proportions intact and raise the numbers proportionally.

- For the cats: 0.86 * 10 ~ 9, and $(9 + 2) (10 + 10)^{H} = 0.55$.
- For the dogs: 0.17 * 10 ~ 2, and $(8 + 2) (10 + 10)^{H} = 0.50$.

And now we find: Also for all pets combined, the proportion of cats in the house is larger than the proportion for dogs. Adding up the subtables into the grand total doesn’t generate a different conclusion on the proportions.

		Cat	Dog	Total
	Female	*Cat*	*Dog*	*Total*
(1)	*House*	9	8	17
(2)	*Outside*	1	2	3
(3)	*Total*	10	10	20
(4)	(1) / (3)	**0.86**	**0.80**	
	Male	*Cat*	*Dog*	*Total*
(5)	*House*	2	2	4
(6)	*Outside*	8	8	16
(7)	*Total*	10	10	20
(8)	(5) / (7)	**0.20**	**0.17**	
	Joint	*Cat*	*Dog*	*Total*
(9)	*House*	11	10	21
(10)	*Outside*	9	10	19
(11)	*Total*	20	20	40
(12)	(9) / (11)	**0.55**	**0.50**	
(13)	(3) / (11)	**0.50**	**0.50**	0.5
(14)	(7) / (11)	**0.50**	**0.50**	0.5

Closure on causality

Perhaps kids at elementary school should not bothered with discussions on causality, certainly not on a flimsy case as this. But perhaps some kids require closure on this, or perhaps the teacher does. In that case the story might be that the kind of pet is the cause, and that the location where the pet is kept is the effect. When people have a cat then they tend to keep it at home. When people have a dog then are a bit more inclined to keep it outside. The location has no effect on gender. The gender of the pet doesn't change by keeping it inside or outside of the house.

Vectors in elementary school

Pierre van Hiele (1909-2010) explained for most of his professional life that kids at elementary school can understand vectors. Thus, they should be able to enjoy this vector graphic [94] by Alexander Bogomolny.

Van Hiele also proposed to abolish fractions as we know them, by replacing *y* / *x* by *y x*^(-1). The latter might be confusing because kids might think that they have to subtract something. But the mathematical constant H = -1 makes perfect sense, namely, check the unit circle and the complex number *i*. Thus we get $y / x = y\, x^H$. The latter would be the better format. See "*A child wants nice and no mean numbers*" (2015). [See Colignatus (2018c) now.]

Conclusions

Some conclusions are:

- What Siegler & IES 2010 call a "common mistake" is the proper approach in serious statistics.

[94] http://www.cut-the-knot.org/Curriculum/Algebra/SimpsonParadox.shtml

- Teaching can improve by explaining to kids what method applies when. Adding fractions of the same pizza is different from calculating a statistical average. (PM. Don't use round pizza's. This makes for less insightful parts.)
- Kids live in a world in which statistics are relevant too.
- Simpson's paradox can be adapted such that it may be tested whether it can be discussed in elementary school too.
- The discussion corroborates Van Hiele's arguments for vectors in elementary school and the abolition of fractions as we know them (y / x) and the use of $y\ x^H$ with H = -1. The key thing to learn is that there are numbers x^H such that $x\ x^H = 1$ when $x \neq 0$, and the rest follows from there.

PM. The excel sheet for this case is: 2017-03-03-data-from-kleinbaum-2003-adapted. [95]

[95] https://boycottholland.files.wordpress.com/2017/03/2017-03-03-data-from-kleinbaum-2003-adapted.xls

Vectors in elementary school 1

Introduction

Vectors can already be introduced in elementary school since pupils already know about co-ordinates. The *Common Core* has co-ordinates in Grade 5 (ages 10-11). [96] Here are some exercises with co-ordinates. [97] When the Pythagorean Theorem is known then pupils can calculate distances, and thus also lengths of vectors.

Vectors are not difficult at all. Pierre van Hiele who was a celebrated researcher on the didactics of mathematics was a strong proponent that they are taught in elementary school. He did not succeed in convincing the world, however. There may have been some stumbling blocks to the discussion of vectors in elementary school:

- Presentation of a subject must respect the Van Hiele levels of insight. These are: concrete, ordering and analysis. [98] Pupils must first feel the water, then create some structure, and then may be open to see the *reason* for that structure. [99] If this didactic approach is not respected, then teaching may be impossible. That Van Hiele did not succeed in getting his proposal accepted has more to do with the training of elementary school teachers than with the difficulty of the subject.
- There may be a *missing link* in the education on geometry, but that was resolved in 2011 by proposing *named lines,* see page 138.
- When the Pythagorean Theorem is not known then one can do little with vectors.

The subsequent chapters consider these steps: (1) The basic geometry of co-ordinates and vectors, (2) The Pythagorean Theorem, (3) Calculating distances and lengths.

It is a bit silly that we repeat the introduction of co-ordinates, but it is useful to create a sandwich with the Pythagorean Theorem in the middle. This introduction into co-ordinates and vectors is largely taken from *Conquest of the Plane* (COTP) (2011). That book targets a higher level audience than elementary school, but it was felt at that place too that there is value in showing how simple the notion is.

The subsections below give a lesson plan for pupils of ages 10-13, thus Grade 5-8, or the last two years of elementary school or the first two years of middle school.

The exercise assumes:

- Hours 9 – 12 AM, 50 minutes per Van Hiele Level (1, 2, 3) with breaks of 10 minutes.
- The pupils have pencil, grid paper, ruler, set square with protractor, calculator with a √ button. They need not "know what √ means" but must have an operational understanding of "input-button-output" with examples "4-button-2" and "25-button-5".
- The teacher has a blackboard.

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[96] http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf
[97] http://mathszone.co.uk/shape/coordinates/
[98] https://en.wikipedia.org/wiki/Van_Hiele_model
[99] Hans Freudenthal mistook the Van Hiele ideas and created his own "realistic mathematics education". This RME misinterpretes the process as "applied mathematics". Pupils are presented with a context from "real life" and have to discover how this might be modeled mathematically. The confusion is that the latter already assumes a mathematical competence that must first be developed. See page 135 below and Colignatus (2014, 2015).

Two axes

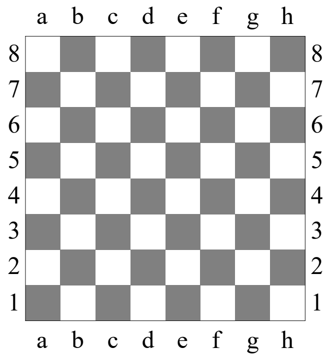
What co-ordinates are

Co-ordinates give information to locate something. For a person it might be a telephone number or an address. When you meet people and want to contact them later then you can ask for their co-ordinates and they will give you their business card.

In the same way for the plane: we use a system of co-ordinates so that every point on the plane can be identified.

A chess board is a familiar system of co-ordinates, see Figure 3. The columns are labelled with the first eight letters of the alphabet (lower case makes for better reading) and the rows are just counted. White starts at the bottom and black at the top. The square at the bottom right hand at h8 will be white. The queen of white will start at d1 and the queen of black will be opposite at d8.

Figure 3. Chess board



X* and *Y

With a ruler on a piece of paper we draw a *horizontal* line and we call it the *x*-axis. Perpendicular to it we draw a *vertical* line and call it the *y*-axis. To identify what axis is what, we label the axes *x* and *y.*

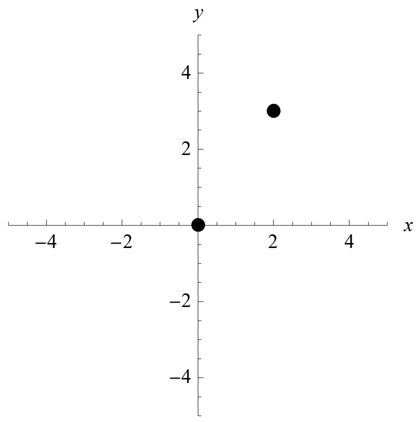
Where the lines cross will be called the *point of origin*. From there we can step right, left, up or down.

We can put numbers on the axes. We copy numbers from the ruler to the axes. The origin will get the number 0. On the horizontal axis we count positive numbers to the right and negative numbers to the left. On the vertical axis we count positive numbers up and negative numbers down. When we go along an axis from 1 to 2, or from 2 to 3, etcetera, then we will call this a full *step*.

We can use curly brackets around two numbers to identify a point on the plane. To start with, {0, 0} will denote the *point of origin*. Then, for example, {2, 3} will mean the point that we can find by moving from the origin, first stepping to number 2 on the horizontal axis and then making 3 steps up.

When you have copied this then you would get a graph like the one below. In this present graph we have put thick dots at {0, 0} and {2, 3}.

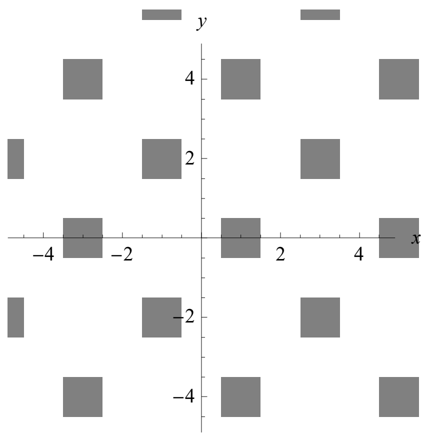
Figure 4. Co-ordinates x and y



Practice makes perfect

It can be good practice to step through this maze in Figure 5 using integer points only and without hitting a square. Start at {1, 2} and try to get to {-4, -3}. [100]

Figure 5. Blocking steps of integer size



Another exercise is to assign letters to points and translate a word into a list of numbers, so that we get a coded message. Try to code FINE using F = {0, 0}, I = {-3, 4}, N = {4, -2} and E = {-4, -3}.

This [101] is a tool for practice, and here [102] is an example of professional use of a grid system for Planet Earth. Well, the Earth is a globe, and henceforth we will only use the plane.

[100] A path is {1, 2} to {2, 2} to {2, -3} to {-4, -3}.
[101] http://www.taw.org.uk/lic/itp/coords.html
[102] http://geology.isu.edu/geostac/Field_Exercise/topomaps/grid_sys.htm

Vectors

When we have a point {*a, b*} and a point {*x, y*} then the novel idea is that we add these two and get {*a + x, b + y*}. That is basically it. It is addition of more things *at the same time*.

For example, count the numbers of pens and pencils that kids in class have, but separately.

Arrows have a direction

Consider a soda can on a deck of a ship. In 10 seconds it rolls 7 meters from port to starboard. In those 10 seconds the ship itself has sailed 67 meters straight forward. People on the ship may see only the movement of the can on the ship. A land-based observer sees a combined movement. The object of discussion is how we could best handle this kind of case.

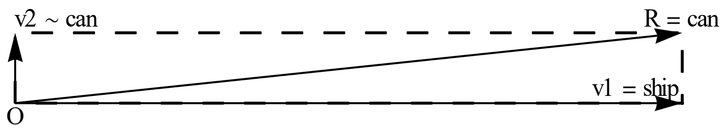
Let us consider two points *P* = {*a, b*} and *Q* = {*x, y*}. We can draw an arrow that starts from *P* and the arrow head ending in *Q.* We shall call that arrow a *vector* and write *v* = {*P, Q*}.

The ship's portside moves along the horizontal axis from {0, 0} to {67, 0}, and this will be vector v1. If the ship would be at rest then the soda can moves along the vertical axis across the deck from {0, 0} to {0, 7}, and this will be vector v2. The resultant movement is *R*. After 10 seconds the can is at position {67, 7}.

Thus the vector from {0, 0} to {67, 7} is *R* = {{0, 0}, {67, 7}}.

The ship moves a distance of v1, the soda can a distance of v2 on the ship, and the soda can has a resulting movement *R* seen by a land-based observer. In those 10 seconds, the soda can moves over a greater distance, and thus it must move faster than the ship.

Figure 6. A can rolls over the deck of a sailing ship



While the earlier discussion used points, we now have arrows, as combinations of points. The news is that we now have a model for motion. Co-ordinates are static, vectors are dynamic. What are the properties of such arrows ?

Before we continue this discussion, we must look at the Pythagorean Theorem.

The Pythagorean Theorem in elementary school

Introduction

Killian (2006)(2012) gives a great way to present the Pythagorean Theorem in elementary school. Some of the innovations are:

(1) to use rectangles rather than right triangles: since pupils are more familiar with rectangles, and the proof uses rectangles anyway

(2) to link up with the formulas for circumference and surface that pupils are familiar with

(3) to avoid exponentiation ("squares") and write out the multiplications: which is what pupils are familiar with.

I asked whether she planned to give an English translation in the foreseeable future. She didn't plan to, and gave me permission to use the ideas here. [103]

You can compare the result below with another more conventional but less accessible treatment for the *Common Core,* designed by the universities of Nottingham & Berkeley, (a) a "discovery" for grade 8 (2nd class of middle school), that is needlessly complex, [104] and (b) a proof for highschool, that is not as straightforward. [105]

Rather than translating Killian's articles my preference is to link up to the ideas by Pierre van Hiele and Dina van Hiele-Geldof. These ideas concern (i) the levels of insight (or abstraction) [106] and (ii) that pupils in elementary school can already deal with co-ordinates (former chapter) and vectors (next chapter).

The Van Hiele levels are: *concrete, ordering and analysis*. Pupils must first feel the water, then create some structure, and then may be open to see the *reason* for that structure. [107]

The subsections below give a lesson plan for pupils of ages 10-13, thus Grade 5-8, the last two years of elementary school or the first two years of middle school.

The exercise assumes:

- Hours 9 – 12 AM, 50 minutes per Van Hiele Level (1, 2, 3) with breaks of 10 minutes.
- The pupils have pencil, grid paper, ruler, set square with protractor, calculator with a √ button. They need not "know what √ means" but must have an operational understanding of "input-button-output" with examples "4-button-2" and "25-button-5". Potentially the class has practiced the table of squares some lessons ago.
- The teacher has a blackboard, with a section where a large table can be constructed and a section for a scratchpad.

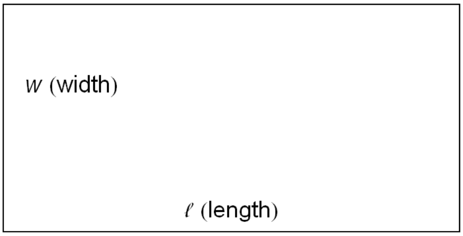
[103] An interview with Killian by me is Colignatus (2012c).
[104] http://map.mathshell.org/lessons.php?unit=8315&collection=8
[105] http://map.mathshell.org/lessons.php?unit=9325&collection=8
[106] https://en.wikipedia.org/wiki/Van_Hiele_model
[107] (a) See footnote 99 on page 109. (b) Van Hiele levels are normally seen as applying to the whole age range (4-18), concerning geometry in general. For this particular topic and stylized case I found it useful to apply the notion of levels to a single class event. Normally, with three Van Hiele levels (moments), there would be two learning phases (periods) inbetween. For this stylized case, I found it useful to associate the level with the period, so that pupils are said to work at a particular level.

Level 1. Concrete. Rekindling what already is known

The lesson opens by telling a bit about Pythagoras (c. 570-495 BC). [108] It may help to recall that a generation might be 25 years, so he lived 100 generations ago. The introduction closes by a statement: Pythagoras did not discover the theorem itself, but the *proof* of the theorem is ascribed to him.

Figure 7 sets the stage.

Figure 7. A rectangle with length and width



Recall the following formulas:

$$c \text{ (circumference)} = \ell + w + \ell + w = 2 \times \ell + 2 \times w = 2 \times (\ell + w)$$

$$s \text{ (surface)} = \ell \times w$$

Example values are in Table 6. Length is measured horizontally and width vertically, for such oriented rectangles. (Otherwise the longest side would be the length.)

Give only values for ℓ and *w,* and let pupils draw the rectangles and calculate the values for circumference and surface. Pupils who have some time left over, before others are finished, can create an additional own rectangle. Check that *c* and *s* have the same outcomes for all.

Table 6. Example values for length and width

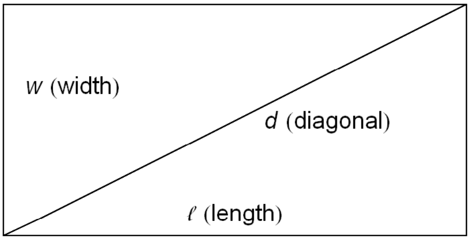
ℓ	w	c	s
3	4	14	12
5	12	34	60

[108] https://en.wikipedia.org/wiki/Pythagoras

Level 1. Concrete. The news

The news is the Pythagorean Theorem. This concerns the diagonal of a rectangle. Figure 8 shows a lower left to upper right) diagonal. Questions: is this the only one ? Are they equal in length ? Why ? (Yes, because of symmetry.)

Figure 8. A rectangle with diagonal



The news consists of the Pythagorean Theorem: the formula for the length of the diagonal:

Diagonal *d* has the formula: $d \times d = \ell \times \ell + w \times w$

The first example of Table 6 can be calculated jointly with the class, the second example can be done by the pupils themselves. With the ruler they check the values on the rectangles that they have drawn. This gives Table 7.

Table 7. Example values for the diagonal

ℓ	w	c	s	$\ell \times \ell$	$w \times w$	$d \times d$	d	*ruler*
3	4	14	3	9	16	25	5	...
5	12	34	5	25	144	169	13	...

Obviously, many pupils who measure the diagonal of their drawn rectangles by using the ruler, may observe differences. This requires the discussion of measurement error.

Collect some measurements and calculate an average, and show that the average is closer to the calculated value.

Overall, a key lesson can be drawn now: There is error in ℓ, there is error in *w,* and there is error in *d.* The Pythagorean Theorem is valuable since it allows to reduce overall error.

PM 1. One observation is that elementary schools have lost the focus on drawing neatly. This exercise shows that there is good value in restoring this.

PM 2. Pupils might accept calculated values as outcomes of their measurements. In that case it might be a learning goal for another lesson to better read off results from a ruler.

Level 2. Sorting. Using approximations

The blackboard can contain Table 7, and new lines can be included. Here on paper it suffices to focus on the new two examples of rectangles, of which the diagonals are no integers. This gives Table 8.

The example $d = \sqrt{5}$ is done jointly in class, and the example $\sqrt{41}$ is done individually.

The pupils are given the length and width, and are asked to draw the rectangles, measure the diagonals with the ruler, and calculate the values from the Pythagorean formula.

It may take too much time to calculate an average for the measurements, but it is always feasible to ask a result from a single pupil.

Table 8. Diagonals with square root values

ℓ	w	$\ell \times \ell$	$w \times w$	$d \times d$	d	*appr. d*	*ruler*
1	2	1	4	5	$\sqrt{5}$	2.23607...	...
4	5	16	25	41	$\sqrt{41}$	6.40312...	...

At this moment it is not a learning objective to deal with the difference between perfect numbers [109] and decimal approximation. It distinction can be just mentioned:

- A lesson is: $\sqrt{5}$ and $\sqrt{41}$ are *perfect numbers* like the integers or fractions like 0.25. The numbers are perfect in the sense that they perfectly tell what the value is without approximation or the need to include an ellipsis (lingering dots). [110]
- For many perfect numbers like $\sqrt{5}$ and $\sqrt{41}$ the decimal expansion creates an infinite number of digits. Cutting of this string – chopping or truncation – always causes an approximation. If you aspire at perfection then you simply write $\sqrt{5}$ and $\sqrt{41}$.

[109] A common phrase is *exact numbers* but *Elegance with Substance* (2009, 2015) explains that this phrase can be confusing. Number theory for https://en.wikipedia.org/wiki/Perfect_number must recode to "ancient Greek perfect number".
[110] See the former footnote again.

Level 2. Sorting. Reversion

Another quesion is: when *d* and ℓ have been given, find *w.*

Pupils are given the option: either first draw this, or first calculate an outcome and then draw it. They will work on this individually.

They will discover that they have to guess the angle of the diagonal, so that it makes more sense to first calculate *w.* (Some may be so smart to use a pair of compasses.)

Calculating the approximate value of the diagonal is not really necessary. It is mentioned here only to size up the number.

Table 9. Reverse calculation

ℓ	*w*	*ℓ × ℓ*	*w × w*	*d × d*	*d*	*appr. d*	*ruler*
1					6		
4					√20	4.47214...	

Filling in the blanks gives Table 10.

Table 10. Reverse calculation (full table)

ℓ	w	ℓ × ℓ	w × w	d × d	d
1	√35 ≈ 5.91608...	1	35	36	6
4	2	16	4	20	√20

Level 2. Sorting. Overview

Lessons about the Pythagorean Theorem are:

(1) It is a welcome addition to the formulas for circumference and surface.

(2) It allows to find one value when two are known.

(3) It helps to check on measurement errors on ℓ, *w* and *d.*

(4) It causes the distinction between perfect √-numbers and their approximations.

(5) The formula ***a*** **×** ***a*** occurs so often that a shorthand notation is a^2, the square of *a.* There is no need to emphasize exponentiation since this may only distract.

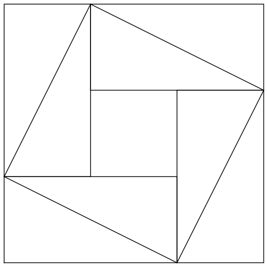
(6) Cutting a rectangle along a diagonal gives a *right triangle.* The theorem holds for such right triangles: for you can always extend it into a rectangle again.

Level 3. Analysis. Proof

The next step is to announce that we will now prove that the Pythagorean Theorem holds for any rectangle: $d \times d = \ell \times \ell + w \times w$. It is called a *theorem* because there is a proof.

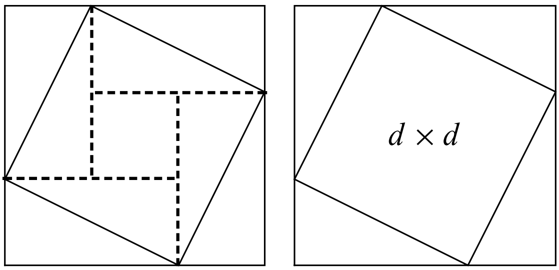
The first step in the proof is to draw four rectangles to create a square as in Figure 9. It is observed in discussion in class that the big square has sides $\ell + w$. Pupils are invited to copy this, each using his or her own rectangle, so that we can later check that the theorem holds for any rectangle that has been drawn today.

Figure 9. Four rectangles with their diagonals



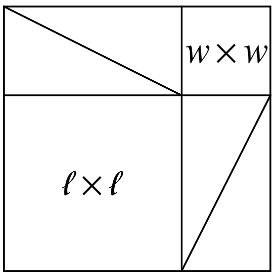
The second step is to erase the lines in the middle, and to recognise the tilted square in the center, see Figure 10. (We skip the proof that it is a square indeed – this can be done in later years.) We indicate its surface.

Figure 10. Erase the lines in the middle, and recognise the center square



The final step is to shift the triangles so that they form two rectangles again, see Figure 11. The pupils will have to draw the new big square again, and are invited to colour or shade the triangles to identify them. (Using labels *A, B, C, D* generates too much text.)

Figure 11. Shifting the triangles



The pupils are now asked:

> *When you look at the last two diagrams, can you give a good reason why d × d must be equal to ℓ × ℓ + w × w ? When you have given a good reason then you have proven the Pythagorean Theorem.*

The pupils will get time to think this over themselves individually. Who has found a good reason may raise a hand, and the teacher can come over to check.

When there are a few verified proofs, or after seven to ten minutes, depending upon progress and prodding, one can start a discussion in class. The objective is to create a list of reasons and to check how convincing they are. One would start with pupils who have not found a proof, and ask for what possible reasons they came up with, and why these indeed are not useful. Eventually the pupils who found the proof are invited up front and asked to explain it to the others.

It is not guaranteed that this order can be kept, since some pupils who have found the proof may be too enthousiastic to be silent about it.

An acceptable proof is:

> In the last two diagrams the areas of the squares with sides ℓ + *w* are the same. We take out the areas of two rectangles. In the first square we are left with *d* × *d.* In the final square we are left with ℓ × ℓ + *w* × *w*. We have taken out the same areas and thus the remainders must be the same too.

Level 3. Analysis. Surplus

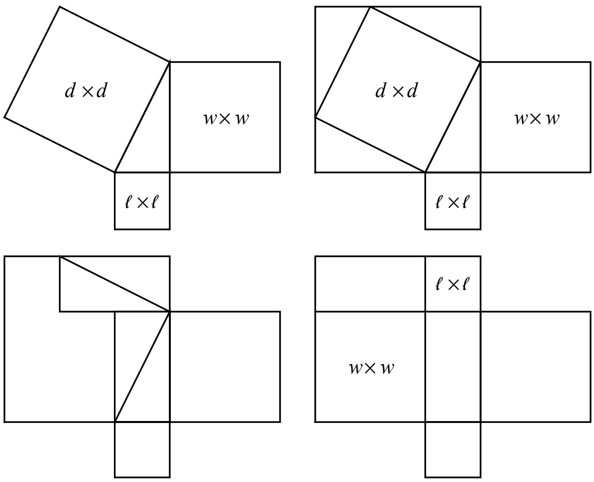
If there is time, or in a later session, one may return to the issue and let the finding sink in deeper by some exercises.

(1) Prove for squares that *d* × *d* = 2 × ℓ × ℓ.

(2) Algebra supported by the diagrams is: (ℓ + *w*) × (ℓ + *w*) = ℓ × ℓ + *w* × *w* + 2 × ℓ × *w*.

(3) Prove that the result holds for right triangles, starting from the drawing on the top left of Figure 12 (and let them find the other diagrams).

Figure 12. Pythagoras for right triangles



Vectors in elementary school 2

We can take up the story at the point where we left it. Pupils can now calculate the distance that the soda can travelled in 10 seconds along the vector from {0, 0} to {67, 7}: 67.36 meters.

For us, this causes a moment of reflection. For design of the curriculum at elementary school and the creation of lesson plans, there are two major steps:

(1) Above outline of a lesson plan for the Pythagorean Theorem shows that this theorem can be presented and that many pupils will find the proof themselves.

(2) The introduction to vectors shows that the concept is simple and that there are useful basic applications.

Thus, Pierre van Hiele was right that vectors can be presented in elementary school.

It is not quite an issue how to proceed and what lesson plans can be developed. The real issue is that decision makers, both on the *curriculum* and the *education of elementary school teachers,* have to decide that these topics better be included.

Circles and measurement of angles

Introduction

The Californian implementation of the *Common Core* has for Grade 4 (ages 9-10):

> "Geometric measurement: understand concepts of angle and measure angles.
>
> 5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
>   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
>   b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.
> 6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
> 7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure." [111]

A better measure for angles is the plane itself, with unit 1.

A right angle would be 4^H = ¼ = 25% of the plane. While notation 4^H shifts understanding of fractions from division to multiplication, it may still be easier for pupils to work with integers than (such) fractions, so that 25% of the plane may be an easier measure for a right angle. Pupils might even appreciate the 250 promille measure.

The original proposal for this is in *Trig rerigged* (Colignatus (2008)). The issue translates directly to elementary school. When pupils in Grade 4 already must handle the protractor to measure angles on a scale of 360 degrees – while this is an illogical number w.r.t. the unity of the plane that they are taking sections of – then the clarity provided by *Trig rerigged* for highschool will surely be relevant for primary education too. I am at risk repeating the issue too much. *Trig rerigged* has been replaced by *Elegance with Substance* (2009, 2015) with principles, and *Conquest of the Plane* (2011) with details.

Pupils in primary education should also know that the angles of a triangle add up to half a plane. This discussion [112] is not targeted at their level but perhaps a version is feasible.

Observe the calculatory overload in the common programme. To understand an angle of 60 degrees for example, a pupil must calculate 60 / 360 to find 6^H = 1 / 6 of a circle. Thus calculation precedes understanding. The fractional form 1 / 6 invites one to continue with the calculator as well, and perhaps needlessly. To imagine what this might be, it may be transformed to 0.166... in decimal form, or 16.7% in common approximation. Instead, when the plane itself is used as the unit, then the angle 6^H stands by itself. Given the identity that 6 6^H = 1 it would be easier to see that 6^H can indeed stand by itself as "(one) per six". A transformation into decimals might not be necessary, since 16.7% is not necessarily informative. If such transformation is desired, to compare with 25%, then such

[111] p32 of http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf
[112] https://boycottholland.wordpress.com/2014/06/29/euclids-fifth-postulate/

a calculation cannot be avoided. Still: it is not required to do a calculation 60 / 360 to understand that 60 degrees is 6^H plane, and this would further understanding.

Please observe the circus: Given the Sumerian 360, there is a convention to consider special angles like 30 and 60 degrees. These have the supposedly "nice" property that sin[30] = cos[60] = ½. There is nothing particularly "nice" about this however. Don't blame the Sumerians. Blame generations of mathematicians who have been telling each other and us that this is "special". [113] But there is nothing nice or special about it. In our case we might say that sin[12^H plane] = cos[6^H plane] = 2^H, and then it is immediately clear that there is nothing special here indeed. If pupils are trained on the decimal system then it may make more sense to focus on 5%, 10% and 25% of the circle. Note that sin[5% plane] = sin[18 degree] = (-1 + √5) 4^H. Now, isn't that *special* ? [114]

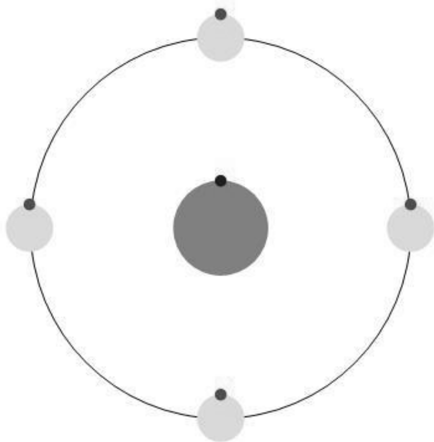
[113] http://www.themathpage.com/atrig/30-60-90-triangle.htm
[114] https://en.wikibooks.org/wiki/Trigonometry/The_sine_of_18_degrees

Einstein versus Pythagoras

2014-10-12 [115]

The following diagram conveys the general notion of relativity. This is not Einstein's relativity due to the constant speed of light, but it is useful to convey the notion of relativity in general.

The diagram gives an imaginary case of the Moon circling the Earth such that Earth and Moon do not rotate themselves but are always in the same orientation to the same distant stars. The Blue and Red dots are observers who remain oriented to those distant stars. An observer on Earth at the Blue dot would be able to see all sides of the Moon – assuming that Earth were transparent. For observer Blue the Moon is rotating, even though it isn't with respect to the distant stars. If the Moon is on the left hand side, Blue will see its right side. If the Moon is on the right hand side then Blue will see its left side. Similarly for top and bottom. Thus, what actually is fixed is observed as rotating. Or, if the Moon were actually rotating and Blue not, subtract one seeming rotation to eliminate this observation effect.



Moon circling the Earth while none are rotating themselves

The principle of relativity may also be explained by comparing a car driving past a house. For the observer in the house, the car has a speed. However, seen from the position of the driver, it is the house that passes by at that speed. This example also conveys that observation is relative to the position of the observer. In this case, however, the example is not that strong. The car has a brake, and the house hasn't. Thus in this case it makes more sense to say that it is the car that is causing the speed difference.

A person who turns his head sees the universe spinning around him or her. It doesn't make much sense to hold that everything is relative and that the universe is spinning around with close to infinite speed and energy. Though it would be difficult to locate, the center of the universe is a more logical point to describe events from.

Above diagram doesn't have the complications of a car brake or the turning of your head or Einstein's use of the constant speed of light. It shows observational relativity in terms of logic. Though the Moon does not rotate itself (Red is always oriented at the same distant stars) it seems to rotate for Blue (with the same orientation).

[115] https://boycottholland.wordpress.com/2014/10/12/einstein-versus-pythagoras/

Pythagoras and the definition of space

Let me quote from my book *Conquest of the Plane* p85:

> There is the paradoxical situation that we may take great pains to prove something that from another point of view is merely a matter of definition. The Pythagorean Theorem [116] is commonly expressed in terms of sides *a, b* and *c*. For the circle $c = r$. Then we get:
>
> Pythagoras convinces us that we have to prove that c^2 = a^2 + b^2
>
> For a distance we now define that c^2 = a^2 + b^2
>
> The solution to this paradox is that Euclid used other axioms than we now do for the distance. Though Pythagoras (ca. 572 – 500 BC) lived before Euclid (around 300 BC), we can say in a figure of speech: Given the Euclidean axioms Pythagoras has to prove his Theorem. Once he got the proof he could define the circle. Without the proof he might define the circle but then would have to prove that it really exists. That said, in analytic geometry it is easier to work the other way around. Starting with formulas is a fast way to get up and running. Using distance we can define parallel lines as lines that have equal distance. With distance the circle arises naturally. The notion of distance is crucial for the Euclidean plane. We surmise that Euclid relied on a notion of distance too by using the compass.
>
> What remains in all this is our notion of Euclidean space: a notion of straightness of lines and flatness of the plane that might derive from everyday experience but that essentially is a concept of the mind, and essentially a definition.

What you should take away from this is: the definition of 'space' is Euclidean space. If you think about 'space' then this is what you think. You cannot change what already has been defined to generate your understanding.

Einstein's historical context

As observational relativity because of the constant speed of light causes measurement errors, Einstein eliminated those errors by adapting 'space'. But can you change the notion of space if it already has been defined by Pythagoras and Euclid ? An elegant way to deal with systematic measurement errors doesn't change 'space'. Something else is happening here.

Let me quote from *Conquest of the Plane* p195 that describes Albert Einstein's historical context.

> A key issue in the theory of science is the issue of measurement. Physics before Newton suffered huge losses in intelligence, time and energy to discussions on unobservables and metaphysics. This in fact lasted partly into the 19th century with discussions on the 'ether'. Their solution was to put a stop to fruitless discussion and concentrate on what can be measured. You don't know what it is, but it moves this way, at that speed, and if you hit it here, then it moves there. This technical approach worked wonders, though it still seems that some theorists assume some 'whats' to derive their theories on the 'hows' (as Bohr's atom model).
>
> (…) A key notion below will be that physics might 'overshoot' by concentrating on measurement and by neglecting definitions and logic.

---

[116] http://en.wikipedia.org/wiki/Pythagorean_theorem

> (…) Einsteins model subsequently seems to confuse the definition of space, given by the definitions of Euclid, and empirical space as measured by the instruments of physicists.
>
> (…) Modern physicists shy away from the possibility that space and time have independent definitions within the mathematical modelling of the world. They regard space and time as what they measure. However, they don't seem to see that they can be hopelessly confused when they measure speed in meters / second while those meters and seconds change under measurement. My impression is that it is better to accept measurement error and try to explain that error.

Education in mathematics vs physics

Please observe that I am no physicist and rely for that on what I remember from gymnasium. The above is a view from the position of the education in mathematics. The views from the education in physics may be different. There may be relativity in education as well.

The above concerns a minor comment in COTP. Its real contribution lies elsewhere. PM. COTP also allows the earlier discussion of derivatives, so that physics education can start using those much earlier too.

The issue might be resolved empirically. A physicist would have to show that it is impossible to describe the measurement error in Euclidean space, so that the use of Riemann curvature is not just a historically understandable way of modelling but also necessary. It would be more interesting of course to see that the Riemann form generated other confusions.

Edward Frenkel [117] holds that the Pythagorean Theorem meant the same to people 2500 years ago as it means to people nowadays. This doesn't seem true to fact, though of course is hard to prove. At least the above shows that we have added shades of understanding that were lacking in the past. Some historians hold that Euclid did not present a cosmology or theory of space but a theory of measuring. However, it seems that the latter presupposed the first, see point (v) here. [118] Also, Frenkel emphasizes the importance of the Riemann model, and thus should admit that modern physicists claim another view of space than Pythagoras and Euclid, so that he cannot uphold that 'sameness'. Overall, Frenkel is a research mathematician and has no background in the empirical science of education, so he is producing a lot of nonsense. More on that later on.

[117] http://math.berkeley.edu/~frenkel/
[118] https://boycottholland.wordpress.com/2014/06/29/euclids-fifth-postulate/

Mathematical constant Archimedes = Θ = 2 π = 6.2831853…

2012-02-18 [119]

My book *Conquest of the Plane* (COTP) uses Θ = 2 π = 6.2831853…. My proposal in supplement to COTP is to use the name “Archimedes” for this particular symbol (“capital theta” with such assigned value). It will be a new mathematical constant. Addendum: Pronunciation "Archi" is better than full "Archimedes".

One Archi thus is the circumference of a circle with radius 1. Another relevant format is 1 / Θ = 0.15915494… When you take a circle with a radius of about 16 cm then the circumference will be about 1 meter. A circle with radius r has circumference $C = r\,\Theta$ and surface $S = r^2\,\Theta / 2$.

In wikipedia (today 2012-02-17) [120] we can read that π is already called “Archimedes’ constant”. However, we commonly speak about “pi” and not about “Archimedes”. Thus the name is free to use as the name of Θ.

There is some momentum in the USA to use tau, thus τ = 2 π. Bob Palais (2001) [121] originated the idea but used an own new symbol (pi with three legs like m), Peter Harremoes and Michael Hartl convinced him to use tau, [122] and Vi Hart has a presentation on YouTube. [123] One argument is that tau refers to “turn” or Greek “tornos”.

However, turns are counted along the unit circumference cirkel $C = 1$ and not along the unit (radius) circle $r = 1$. Thus this association of tau would be confusing. Also, there is not much difference in writing r or τ. This can create a lot of confusion in handwriting, doing homework or checking exams.

Independently from Bob Palais I also came up with the idea that 2 π is the proper unit of account. Looking at the various symbols available on the keyboard I rejected tau because of the similarity to r, and settled for Θ since it neatly looks like a circle. I wasn’t quite happy with its uninformative name Theta but we had that also with pi or meter. Vi Hart pointed out that lower case theta is often used for angles which causes the problem of “theta Theta”. This disappears when we use “Archi”.

The proposal is to take the plane itself as the unit of account for angles. We know how to cut up a pie in those pointy bits radiating from the center, and we can do the same with the whole plane, getting a half plane, a quarter plane, etcetera. All those pointy bits add up to 1 plane. When we make circles we can find one with a circumference of 1 by which we can measure the angles. Comparing circles, the Archi unit shows up as a proportionality factor.

We need empirical tests whether this indeed works out better for students.

Unit circumference circle = Angular circle	Unit radius circle = Unit circle
$C = 1$ $r = 1 / \Theta$ angles α, β functions Xur and Yur	$C = \Theta$ $r = 1$ arcs $\varphi = \alpha\,\Theta$ and $\psi = \beta\,\Theta$ functions Cos and Sin

See COTP page 41. Here Xur[α] = Cos[α Θ]. Angles can be measured by arcs or possibly be identified by them. It helps to separate the notions somewhat by putting emphasis on angles on the angular circle and arcs on the unit circle.

[119] https://boycottholland.wordpress.com/2012/02/18/mathematical-constant-archimedes/
[120] http://en.wikipedia.org/wiki/Mathematical_constant
[121] http://www.math.utah.edu/~palais/pi.html
[122] http://www.tauday.com/
[123] http://www.youtube.com/watch?v=jG7vhMMXagQ

Archimedes revisited

2012-3-06 [124]

My proposal to use the name "Archi [-medes]" for Θ = 6.283… got a reply from Peter Harremoës from Denmark. Peter argues that engineers and artisans in Archimedes' time found it more efficient to measure circles by their diameter *d* and not with the radius *r* = *d* / 2, so that Archimedes calculated π = Θ *r* / *d* = Θ / 2 = 3.141…. Hence the latter number is called Archimedes' number, historically. Peter discovered that the Persian mathematician Jamshid Al-Kashi in 1424 apparently was the first to use 6.283… as a separate number. Hence Peter suggests [125] to use Al-Kashi's constant τ, where he also adopts the symbol tau as do Robert Palais, Michael Hartl and Vi Hart as shown on my proposal page.

Bear with me. I have been aware of Archimedes' historical position, see the proposal text indeed. The point is that there is only one mathematical constant. The values 6.283… and 3.141… are mere transforms of the same constant. Thus we should select only one name. Moreover, Θ / 2 would be vocalized as "one half Archimedes" such that Θ is a unit of account and not just a number discovered by some person.

It may be fun to say that Isaac Newton discovered one Newton and Alessandro Volta discovered one Volt while Archimedes discovered only one half Archimedes, but that would stretch what we mean by a mathematical constant. Archimedes really was the first to determine the mathematically correct way to catch that mathematical constant. So, there is no conflict between using the Archimedes as the unit of account and accepting that 3.141… was historically seen as Archimedes' number.

Subsequently, Archimedes' reasoning was didactical, since he adopted the common usage in his day of the diameter. We have switched to the radius so let us switch consistently. Perhaps Al Kashi was instrumental in that switch but he was aware of Archimedes' important discovery and I like to think that he would agree that Archimedes receives all honour.

I have really thought deeply about tau. I really don't mind what is actually chosen as long as it works best in education. I considered tau independently from the others but rejected it because it looks too much like *r*. The capital theta looks nicely like a cirle. The little mark in the center is not a slash like for the diameter or crosssection Ø. My proposal is that we research what works empirically best in education.

It might be a nice idea to put the choices up for an opinion poll. The true vote would be to use either current π or one of the alternatives for 2 π. But this vote would be biased when there is a difference in opinion about what that alternative will be. A vote now cannot be decisive since it is a matter of empirical research. However, voters can have an opinion about what should be tested in that research, or have a forecast about what would work best, at least for themselves. Thus, an opinion poll can be somewhat informative.

See **this page** for the vote. [126]

[124] https://boycottholland.wordpress.com/2012/03/06/archimedes-revisited/
[125] http://www.harremoes.dk/Peter/Undervis/Turnpage/Turnpage1.html
[126] http://www.easypolls.net/poll.html?p=4f5619a1c2e1b0e4901bc494

An Archi gif, compliments to Lucas V. Barbosa

2014-07-14 [127]

Given the last weblog on radians, [128] I noticed that Wikipedia had a nice gif animation created by Lucas V. Barbosa. [129] The article even mentions: “This is a featured picture on the English language Wikipedia and is considered one of the finest images.” Barbosa even made a version with tau = 2 pi. [130] The latter is less appealing since it does not mention pi, while, of course, tau reads like radius *r*, and then can cause confusion (indeed, run that gif too).

It appears that Barbosa put his gif into the public domain. Thus I adapted it for Archi = Θ = 2 π, including a note of reference that he did most of the creative work.

(animation: https://boycottholland.files.wordpress.com/2014/07/circle_radians_archi1.gif)

A radian is an angle measured by an arc of a circle with the same length as the radius of that circle. A full circle corresponds to an angle of 1 Archi = 2π radians. Use 1 Turn ⇔ Θ radians, so 1 radian ⇔ 1 / Θ Turn ≈ 16% Turn.

Interestingly, Barbosa’s original gif has a small shaded disc in the center. If we take the radius of the larger circle as $r = 1$ then we get the smaller Angular Circle [131] in the center with $r = 1 / \Theta$ and circumference 1. My proposal is to speak about “angles” on the Angular Circle (use α and β), and to use “arc” for the radians on the Unit Circle (use φ and ψ). Of course, angles as measured on the Angular Circle are arcs too, but it helps being able to say that angles add up to 1 Turn and Unit Circle arcs to 1 Archi rad.

PM. The Wikipedia article [132] I referred to has a wrong statement on dimensions (today, July 2014). For a discussion of this, see the earlier weblog entry on radians. [133]

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[127] https://boycottholland.wordpress.com/2014/07/14/an-archi-gif-compliments-to-lucas-v-barbosa/
[128] https://boycottholland.wordpress.com/2014/07/07/why-are-radians-not-more-natural-than-any-other-angle-unit-q/
[129] http://commons.wikimedia.org/wiki/File:Circle_radians.gif
[130] http://commons.wikimedia.org/wiki/File:Circle_radians_tau.gif
[131] https://boycottholland.wordpress.com/2012/02/18/mathematical-constant-archimedes/
[132] http://en.wikipedia.org/wiki/Radian
[133] https://boycottholland.wordpress.com/2014/07/07/why-are-radians-not-more-natural-than-any-other-angle-unit-q/

Trigonometry rerigged 2.0

2016-09-05 [134] [135]

Trig Rerigged 2.0 (draft) [136] proposes a new didactic approach to trigonometry. The proposal has the form of a booklet since it reprints some pages from *Elegance with Substance* (2009, 2015) and *A child wants nice and no mean numbers* (2015). The format might change in the future, like the earlier discussion of Trig Rerigged 1.0 of 2008 (now legacy) was absorbed in *Conquest of the Plane* (2011).

The reader might start with page 15 with the main idea, and page 16 with the main graphs. When these make sense, then restart at the beginning. Trig Rerigged 2.0 is targeted at researchers in mathematics education, teachers of mathematics and trainers of teachers. Well, science journalists might step in too. When you are none of these, then you are advised to be satisfied with the following.

Angular circle with circumference 1. Hook disk with area 1

- A ***circle*** is defined as the collection of points at a given distance from a center. This distance is called radius. The circle is a concept of circumference. There is proportionality with the radius. With radius r we have circumference $r\,\Theta$.
- A ***disk*** is defined as the collection of points at a given distance or less from a center. The disk is a concept of area. Area depends upon the square of the radius. The general disk area is $\pi\, r^2 = 2^H\, \Theta\, r^2$ with $\Theta = 2\pi$.
- The ***unit circle*** has radius 1 and circumference Θ ("archi") and disk area π ("pi").
- The ***angular circle*** has circumference 1. Angles can be measured as arcs on the angular circle, as percentages of 1. The angular circle has radius Θ^H. It is common to use the algebraic symbols instead of their numerical values $\Theta \approx 6.28\ldots$ and $H = -1$ ("eta").
- The ***hook disk*** has area 1. Angles can be measured as sectors on the hook disk, as a percentage of 1. Thus this is a surface measure. The hook circle has radius $\sqrt{\pi}^H$.

Main conclusions

- It is immaterial whether angles are measured as arcs on the angular circle or as sectors on the hook disk. In both cases we have perunages or percentages of 1. The unit of measurement is actually the plane itself. Another formulation is the number of turns around a circle.
- Both Θ and π are useful symbols to denote these relationships. They support a rich didactic environment, that allows students to grasp the notions that are closer to their understanding, and develop from there.

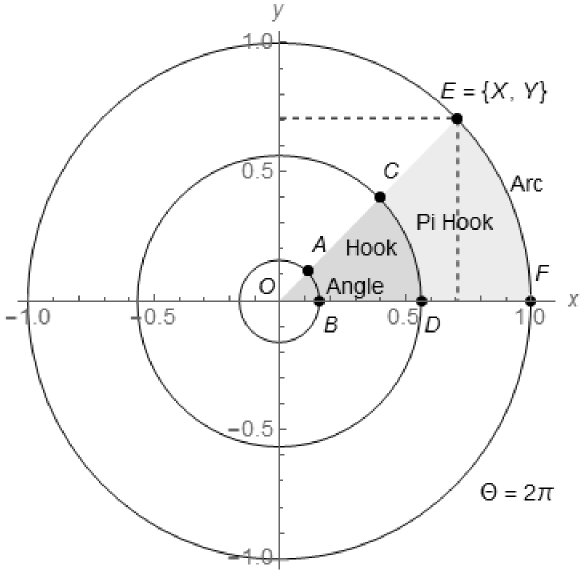
Graph of angular circle and hook disk

The following graph from page 16 gives the notions in a nutshell.

[134] https://boycottholland.wordpress.com/2016/09/05/trigonometry-rerigged-2-0/
[135] Dutch: https://boycotholland.wordpress.com/2016/09/10/draaicirkel-en-draaischijf-met-maat-1-voor-goniometrie/
[136] http://thomascool.eu/Papers/Math/2016-09-04-Trig-Rerigged-2.pdf

- The ***angle*** α is the ***arc*** *BA* along the angular circle, or the ***sector*** *OCD* on the hook disk. When the sector is extended from the hook disk onto the unit circle, then this sector might be called a “Pi hook”, for its value is α π.
- The ***arc*** *FE* is the angle in radians, with the value α Θ.

The point {*X*, *Y*} = {*x*, *y*} r^H has the property that $X^2 + Y^2 = 1$. It is useful to use the separate symbols *X* and *Y* for this point, since it determines the length of arc from {1, 0}. The point on the unit radius (*ur*) circle can also be described as a function of the angle α, as {*X*, *Y*} = {Xur[α], Yur[α]} = {Cos[α Θ], Sin[α Θ]}.



Potential implementation

Since these are suggestions for improved didactics, there must be empirical testing to determine whether these are improvements indeed. It are the students who must show that it works.

Earlier I discussed the US Common Core. [137] This new development on the didactics of trigonometry can be included. See *Trig Rerigged 2.0* for more on the relationship to the US Common Core.

I am not qualified for primary education, but the above would seem to be helpful. For example, young pupils could colour sectors of the hook disk, and determine that hooks are additive. At a next stage, they may see the frailer circumference, and see that e.g. 25% of hook matches with 25% of angle. The pupils would be able to determine the radii of the hook disk and the angular circle, so that they grasp proportionality, and that area goes by the square of the radius, and the relationships to Θ and π. [138]

There is an intermediate stage at which {*X*, *Y*} = {Xur[α], Yur[α]} = {Cos[α Θ], Sin[α Θ]} will be discussed, and their inverses. Parts might already be done in elementary school, but it would surely be done (repeated) in the early phase of secondary education.

[137] https://boycottholland.wordpress.com/2016/03/22/looking-beyond-the-ccss-m/
[138] Dutch: http://thomascool.eu/Papers/AardigeGetallen/2014-01-09-Cirkels-in-vierkanten.pdf

PM. The animation [139] at wikipedia for the sine is fairly good, but one would want to be able to manipulate the position, and the choice of yellow for the vertical position is too light.

At the end of highschool, students should be able to deal with radians and sine and cosine. Those functions remain key because of the derivatives. However, the working horses will likely be Xur and Yur, for in trigonometry it is natural to work with turns.

Acknowledgement and word of protest

Above idea basically builds upon Trig Rerigged 1.0 from 2008. The issue here is didactics of trigonometry.

Michael Hartl published a *tau manifesto* [140] in 2010, and SpikedMath (MSC) published a reply *pi manifesto* [141]. The issue here is rather curious. Hartl explains his approach: "π is a confusing and unnatural choice for the circle constant." This doesn't concern didactics but concerns some notion of naturalness in some mathematical universe, as if criteria in mathematics itself would force a choice between either Θ or π. I don't think that this is a relevant way to formulate the issue or discuss it. There is no need for an "archi manifesto" since the relevant issue has been stated in terms of didactics of mathematics in above books. Also, tau (τ) is an awkward choice of symbol, for it looks too much like the symbol *r* for the radius, especially in the handwriting of students.

Still, I read these manifestos and benefited from aspects of them, notably since this gave me the idea to define the hook disk as the disk with area 1, so that we can better see the underlying unity of the notion of angle or hook. Thus I acknowledge the contributions, but also must protest that it doesn't help when these manifestos divert attention away from the proper question on didactics.

Earlier weblog texts on this issue have been here [142]and here [143] and this animation. [144] On the use of *H*, see here, [145] now also Colignatus (2018c).

[139] https://en.wikipedia.org/wiki/Trigonometry
[140] http://tauday.com/tau-manifesto
[141] http://www.thepimanifesto.com/ See also http://spikedmath.com/info.html
[142] https://boycottholland.wordpress.com/2012/02/18/mathematical-constant-archimedes/
[143] https://boycottholland.wordpress.com/2012/03/06/archimedes-revisited/
[144] https://boycottholland.wordpress.com/2014/07/14/an-archi-gif-compliments-to-lucas-v-barbosa/
[145] https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/

A key insight in the didactics of mathematics

Adapted from §15.2 of *Conquest of the Plane* (2011)
Also included in *Elegance with Substance* (2015)

Introduction

It was an option to start the composition of this book with an introduction to didactics. In that case the reader could see how the subsequent parts fit the didactics. This would have been a top-down approach.

However, it seems more likely that the reader would not have understood this introduction to didactics. It is better to work bottom-up. Reading the book, the reader meets various arguments that argue for a particular approach. The arguments should make sense at the particular points when they are presented. Only in hindsight it appears that there is a method underlying it all.

The method is: that pupils first must become familiar with information at the bottom, before they can make the conceptual leap up to a higher level.

There is more to it: this didactic approach closely relates to thinking itself.

This chapter has been adapted from §15.2 of *Conquest of the Plane* (COTP). COTP is a primer for highschool and first year of higher education. Readers interested in elementary education will not quickly read COTP. But the discussion is relevant for all education.

The didactic approach

Learning goals are generally knowledge, skills and attitude. The didactics are guided by the Van Hiele levels: *concrete, sorting, analysis,* or, with the latter split w.r.t. formality:

> Level 0: visualization and intuition
> Level 1: description, sorting, classification
> Level 2: informal deduction
> Level 3: formal deduction

Importantly, at each level the *same* words may be used but with *different* intentions, complicating mutual understanding.

Van Hiele (1973:177) gives the following example, and (1973:179) explains: “At each level we are explicitly busy with internally arranging the former level.” (my translations):

(0) An isosceles triangle is recognized like an oak or mouse are recognized.
(1) It is recognized that this triangle has the property of at least two equal sides or angles.
(2) Relations between properties are recognized: at least two equal sides *if and only if* at least two equal sides.
(3) The logical reasons for these relations are considered: why *if*, and what does it mean to reverse an implication ?

Van Hiele (1973:179) on geometry:

> “At the base level we consider space like it appears to us; we can call this *spatial sense* (like *common sense*). At the first level we have the geometric spatial sense. (E.g. measuring degrees of an angle / TC.) At the second level we have

> mathematical geometric sense; there we study what geometric sense involves. At the next level we study the mathematical logical sense; it then concerns the question why geometric manners of thought belong to mathematics."

The levels do not provide information about the boundaries of topics, and they are not strong when it comes to finalizing a topic and switching to a next one (that builds upon the earlier). In this book we mostly look at Level 1 and 2, and there are some patches that peek into possibilities for Level 3. The reader should be able to identify the spots.

In moving from one level to the next, Van Hiele (1973:149+) identifies phases:

(1) intake of information (examples)
(2) bounded orientation (direct instructions)
(3) explicitation (making explicit, verbalization in own words of what is known)
(4) free orientation (extending the relationships in the network)
(5) integration (summarizing and compacting what has been learned, often old fashioned learning).

Van Dormolen recognizes similar stages: Orientation, Sorting, Abstraction, Explicitation, Processing & Internalisation (OSAEP/I).

We reject Freudenthal's "realistic mathematics education" (RME) in its more extreme interpretation. This is best discussed in separate paragraphs.

It hinges on what counts as experience

Van Hiele and Freudenthal overlap in the starting point in experience. The question remains what kind of experience we choose:

- Working in the plane itself is seen by Freudenthal as too abstract
- while Van Hiele in principle allows the notion that it might be experience too. Mental thought is an abstract process by nature and we can have experience in that.

Modern research on the brain clarifies many aspects of mental processes. Operational definitions of thinking and consciousness however cannot replace the definition of thinking as experienced by the conscious self. When we look for a definition of what thought is, in that experience of being conscious, then we quickly arrive at a Platonic version of ideas. In the mind's eye a triangle has a purity about it that is not caught in any drawing. Also mudd becomes perfect mudd. There is no difference between an image of a triangle and an image of mudd, or even an image of a sunset, in the sense that they are constructed out of the same mental elements that can only be pure. It are these mental ideas that education deals with, and experience in reality is only a tool to reach them. This does not mean that we have to be full Platonists in assigning an indestructible and immortal quality to these ideas. Thought and thinking, consciousness and awareness, are primitive notions for the thinking intellect itself, and up to this day and age of human history they do not generate any additional information for more conclusions than their very experience. [146]

The paradox – seeming contradiction – is that Freudenthal was an abstract thinking mathematician who developed an abstract notion of "realism", while Van Hiele was a practicing school teacher who was open to the relevance of abstraction itself.

There is a difference between:

- designing a mathematical model, as in applied mathematics, by someone who already has a command of mathematical concepts, with the aim to match properties,
- learning to understand and developing a command of those mathematical concepts.

[146] 2015: Lee Smolin (2015) also presents the naturalist view that shows that Platonism is not necessary, and can be eliminated with Occam's razor.

What we can assume and build upon

Students and puplis have sufficient experience with the plane since making drawings in kindergarten. When they think about a triangle it is as abstract as it can get because such thought is abstract by nature. We can draw many triangles on paper but the notion of a triangle in the mind is an entirely different matter, and when the student or pupil thinks about a triangle then it is that notion that is in the mind and not the drawing on the piece of paper. What counts are the lingering notions in their abstract imagination that have to be activated. When we put labels to angles on paper and draw supporting lines then we use paper images to enter new concepts into the mind. It remains an essentially abstract activity, with pen and paper only tools for communication. It distracts and confuses when mental clarification is mixed with the application to reality. Application to reality is relevant but should be dosed wisely.

Finding the proper dose and perspective

My book *A Logic of Exceptions* maintains that the force of logic derives from reality. If a truck approaches and if you do not jump aside then it will hit you. Mimicking this, *A Logic of Exceptions* starts with electrical switches to clarify the constants of propositional logic. In this case we do not need to explain these constants since we presume that students already know them. We only help making them explicit. The empirical examples are only intended to highlight the properties and to pave the road towards formalization. Here the electrical switches do not distract since the case is not presented as an exercise in building electrical circuits. The examples help to focus on the logical properties. Electrical switches are as good an example as language, and in a way a better example since the focus in logic is already so much on language that it helps to provide another angle.

For analytic geometry it may be argued that a bucket and a faucet that adds a liter per minute would be a similar good starting point. This is dubious however. If the objective is to distinguish linear processes from other processes then indeed examples in reality are the stepping stones, but that is another issue than linking up with geometry. The example distracts from the very abstract notion that we want to establish. “Realistic math” might require a student to spend a sizeable part of the lesson time on realistic examples trying to figure out what is the point. When supporters of “realistic math” argue that students of geometry do not understand a linear process without such examples as the bucket, then the reply is that those teachers have not spent sufficient effort in providing the abstract tools to perform the mental process.

It are different mental processes: imagining a bucket and faucet and imaging a graph of a linear function.

- The bucket and faucet have been learned in kindergarten.
- The graph and its geometric interpretation first have to be learned before they can be imagined and linked up to the bucket and faucet.

Once we have the graph then it is OK to say, and indeed we ought to say, that the bucket and faucet are an interpretation and application, and only then there can be that flash of understanding that shows that the link has been achieved. Once an aspect of the plane has been conquered then abstract understanding can be easier related to those other cases from reality, which means that those other examples are relevant for the Van Dormolen Processing & Internalisation stages. *But first we must develop the geometry of that graph, using the mental images of geometry itself.*

The challenge

There is a challenge though. Eudlid’s *Elements* and his axiomatics have been the standard for more than two millenia. They are at Van Hiele’s highest level. Perhaps 12-

year olds can deal with those abstractions, as they actually are rather simple. But it becomes a bit different when we try to incorporate the advances in analytic geometry and calculus. Here are concepts that better be developed at a lower level and Van Hiele then wins from Euclid. Here Freudenthal steps in and resorts to the richness of reality, and at first that seems like a golden solution. Indeed, axiomatic geometry is at Level 3 and not at Level 0 ! However, as explained Freudenthal's approach is not convincing since it neglects that thought is abstract by nature. Rather than going sideways into reality we should focus more on the processes of thought and thinking itself.

A missing link

We should provide for an abundance of words and concepts in the abstract plane, so that the student has enough to hold on to for visualization and intuition. A key observation is:

A missing link in geometry appears to be that those anchors are rather absent.

When you visit a new city then you tend to like it when the streets already have names. Suppose that you would be forced to invent your own labels, like "that crooked street with the blue shop" and then hope that other people understand you. Current textbooks on geometry send out students to conquer the plane but present it as a verbal desert, without conceptual guidance other than the x and y co-ordinates. The Van Hiele Level 0 requires them to visualize and to activate their intuition, yet that also requires a richness of words and concepts – that currently are lacking. Euclidean geometry has a poverty of points and lines that can intersect, be parallel or overlap: and though it is a great exercise in logic it must be admitted that Freudenthal has a point that Euclid's approach is not so appealing to the average student over the last two millenia. Conventional analytic geometry is an improvement since drawings are supported with formulas, and vice versa, yet again, its richness is only developed over time, and at the Level 0 and 1 there still isn't much to visualize and intuite and verbalize.

In particular, it will be useful to extend the plane with a nomenclature of *"named lines*". Chapter 4 of *Conquest of the Plane* opens with them and then builds up – see there to check what this means. A quick reply will be that we already have names, such as $x = 1$, $x = 2$, ... for vertical lines for example. Those names derive from a formal development however. Instead we rather first create standard names that fit the experience with the plane. This will provide the fertile ground, where the coin can drop when experience is morphed into abstract understanding.

It may be argued that it is fairly simple to draw a line and determine the starting value on the vertical axis and its slope. Exercises and realistic examples then provide for learning. However, experience shows that students later have difficulty with the horizontal and vertical lines. Why a line works as it does tends to remain elusive for them. A conclusion is that it is better to start with named horizontals and verticals and then awaken the motivation that a general formula will be useful.

Thus the didactic suggestion here is that the notion of *"named lines"* can be the missing link that resolves the issues in the choice between dropping Euclid and moving towards analytic geometry and calculus (and not just Descartes but along the lines of Van Hiele). The notion of these named lines caused the very layout of Chapter 4 on lines and subsequently from there the layout of the whole *Conquest of the Plane.*

Co-ordinates and vectors

Pierre Marie van Hiele argued most of his life (May 4 1909 - November 1 2010) in favour of the use of vectors already in elementary school. Though he has been greatly valued for his ideas on the didactics of mathematics, he never succeeded in overcoming the opposing views. Vectors even appear late in highschools. The missing link suggested here of *named lines* is hopefully helpful. Logically, if this is indeed the missing link that has been provided only now, then teachers seem to have been right in resisting Van Hiele's suggestion, since the picture is complete only now. Alternatively, the suggestion of *named lines* is not really a missing link and only one of the possible bridges, and we are underestimating the capabilities of pupils and students all over the board.

Clearly, the proof of the pudding is in the eating, and only empirical testing will show whether students indeed learn faster following the didactic approach presented here. If this book would be mistaken, and "realistic mathematics education" would still be needed to propel the more practically minded students, then, the lame argument becomes, it would suffice to include it in this book as well, and the advantage of this book would remain to be its logical order and novel concepts.

The importance of motivation

A final point of note is that I do not have clear ideas about what would motivate a pupil in elementary school to be interested in arithmetic and geometry, or a 12 or 14 year old kid to be interested in analytic geometry and calculus. Van Hiele (1973) rightly remarks that students and pupils hardly can be motivated for what they learn since they do not know yet what they will learn. A common ground is that man is a curious ape and cherishes the flashes of insight. Pupils recognise the moments when they grow in competence. Mathematics is a language and it can be fun to learn a new language and a new world. Paul Goodman (1962, 1973) *Compulsory miseducation* remains sobering though. While my books on mathematics education concentrate on knowledge the didactic setting naturally is a complex whole, in which motivation plays a key role, and it is mandatory to keep that in focus too.

Relating to the Common Core (USA, California)

The USA in 2010 installed the *Common Core.* Implementations can differ per State, and my frame of reference is California, given my attendance at Burbank Highschool in 1972-73. The Common Core CA (henceforth CCC) is here. [147]

This book relates to the Common Core at various points.

Decimal positional system

The information for Grade 1 is ambiguous about the decimal positional system. The discussion of CCC:14 does not mention it, but the Overview of CCC:15 explicitly states:

> "Understand place value. Use place value understanding and properties of operations to add and subtract."

My impression is that CCC:15 presents the ambition but that CCC:14 presents the reality that addition and subtraction are regarded as more important than the awareness of the structure of the number system. The suggestion of this book on page 17 above is that a better pronunciation of the numbers will allow to make progress.

Grade 2 is more ambitious on the decimal positional system:

> "Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones)." (CCC:18)

My suggestion is that they learn instead that 853 is 8 × hundred + 5 × ten + 3 × one. The multiples "hundreds", "tens" and "ones" are confusing. This invents some baby-language as if this would help. A teacher better asks "how many groups of a hundred are there ?", by which the notion of *grouping* (multiplication) is emphasized.

If Grade 1 succeeds in understanding 99 as 9 × ten + 9 × one then Grade 2 will quickly see that it is mere repetition to include hundred or thousand.

Co-ordinates and vectors

If pupils can count to 1000 in Grade 2 then they will also understand yardsticks and number lines, a city grid, a chess board, and thus also the system of co-ordinates. Didactics namely requires a unity of text, formula, table and graph. There is no need to wait till Grade 5 as happens now.

In Grade 3 (age 8-9):

> "By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle." (CCC:23)

Perhaps Grade 3 but then certainly Grade 4 (age 9-10) would be able to understand and likely prove the Pythagorean Theorem, if presented in the manner by Killian, see page 113 above.

[147] http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf

Geometry and angles

On the measurement of angles in Grade 4, see page 123.

One can do a bit more geometry once the Pythagorean Theorem is known, see page 113.

Remarkably, only Grade 6 learns how to calculate the area of a triangle (CCC:40):

> "They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles."

One contributing reason for this delay is that texts on geometry tend to be rather prim by presenting special formulas for special forms. The formula for the triangle uses *height* while this is a 3D and not planar term. However, there is also a general formula, see Table 11. (This is an idea of Killian too.) The didactic set-up then would be:

- develop the notion of an *average,* also an average length of a trapezoid: $(\ell + k) \times 2^H$
- develop the algebraic skills to work with the general formula
- discuss each particular formula but also the relation to the general formula
- suggest that pupils only need to remember the general formula.

Table 11. The general formula using the average length (base ℓ, across *k*)

	length	*width*	*surface s*	$s = 2^H \times (\ell + k) \times w$
square	ℓ	$w = \ell$	$\ell \times \ell$	$k = w = \ell$
rectangle	ℓ	w	$\ell \times w$	$k = \ell$
triangle	ℓ	w	$\ell \times w \times 2^H$	$k = 0$
parallelogram	ℓ	w	$\ell \times w$	$k = \ell$
trapezoid	ℓ, k	w	$(\ell + k) \times w \times 2^H$	also when $0 \neq k \neq \ell$

Fractions

Professor Wu from Berkeley gives much attention to the CCC. Consider his text on fractions, that is advised reading for anyone looking at arithmetic in primary education. [148] His objective is to accurately present the traditional approach. This differs from my objective to find possible sources of confusion in that traditional approach.

Wu:9 quickly moves to the number line, but this causes a rather complex discussion that slows down again, taking space till page 15. I would rather first introduce 2D co-ordinates on whole numbers, and then introduce Proportion Space (see COTP) so that the number line and equivalent fractions are immediately clear.

Wu:18 repeats CCC goal 4.NF 4: "Understand a fraction *a*/*b* as a multiple of 1/*b*." When we write this as $a\ b^H$ then it is clear that you can only understand the expression as a multiple of b^H. The 4.NF learning goal is provoked by a particular notation. What one should learn is that the notation is awkward. One might consider that $a \div b$ is an operation and *a*/*b* is a number, but this is awkward too since these are all numbers. The only thing of interest for $a\ b^H$ is what its decimal expansion is, for a location on the number line.

[148] https://math.berkeley.edu/~wu/CCSS-Fractions_1.pdf

Wu:26 states without hesitation on mixed numbers:

> There is a **notational convention** associated with mixed numbers: the + sign is omitted. Thus we write $5\frac{3}{4}$ in place of $5 + \frac{3}{4}$ and $7\frac{1}{5}$ in place of $7 + \frac{1}{5}$.

On p100 above we have mentioned that it is confusing to omit the plus-sign. This namely conflicts with the notation of multiplication. (Consider handwriting, not prints.)

These are but a few comments on the traditional programme on fractions. It is useful that Wu discussed the CCC programme so extensively, and also gave his own critiques (e.g. that a pizza is not a good learning example). But we should hope for change.

Higher mathematics standards: the notion of proof

CCC:58+ discuss the higher mathematics standards. Let me refer to *Elegance with Substance* (2009, 2015), *Conquest of the Plane* (2011) and *Foundations of Mathematics. A Neoclasssical Approach to Infinity* (2015) for discussion of this area. There will be other consequences for primary education, but for now it suffices what has been said already.

A major point of course is that when the notion of *proof* is established in primary education – see above p98 on *H* and p118 on the Pythagorean Theorem – then this will be greatly advantageous for mathematics education and mathematical competence in general.

Logic is only mentioned for Grade 8 in CCC:52, and set theory has a vague existence between middle and high school. It is more logical to introduce these in primary education, avoiding the New Math disaster of 1960-70 of course.

International standards, TIMSS and PISA

The Common Core programme has been based upon international standards too.

> "In mathematics, the standards draw on conclusions from the Trends in International Mathematics and Science Study (TIMSS) and other studies of high-performing countries that found the traditional U.S. mathematics curriculum needed to become substantially more coherent and focused in order to improve student achievement, addressing the problem of a curriculum that is "a mile wide and an inch deep."" [149]

I have only superficial understanding of TIMSS [150] and PISA [151] and draw a blank here. [152]

However, a critical comment is possible from the angle that we have mentioned the difference between the Van Hiele and Freudenthal approaches. The Freudenthal approach was institutionalised in the Freudenthal Institute in Utrecht, and its director Jan de Lange has been chair of the math working group of PISA. One of the issues is whether "arithmetic sums" are really arithmetic, and whether they are not "exercises in reading well". The Dutch position on the PISA list might reflect that Dutch mathematics education is an early adapter to the Freudenthal mold, and it might not reflect mathematics competence per se. Thus a general warning to be critical is no luxury.

The most relevant remark that I can make is to mention the website by Ben Wilbrink, a psychologist specialised in testing. [153] Apparently he values this paper [154] and he warns

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[149] http://www.corestandards.org/about-the-standards/myths-vs-facts/

[150] http://timssandpirls.bc.edu/timss2011/downloads/T11_IR_Mathematics_FullBook.pdf

[151] https://www.tes.co.uk/article.aspx?storycode=6360708

[152] http://www.mathunion.org/ICM/ICM2006.3/Main/icm2006.3.1663.1672.ocr.pdf

[153] http://benwilbrink.nl/projecten/pisa.htm

about the involvement by big corporations like Microsoft, Cisco and Intel in a project like "Assessment and Teaching of the 21st Century Skills" (ATC21S).[155]

Let me refer to *Elegance with Substance* (2009, 2015) for the political economy of the mathematics industry. The proposal there is that nations create national institutes on mathematics education, under democratic control, with involvement of participants. For the US, such institutes at the State level might work well too.

The Dijsselbloem confusion

The Common Core approach tends to follow the distinction that was also adopted by a Parliamentary committee in Holland, led by Jeroen Dijsselbloem, now President of the Eurozone ministers of Finance. This is the distinction between *what* and *how.* The idea is that policy makers (Parliament) decide what subjects shall be taught in education, for example arithmetic and geometry, and that teachers decide how it shall be taught. This seems fine for subjects like geography and biology (that I am not qualified for). However, for mathematics we run into the problem that mathematicians sell as "mathematics" which really is not very much of mathematics, when we look at it from the angle of didactics. For example, 2½ is rather crummy when it should be at least 2 + ½ and at best $2 + 2^{H}$. The list of errors is huge, including the major mishap that Freudenthal breached scientific integrity w.r.t. Van Hiele. Thus the *what* and *how* distinction doesn't work for mathematics, and nations need parliamentary investigations into mathematics education to sort out the mess and make funds available to re-engineer not only the dust of ages but also a culture that works against didactics. [156]

New in 2018

Colignatus (2018bcd) pertain to the US Common Core, and are structured on it.

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[154] http://www.utwente.nl/bms/omd/medewerkers/artikelen/vdLinden/IJER%201998%2C%20569-577-1.pdf

[155] http://blogs.msdn.com/b/microsoftuseducation/archive/2012/01/10/the-importance-on-assessing-students-21st-century-skills-not-just-math-science-and-reading.aspx

[156] https://boycottholland.wordpress.com/2013/03/27/jeroen-dijsselbloem-on-money-and-math/

Letter to the makers of CCSS and the makers of TIMSS

2018-05-31 [157]

The US National Governors Association (NGA Center for Best Practices) and the Council of Chief State School Officers (CCSSO) are the makers of the US Common Core State Standards (CCSS).

The CCSS refer to the Trends in International Mathematics and Science Study (TIMSS) (wikipedia).

TIMSS is made by the International Association for the Evaluation of Educational Achievement (IEA) (wikipedia). It so happens that IEA has its headquarters in Amsterdam but the link to Holland is only historical.

I am wondering whether CCSS and TIMSS adequately deal with the redesign of mathematics education.

There are conditions under which TIMSS is invalid.

There are conditions under which TIMSS is incomplete.

See my letter to IEA (makers of TIMSS) and NGA Center and CCSSO (users of TIMSS, makers of CCSS).

❧

To the International Association for the Evaluation of Educational Achievement (IEA) (**makers of TIMSS**) and to the US Common Core State Standards (National Governors Association (NGA Center) and the Council of Chief State School Officers (CCSSO)) (**users of TIMSS**)

IEA executive director Dirk Hastedt [... @ ....]
NGA Center director R. Kirk Jonas [... @ ....]
CCSSO executive director Carissa Moffat Miller [... @ ....]

[anonimised June 27 2018]

May 31 2018
Concerning: When TIMSS may be invalid and when TIMSS is incomplete

Dear Mr Hastedt, Mr Jonas and Ms Moffat Miller,

Thank you for all your good work.

The following is intended to help improve TIMSS and the CCSS.

[157] https://boycottholland.wordpress.com/2018/06/27/letter-to-the-makers-of-ccss-and-the-makers-of-timss/

(1) Introduction

I am writing as an econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008). It is fortunate that Mr Hastedt has a masters in mathematics and a Ph.D. in education research. Mr Jonas has a masters in political science and a Ph.D. in public policy. Ms Moffat Miller has a masters in sociology and a Ph.D. in education research. Let me advise you to consider reading Pierre van Hiele (1986), "*Structure and insight: A theory of mathematics education*", Academic Press. Van Hiele's theory is often misunderstood as relevant for geometry only, but Van Hiele presented it as a general theory for any subject, with geometry only as (aptly) *demonstration*. [158]

The impetus to write you precisely today is that I completed the paper *Arithmetic with H = -1*, [159] that highlights the issues below that are relevant for you more generally. The paper clarifies a.o. that negative numbers are important for number sense, understanding of the world, and pupil competence in mathematics. The paper deconstructs a curious confusion in traditional didactics of math. CCSS puts negative numbers very late in Grade 6, while Holland and the UK have them in Grade 7 (junior high). Would CCSS allow for change, and would TIMSS be able to record improvement, or, might reference to TIMSS make it harder to change at CCSS ?

(2) When TIMSS may be invalid

Arithmetic in elementary school prepares kids for later life. The highest aim is for algebra in highschool. If you do not master the classic algorithms in arithmetic (e.g. 1 / 7 + 1 / 8 = ...) then you will not be able to do proper algebra (e.g. 1 / *a* + 1 / *b* = ...). When there is a diagnosis for some pupils that they really cannot do better than with a calculator, and some trial and error, then these kids should get the education that serves them for later life, and they should not be subjected to later teaching of algebra that they would not be able to understand. I presume that these notions are so obvious that you will agree with them.

It has appeared in Holland that psychometricians (at Leiden University but also at Dutch CITO) can lack expertise in mathematics education (ME) and its research (MER). Some of them also explicitly say that they are not interested in ME & MER, and that it is sufficient for them that they would be competent on "testing" itself. They presume that the sums in K12 are so simple that they, with their expertise in Item Response Theory (IRT), would be competent to deal with those. This however is an unwarranted presumption. These psychometricians *count the correct outcomes of sums* and neglect *how* the outcome was found, by technique or calculator or trial and error. This kind of measuring runs against the very purpose of teaching mathematics: that it matters (for later stages in the curriculum) how an answer was found.

If this attitude and/or phenomenon also occurs at TIMSS, TIMSS would be invalid.

Around 1970 there was a "math war" [160] w.r.t. what was called the "New Math". [161] Hans Freudenthal and later the Freudenthal Institute at Utrecht University advocated an approach to didactics of mathematics, baptised as "realistic mathematics education" (RME), and likely also known in the USA as "reform math", that indeed allows for such trial and error and the use of the calculator. But trial and error and the use of the calculator are not the way to do algebra in highschool. Proper testing should expose RME as inferior for such purpose. The Freudenthal Institute should be abolished as unscientific, and motivated by ideology, like homeopathy.

[158] https://boycottholland.wordpress.com/2015/11/24/a-general-theory-of-knowledge/
[159] https://zenodo.org/record/1251687
[160] https://en.wikipedia.org/wiki/Math_wars
[161] https://en.wikipedia.org/wiki/New_Math

However, there now is the “perfect storm”, that invalid testing by the psychometricians allows this teaching philosophy to survive. Currently, Holland features somewhat high in TIMSS because many kids use trial and error and the calculator, but universities set up remedial teaching since students are lacking in technique. Given this criticism, the KNAW / Royal Dutch Academy of Sciences in 2009 supported a report on arithmetic education. [162] The committee was dominated by mathematicians without a background in empirics or K12 itself, and they relied upon psychometricians to provide for “the empirics”. Their report is invalid, for above reason.

The issue might be rephrased in this manner: whether the test criterion is the *correct outcome* for some types of sums or whether proper tests should be developed to measure student competence in technique. This rephrasing however would change the subject. The latter namely should be obvious, see the first paragraph in this section. Highschool teachers grade exams by checking on technique. The true problem is that psychometrians are lacking in compentence on ME & MER and that they don’t care about this.

This “perfect storm” may somewhat be rephrased in the following manner: That RME has succeeded in advocating that “sums with context” would provide an excuse for allowing the use of a calculator. Pierre van Hiele developed the theory that context would be important for the early levels of insight, but he also pointed to the need of abstraction for advancement. Basically, the methods (technique & algebra / trial and error / calculator) and the situations (context / no context) are independent of each other. It is only a confusion by RME to suggest that a context justifies the use of a calculator. It is true that education should also involve the use of the calculator, including when there are awkward numbers, but we should make sure that its use does not prevent the learning of technique required for later algebra (when you might avoid such awkward numbers). In this view on the issue, it are advocates of RME who have dominated with their confusion that context justifies the calculator, with the decision to include such sums in the tests, such that the psychometricians are not primarily responsible. The problem remains that the psychometricians support something that they do not understand, and that they are instrumental in tests that fail to expose an inferior didactics of mathematics.

There is a huge scandal here in Holland, with an experiment on kids in elementary school without the proper protocol on experimentation on human beings. A teaching method was introduced that allowed trial and error and the use of the calculator, and it survived in psychometric testing because those tests are invalid. Psychometricians neglect critique on their failure. This constitutes a breach of scientific integrity. [163] My reason to write you (the reason, and not the impetus above) is that the Dutch system on Research Integrity apparently is failing as well. Leiden University regards this issue as a “scientific dispute”. KNAW, supervising itself on research integrity, allows this to happen. We live in a “knowledge society” but the safeguards on what “knowledge” is are underdeveloped. Research integrity however requires scientists to correct an error when it is clarified to them, and to first study a field before meddling with it.

(3) When TIMSS is incomplete

In ME & MER there is a distinction between (i) those who regard K12 “mathematics” as given (tradition) and who only look for better ways of teaching tradition, and (ii) those (me) who hold that mathematics would be clear in itself, so that problems in didactics are caused by the empirical fact that tradition is not clear but rather crooked. Improvement of didactics is another term for that mathematics education must be re-engineered.

[162] https://www.knaw.nl/nl/actueel/publicaties/rekenonderwijs-op-de-basisschool
[163] http://thomascool.eu/Papers/Math/CWI-Leiden/2016-09-30-Letter-to-CWI-anonimised.pdf

Mathematics may have a reputation of being clear but the reality is that it has come about in a process of 5000 years of, indeed, trial and error. A critical mind will find a paradise for improving didactics (re-engineering of mathematics education). A main criterion is that we must remain *practical.* What could be handled by the system of standards, teacher training, and so on ? Would IEA and CCSS e.g. support a new ISO standard on the pronunciation of integers ? [164]

Perhaps the matter might be rephrased in this manner: What does TIMSS want to measure: whether kids learn the current crooked "mathematics" or whether they learn mathematics ? However, this opens the can of worms who decides what math would be, with presumably a key role for mathematicians themselves who are blind to the empirics of didactics. However, it is proper to look at this way: When TIMSS encodes current tradition in concrete, so that countries with a low score try to copy the higher-up countries, and when TIMSS does not allow for improvement beyond this, then TIMSS indeed turns current tradition into a prison, on the assumption that the last 5000 years of history indeed would have created a perfect mathematics.

Let me invite you again to read the paper mentioned above on arithmetic with H = -1 on the treatment of negative numbers and rational numbers in elementary school. [165] This deconstructs some misconceptions amongst teachers and educators in elementary school. Apparently mathematicians have been so focused on their abstract theories since John Wallis in 1673 that they could not understand those misconceptions by teachers and educators in elementary school. (I am open for better explanations.) Subsequently, CCSS give standards that put fractions first and that postpone negative numbers to Grade 6. However, this curriculum appears to be based upon a confusion, and, apparently, deliberate efforts to eliminate contradictory evidence from discussion. Read the paper and wonder how this ever could have happened.

Well, the likely explanation is: Within this group (i) above, who take tradition as given, there are (i-a) mathematicians without a background in didactics and empirical research, who mistake mathematics education for mathematics itself, and (i-b) pedagogues without a background in mathematics (who might hold that a calculator does a goed job for daily life). The latter implies that these pedagogues have no access to empirical methods that use techniques based in mathematics and statistics. There might be (i-c) mathematically competent people who employ empirical techniques, like those psychometricians, but still within said tradition. The latter means that they have not been trained on proper empirics, since proper *observation* of mathematics education leads to (ii) with its rejection of tradition. Researchers who are more competent on statistics may also have agendas of their own, like those psychometricians.

CCSS states with a reference to also TIMSS: "Fact: The mathematical progressions presented in the Common Core State Standards are coherent and based on evidence." [166] But, this "evidence" is based upon input that has been generated by the very tradition that ought to be tested. You will only find what you put in there before. CCSS thus sets up a circular argument.

In answer to this, TIMSS and CCSS should better allow for a decent degree of experimentation at the frontier of innovation and re-engineering of mathematics education. TIMSS might say that countries are free to experiment and that TIMSS will duly record the results. I doubt whether such response would really fit IEA. IEA started with the assumption that countries could learn from each other, but the focus now shifts from *countries* to *factors* that drive success. One of the factors for future success will be proper experimentation at the frontier. Subsequent improvements in countries could be traced to the adoption of methods that appeared successful at the frontier. TIMSS might

[164] https://doi.org/10.5281/zenodo.774866
[165] https://zenodo.org/record/1251687
[166] http://www.corestandards.org/about-the-standards/myths-vs-facts/

hold that it looks at *current* and not *future* success: "(...) to investigate how the participating countries are providing educational opportunities in mathematics and science to students, and the factors related to how students are using these opportunities". However, also a *current* success *now* might be explained by investments in experiments in the past. Also, some of the opportunities provided to students are the very experiments at the frontier, that allow some students to escape from the shackles of tradition.

This warrants the statement: TIMSS would be incomplete.

TIMSS needs a frontier for redesign of mathematics education and its research.

(4) Supplementary 1. Mathematics by computer

Let me alert you to an issue for State regulators w.r.t. computers and computer languages: "Everyone will be served by clear distinctions between (i) what is in the common domain for mathematics and education of mathematics (the language) and (ii) what would be subject to private property laws (programs in that language, interpreters and compilers for the language) (though such could also be placed into the common domain)." [167]

(5) Supplementary 2. Statistics education and government classes

Let me alert you to that something goes horribly wrong [168] in apparently many countries in the world, with both (i) education in statistics and (ii) government classes on democracy w.r.t. the application of statistics in the political science of electoral systems and votes and seats. The link to IEA / TIMSS and CCSS is: this branch of political science mistakes *words* (the language program) for proper *observation* (mathematics program on measurement and data).

On these latter two alerts: I can only observe that it is relevant for education but I have not elaborated how it could be relevant for IEA and NGA and CCSSO. I assume that you would spot the relevance when you would consider the given links.

(6) Warning on Holland

I started writing about didactics of mathematics in 2008, and was quite surprised by both my findings and reactions by others to those. My background is econometrics, with is a generalisation and not specialisation, and perhaps this helped to maintain common sense and roots in science and observation. Let me warn about the pervasive influence of ideology. Mathematics concerns thinking and people have a close attachment to how they think, or are trained to think, and mathematics even comes with some claim of being a better way of thinking. Regrettably I must report that Holland apparently since about 2000 has a "math war" to such extent that you cannot trust anyone from Holland except me on the issues in this letter. Each person might be correct for 99% but then there is always this 1% that subverts it, while it takes scrutiny to pinpoint and deconstruct that 1%. You can check how my analysis since 2008 has been treated in Holland: a wonderful opportunity to improve mathematics education and its didactics is maltreated by RME and traditionalist mathematics and psychometric "testing" and whatever 1% of bottleneck confusion. IEA has its headquarters in Amsterdam, and some of its staff may have heard about this "math war". This letter should provide you with a reality check.

[167] https://boycottholland.wordpress.com/2018/05/30/terminology-of-mathematics-by-computer/
[168] https://zenodo.org/record/1228640

(7) Closing

I have tried to alert you to the notion of “re-engineering of mathematics education”. Let me invite you to read said paper on the negative and rational numbers and also my book *Elegance with Substance* (PDF online), [169] and also *A child wants nice and no mean numbers* (PDF online), [170] (with now above [amendment] on the pronunciation of numbers) and allow me to suggest that you invite the participants in TIMSS and CCSS-Math to also give the issue a chance. Best is that you set up workshops on the many examples given, and that you invite people to write reviews on those examples and the general issue. When you see emerge some critical mass then you could proceed from there. I hope that this works for you.

Sincerely yours,

Thomas Cool / Thomas Colignatus
Econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008)
https://zenodo.org/communities/re-engineering-math-ed
http://econpapers.repec.org/RAS/pco170.htm
[.... Scheveningen, Holland ...]
http://thomascool.eu

[169] https://zenodo.org/record/291974
[170] https://zenodo.org/record/291979

Computer algebra is a revolution. But 21st century skills ? With protest against misrepresentation by Koeno Gravemeijer

December 5 & 7 2015 [171]

[Using Google Translate 2018 for some Dutch. See the pdf for the original quotes.]

Abstract

When you do mathematics on the computer then it is called "computer algebra". Since it is mathematics, it must also be studied in didactics for mathematics. For mathematics education the challenge is to bring computer algebra into the textbooks and the schools in ways that work. Applications of computer algebra to particular fields must be distinguished from those for learning mathematics proper. My three books that qualify on both issues simultaneously are "*Voting Theory for Democracy*" (2001, 2014), "*A Logic of Exceptions"* (2007, 2011) and "*Conquest of the Plane*" (2011), all applications of *Mathematica*. Instead of a fruitful exchange of ideas and experiences on education and didactics, the decision making discussion is haunted by ghosts from the past. Hans Freudenthal (1905-1990) created "realistic mathematics education" (RME). This RME was not tested in experimental manner but introduced generally in Dutch education. It appears to be a failure, and not a theory but an ideology. The Dutch government has set up additional courses and exams ["Rekentoets"] for secondary education to correct for what now has gone lacking in elementary school. In 2014 it appeared that Freudenthal also committed intellectual fraud on RME by appropriating and misrepresenting ideas from Pierre van Hiele (1909-2010). Koeno Gravemeijer (1946) has been promoting RME since around 1980 apparently without real interest in testing it, without discovering this (obvious) fraud, and has since 2008 not explicitly accepted its failure. Since at least 2001 he argues for "21st century skills", and uses the same arguments as for RME. Gravemeijer has written on computer algebra and supervised the Paul Drijvers (2003) thesis: yet, his wrong handling of didactics makes his expertise on didactics of computer algebra questionable too. Gravemeijer's lecture for the 2015 NVvW annual convention of teachers of mathematics in Holland neglected the failure of RME, was scare-mongering about the risks of the 21st century, and disinformative about the really interesting challenges with respect to computer algebra. The current decision making framework puts teachers in a powerless position, and this can be amended by a Simon Stevin Institute (SSI).

[171] http://thomascool.eu/Papers/Math/2015-12-05-Computer-algebra-21st-century-skills-Gravemeijer.pdf with a summary at https://boycottholland.wordpress.com/2015/12/08/computer-algebra-is-a-revolution-but-21st-century-skills-q/

Introduction

There is a seminal revolution on computer algebra but the discussion is dragged down into a morass. Distractive is "realistic mathematics education" (RME). This was presented in the 1970s as the answer to the New Math disaster in the 1960s. RME turns out to be a disaster itself too. It appears that Hans Freudenthal (1905-1990) who introduced it also committed fraud, see Colignatus (2014, 2015). [172] Distractive are the "21st century skills" as the present answer to the RME disaster. It appears that "21st century skills" that apply to mathematics are RME in disguise. What is happening here ? What gives a rational framework to handle the confusions ?

In Holland, a key supporter of both RME and "21st century skills" is Koeno Gravemeijer (born 1946, in 2015 69). There is no particular reason to single out Gravemeijer except for his speech to the 2015 NVvW annual convention of the Dutch association of teachers of mathematics that I attended. [173]

The following reviews the general argument and is also my protest against the abuse of science and logic by Gravemeijer. This memo provides a rational framework and deconstructs fallacies by RME and "21st century skills", for example on the need of teaching arithmetic that fits the so-called "21st century skills". Other texts by Gravemeijer are from 2002, 2013, 2014.

The **Article-Appendix** deconstructs Gravemeijer (2014) by paragraph: Dutch orginal on the left [Google Translate 2018 into English], comments in English on the right. This tabular format allows to see the fallacies, rhetorical techniques (like the straw-man) and disrespect for the intelligent reader.

We first need to develop basic notions before we can do the discussion.

Basic notions

Seminal revolution

We are living in a period with a **seminal revolution** similar to the invention of the wheel, the alphabet and positional number system:

> we can do mathematics on the computer – and it is called "computer algebra".

We know this since 1963 and Project MAC. [174] This is doing mathematics, rather than mere programming or punching buttons. By comparison:

- The arrival of calculators is not much different from the invention of ruler and compass, or the later arrival of tables for trigonometry and logarithms (recovered exponent, rex rather than log). Those are techniques, with the didactic balance of drilling and understanding.
- Doing mathematics on the computer is a game changer.

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[172] Update; For Dutch readers: http://www.wiskundebrief.nl/718.htm#7

[173] NVvW website in Dutch, summary of the Gravemeijer 2015 speech: [Google Translate 2018: "Globalization, digitization and automation are becoming more and more explicit in the news. Society is changing rapidly and with it, what is needed to be able to participate successfully. The education will have to be adjusted accordingly. It is therefore high time for a reflection on mathematics education. On the one hand because mathematics plays an increasingly important role in our society. On the other hand, because more and more mathematical and mathematical operations can be performed by devices. This requires a reconsideration of the goals of mathematics education. Moreover, it will be necessary to think about what digitization means for the design and implementation of mathematics education."]

[174] https://en.wikipedia.org/wiki/MIT_Computer_Science_and_Artificial_Intelligence_Laboratory #Project_MAC

In this, there is nothing special about the calendar, the year 2000 and the 21st century.

Computer algebra became mature in the 1980s. For mathematics education the challenge is to bring computer algebra into textbooks and schools in ways that work. Applications of computer algebra to particular fields must be distinguished from those for learning mathematics proper.

Bottlenecks in mathematics education

The real problem is this:

> **Proposition 1:** Mathematicians are trained for abstraction. When they come into the classroom, then they see real live students. They resolve cognitive dissonance by sticking to traditional views. Those views are not designed for didactics. Mathematical formats even appear to be crooked, and mathematicians are trained not to see this.
>
> **Proposition 2:** Mathematicians tended to despise computer algebra in 1980-2015, even though it was highly relevant for education. Nowadays there is more acceptance, but not necessarily for education. There is little need for teachers and educators of mathematics to use computer algebra, hence see its value.

In itself it is surprising that computer algebra isn't used so much yet. One supposes that the wheel or the alphabet also had to compete with alternatives. Once it is used, it becomes difficult to imagine how people could have lived without it. The market share for the use of computer algebra seems to have stopped at early adopters. There is however an explanation for the current stagnation.

A failed revolution since the 1970s: realistic mathematics education (RME)

In 2015, Holland is trying to recover from "realistic mathematics education" (RME). Correlation is no causation, and there willl be other factors at work, but there still is a causal connection:

- In the period 1970-2015, RME became dominant, with 75% coverage in 1994 and a peak of 100% in 2002-2010.
- In 2015, the government requires separate courses and tests on arithmetic for students in highschool and vocational schools, since they don't adequately learn this in elementary school anymore. [175]

Kees van Putten (2008) answers Adri Treffers, in the latter's denial on the worsening outcomes since 1987 in educational results on arithmetic. The thesis by psychometrician Hickendorff (2011) suggests that RME and "traditional arithmetic teaching" are "equally good" but this research suffers from invalidity since it neglects that arithmetic is relevant for later algebra, see Colignatus (2015c). (A quick test is that the words "algebra" and "algebraic" do not occur in the Hickendorff thesis.) [See this book p175.]

> **Proposition 3:** RME was created by abstract thinking mathematician Hans Freudenthal (1905-1990). Apparently the New Math and behavioral psychology (drilling pigeons) were a disaster, so that Freudenthal got a platform. But RME still is ideology and not empirical science. [176]
>
> **Proposition 4:** Freudenthal also committed intellectual fraud by taking ideas from practical teacher Pierre van Hiele (1909-2010), distorting them and presenting those as his own (while the distortion doesn't reduce the theft). See Colignatus (2014, 2015). [177]

[175] http://www.wiskundebrief.nl/721.htm#4
[176] Update: https://boycottholland.wordpress.com/2015/09/30/pierre-van-hiele-and-gerald-goldin-1/
[177] http://arxiv.org/abs/1408.1930 and above short http://www.wiskundebrief.nl/718.htm#7

In Utrecht there is the "Freudenthal Institute". This term suggests scientific neutrality. This is not deserved. It is better to speak about "Freudenthal Head in the Clouds Realistic Mathematics Institute" (FHCRMI). This denomination is no demonisation but an invitation to study the evidence in the given references.

Seeming revolution: 21st century skills

Some educationalists, also on mathematics, speak about "21st century skills", with notions like: communication, collaboration, "ict literacy", creativity, critical thinking, problem solving abilities, social and cultural values. [178] These are mostly goals of teaching since antiquity (except that parents may buy education for their children to give them a competitive edge over others), and there is nothing special about the 21st century, except the onset of computer algebra after 1963 that undeniably also continues in the present century. The phrase of "ict literacy" distracts from the real issues (see the Propositions).

If "21st century skills" *on mathematics* merely meant the decent introduction of computer algebra into education, including adaptive testing and assessment, then the discussion would be different. Instead we find a whole range of topics that are rather distractive. Someone is trying to set the agenda, and this someone is not necessarily the teaching community. It is not clear whether this "21st century skills" platform has an origin in industry or that it morphs various educational philosophies like RME.

FHCRMI has also been involved with "21st century skills" with special attention to mathematics. [179] There are now texts under the label of "21st century skills", also presented at the OECD, that are quite similar to RME. You may understand the feeling: plugging the hole in front of you, then another pops up behind you, with the same freezing water.

The movement for "21st century skills" is more diverse than only mathematics but doesn't seem to be properly critical about (i) abstract thinking mathematicians, (ii) RME, (iii) itself. Both RME and "21st century skills" are highly ideological and neglect that didactics is an empirical science – in this case of mathematics.

Abuse of fancy phrases

FHCRMI has a tradition of coining terms, to distract from already known concepts and to suggest something new. It comes with the advantages that you can hide that you have no new insights yourself, and that you do not have to refer to others who already have shown that you were incompetent in the first place.

- Hans Freudenthal coined words like "guided reinvention" and "anti-didactical inversion" so that he didn't need to refer to Van Hiele (1909-2010).
- "Literacy" is a term from the education in language, and the term has been applied by Jan de Lange of FHCRMI to Mathematics, and it has been adopted by OECD PISA. [180] Now there is "ict literacy" as if the notions would be so new.
- Another example is the phrase "think activities". [181] Activities are related to drilling and thinking tends to require that you sit down (Kahneman (2011), *Thinking, Fast and Slow*). Didn't Van Hiele in his 1957 thesis not already discuss the notion of insight ?
- Gravemeijer introduces the phrase "global arithmetic".

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[178] http://benwilbrink.nl/literature/21st_century_skills.htm
[179] http://www.fisme.science.uu.nl/wiki/index.php/21ste_eeuwse_vaardigheden and http://www.fisme.science.uu.nl/wiki/index.php/21ste_eeuwse_vaardigheden_en_WDA%27s
[180] http://www.oecd-ilibrary.org/education/pisa-2012-assessment-and-analytical-framework_9789264190511-en
[181] Update 2018: http://thomascool.eu/Papers/Math/2016-10-31-MTA.pdf with Dutch summary at http://thomascool.eu/Papers/AardigeGetallen/2016-11-20-Samenvatting-analyse-tav-WDA.pdf

Gravemeijer on the combination of RME and 21st century skills

Holland is recovering from the RME disaster. Gravemeijer has criticised this Dutch discussion on recovering from RME as being too much focused on arithmetic and neglecting the preparation of students for the modern information society. [182] This turns the true situation upside down. It neglects that it have been RME and he himself who created chaos. There still is no stated admission by Gravemeijer that RME is a failure, and explanation why. This denial causes that other people lose time now only to restore mathematical competences known since antiquity. A good basis in mathematics is required to deal with computer algebra. It is not sufficient to just have computer algebra: students need knowledge, skills and attitudes also with pen and paper. There are also developments w.r.t. computer algebra that Gravemeijer in this statement neglects, like my three books for mathematics education that use computer algebra.

My background

I myself have been a user of the computer algebra program *Mathematica* (by Wolfram Research Inc.) since 1993. I stated already at that time that computer algebra is like the invention of the wheel and the alphabet. I sell *"The Economics Pack. Applications of Mathematica",* my collection of applications. My book *"Elegance with Substance"* (2009, 2015) contains a discussion from 1999, *"Beating the software jungle",* with arguments that are repeated here. My three books are *"Voting Theory for Democracy"* (2001, 2014), *"A Logic of Exceptions"* (2007, 2011) and *"Conquest of the Plane"* (2011), all applications of *Mathematica*. These are applications to fields but also generate mathematical understanding.

Thus I write also from own experience – which the reader may see as a disclaimer too. Propositions 1 & 2 are based also upon observation as an eye witness. As far as I know, FHCRMI including Gravemeijer haven't shown an interest in my books that use computer algebra.

Governance of mathematics education and the Simon Stevin Institute (SSI)

Thus, what Holland tries to repair on RME may be re-introduced under international pressure before teachers get involved. It is crucially important to be aware of the power unbalance w.r.t. education and didactics. While the demand side is organised – governments are in the driver seats as to what should be taught - there is institutional chaos on the supply side. A key problem is that teachers of mathematics have no proper platform to discuss issues in scientific manner, with this harrassment by ideologues and non-empirical mathematicians. [183]

> **Proposition 5:** The evidence about the failure of RME is also evidence of the disastrous impact of the lack of *organised* influence by teachers on mathematics education. This failure of RME warns about the prospect for "21st century skills", and in Holland the Onderwijs2032 discussion. There is need for a Simon Stevin Institute (SSI). [184]

The Dutch Council on Education Onderwijsraad (2014) also speaks about "21st century skills" and concludes to the need of a new national authority to set the curriculum. Apparently the curriculum must be set by specialists, who need not be teachers. My suggestion instead is to have this SSI, such that teachers do also scientific research, and have their say about educational values, curriculum and didactics. Currently, it is a

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[182] http://www.fisme.science.uu.nl/wiki/index.php/21ste_eeuwse_vaardigheden [Google Translate 2018: " In his lecture on 7 March 2014 (National Competence Coordinator Day), Koeno Gravemeijer stated that it is strange that in all educational renewal around the learning pathways an orientation on mathematical knowledge that people need to function in the information society is lacking."]
[183] https://boycottholland.wordpress.com/2015/10/31/the-power-void-in-mathematics-education
[184] http://thomascool.eu/Papers/AardigeGetallen/2015-10-17-Aan-TK-commissie-OCW.html

problem indeed that most teachers know little about computer algebra: since they don't use it. The focus of teaching is confined to given curricula and the classroom, while my proposal is to have Academic Schools that allow teachers to do research on education, the curriculum and didactics.

In Holland, the Inspectorate is the guardian appointed by the government to check whether schools do what they are supposed to do (and for which they receive funds). Currently the Inspectorate ressorts under the government, but when the power unbalance in mathematics education is resolved, it would ressort under the national institute for mathematics education - proposed name: Simon Stevin Institute (SSI) - since the key control parameters are educational values, curriculum and didactics. [185] Obviously, this SSI would be based in the empirical science of educational research, and be responsive to practical teachers rather than ideologues.

As an example of the current power unbalance, consider the report by the Dutch Inspectorate of Education Onderwijsinspectie (2002), on ICT and education on mathematics and arithmetic. It is a horror show. I will say a bit more on this below. The Inspectorate of Education seems to rely on FHCRMI and its associates. FHCRMI has a tradition to program in single-purpose menu-driven push-button java-applications, instead of using an integrated computer algebra package that allows applications to build upon each other, and that allows teachers and pupils to further develop and adapt for suitable purposes.

In the current situation of unbalance we must perhaps wait for the application that teachers can use professionally – adaptive testing and assessment (Maple TA) – after which they can understand more about the seminal revolution that is taking place w.r.t. doing mathematics on the computer. But it also appears in this memo that policy makers on education do not get the proper information from said blind mathematicians and related educators (like Gravemeijer).

Structure of the argument by Gravemeijer (2014) on 21st century skills

Koeno Gravemeijer holds with hardly any evidence:

(1) that there *are* 21st century skills, also indicated by economists (though he is no economist and doesn't refer to views by critical economists)
(2) that the arrival of calculators and computers require a (vague) change in the teaching of mathematics (Sputnik 1957, computers were already a hot item in 1963, [186] Microsoft founded in 1975, Wolfram Research Inc. founded in 1987 for *Mathematica,* a system for doing mathematics on a computer [187])
(3) that calculators and computers are even more important after 2000 since there is now talk about "21st century skills" (as if the calendar matters) and this provides a welcome bandwagon to save RME
(4) that the solution is "realistic mathematics education" (RME), originally presented somewhat later than 1957 for somewhat other reasons than calculators or computers: but computers and such "21st century skills" can be usefully included in the arguments for RME, even though RME has shown to be disastrous
(5) that it is possible to *say* that some things in education must change (by implication also in RME) and to *hold* at the same time that RME should not be adapted, which is a remarkable agility with dealing with veracity.

[185] https://boycottholland.wordpress.com/2015/10/31/the-power-void-in-mathematics-education
[186] https://www.youtube.com/watch?v=Q07PhW5sCEk
[187] See various computer algebra systems: https://en.wikipedia.org/wiki/Computer_algebra_system

I will reply by giving evidence, both as an econometrician and teacher of mathematics. Note that I am qualified for teaching at secondary and tertiary education but not at primary education, while a key question in this discussion is whether (small) children should learn counting and arithmetic with their fingers or on the computer (tablet). My suggestion is an enquiry by parliament, that had authority and ample funds and can call in the help of other scientists versed in experimental designs involving children.

Neoclassical view on economics and mathematical skills

The argument about disappearing jobs due to technology and international competition with low wage countries (also low wage engineers) is scare-mongering, since technology and trade are sources of welfare. The real issue is how goverments are distorting markets with regulations and taxes. Fellow-economists writing on the impact of trade and technology (not only computers) better stop writing and first read my books (PDFs online):

(i) *"Definition & Reality in the General Theory of Political Economy"* (DRGTPE) on economics [188] and

(ii) *"Elegance with Substance"* (EWS) on mathematics education and the political economy of the mathematics industry.

DRGTPE presents new insights in economics. EWS has new insights on its subject too. These books present the evidence, and there is no need to further discuss this here. For Holland I propose two parliamentary enquires to study on the evidence and draw policy conclusions.

Gravemeijer neglects these books though EWS was reviewed in 2010 in *Euclides,* the Dutch journal for teachers of mathematics. Thus in the small research community of Holland, Gravemeijer writes about the economic impact and supposed need for educational changes, but neglects a different view close at hand without dealing with the arguments. (Ben Wilbrink lists various sources and other criticisms. [189])

On Gravemeijer (1994) and Van Hiele's theory of levels of insight

When a person is affiliated with the Freudenthal Head in the Clouds Realistic Mathematics Institute then it is advisable to check how he or she writes about Van Hiele's work. The Gravemeijer (1994) thesis is online. [190] P22-23 correctly summarizes Van Hiele's level theory. He also correctly quotes Van Hiele (1973) *"Begrip en inzicht"* p182-183.

> "Whereas at ground level the concept 'four' may be tied to visible entities, e.g. to the vertices of a square, and features as a word in the series 'one, two, three, four, five, ...', on the first level it is a junction in a relational framework. On this level it might be two plus two, or two times two, or possibly five minus one. In any case it has already disengaged itself from the realm of the concrete." (Van Hiele p182 quoted by Gravemeijer 1994:23)

Gravemeijer p23 concludes fairly:

> "For the authors of R&W [a textbook developed by him and others], the significance of the level theory did not reside in its theoretical use, for example a sharp classification in levels, but in its practical implications. First, mathematics has to start on a level at which the concepts used have a high degree of familiarity for the students, and, secondly, its aim has to be the recreation of a

[188] Dutch readers can benefit from D&S: http://thomascool.eu/SvHG/DenS/Index.html

[189] http://benwilbrink.nl/literature/21st_century_skills.htm

[190] http://repository.tue.nl/443094

> relational framework. The selection of Van Hiele's level theory also had consequences for the curriculum goals: rather than aiming for isolated skills or basic facts, courses would be aimed at the creation of relational frameworks. In more concrete terms, numbers up to 20 would eventually function as junctions in a relational framework."

In the **Article-Appendix** we will see Gravemeijer consistently argue for a "network", required for the development of proper number sense and algebraic sense. However, the errors are:

(i) allowing pupils to rediscover relations: allowing them to get lost or take too much time
(ii) overindulging: too many exercises, again and again requiring the same discovery.

This overindulging is an abuse of the work of Van Hiele, since these two errors cannot be logically tried to this work. Reading the work of Van Hiele one gets the impression that he is rather traditional in terms of guiding pupils along the path of the traditional algorithms like long division. Thus, while it is important to develop relations, it is important to see that the tables of addition and multiplication already provide such relations, and that awareness of those can be generated by discussing and memorising and gradual rising experience. Yes, one can bring a horse to the water but not make it drink. The pupils should have freedom for their own creativity so that the penny can drop. But the errors above are in the principles that cause excess – caused by Freudenthal's misrepresentation of Van Hiele, and duly copied by Gravemeijer.

Gravemeijer (1994:25-26) takes only three levels. With the third level given as group theory and its feature to enter into the didactics of proof, he is forced to assign the associative, commutative and distributive rules of arithmetic to the first level of relations (above the basic level). It is more useful to have four levels, with those rules as a separate intermediate level. In that case the shift from numbers to variables is gradual and natural, since variables are handled more via rules than numbers. Van Hiele (1973:199-200) proposes the introduction of the Abelian groups for addition and multiplication early in education, since it is easier to discuss the notion of proof with arithmetic than with geometry. It is not clear to me whether Gravemeijer discussed this. It does seem that the years 1957-2015+ have been lost for didactic improvement according to Van Hiele.

On Gravemeijer (1994) and RME

This present deconstruction of Gravemeijer and "21st century skills" somewhat overwhelms me. At first I thought that the deconstruction in the **Article-Appendix** should be sufficient. But Gravemeijer's argument on "networks of relations" reminded of Van Hiele and caused me to look at his 1994 thesis, see the discussion above. For the rest I looked at it only diagonally. Perhaps I should look into it deeper – but there is also lack of time and urgency. For this memo my position is: (i) RME is bankrupt, given the Dutch evidence, (ii) it is fair to take some points from the 1994 thesis to give some indicative links for who wants to delve deeper.

Gravemeijer (1994) Section 6.1 on "evaluation research" gives his view on empirical tests on RME. He distinguishes curricula (his topic) from other issues (practical teaching). A statement:

> "My suspicions are that the realistic curricula in The Netherlands will surpass their competitors in the area of learning results."

There is a reference to Wijnstra 1988 (PPON by CITO), and he calls it "quite convincing" (with only reference on p136 to Treffers 1988 that the distinction would be positive for RME). He might have regarded CITO as better equiped rather than himself to do the actual testing.

When we observe in 2015 that RME doesn't work, Gravemeijer (1994) Section 6 also indicates the true RME way out. RME might not be properly executed. It might be that teachers use RME books but continue teaching in traditional manner, for example pick out only the sums and start drilling again. He sees two possible remedies: either fully work out proper RME didactics and put this in the textbook fully, so that the teacher may also be an actor playing a script, or resort to indoctrination, such that teachers know RME by heart and will not deviate from the true gospel. Gravemeijer (1994:175):

> "For the time being, two paths are available for improving the implementation:
> - directly influence the teachers' views, knowledge, insight and skills (...),
> - choose a more directed form of realistic mathematics education and adapt the textbooks accordingly."

It remains remarkable that the option that it doesn't work isn't mentioned. This however can be explained by the expectation that the method will be successful, which expectation is so great that the method was introduced without proper testing on lab rats. [See p186 for an indication of the evidence post hoc.]

The degree of control that Gravemeijer specifies is proper for a controlled experiment, and generally inadequate for a field test. When he requires such a degree of control, why didn't he design such a controlled experiment ?

Gravemeijer's lack of teaching practice and empirical testing

The 1994 thesis by Koeno Gravemeijer's informs us that he first studied mathematics and physics, majoring in nuclear physics, in Amsterdam, and subsequently education, majoring in *structural design*, in Leiden. His curriculum vitae does not mention practical experience in teaching mathematics at any level (primary, secondary, tertiary). There is no stated evidence of having been involved systematically in proper empirics, e.g. empirical modeling, testing or experimental design. The thesis mentions an "experiment" on a number line, a test at an American school, and some schools are called "experimental schools". Chapter 5 contains a few statistics on curricula but I have not looked at it to see what it means and whether it is relevant or valid, either in 1994 or 2015. I haven't looked at his list of publications.

Having listened to his 2015 NVvW lecture, my impression is that Gravemeijer is more an abstract thinking mathematician than an empirical researcher.

He worked since 1986 at Freudenthal Head in the Clouds Realistic Mathematics Institute in Utrecht and retired as professor in Eindhoven and Utrecht. [191] He has been long involved [192] in RME, say with the MORE (1993) abuse of the work by Pierre van Hiele, [193] and this 1994 thesis is under supervision of Adri Treffers, another pillar of RME. [194]

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[191] https://www.tue.nl/en/university/about-the-university/eindhoven-school-of-education/about-esoe/staff/detail/ep/e/d/ep-uid/20072724

[192] http://benwilbrink.nl/projecten/positionering.htm

[193] http://thomascool.eu/Papers/Math/2015-09-15-Breach-by-Jan-van-de-Craats-and-Ben-Wilbrink-wrt-scientific-integrity.html

[194] Treffers's thesis "Wiskobas doelgericht" is from 1978, when supervisor Freudenthal (1905-1990) was 73 years of age. Second supervisor was Jelle Sixma (1918-2010) an educator known for his work on reading conditions ("Leesvoorwaarden", 1973). I haven't checked this thesis. One can only hope that Sixma wasn't bullied by Freudenthal into believing that the educational ideas on mathematics were correct. Incidently, there are stages of reading, that remind of the levels of Pierre

Rather than admitting failure on RME (the evidence mentioned above), Gravemeijer points to this "21st century skills" discussion, perhaps in real concern about the 21st century, but just as likely to save RME.

On Gravemeijer and didactics of computer algebra

Gravemeijer has written on computer algebra and supervised the Paul Drijvers (2003) thesis, alongside other superviser Jan de Lange. [195] Yet, Gravemeijer has shown a wrong handling of didactics with an emphasis on RME ideology and less interest in empirical mathematics research: thus his expertise on didactics of computer algebra becomes questionable too. With emphasis: *questionable*. This is a new field and all researchers are handicapped. Interactivity with a computer reminds a bit of private tutoring, but the computer can be stupid on questions and fast on complex results; and so on. Doing mathematics on the computer is a game changer. It might be that one must first learn mathematics in the traditional manner before using the computer. Thus there is every reason to be careful.

I was involved in college education 1997-2001 and highschool after 2007 (with a first degree in 2008). There was no cause for me to look at Gravemeijer or Drijvers (2003) on computer algebra, see my objectives in EWS (2009, 2015).

- I do protest that Drijvers in 2012 as editor of the "Handboek Wiskundedidactiek" allowed Gerrit Roorda to be silent on my suggested algebraic approach to the derivative. This wasn't resolved in 2012 and hence I also protest since 2014 that he was appointed professor in mathematics education research in 2014. [196]
- Given the 2014 discovery of Freudenthal's fraud on RME, that is obvious when one starts studying the works by Freudenthal and Van Hiele, it is curious that Drijvers didn't discover this himself, and hasn't responded yet.
- Given Drijvers's stated academic interest it is curious that he hasn't looked yet at my books that are written in the environment of Mathematica (dates 2001, 2007, 2011), and hasn't even stated why he has shown no interest (whether it is because of RME ideology or other). He might also have played a positive role like Christiaan Boudri w.r.t. an improper "review" but apparently did not. [197]

The mentioning of these various names should not distract. My proposal has been an enquiry by Dutch parliament and the creation of a Simon Stevin Institute that would create an environment to discuss such issues in proper scientific fashion.

A problematic text by the Dutch Inspectorate of Education 2002

Looking into this subject, Google generates also this report by Onderwijsinspectie (2002), that explicitly deals with ICT (information and communication technology) and "arithmethic and mathematics for the 21st century", in which they refer on p4 to Gravemeijer (2001, 2002).

There are various institutional connections and flows of funds. Getting rid of the RME and "21st century skills" confusions is one thing, but these institutions must also appear to be willing to blink.

There has been a huge waste of public funds, with the finance of all small applets and other computer projects, instead of adopting a fully integrated computer algebra system

van Hiele, see https://en.wikipedia.org/wiki/Jeanne_Chall And apparently there are stages in language acquisition https://en.wikipedia.org/wiki/Roger_Brown_%28psychologist%29

[195] http://jdlange.nl/about (thesis 1987 with supervisors F. van der Blij en A. Treffers) http://www.fisme.science.uu.nl/wiki/index.php/Mathematics,_Insight_and_Meaning

[196] http://thomascool.eu/Papers/BHRM/Index.html

[197] http://thomascool.eu/Papers/COTP/LOWI/Index.html

(that doesn't deal with gadgets but concentrates on what matters: doing mathematics on the computer).

Heck et al. (2008) modestly state: "The main drawback of the generated Java applets is that there is no real computer algebra system behind it yet." A current remedy is to use the Java applets as front-ends, and create a link to the CA system in the back, but, why not use CA directly ?

Disclaimer on Mathematica

In the USA there is a general reliance on private enterprise and distrust of big government. In Europe there may still be preference for government intervention under democratic control. Mail delivery is an old government licence, and paradoxically U.S. Mail still exists because it is mentioned in the US Constitution while Holland has now privatised mail delivery.

As a scientist I use a computer with an operating system that are both produced by private companies. Thus I am also reasonable at ease with WRI as a technology firm that develops *Mathematica.* Still, mathematics should be free for all, and there are awkward issues when part of the mathematical language would be claimed as format in a particular computer application. For example, *Mathematica* uses = for Set, == for Equal and === Identical, while Algol (Edzger Dijkstra) used := for Set. Mathematics is also communication, also with a computer, and one must make choices. One can imagine that artists may have some claim on some form that they invent, but for mathematics such notions arise in the literature and it shouldn't be that a technology firm actually uses the need for a convention to create a platform that subsequently is claimed to be their property. Creativity and endurance should meet with rewards but not block such efforts by others.

Elegance with Substance (2009, 2015) discuss the issue on beating the software jungle. A new analogy is the business model by CITO, the assessment company that derives from psychologist A.D. de Groot. They have both a not-for-profit foundation and a for-profit company. CITO does testing for the government, say all kids graduating from elementary school, which can be seen as a public service which also requires involvement and open access for science. Apparently there are gate-keepers who guard the exchange of R&D knowledge between the two legs of CITO. [198] My suggestion to WRI is to look into this model. When all computer algebra systems can use the same language, then programs can be exchanged, and then competition shifts to relevant areas as it properly should.

Admittedly, the phrase "language" may be too simple. This isn't just the use of the alphabet. Communication between people is not just by talking and (sometimes) listening, but also uses gestures, (motion) pictures, and so on. For computers there is the interface – in *Mathematica* called the Front End. This uses menu's and conventions on what to project on the screen. Apparently there is a growing legal body on the "look & feel" of computer programs. However, one can imagine that education should be able to specify requirements, and that those would create a platform for competition.

Consider the role of Microsoft Word. Admittedly, dedication generates stability. (But it was integrated with Excel into Microsoft Office.) Still, if Microsoft had made Word a public domain program, then it could have been the basis of PDF, e-readers and browsers too, and there would have been less need for other dedicates. An environment for doing mathematics on the computer also requires an environment for text editing, if only to type in answers for an assessment, but also to write books. Currently programmers are forced to recreate the same functionality of Word, and take the advantage of giving their own formats a commercially exploitable "look & feel". This is what I call the "software jungle".

[198] http://www.cito.nl

Instead, with a common foundation competition is not between English and French, Apple or Microsoft, but on extras and the "je ne sais quoi" that generate productivity growth. Potentially governments have a vested interest for education to create such a foundation: but now crucially with the feature of doing mathematics on the computer.

Conclusions

Our conclusions are:

(1) Once Freudenthal (1905-1990) as a mathematician accepted the 1957 thesis by Pierre van Hiele (1909-2010) on didactics of mathematics, Van Hiele should have become professor in mathematics education research, and Freudenthal should have stopped peddling his educational views unless following proper methodology of science. Freudenthal was already deep into fraud, see Colignatus (2014, 2015), when he promoted Treffers in 1978, who again promoted Gravemeijer in 1994, who promoted Drijvers in 2003. It is absurd that Treffers, Gravemeijer and Drijvers studied the works by Van Hiele and Freudenthal and did not discover Freudenthal's fraud. It doesn't seem likely that the thesis and professorships by Gravemeijer derive from real science.

(2) Gravemeijer should give a clear explanation of the dismal results of RME in Holland, and not dodge the question. Given his stated expectation in 1994 of a success, it is strange that he is not curious about the real outcomes. The more he dodges the issue, see Van Putten (2008) in answer to Treffers, the more he appears to be an ideologue. It becomes ever more likely that he wasn't interested in real outcomes in the first place. It may also be difficult for him to judge on this, because he lacks qualification and actual practice in teaching mathematics at elementary or secondary level (and only non-mathematics at tertiary level).

(3) Gravemeijer (2015) 's presentation at the 2015 NVvW annual convention was a repeat of earlier misconceptions and misrepresentations. Apparently he regards math teachers as people who can be told such stories. Personally I was a bit amazed about the more than polite applause but I also suppose that teachers of mathematics tend to be lacking in knowledge of economics. (See EWS w.r.t. some confusions on economics in textbooks on mathematics. Please remember that I am critical of mathematics education.)

(4) Key challenges for mathematics education that don't depend upon the calendar are discussed in EWS (2009, 2015), and neglected by Gravemeijer. Let he explain this neglect. Holland is a small country and foreigners would tend to suppose that locals communicate. Since I listened to Gravemeijer's lecture and made this deconstruction, let he look at EWS.

(5) This discussion is not about ideology but about scientific standards. Naturally there are many other challenges but a core issue is the resolution of the power unbalance in mathematics education, i.e. for Holland the need to create a Simon Stevin Institute. [199] Without such a national body that provides a foundation for this kind of discussion within the empirical science of education research, and that links theory and empirical foundings to educational practice, this present discussion is rather hopeless because quickly soured by ideology, as it apparently has been since Sputnik 1957 and the New Math in the 1960s.

[199] https://boycottholland.wordpress.com/2015/10/31/the-power-void-in-mathematics-education

Article-Appendix: Deconstruction of Gravemeijer (2014)

Tijd voor ander rekenonderwijs	***Comments on this article***
Dutch text by Koeno Gravemeijer (around November 2014 ?) [Google Translate 2018] http://www.didactiefonline.nl/blog-blonz/12171-tijd-voor-ander-rekenonderwijs and also http://www.rekenenwiskunde21.nl	English Comments by Thomas Colignatus, December 5 2015 http://thomascool.eu and https://boycottholland.wordpress.com/2015/10/20/hans-freudenthal-petty-crook-no-demon-1
Virtually all calculations that children now learn are performed in the world outside the school by computers. But that does not make math education superfluous. You also need calculation skills to be able to work with devices that perform all kinds of calculations.	The reference to computers is a bit silly. The TI-83 was "advanced" when it was introduced in 1996, see http://mic.com/articles/125829/your-old-texas-instruments-graphing-calculator-still-costs-a-fortune-heres-why (reference thanks to Raymond Johnson)
But these are other mathematical skills than those that arithmetic education is now focusing on.	Current education has also been targetted at handling the calculator in a decent fashion. One can agree that this education must be changed, but not because of the argument that computers are a novel phenomenon.
This involves being able to apply math knowledge, understanding what the computer does and being able to monitor the computer globally. In addition to more attention for application and understanding, this also requires a change in subject matter.	• Current mathematics education is already quite focused on applications – see the use of contexts. • Van Hiele indicates that learning goes from concrete to abstract, and Freudenthal misread this as applied mathematics. • The new phrase "global control" for computer use just rephrases what is known since the advent of computers.
Society Computers change society in two ways, on the one hand by making labor superfluous, on the other hand by creating new jobs. This way, computers take over all kinds of tasks; especially in industrial processes, but also by calculating the price of the vegetables that you weigh in the supermarket, or by making bank employees superfluous when withdrawing money.	• It is well known that computers change society. Microsoft was founded in 1975. • Giving such examples is a bit silly. • Challenges for the future are a bit different than this early phase of computerisation. Challenges to the legal system are for example: privacy and other protection against abuse.
But the computer also creates new possibilities, such as the use of 3D printers, the analysis of big data and the calculation of simulations. In the first case the computer is a competitor, in the second	• It is again silly to argue that the computer is a tool. It isn't intelligent yet, is it ? • Wolfram Research was founded in

case a tool that complements human action.	1987. Perhaps G adapted this text a bit from a text from 1990 ?
Translated into education, this means that we should not focus so much on skills that the computer takes over.	This can only be a deliberate confusion and hence a fallacy: (a) Nobody claims that people should be trained to beat computers (Kasparov vs Deep Blue 1996). (b) People still need education on mathematics and so on.
More important are skills that you need when working with computers or computerized devices or that are more important in a broader sense for participating in a computerized society.	Yes, people need an education in mathematics and an education in how to deal with computers and an education on doing mathematics with a computer system. (But G tends to forget the latter.)
For calculating, we then come to issues such as recognizing arithmetic problems, translating such problems into computational tasks for a computer, understanding these operations and interpreting and evaluating answers. This is roughly about applying, understanding and global computing.	• This is what the training on the use of the calculator has been about. • But beware: G claims to introduce the new magic phrase "global arithmetic". A term before 2000 was "computer savvy". • Computer algebra is a game changer, and indeed doesn't get sufficient attention. Why don't you explicitly say this ?
Understanding and applying belong to accepted calculation goals. But a choice for global arithmetic leads to a thorough change of the subject matter. To be able to calculate globally, you must have networks of number relations and be able to deal flexibly with properties of calculation operations.	G's misrepresentation is: • This supposedly novel concept of "global arithmetic" isn't included in current mathematics education – neither in "realistic mathematics education" (RME). • Supposedly the "computer" would be the new phenomenon causing this change.
Number relations When evaluating calculations, it is sufficient to be able to determine globally what the answer should be. To give a simple example: for a problem such as 4 x 27, this means that a student thinks that the answer is more than 100 (4 x 25), another that it is less than 120 (4 x 30). And yet another student can think that there is 108 (2 x 54). Ideally, it should be the case that students use those number relationships that they are familiar with.	• This "global arithmetic" turns out to be the competence to guesstimate what a model would generate. • It so happens that part of this is already included, both for handling a calculator, and in elementary school RME. • Unfortunately, in this RME, much of this guessing needlessly tends to replace proper accuracy when such accuracy should not be a problem. (The given examples can be calculated simply.)
If we want students to be well-placed in this respect, then we have to invest in practicing and playing with number relations that you can use a lot.	What about learning the tables of addition and multiplication by heart ?
For multiplication, for example, we can think of multiples of 25, 75, 125 and so on,	• It is dubious whether it should be a learning goal to memorize what

and being able to relate these numbers to decimal numbers, fractions and percentages. It is ultimately about networks of number relations on the basis of which students can for example think that 4 x 1.25 = 5, because 4 x 25 = 100, and thus 4 x 125 = 500, or, because 4 x 1.25 equals 4 x 1¼.	numbers are easy to calculate. • It is a learning goal however that students develop a sense of numbers and algebra, so that they can proceed to the next stage of quantification: handling unusual quantities and without getting lost.
For the sake of clarity: I do not plead for all kinds of rules for handy math that the pupils should learn to apply. When the pupils have a network of number relations, they can, as it were, view these number relations as puzzle pieces that they can combine in such a way that they find an answer.	• If you do not plead for this, why give those misleading examples ? • If you agree to abolish RME and return to more traditional education, why not say so ?
Consider, for example, the calculation of 7 + 8. If we present this task to young children who have a suitable network of number relations, the numbers 7 and 8 will call up different number relations with them. Such as, for example: 7 + 3 = 10, 7 + 7 = 14, 8 = 7 + 1, 7 = 5 + 2, and 8 = 5 + 3.	• There is no criterion why 7 + 8 should associate with 7 + 3 = 10. Perhaps in the early grades of elementary school when the tables of addition haven't been learned yet, one might have a discussion on using 7 + 3 + 5. But such a pons asinorum better soon be replaced by learning the tables of addition and multiplication. • It is silly to compare such elementary outcomes with the understanding and skill of dealing with calculators and computers.
They can combine these in various ways into a calculation sentence that provides the right answer. As: 7 + 8 = 5 + 5 + 2 + 3 = 10 + 5, of: 7 + 8 = 7 + 7 + 1 = 14 +1, of: 7 + 8 = 7 + 3 + 5 = 10 + 5.	• Yes, back to first grade. • Still a confusion between calculator and computer algebra. • This is supposed to be the argument for a novel approach required for dealing the revolution of computer algebra ?
From a mathematical point of view, the pupils use the 'associative' and the 'commutative' traits. In the aforementioned example of 5 x 25, they use the 'distributive' property: 5 x 25 = 4 x 25 + 1 x 25.	• Yes, for the development of a good sense of number and algebra: a discussion of the properties of association, commutation and distribution are advisable. • This would be in the current programme if RME hadn't created such a havoc.
It is therefore about using calculation properties and number relations, and not about choosing from a repertoire of 'handy solution strategies'.	This is a mispresentation. • Yes, distribution is implicit in this example. • The example above was introduced as coming from a "network of relations": which still is RME trying to allow students to develop number sense by

	trying. • Thus G has the implicit false suggestion that RME methods train students in understanding association, commutation and distribution. • It can be observed that Jan van de Craats has been criticising RME in this manner, mistaking Van Hiele networks of relations as handy solution algorithms. Thus Gravemeijer's article is partly a correct defence against this shallow critique by Van de Craats. Both are wrong however.
This requires a different primary school program than mathematical education that trains students to quickly and routinely solve math problems.	The misrepresentation is: RME is defended (a) via the supposedly new phenomenon of the calculator or computer, (b) to answer to the pleas for more attention for algebra. (c) to downgrade traditional education on arithmetic as routine drilling.
The strength of standard procedures is that you do not take into account specific characteristics of numbers: the procedure always works and you do not have to think about the numbers. The downside is that you do not develop the number and calculation knowledge described above.	The misrepresentation is that traditional education would only be interested in routine drilling, and not in the development of other aspects, such as the development of sense of number and algebra, and transfer to applications. RME follows one particular road to get to sense for number and algebra, and doesn't see any alternative except drilling.
Another advantage of standard procedures is that they are efficient in the knowledge they use. You have the basic automatics for addition and subtraction and the multiplication tables sufficient for performing all digit algorithms. But here too there is another disadvantage. Multiplication knowledge that exceeds the standard tables - such as multiples of 12, 15 and 25 - are not discussed in [cyphering].	• Traditional algorithms (like long division) are called "cyphering". This is denouncing writing as "lettering". • It is suggested that traditional didactics would only be interested in drilling. • It is false that the tables of multiplication up to ten and the traditional algorithm would cause a problem for numbers like 12, 15 or 25. It is false that you would have to learn a table of multiplication for each number.
Unnamed numbers Networks of number relations have another function: they play an important role in the transition from named to unnamed numbers. When calculating with natural numbers, this goes more or less automatically. Numbers which initially only have meaning in combination with concrete quantities, such as in 'four marbles', gradually acquire the character of objects,	• G refers to algebra as "unidentified numbers" but algebra is more than the possible interpretation by means of numbers. Algebra also concerns formal patterns. • This repeats the confusion as if "networks of connections between numbers" other than the tables of addition and multiplication would be a serious objective in the development

which derive their meaning from networks of number relations. The number 4 is then associated with 4 = 3 + 1, 4 = 2 + 2, 4 = 5 - 1, 4 = 8: 2, etc.	of a sense of numbers and algebra. Perhaps this might be true for RME that doesn't rely on traditional algorithms and that forces students to try to find solutions. But then it should be presented as a disastrous consequence of RME and not as a valuable educational target.
There is a problem here with fractions. Research by Bruin-Muurling shows that pupils in the PO [Primary Education] work almost exclusively with fractions as named numbers, while in the VO [Secondary Education] it is assumed that the inflowing pupils have already reached the level of the unnamed numbers. For a good connection, the fractions must also derive their meaning from number relations. For example, at ¾ this may be: ¾ = ¼ + ¼ + ¼ = 3 x ¼, ¾ = 1 - ¼, ¾ = ½ + ¼, or ¾ + ¾ = 1½, but also 3 : 4 = ¾, ¾ = 6 / 8 = 9/12 = 12/16 = ..., and ¾ of 100 is 75, etc.	• Yes, elementary school in Holland has made it easy for itself by dropping algebraic understanding of fractions from the learning goals. They are happy when students can calculate sums, e.g. by using "tables of proportions" [“equivalent fractions”], even when they have no insight in the formal form. • No, you cannot misrepresent and rephrase this as if this is related to such a "network". The objective to master algebra of division doesn't require the acquistion of such networks first.
A good base At the moment there is a lot of attention for the 'basic skills'. In doing so, it is assumed for the sake of convenience that basic skills for calculation are unchangeable. But what are the real skills that today's students need? I do not think that the smooth and routine multiplication of numbers of three or four digits is included. You use the calculator for this outside education. But you also do not need this skill for mathematics in secondary education.	This misrepresents the learning goals w.r.t. the fast and routine multiplication of figures with three or four digits. G suggests that people who advance this learning goal would negate the existence of the calculator, and would overstate the requirement for later algebra in highschool. Instead, the learning goal w.r.t. this activity is not in the result of the calculation, but is the command of the underlying operations, the understanding of the positional system, the use of memory for the various steps, the development of sense of number and algebra. It is a gross misrepresentation as if the learning goal for traditional didactics are that society would need that all people can do such calculations routinely. It is merely one of the useful *test formats* at the end of elementary school, for the stated purposes.
When you ask yourself what math knowledge and skills lay the foundation for algebra, you come across other things, such as insight into the properties of calculation operations. Students must be able to handle these properties flexibly.	In these lines G reponds to criticism: that the RME method "try to find an answer" doesn't lay a foundation for algebra. He misrepresents this criticism, for he suggests that traditional didactics would think that routine calculation would provide such a foundation. The traditional didactics is to make students aware why the traditional algorithms work: which cause the awareness of the properties of association, commutation and distribution.

For example, the multiplication of tweetermen, ((a + b) x (c + d) = ac + ad + bc + bd), and the related curious products, are based on the repeated application of the distributive property. This flexible use of properties of calculation operations is not new. It has a long tradition in the Netherlands in the so-called mental arithmetic.	• G suggests that such algebraic understanding derives from calculation by heart (in one's head, without pen and paper), whence in his opinion this is better learned by RME, which would generate better number sense ("networks" other than tables of addition and multiplication). • But no, such algebraic understanding is based upon knowledge of the traditional algorithms and subsequent didactics on algebra itself (e.g. geometry of rectangles).
We will therefore have to provide a central place for the development of networks of number relations and the flexible use of calculation properties. Only then arithmetic education arises that prepares students as well as possible for the future.	• G reformulates the objectives of RME but uses abstract terms so that readers do not see that RME is reformulated again. • In this way RME is linked to 21^{st} century skills, to provde a "good future for students". Students who get this education will not understand what computers do. • G doesn't take the opportunity to say that RME has cause huge problems in mathematics education in Holland.
Text by Koeno Gravemeijer, emeritus professor Eindhoven School of Education. More articles on mathematics and mathematics education for the 21st century can be found on www.rekenenwisk21.nl	Gravemeijer has been a pillar of RME and wrote his 1994 thesis with supervisor Adri Treffers, another pillar of RME. Which RME fails. Why doesn't he openly say so ?

Letter to the Leiden University committee on research integrity

Book version with “Google Translate Letter-Apppendices”.

To Commissie Wetenschappelijke Integriteit (CWI)
Leiden University
Postbus 9500
2300 RA Leiden

Internet version – anonimised as far as relevant.

September 30 2016

Concerning: Breach of integrity of science in 2009-2016 by Marian Hickendorff, Cornelis (Kees) van Putten (recently retired), Willem Heiser (emeritus) and Rob Tijdeman (emeritus), and after November 1 2016 potentially also Hester Bijl (appointed as vice-rector magnificus)

Dear Sir, Madam,

Your website still provides a regulation from 2014 that specifies mediation. [200] I have contacted professor Tieken-Boon in January 2016 with the request whether she could mediate, but she wrote me, and confirmed this in a conversation, that your University Board does no longer allow mediation (**Letter-Appendix A**). My suggestion is that you advise the Board to revise the published regulation or still allow for mediation first.

I have set myself a deadline of October 1 2016, because Marian Hickendorff defended her thesis in October 2011 and it seems a common rule to use a time-window of five years. [201]

I also looked for other ways since January. A potential solution was to ask professor Tijdeman in his capacity as member of the committee of the KNAW report of 2009 on arithmetic education in The Netherlands. [202] However, Tijdeman's response was such, that I must include him as breaching integrity of science himself now too. It is this recent response that is most problematic. If we allow a time window back to 2009 then we can observe that Tijdeman is a mathematician and no teacher of mathematics, so that his participation in the KNAW report of 2009 on arithmetic education comes with the conclusion that he is advising on an issue for which his isn't qualified, and it is a problem that he still does not recognize this in 2016.

With this time window and dead end roads, I see myself forced to submit this report to you now.

For me, there has been a clear breach of integrity for a while now. You must still consider the evidence, and hence for you this formally will be a suspicion of a breach.

[200] http://media.leidenuniv.nl/legacy/klachtenregeling-wi-universiteit-leiden-versie-14-11-14.pdf
[201] Hickendorff (2011). "Explanatory latent variable modeling of mathematical ability in primary school : crossing the border between psychometrics and psychology", https://openaccess.leidenuniv.nl/handle/1887/17979
[202] KNAW 2009, "Rekenonderwijs op de basisschool", https://www.knaw.nl/nl/actueel/publicaties/rekenonderwijs-op-de-basisschool

The VSNU code is of 2012 and one might hold that work for a report in 2009 or a thesis of 2011 might not be subject to it. However the major problem is that the researchers reacted wrongly after 2014 when their conceptual errors were pointed out to them.

An additional complication is: member of the KNAW 2009 committee was prof. dr. ir. Hester Bijl, who will be vice-rector magnificus of Leiden University per November 1 2016 in charge of education. She is not qualified as mathematics teacher either. When I contacted her on the KNAW report she did not respond. In Leiden her position will be administrative and not as a scientific researcher. Perhaps one might say that I should submit a report to TU Delft as well, but let me concentrate on Leiden.

This also brings me to a word of protest. Your current regulations require me as a reporter of a problem to do all kinds of effort for you to identify the issue. It really takes a lot of time to trace an issue, query researchers, talk to people, and develop the proper approach. I also get the impression that you require that the burden of evidence would be on me. While I am convinced that there is a breach in integrity of science, it is not impossible that some lawyer at Leiden University finds some loophole such that suddenly it would be me who has filed a false accusation. Let me also refer to the weak rules at KNAW / LOWI, that apparently are targeted at allowing universities to defend themselves institutionally rather than at defending science itself. [203] My advice is the creation of a national body of investigation of scientific issues, that resolves issues, and that might also deal with cases of integrity when such arise. The proper approach is to start looking at issues from science, and not what you do, forcing me to submit a case of integrity as if I would have all information and other means of investigation.

(1) The issue

The problem has been described succinctly in the mathematics education newsletter: "Het rekenexperiment op kinderen moet en kan worden beëindigd". [204]

There I refer to the work by Van Putten and Hickendorff, and explain what conceptual error they make, so that the "conclusions" that they draw from their research are invalid.

My suggestion there is that the Inspectorate of Education resolves what problem this has caused for education itself.

For you, the issue must be looked at from the integrity of science.

When Van Putten and Hickendorff are presented with criticism of their work, should they not reply ? Shouldn't they either correct or specify why this criticism would not apply ?

Hickendorff, in supporting work for the KNAW committee in 2009, then her thesis in 2011, and later public presentations on this, notably a KNAW conference in 2014, deals with education in arithmetic, which is an issue in didactics of mathematics *and its research*, but she states in an email to me in 2014 that she tries to keep a distance from didactics of mathematics "as much as possible" (**Letter-Appendix B**). Apparently she regards my questions as part of what is "possible to keep apart from".

Thus, arithmetic education is an issue of didactics of mathematics *and its research*, but we see an involvement of mathematicians and psychologists / psychometricians who are not qualified for the issue, and who draw conclusions and provide policy advice, claiming that this would be based in science !

[203] https://boycottholland.wordpress.com/2015/11/26/allea-defines-research-integrity-too-narrow
There is a curious exchange with former KNAW president Pieter Drenth who also happens to be a psychologist involved in testing.

[204] Colignatus (2015), " Het rekenexperiment op kinderen moet en kan worden beëindigd" http://www.wiskundebrief.nl/721.htm#5

And when I contact them on the problems that they cause in this field, then they argue that they are not interested in this field. This is Kafka and not science.

Historically, we might understand these developments. Mathematicians have a tendency to meddle in mathematics education even though they have no training on this. There is the phenomenon of "math wars" that make it no fun to get involved in this field. [205] There is the paradigm of psychometrics that knowledge comes from measurement. As an econometrician I am very sympathetic to this. However, it still isn't science when key information and criticism is neglected. PM. The situation is also somewhat complicated since I present a paradigm shift, which causes additional discussion with other educators and researchers. [206]

Thus, the conclusions of that KNAW report and by Van Putten (member of the KNAW report and in the thesis supervising commission) and Hickendorff (both supporting work for the KNAW report and thesis) are invalid. They neglected relevant information. They didn't respond adequately on criticism afterwards.

Curiously, professor Heiser, as promotor of Hickendorff, has supported this thesis and invalid approach. Perhaps the thesis supervisor can indoctrinate the Ph.D. student in thinking alike, but both would still be in error.

The thesis committee was unbalanced. Perhaps there was an effort to include a view from didactics in mathematics ? Lieven Verschaffel from Leuven might perhaps come closest to having some knowledge about didactics of mathematics, but he has no background in this either. When he wrote a book review of the thesis by La Bastide – Van Gemert about Hans Freudenthal, Verschaffel overlooked a major inconsistency in that thesis, which is another example that shows that he is not adequately knowledgeable in this area. [207]

(2) Van Putten, Hickendorff, Tijdeman and Bijl at Leiden w.r.t. KNAW 2009

(2a) In 2009

The list of references of the KNAW 2009 report mentions the article by Van Putten and Hickendorff in Tijschrift voor Orthopedagogiek (TvO), may 2009 (no 5), but *doesn't mention* the critical article by Liesbeth van der Plas in the same issue – and at the same conference – that shows that the scoring method by Van Putten and Hickendorff is invalid, because they only consider the outcomes of sums and not the way of solution, while that way is relevant for learning algebra in subsequent education. [208]

Relevant letters are:

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[205] https://en.wikipedia.org/wiki/Math_wars and see also https://boycottholland.wordpress.com/2016/01/24/graphical-displays-about-the-math-war

[206] http://thomascool.eu/Papers/Math/2016-05-09-Letter-to-VOR-and-Trainers-of-teachers.pdf

[207] https://lirias.kuleuven.be/handle/123456789/282138?mode=full&submit_simple=
Show+full+item+record [The original letter switches to Dutch here. Google Translate 2018:]
His discussion of the thesis of La Bastide - Van Gemert [2006] shows:
(1) He does not mention the inconsistency in the chapter about Van Hiele, in which Freudenthal in fact appropriates the discovery by Van Hiele. He did not notice, or he did not really read this part.
(2) He does not mention that there is criticism of Freudenthal's realistic mathematics. The publication is from 2010, and perhaps written before Verschaffel participated in the KNAW committee, because otherwise you would expect that he would have referred to it. Yet there was already criticism that should have been mentioned. The thesis was from 2006 when Jan van de Craats also gave all this criticism.
It fits with a background as an educational expert who is not a didactic mathematician. For the time being, however, I am the only one who protests against the fraud by Freudenthal, although we may be glad that the editors of the WiskundE letter allowed me to raise this.
http://www.wiskundebrief.nl/718.htm#7

[208] http://www.liesbethvanderplas.nl/rekenvaardigheid-in-relatie-tot-wiskunde

http://thomascool.eu/Thomas/English/Science/Letters/2016-05-17-Letter-to-KNAW-and-CPB.pdf
http://thomascool.eu/Thomas/English/Science/Letters/2016-05-25-Letter-to-KNAW-and-CPB-supplement.pdf

According to the VSNU code: [209]

> "4.5. Een wetenschapsbeoefenaar is pas verdediger van een bepaald wetenschappelijk standpunt als dat standpunt voldoende wetenschappelijk is onderbouwd. Rivaliserende standpunten dienen daarnaast te worden gemeld en toegelicht."

Potentially the psychometricians might argue that they neglected the article by Van der Plas because it doesn't feature statistics, but, the issue concerns validity. When studying a topic one cannot neglect issues on validity with the fallacy of requiring statistics.

(2b) In 2014-2016

> "6.2. Wetenschapsbeoefenaren laten zich eerlijk en loyaal de maat nemen over de door hen geleverde kwaliteit. Zij werken mee aan in- en externe beoordelingen van hun onderzoek."

Van Putten didn't respond. Hickendorff stated to me that she is basically unqualified for education in mathematics *and its research,* but this is not clarified in the thesis itself, and not told to the minister of education who might look differently at the KNAW report of 2009 now.

Relevant texts for Van Putten and Hickendorff are:

http://thomascool.eu/Papers/AardigeGetallen/2016-01-17-Meta-opmerkingen-over-psychologie-en-wiskunde.pdf
http://thomascool.eu/Papers/AardigeGetallen/2016-01-31-Enkele-emails-rekentoets-psychometrie.pdf
http://thomascool.eu/Papers/AardigeGetallen/2016-02-10-Basisprobleem-in-pedagogie-onderwijs-en-didactiek-van-wiskunde.pdf

For Tijdeman:

http://thomascool.eu/Papers/Math/2016-05-25-Email-exchange-with-Kool-Noteboom-Tijdeman.pdf

(3) Heiser as thesis supervisor, Van Putten as co-promotor and Hickendorff as Ph.D. student

(3a) In 2011

The same problem of not-mentioning Van der Plas (2009), breaching VSNU 4.5.

(3b) in 2014-2016

The same problem of breaching VSNU 6.2.

For professor Heiser there is the curious email in **Letter-Appendix C**. My query was a bit wider than only the thesis since I was also interested in how CITO dealt with the issue. Apparently Hickendorff is active there w.r.t. testing and there is insufficient attention for validity. However, though my query was wider, I clearly addressed Heiser as promotor of Hickendorff. See the emails mentioned above.

[209] http://media.leidenuniv.nl/legacy/vsnu-code-wetenschapsbeoefening-2004-%282014%29-def.pdf

It is not impossible that other aspects of the regulations on integrity are relevant too. I find the code(s) questionable since they formulate some general statements and then specify some points, which makes one wonder whether its generality is actually replaced by the specifics. Overall, I find these Leiden researchers w.r.t. this issue not helpful, careful, reliable, impartial and responsible.

Kind regards,

Thomas Cool

Econometrician and teacher of mathematics

[...] Scheveningen, The Netherlands
[...]
[...] http://thomascool.eu

Letter-Appendix A. Email of 2016 by prof. Tieken-Boon that declines mediation

Google Translate 2018: “Thank you very much for your message. I am happy to hear that you enjoyed [the interview]. As far as your expectations of what I could do for you and whether or not it would be worthwhile for you to come to Leiden for an interview: I already told you in an earlier email that you probably expected too much of it. It is therefore not the case that I reject your request for mediation: that simply does not belong to my duties.

This also means that I can not meet the demand in your other mail: it is not my job to do that sort of thing. As I said, my job as a confidential adviser is to listen, give advice and guide people in the procedure they could possibly follow.”

Dutch original:

From: "Tieken, I.M." [...]
To: "'Thomas Cool / Thomas Colignatus'" [...]
Subject: RE: Na het gesprek - RE: Wetenschappelijke integriteit t.a.v. Marian Hickendorff en onderzoek aan onderwijs in rekenen
Date: Mon, 18 Jan 2016 [...]

Beste meneer Cool,

Hartelijk dank voor uw bericht. Ik ben blij te horen dat u het een prettig gesprek vond. Wat betreft uw verwachtingen van wat ik voor u kon doen en of het wel of niet de moeite waard voor u zou zijn om naar Leiden te komen voor een gesprek: ik had u al gezegd in een eerdere email dat u daar waarschijnlijk teveel van verwachtte. Het is dus ook niet zo dat ik uw verzoek om te bemiddelen afwijs: dat hoort gewoon niet tot mijn taken.

Dat houdt ook in dat ik niet kan voldoen aan de vraag in uw andere mail: het is namelijk niet mijn taak om dat soort dingen te doen. Zoals gezegd, mijn functie als vertrouwenspersoon is om te luisteren, advies te geven en mensen de weg te wijzen in de procedure die ze eventueel zouden kunnen volgen.

Vriendelijke groet,

[...] Tieken

Letter-Appendix B. Email of 2014 by Marian Hickendorff

Google Translate 2018: “Dear Thomas Cool, Thank you for your mail. I fear that I can not find the time to view everything you send. In addition, I also wonder if you have come to the right place for me: I am not a didacticist but a psychological researcher, and I also try to stay out of the discussion about didactics as much as possible because I do not believe that that is my expertise. Kind regards, and until a.s Monday, Marian Hickendorff”

Dutch original:

From: "Hickendorff, M." [...]
To: "Thomas Cool / Thomas Colignatus" [...]
Cc: J.A.Bergstra [at] uva.nl, "Craats, Jan van de" [at] uva.nl
Subject: RE: T.b.v. a.s. maandag (KNAW reken-onderwijs)
Date: Fri, 27 Jun 2014 [...]

Beste Thomas Cool, Bedankt voor uw mail. Ik vrees dat ik niet de tijd kan vinden om alles wat u stuurt te bekijken. Daarnaast vraag ik me ook af of u bij mij hiervoor aan het juiste adres bent: ik ben geen didacticus maar psychologisch onderzoeker, en probeer ook zo veel mogelijk buiten de discussie over didactiek te blijven omdat ik niet meen dat dat mijn expertise is. Vriendelijke groeten, en tot a.s. maandag, Marian Hickendorff

Letter-Appendix C. Email of prof. Heiser that declines a response

Google Translate 2018: “My maxim is that everyone is free to send me something, but that I am free to not go into that.”

Dutch original:

From: "Heiser, W.J." [at] FSW.leidenuniv.nl
To: "Thomas Cool / Thomas Colignatus" [...]
Cc: "Hickendorff, M." [...]
Subject: RE: Ethiek van het toetsen op rekenen (PPON of LVS)
Date: Tue, 13 Oct 2015 [...]

Geachte heer Cool:
Mijn stelregel is dat het iedereen vrijstaat om mij iets op te sturen, maar dat het mij vrijstaat om daar niet op in te gaan.

Gegroet, Willem Heiser

The Math War between traditional and “realistic” mathematics education and its research. An analysis in institutional economics on research on education in arithmetic and algebra, with a focus on long term memory of pupils and using a causal model for valid testing on competence

September 4 & 14 2018 [210]

Journal of Economic Literature: JEL
P16 Political economy
I20 General education
D02 Institutions: Design, Formation, Operations, and Impact
O17 Formal and Informal Sectors • Shadow Economy • Institutional Arrangements

American Mathematical Society: MSC2010
00A35 Methodology and didactics
97-XX Mathematics education
97F02 Arithmetic, number theory ; Research exposition
97G70 Analytic geometry. Vector algebra
97H20 Elementary algebra

Abstract

Institutional economics investigates how institutions affect empirical events. The term “institution” can be taken widely, and may also represent engrained mental conceptions by organised groups of actors. There is a curious but counterproductive combination of three groups also at universities in Holland w.r.t. research on education in arithmetic and algebra: (1) adherents of "realistic" mathematics education, an ideology that compares to astrology or homeopathy, (2) traditional mathematicians, who have no expertise on the empirical science of didactics of mathematics either, (3) psychometricians, who look at statistical data but who have no expertise on the empirical science of didactics of mathematics either. This combination needs deconstruction and the present paper focuses on (3), though with influence from (1) and (2). Some psychometricians seem to have a sound dislike of both the ideologues from (1) and the discussion between (1) and (2), but they are less aware that (2) are ideologues too. Some psychometricians also throw away the child with the bathwater by disregarding (4) the proper science of didactics of mathematics. Measuring competence in arithmetic and algebra requires consideration of long term memory of students. What you learn in elementary school tends to stay with you for the rest of your life. What you learn in highschool has the property of “use it or lose it”. Algebra in highschool requires competence in the traditional algorithms of arithmetic, best learned in elementary school. “Realistic” mathematics education has reduced the competence of students at elementary school which affects them not only for algebra in highschool but also for the rest of their lives in both arithmetic and algebra. Inadequate testing by psychometricians allows this detrimental state to continue. The paper presents a causal model that identifies the engrained mental conceptions by psychometricians and where they would have to accept insights from didactics of mathematics. There is also a role for the Dutch Academy of Sciences KNAW that supported an inadequate report in 2009.

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[210] https://mpra.ub.uni-muenchen.de/88810/

Introduction

The issue: education in arithmetic and algebra

Primary education has a window of opportunity.

- What you learn in elementary school tends to stay with you for the rest of your life.
- What you learn in highschool has the property of "use it or lose it".

We now look at arithmetic and algebra:

- Above two properties hold. Learning arithmetic in highschool comes with the property of "use it or lose it".
- Algebra at highschool requires pre-algebra training at elementary school on the algorithms of arithmetic. If you don't properly learn how to manipulate 1 / 2 + 1 / 3 or $2^H + 3^H$ at an early age then you will tend to fail on 1 / a + 1 / b or $a^H + b^H$ at a later age (using H = -1, see [211]).
- If arithmetic at elementary school relies on the calculator or trial and error, then this will be your standard on arithmetic, while the window of opportunity on algebra closes. Highschool may try remedial teaching on arithmetic but your level of algebra will tend to remain low.
- For example: The teaching method of "equivalent ratios" using tables only [212] is called "pre-algebra" but might also be perused as "never-algebra". Proper didactics requires integration of text, formula, table and graph.

Let us look how how these phenomena are dealt with by mathematics education research (MER) and policy making. I already discussed main aspects in *Elegance with Substance* (2009, 2015), also see its website, but now we look at the window of opportunity for arithmetic and algebra occurring in primary education. See the preface of *A child wants nice and no mean numbers* (2015, 2018) for my lack of expertise on primary education.

The situation in Holland could be interesting to the world (see the AAAS Project 2061 [213]). Holland is a middle sized country of 17 million people with data collection for the population of students (PPON) and not only samples (TIMSS). The population and education characteristics are not too heterogenous. Important is also that the "reform in mathematics education" in the whole world had a key impulse from Hans Freudenthal (1905-1990) from Utrecht University, to the extent that ICMI now features a Freudenthal Medal, see also Colignatus (2014, 2015).

For the international context, a common reference is to Slavin & Lake (2008). Their p445: "More research is needed on all of these programs, but the evidence to date suggests a surprising conclusion that despite all the heated debates about the content of mathematics, there is limited high-quality evidence supporting differential effects of different math curricula."

Institutional economics and the context in Holland

In economics, there is the branch of "institutional economics" that investigates how institutions can affect empirical events. The term "institution" can be taken widely, and may also represent engrained mental conceptions or ideologies. [214] Below we will

[211] https://doi.org/10.5281/zenodo.1251686
[212] https://www.khanacademy.org/math/pre-algebra/pre-algebra-ratios-rates/pre-algebra-visualize-ratios/e/solving-ratio-problems-with-tables
[213] https://www.aaas.org/program/project2061/about
[214] https://www.cambridge.org/core/journals/journal-of-institutional-economics/article/what-is-an-institution/3675101CE15BE2A7681CD5783C01F6D0

mention some formal institutions that apply here but it appears that developments are more dominated by such engrained mental conceptions.

In Holland there is a "math war" [215] [216] that caused the Dutch Academy of Sciences KNAW to set up a committee, that produced the KNAW (2009) report. [217] This math war provides context to our issue, and it must be discussed to prevent confusion about what this paper achieves.

1. The Dutch math war is between "traditional mathematics education" (TME) and "realistic mathematics education" (RME) a.k.a. "reform mathematics", as proposed by the Freudenthal Head in the Clouds Realistic Mathematics Institute (FHCRMI). [218]
2. My position is the third approach, consisting of scientific research, with re-engineering of mathematics education. [219]
3. The problem with TME and RME is that they derive from mathematicians trained on abstract thought who have little grasp of empirical research.
4. Both TME and RME have delegated empirical research to psychometricians, often at CITO. The subsequent problem is that psychometricians have no training in didactics of mathematics and mathematics education research (MER), whence such psychometric research runs the risk of invalidity. (See the present paper.)
5. In empirical science, when there are competing paradigms, then researchers set up a distinguishing experiment that shows which paradigm provides the best explanation. Adherents of TME and RME did not do so. However, a critical look at the available evidence would provide for such a decision, see Colignatus (2015c). [220] At issue is not which paradigm would be "right". There are useful ideas in both TME and RME, and it depends upon time and place what is most relevant, often decided by the teacher. At issue is to get rid of blinding effect of ideology. If the distinguishing experiment shows that the TME (RME) textbook is best, then it can provide the baseline, and RME (TME) alternatives can be tested on a case by case basis.
6. Policy makers interfered and increased the chaos. Holland observed a reduction of competence in arithmetic and math, and TME claimed that this was being caused by RME. The minister of education imposed a separate test on arithmetic ("Rekentoets") as part of the highschool diploma. [221] While the problem was being caused at elementary school, requiring the re-training of 140,000 elementary school teachers, the minister approached the issue as end-of-pipe and put the burden on perhaps 12,000 math teachers in highschool. This neglected that incompetence in arithmetic at primary school would mentally maim students for algebra for highschool and the rest of their lives. (Students unable to do algebra are transferred to vocational schools.) (In 2018 the new state secretary adopted a new format for testing on competence, but I haven't had time to look into this.)
7. Because of the national debate a publisher created a textbook that uses "the best of TME and RME". They did so without scientific research to back this up, and without using a distinguishing experiment. This new mixture tends to make it more difficult to get such an experiment.

Thus we have two warring factions on the field, one playing soccer and the other playing American football, with an arbiter from basketball, and with the public throwing darts onto the field. My research hopes to help clean up the mess, while also hoping that others will be grateful for the clarity.

[215] https://en.wikipedia.org/wiki/Math_wars
[216] https://www.theglobeandmail.com/opinion/article-in-the-ongoing-math-wars-both-sides-have-a-point/
[217] https://www.knaw.nl/nl/actueel/publicaties/rekenonderwijs-op-de-basisschool
[218] https://boycottholland.wordpress.com/2016/01/24/graphical-displays-about-the-math-war/
[219] https://zenodo.org/communities/re-engineering-math-ed/about/
[220] http://www.wiskundebrief.nl/721.htm#5
[221] http://www.wiskundebrief.nl/512.htm#1

Formal and informal institutional setting

There is a large list of institutions for our issue. [222] [223] Key ones are:

- Onderwijsraad (Education Council), an advisory body for the minister of education [224]
- Inspectie voor het Onderwijs (IvhO) (Inspectorate for Education) [225]
- CITO, that provides for tests at the end of primary education [226]
- National board for education research (NRO) [227]
- Association of education researchers (VOR), commonly from the universities [228]
- Teachers and educators of teachers, [229] association of teachers of mathematics (NVvW) [230] and association on arithmetic (NVORWO) [231]
- Publishers

While TME was the original standard in the 1960s, the takeover by RME was gradual. Dutch elementary school teachers started adopting RME and at some point the Inspectorate pushed for it. By 2009, all Dutch primary school textbooks used the RME method. (By comparison, the USA still has variety in TME and RME, see below.)

The distinguishing experiment between TME and RME has these aspects.

- The main aspect consists of pure logic. Preparation for algebra requires command of the traditional algorithms for arithmetic. Since RME spends much less attention on those (and aspires at their “guided reinvention” which is merely a hope and not proven), we can expect that RME performs less well on those algorithms. In Holland, this logic is not understood. TME has been singularly ineffective in bringing this logic into attention.
- The statistical aspect consists of the actual tests at the end of primary education, administered by CITO, with application of psychometric techniques and diagnostics. The focus of a group of education researchers has shifted to statistical testing.
- Let us use an analogy to compare logic and statistics. In 1950 there were no actual (statistical) observations about the other side of the Moon. A statistician could have hold that one can’t infer the existence of this other side because statistical evidence was lacking. Hopefully such statistician would not defy the logic by physics. The relevance of statistics for TME and RME is only for the particular value of the effect size, graduated by the talents of pupils. Perhaps some students will never learn algebra and then are better served by a good command of the calculator.

The statistical aspect not only defies the first element of logic, but shows two other illogical phenomena.

(1) KNAW (2009) has documented that the shift from TME to RME has not been supported by statistical testing. Thus, the shift towards statistics did not come along with this notion of rigour. (The same happened in the USA, see below.) The CITO tests have shifted over time in favour of RME, with the use of the calculator or trial and error, but this shift itself was not corroborated by tests.

(2) KNAW (2009) supports teacher experience but does not investigate whether teacher experience was the cause for the shift from TME to RME, thus without such

[222] https://boycottholland.wordpress.com/2015/10/31/the-power-void-in-mathematics-education/
[223] See also the Presmeg chart at https://boycottholland.wordpress.com/2015/10/15/pierre-van-hiele-and-annie-selden/
[224] https://www.onderwijsraad.nl/english/item34
[225] https://www.onderwijsinspectie.nl/over-ons
[226] http://www.cito.nl/
[227] https://www.nwo.nl/over-nwo/organisatie/nwo-onderdelen/nro
[228] https://www.vorsite.nl/en/content/about-netherlands-educational-research-association
[229] http://www.lerarenopleider.nl/velon/wp-content/uploads/2016/12/K2_6KempenDietzeCoupe.pdf
[230] https://nvvw.nl/
[231] http://www.nvorwo.nl/

(perceived) need for statistical testing. The main cause may still be ideology (and the argument of authority of Hans Freudenthal).

I cannot avoid the conclusion that ideology has had a strong influence. Normally there would be strict rules on experimenting on humans. When there are two methods TME and RME, then you are supposed to develop a distinguishing experiment. When one method is shown to be superior then you abort the experiment and switch all subjects to the better method. (This holds per topic and may be extended to paradigms.) Curiously, TME were not able to convince the education community by merely pointing to the logic of the argument. Somehow, statistical testing by CITO started weighing in. The use of statistics, adopting some standard out of thin air, allowed a distinction between kids performing well and kids performing less, and who oh who was to argue that "well" wasn't "enough", or that the lesser performing kids could do better on the other method ?

Jan van de Craats is a professor of mathematics (now retired) without a degree for teaching at elementary school. Since about 2005 he started defending TME. He created the SGR foundation and got support also from some researchers who were later appointed in the Education Council (Onderwijsraad). Van de Craats was invited to participate in a committee that identified levels of competence (comparable to the US Common Core).. [232] They identified fundamental and target levels (abbreviated as F and S). In this letter [233] Van de Craats states that the committee intended that the targets be adopted, while the Ministry of Education embraced only the fundamental levels. By this move, the lower level of competence became the new official level in Holland. There is now less need of criticism on the Ministry that the official level is not attained.

Van de Craats and his SGR have a strong argument because of the logic mentioned above, that should be sufficient to reject RME. However, in advocating TME they basically have a position in ideology because they lack expertise for education at primary school. [234] It should be qualified teachers and researchers who should make that decision. Van de Craats and his SGR supported the creation of a new TME textbook "Reken Zeker" (Noordhoff), that was introduced in 2010. This textbook was written by elementary school teachers Piet Terpstra and Arjen de Vries, [235] and thus satisfied the criterion that it was backed by their degrees and experience. Some 20 schools started with it. It still came without scientific support and testing that we would like to see nowadays. [236] [237] SGR claims: "Their [textbook "Reken zeker"] combines the best of the two worlds of traditional [arithmetic] and realistic [arithmetic]", [238] without explaining how this "best" combination has been corroborated.

In 2015 I suggested the following idea to CITO [239] [240] and Dutch Parliament. [241] In 2016, the pupils taught with "Reken Zeker" would finish their primary education. Thus, if CITO would keep their tests apart, and perhaps test them additionally in comparison with a random selection of other kids that used the prevailing RME methods, then there would be a (natural) distinguishing experiment with adequate test results. Obviously, the school teams that opted for "Reken Zeker" would be motivated for TME and thus we should require a larger difference in success to warrant its claim on superiority.

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[232] http://www.steunpunttaalenrekenenvo.nl/sites/default/files/Over%20de%20drempels%20met%20rekenen.pdf
[233] https://staff.science.uva.nl/j.vandecraats/Mails_aan_Victor.pdf
[234] http://thomascool.eu/Papers/AardigeGetallen/2015-09-23-SGR-deugt-ook-al-niet.html
[235] http://www.goedrekenonderwijs.nl/wp-content/uploads/Interview-Reken-Zeker-1.pdf
[236] http://www.few.vu.nl/~jhulshof/2011bib2.pdf
[237] http://benwilbrink.nl/projecten/reken_zeker.htm
[238] Dutch: "Hun methode "Reken zeker" combineert het beste uit de twee werelden van traditioneel rekenen en realistisch rekenen." http://www.goedrekenonderwijs.nl/reken-zeker/ Google Translate tends to translate "rekenen" as "calculation" while "arithmetic" would be better here.
[239] http://thomascool.eu/Papers/AardigeGetallen/2015-10-18-Aan-CITO.html
[240] http://thomascool.eu/Papers/AardigeGetallen/2015-10-18-Tweede-brief-aan-CITO.html
[241] http://thomascool.eu/Papers/AardigeGetallen/2015-10-17-Aan-TK-commissie-OCW.html

In its reply, CITO shifted responsibility to the Inspectorate of Education (IvhO). [242] [243] In this reply, CITO abused the distinction between principal and agent. Indeed IvhO was the new principal: it may not do research itself but contracts external researchers or consortia. [244] Instead CITO should have taken research responsibility, by supporting my suggestion at IvhO. Nevertheless I wrote to IvhO, my letter officially recorded as number M0155149, [245] and also published my suggestion in a newsletter for teachers of mathematics, as Colignatus (2015c). [246] In its reply, the Inspectorate rejected its role and reponsibility in this, by interpreting my suggestion as if this would only concern a scientific experiment whether TME or RME would be right. [247] I protested that this was a false interpretation, and that the task of protecting children lies with IvhO (and not merely NRO). This protest at IvhO did not receive a reply. By January 2016 I looked deeper at the role of psychometricians, also at CITO. [248] I approached NRO only in 2016, with the practical point that my kind of research was excluded by their choice of criteria. [249] [250] I

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[242] Google Translate 2018: "Through the PPON research we have provided insight in the past into the management of various domains within arithmetic. This research is no longer under the responsibility of Cito since 2015, but below that of the Inspectorate of Education. I want to refer you to that."
Dutch: At 2015-10-15, Strijp wrote:
Geachte heer Cool,
Bedankt voor deze en uw eerdere uitgebreide email.
Doormiddel van het PPON onderzoek hebben wij in het verleden inzicht gegeven in de beheersing van verschillende domeinen binnen rekenen. Dit onderzoek valt sinds 2015 niet meer onder de verantwoordelijkheid van Cito maar onder die van de Inspectie van het onderwijs. Daar wil ik u dan ook naar verwijzen.
Ik hoop uw hiermee voldoende informatie te hebben gegeven.
Met vriendelijke groet,
Ineke Strijp
[243] https://www.onderwijsinspectie.nl/onderwerpen/peil-onderwijs
[244] Google Translate 2018: "Since 2014, the Inspectorate of Education has been in charge of the surveys under the name Peil.onderwijs. The surveys have been launched via the NRO since 2016." https://www.nro.nl/onderzoeksprojecten/peil-onderwijs/ For example, for 2017, IvhO / NRO contracted Marian Hickendorff at Leiden for another review study. https://www.nro.nl/nro-projecten-vinden/?projectid=405-17-920-rekenen%20op%20de%20basisschool
[245] http://thomascool.eu/Papers/AardigeGetallen/2015-11-23-Het-rekenexperiment-op-kinderen-moet-en-kan-stoppen.html
[246] http://www.wiskundebrief.nl/721.htm#5
[247] Google Translate 2018: "It is not the duty of the inspectorate to settle the discussion between supporters of the various [arithmetic] methods. Thank you for your interest. Perhaps there is interest in the National Education Research Foundation (NRO)"
Dutch: Date: Fri, 18 Dec 2015
From: Loket Onderwijsinspectie
To: Thomas Cool / Thomas Colignatus
Subject: M0155149 Memo: "Het reken-experiment op kinderen moet en kan stoppen"
Geachte heer Cool,
Het is niet de taak van de inspectie de discussie tussen aanhangers van de diverse rekenmethodieken te beslechten.
Dank voor uw interesse. Misschien is er belangstelling voor bij de Nationaal Regieorgaan Onderwijsonderzoek (NRO)
Met vriendelijke groet,
[XYZ]
Loket Onderwijsinspectie
....
Inspectie van het Onderwijs
Ministerie van Onderwijs, Cultuur & Wetenschap
www.onderwijsinspectie.nl
[248] http://thomascool.eu/Papers/AardigeGetallen/2016-01-31-Enkele-emails-rekentoets-psychometrie.pdf
[249] http://thomascool.eu/Papers/Math/2016-04-15-Letter-to-NRO.pdf
[250] http://thomascool.eu/Papers/Math/2016-07-12-Second-Letter-to-NRO.pdf

also informed the Dutch education researchers (VOR) about the paradigm shift w.r.t. mathematics education research (MER). [251] [252]

There may be reason to regard 2016 as a crucial year to test. The authors of "Reken Zeker" have retired and there was some rumour that the textbook might be stopped. Yet in 2018 the textbook is still available and it is unclear who took over. [253] Since the KNAW 2009 report, RME textbooks and teachers have started including more TME elements, though in unknown ways, making it less clear what "real" RME is, and making it more difficult to arrive at a distinction. Graduation year 2016 would be the least untainted one.

In 2018 the Inspectorate started an evaluation targeted to show results in 2020/21. [254]

> Google Translate 2018: "An investigation will be conducted into the cause of the declining level of mathematics and mathematics education in the Netherlands. The research was announced in the annual work plan 2018 of the Education Inspectorate. This reports NU.nl. [...] international research (TIMSS) showed last year that 99 percent of primary school students in [grade 4] in the Netherlands master the basic skills in the field of arithmetic. The basic level is perfectly fine. At the same time, relatively few Dutch students achieve a higher level in comparison with other countries. The inspectorate wants to know how this is done and what can be done about it. [Arithmetic] education has had more narrative calculations since 2004. Some believe that this 'realistic [arithmetic]' is partly responsible for the decline of mathematical education. The Education Inspectorate thinks this is an outdated discussion: "You can see that in recent years the realistic [arithmetic] and the old form of arithmetic, ie [drilling] tables, are growing closer together."" [255]

The latter is a confused statement, as if the choice between RME and TME finds its proper answer in mixing those (in unknown ways), like the choice between astrology and homeopathy finds its answer in mixing those (in unknown ways), or like the choice between astrology and astronomy finds its answer in mixing those (in unknown ways). Basically the Inspectorate itself failed to take advantage of the "Reken Zeker" opportunity in 2016. Due to the current mixture we might be less able to deal with the ideologies.

The PPON results come in two batches. Alongside the annual results on the population for the scores only, there are periodic samples that also collect data on textbooks used, social-economic-status (SES) and such. The PPON report on 2016 only gives the population. [256] An enquiry at CITO confirmed that 2016 had no collection of data on textbooks, SES and other factors. For a comparison of RME and TME such would have to be reconstructed from the school archives. The 2018 competition for research grants to do a periodic sample for 2018/2019 (but not 2016) was won by a consortium with participation by psychometrician Marian Hickendorff. [257]

These formal institutions obviously have their role in these developments. My tendency is to think that agents at these formal institutions might be more influenced and motivated by their views on the role of science and the informal ideologies of TME and RME. Whatever this may be, there still is reason to look at the latter anyhow. This attention for the informal institutions brings us to the present paper.

[251] http://thomascool.eu/Papers/Math/2016-05-09-Letter-to-VOR-and-Trainers-of-teachers.pdf
[252] http://thomascool.eu/Papers/Math/2016-07-15-Second-letter-to-VOR-and-Trainers-of-teachers.pdf
[253] https://www.noordhoffuitgevers.nl/basisonderwijs
[254] https://www.onderwijsinspectie.nl/actueel/nieuws/2018/07/10/onderzoeken-rekenen-wiskunde-en-schrijfvaardigheid-voor-peil.onderwijs-gestart
[255] https://blog.sbo.nl/onderwijs/onderzoek-dalende-niveau-reken-en-wiskundeonderwijs/
[256] https://www.onderwijsinspectie.nl/onderwerpen/peil-onderwijs/documenten/rapporten/2018/04/11/taal-en-rekenen-aan-het-einde-van-de-basisschool-2016-2017
[257] https://www.universiteitleiden.nl/nieuws/2018/05/subsidie-marian-hickendorff

Causal modeling

Didactics concerns the study of what an issue might be, what students might handle, how they might learn it, and how you would test this. Let me refer to Van de Grift (2010:16) (in Dutch) for the activities of a succesful teacher, and check that these activities are targeted at affecting learning behaviour. [258] Van de Grift and KNAW tend to refer to Hattie. It remains important to be aware of Slavin's criticism w.r.t. Hattie's approach. [259]

This issue on arithmetic and algebra also created some insights on causal modeling on didactics, psychology and student results. There is the distinction between instruction / direction (what teachers do) and learning behaviour (what students do). Some psychometricians seem to suggest that they study learning and that didactics studies teaching without studying learning. However, didactics obviously looks at learning.

In this discussion, there is the key point of grading exam questions. This obviously pertains to the psychometric measurement of test results. The point should be clear by itself, but apparently still contributed to confusion, and thus is best discussed.

This paper thus has the following structure: After clarifying grading and the effect measure, we summarise the Dutch situation, and then look at the causal modeling.

Grading and the effect measure

Table 12 considers a problem and answers provided by students. How can we grade those answers ?

Table 12. A problem and its answers by two students

Problem: What is 100 / 4 ? Answer: Traditional or reform.

John: Traditional algorithm: long division	Susan: "Realistic" trial and error
4 / 100 \ 26 8 ---- - 20 20 ----- - 0	100 20 = 4 x 5 ------- - 80 20 = 4 x 5 ------- - 60 20 = 4 x 5 ------- - 40 = 4 x 10 3 x 5 + 10 = 25

There are at least two possible effect measures, or methods of scoring:

- A simple method is to "only check the final result": John gets a Fail. He used the advised algorithm but made a calculation error, with likely an oversight and lack of discipline to check up. Susan gets a Pass. She made various correct but simple calculations.
- Didactics of mathematics tends to advise: "also intermediate steps can show insight".

[258] https://www.rug.nl/education/lerarenopleiding/onderwijs/oratie-van-de-grift.pdf
[259] https://robertslavinsblog.wordpress.com/2018/06/21/john-hattie-is-wrong/

Suppose that you can earn 5 points on this sum.

- Then John might lose 2 points, for his answer is false, and the student should have checked the answer by multiplying 4 x 26. But otherwise the method is applied properly. For example, when John does another long division, and performs the algorithm again but then makes a calculation error at another point, then we verify that he knows the algorithm but should practice more on his tables of multiplication or rather his discipline on checking up. [260]
- Susan might earn 5 points simply because trial and error generated the right answer. I am not at home in “realistic” conventions (like astrology or homeopathy). I would find the steps too small, or the student should have recognised 80 as 4 x 20. Thus a score of 4 would make more sense. If the student would give a wrong answer, then I would find it hard to judge the trial and error process, since it might go anywhere.
- Thus the Pass / Fail method has scores 0 & 5 while didactics has scores 3 & 4.
- In general, didacticians have discussions about such grading steps, since it depends upon what students have been trained for and what they are being tested about.

These issues are fundamental for didactics and psychometrics. The definitions and observations are closely connected. There isn't just measurement but this depends upon the definitions. In the 100 / 4 example it seems as if the algorithms might be well defined. But when you don't score the steps properly, then it might still be trial and error. See **Table 13** on a contrived case that might only occur seldomly but that highlights the aspects.

Table 13. Why grading steps tends to be advisable

Jack: "Traditional algorithm: long division"	
4 / 100 \ 25 8 ---- 180 180 ----- 0	

A simple scoring method would only look at the right answer 25 and give Jack a Pass. If Jack found the right answer by trial and error, but also has learnt that the teacher is only happy when shown a semblance of a long division, then he might mimic this. Categorising him as following the traditional method could be wrong. The categorisation is less relevant because the relevant measure of using the method of long division requires that you also grade the intermediate steps. In this case Jack might get 1 out of 5 points because 2 times 4 is 8, while the right answer of 25 is judged as deriving from trial and error and not from following the algorithm.

This discussion only exemplifies the key importance of defining your measurements. **Table 14** gives an overview of the possible combinations. Psychometricians Van Putten & Hickendorff (VPH) (2009) classify answers by students on strategies but they still score on outcomes only. It is not clear to me whether they would still categorise the semblance of long division in **Table 13** as that Jack would have really worked in traditional manner. Nevertheless, when they score on outcomes only, then they don't really score on strategies, because they do not assign points per step. A (vertical) categorisation on

[260] Referring to grading on scale of 0-10, Henk Boonstra thinks that the grades 7+ are more indicative of discipline in execution rather than understanding of principle. Current testing is deficient in making this distinction and giving pupils the proper feedback. Boonstra also calls attention to the fact that students are heterogeneous, in primary and secondary education alike. See https://henkboonstra.blogspot.com/2010/01/de-ongelijkheid-van-kansen-in-het.html

semblance on strategy is not the same as (horizontal) assigning points for the various steps. A vertical comparison is at risk of invalid conclusions because the strategies are not scored properly. A conceptual base for algebra is not merely arithmetic success of getting the right outcomes, but requires command of the traditional algorithms. These considerations are so blatantly obvious to teachers and researchers on testing that it is almost painful to restate them here again, but surprisingly there is confusion about them in Holland.

Table 14. Categorising and scoring

Scoring	*Outcome only*	*Score on steps*
Traditional algorithm	VPH (2009)	Valid
RME, trial and error or context allows calculator	VPH (2009)	No standardised steps
No distinction	CITO	-

By 2009 all textbooks used in Dutch elementary schools were of the RME kind. The VPH (2009) categorisation only concerned a distinction by *technique* and not by *didactics.* All pupils using the traditional algorithm had received their training in an environment of RME. Thus, a conclusion, based upon this classification, that there was no real difference in performance cannot be translated into a conclusion about RME and TME. The VPH (2009) classification thus doesn't allow for a test on didactics (and thus the link to later algebra and the TME claim that it are precisely the less talented pupils who would benefit from a training on the algorithms without distraction from other solution techniques).

The math war in Holland and the KNAW 2009 report

Declining competence in arithmetic

The competence of students in arithmetic has been deteriorating over the years, with CITO duly recording this, as they provide official tests at the end of primary education. We can be grateful to CITO, because they actually monitor this, while the ideologues of "realistic" mathematics education don't do so, and while the traditional mathematicians actually don't do so either (for they are trained to think abstractly and they don't like empirical methods) - with the exception of A.D. de Groot (1914-2006) [261] [262] who with a BSc in mathematics switched to psychology and was key in founding CITO. However, if CITO had been measuring with the proper effect measure (highlighting the preparation for algebra) then we should have seen the deterioration much earlier and much larger. (Obviously, this is a counterfactual based upon logic without statistical evidence.)

Van der Plas (2009:210-211) [263] explains that the shift to "context questions" has obscured the lack of algebraic competence, i.e. the arithmetic competence of methods that are also relevant for algebra.

It is an innovation by Kees van Putten that he looked at student strategies, which CITO neglected, as it only looked at the outcome of sums. If I understand this correctly, it was for this project that Marian Hickendorff was recruited for, for her Ph.D. thesis. Van Putten to Jan van de Craats 2008-01-28: [264]

[261] https://en.wikipedia.org/wiki/Adriaan_de_Groot
[262] https://boycottholland.wordpress.com/2015/11/24/a-general-theory-of-knowledge/
[263] http://www.liesbethvanderplas.nl/rekenvaardigheid-in-relatie-tot-wiskunde
[264] http://www.onderwijskrant.be/kranten/ok146.pdf page 22: "In 2006 hebben Marian Hickendorff en ik samen met zes studenten bijna 10 000 vermenigvuldigopgaven van ruim 1500 leerlingen bekeken in de PPON-toetsboekjes die door het Cito aan de Universiteit Leiden ter beschikking zijn gesteld. Dit zijn de eerste resultaten en mijn AIO Marian gaat binnenkort beginnen met gedetailleerdere analyses. De traditionele vermenigvuldiging 'onder elkaar' (zoals opa het deed) komt nog steeds

Google Translate 2018: “In 2006, Marian Hickendorff and I, together with six students, looked at almost 10,000 multiplication assignments of over 1,500 pupils in the PPON testbooks that were made available by the Cito at Leiden University. These are the first results and my AIO Marian will soon start with more detailed analyzes. The traditional multiplication 'under each other' (as the grandfather did) is still very common (in contrast to the tail division [long division]), but it is declining in 2004 compared to 1997. I have specifically zoomed in on the task '99 × 99 = ? ' because it lends itself so well to the so-called realistic approach. I inspected a large number of testbooks with this assignment from PPON 2004 one night and slept exceptionally badly that night: as long as the students counted 'according to grandpa', it usually went well, but realistic approaches via for example 100 × 99 or 100 × 100 provided a battlefield with erroneous effects and answers. It already started with errors in 100 × 99 or 100 × 100 (with errors like 990 and 1000 or 100 000 respectively), and then the problem how much to subtract (compensate) with errors like 1 or 2, 100 or 200 off. In fact, only the traditional approach was successful here and only the strong calculators (best 33%) could afford a realistic approach; all other combinations had no chance.”

However, Van Putten should also have realised the key notion of measurement (psychometrics), i.e. that definitions matter about what you observe. Categorising strategies into either traditional or "realistic" or both or none, is one step, but it matters whether one measures (scores) the intermediate steps, to indicate to what extent such strategies are actually pursued (for it might also be just trial and error).

An observation by Van Putten and Hickendorff (VPH) was that students who used pen and paper did better than students who did not (relying on mental calculations only). In itself teachers know this already, but one still needs to check what it actually means. Perhaps students who did not write much were mostly deficient anyway (excepting those who got the right answer). (I did not check this part of their analysis.) But, if you grade intermediate steps, then students are aware that they should also record intermediate steps, and then there is an automatic reward for recording these steps. Thus the very way of measurement would affect whether students actually perform better or worse.

The main claim of lack of evidence on a difference

The “math war” caused the Dutch Academy of Sciences KNAW to set up a committee, that produced a KNAW (2009) report. The KNAW committee consisted of mathematicians and psychometricians. Key researchers were psychometricians Kees van Putten and (non-member and Ph.D. student at the time) Marian Hickendorff. The Hickendorff (2011) thesis [265] partly refers to her research for the KNAW report. KNAW (2009:10) gives a summary in English and its mission and conclusion 2 are:

“The Committee’s mission was the following: To survey what is known about the relationship between mathematics education and mathematical proficiency based on existing insights and empirical facts. Indicate how to give teachers and parents

veel voor (in tegenstelling tot de staartdeling), maar is wel aan het teruglopen in 2004 vergeleken met 1997. k heb speciaal ingezoomd op de opgave ‘99 × 99 = ?’ omdat deze zich zo goed leent voor de zogenaamde realistische aanpak. Ik heb een groot deel van testboekjes met deze opgave uit PPON 2004 op een avond zelf nagekeken en heb die nacht bijzonder slecht geslapen: zolang de leerlingen maar ‘volgens opa’ rekenden, ging het meestal goed, maar realistische aanpakken via bijvoorbeeld 100 × 99 of 100 × 100 leverden een slagveld aan foutieve uitwerkingen en antwoorden op. Het begon al met fouten in 100 × 99 of 100 × 100 (met fouten als 990 respectievelijk 1000 of 100 000), en vervolgens het probleem hoeveel daarvan af te trekken (compenseren) met fouten als 1 of 2, 100 of 200 eraf. Eigenlijk was alleen de traditionele aanpak hier succesvol en konden alleen de sterke rekenaars (beste 33 %) zich een realistische aanpak veroorloven; alle andere combinaties waren kansloos.”

[265] https://openaccess.leidenuniv.nl/handle/1887/17979

leeway to make informed choices, based on our knowledge of the relationship between approaches to mathematics teaching and mathematical achievement."

"2. The public debate exaggerates the differences between the traditional [TME] and realistic [RME] approaches to mathematics teaching. It also focuses erroneously on a supposed difference in the effect of the two instructional approaches whereas in fact, no convincing difference has been shown to exist."

This basically fits the Slavin & Lake (2008) methodology and conclusions on the USA.

The situation in Holland

Hickendorff's review study, chapter 1 in the thesis, selects 25 studies (18 experimental and 7 curriculum) that would relate to the Dutch situation. Test-psychologist Ben Wilbrink would like to impose stricter criteria:

> Google Translate 2018: "The committee therefore seems to be a bit sloppy: there is no research available that makes it possible to say something sensible about the effectiveness of different didactics. This is certainly something different from the first two sentences in the report, cited above, suggesting that research would have been done, aimed at the existence of differences, where no differences were found." [266]

Let me shortly indicate the three most relevant curriculum studies. The Harskamp 1988 thesis found a modest effect size of 0.09 at CITO in favour of RME, apparently not looking at later algebra. The Gravemeijer et al. 1993 MORE study found an effect size of 0.32 in favour of TME. (Wilbrink criticises this MORE study [267] but I find that he misrepresents the work Van Hiele. [268]) PPON 1997 still had some TME textbooks, and RME had an effect size over TME in the range of 0.22 to 0.53 depending upon textbook (p43). There again is no discussion of the relation to algebra later in highschool.

KNAW (2009) should have advised to abolish the Freudenthal Head in the Clouds Realistic Mathematics Institute in Utrecht, that pushed RME without proper testing on arithmetic and its relation to later algebra. Freudenthal and his institute clearly were motivated by ideology and not science. As said, logic also points to TME as the base line. KNAW (2009) however seems to have followed the reasoning by the psychometricians that you cannot say that the Moon has another side when statistical evidence is lacking.

Selective use of sources

The VSNU (Dutch joint universities) and Leiden code of research integrity has:

> Google Translate 2018: "4.5 A scientific practitioner is only a defender of a certain scientific point of view if that position has been sufficiently scientifically substantiated, and in addition, rival positions must be reported and explained." [269]

There was a conference in 2008 [270] that resulted in a special May 2009 issue of the peer review *Tijdschrift voor Orthopedagogiek* (TvO). The issue contains VPH (2009) and the paper by astrophysicist and teacher of mathematics Liesbeth van der Plas (2009). [271] The report KNAW (2009) was published on November 4. In the KNAW (2009) list of references on p91-94 we find VPH (2009) who neglect above window of opportunity for algebra. In the list of references of KNAW (2009), p91-94, *we do not find reference* to the

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266 http://benwilbrink.nl/projecten/realistisch_kolomrekenen.htm, search on Lenstra
267 http://benwilbrink.nl/projecten/more.htm
268 https://boycottholland.wordpress.com/2015/09/05/pierre-van-hiele-and-ben-wilbrink/
269 http://media.leidenuniv.nl/legacy/vsnu-code-wetenschapsbeoefening-2004-%282014%29-def.pdf "4.5. Een wetenschapsbeoefenaar is pas verdediger van een bepaald wetenschappelijk standpunt als dat standpunt voldoende wetenschappelijk is onderbouwd. Rivaliserende standpunten dienen daarnaast te worden gemeld en toegelicht."
270 http://www.wiskundebrief.nl/471.htm
271 http://www.liesbethvanderplas.nl/rekenvaardigheid-in-relatie-tot-wiskunde

Van der Plas (2009) paper and contribution to the 2008 conference, who warns about above window of opportunity for algebra and the effect measure. [272]

Thus the KNAW (2009) report, that was intended to deal with the math war, appears to be biased itself, and appears to be in violation of the VSNU and Leiden code of research integrity. Reference to Van der Plas (2009) is also missing in the Hickendorff (2011) thesis. The word “algebra” is entirely missing in the thesis too.

Van der Plas (2009) shows *in a didactically valid manner* that the scoring method also used by VPH (2009) is invalid. Van der Plas doesn’t refer to VPH but she discusses the scoring method of CITO that VPH use too. This CITO method only considers the outcomes of sums and not the algorithm, while the latter is relevant for learning algebra in subsequent education. Pupils might score better by the use of the calculator and trial and error as allowed by RME but this would still maim them mentally for highschool and the rest of their lives in the compentence w.r.t. algebra. Yet this criticism by Van der Plas was neglected by VPH and “thus” the KNAW committee.

It isn't only that the research record is tainted by (deliberate) neglect (exclusion). Let me add that there have also been costs to Van der Plas for not being referred to properly. What would have happened when VPH (and subsequently the KNAW report) had referred properly ? Then others would have seen the key relevance of this paper too. In a specialising world: when you are excluded from the key overview, then you likely aren't noticed anymore.

The conference paper VPH (2009) does not refer to Van der Plas (2009) either. It might be seen as fair that papers presented at a conference in 2008 don’t refer to each other. On the other hand there were months between the conference and the actual publication. The idea of a conference with peers is that when there is criticism that invalidates your analysis, then you would at least adapt the paper with a discussion of the criticism.

Perhaps after the conference in 2008 VPH were so busy with the KNAW committee that they did not have time to listen to criticism ? Perhaps the only reason for VPH and thus KNAW (2009) to neglect the argument by Van der Plas (2009) may have been that she did not use statistics ? This is unclear, and as far as I know VPH publicly neither discussed it nor explained why they excluded it. For completeness: we can guess at other confusions [273] but none of these confusions would be valid either. The Hickendorff (2011) thesis refers to “empirical studies” but it may be that she confuses this with statistical studies only. Van der Plas (2009) clearly is an empirical study too, since it looks into the issue and its effect measure. A problem is that these psychometricians have no background in the didactics of mathematics and may not recognise the validity of the argument by Van der Plas (2009).

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[272] In the same issue of *Tijdschrift voor Orthopedagogiek* in May 2009 there is an article by Gerard Verhoef who has a similar point on the effect measure. There is also an article by Jan van de Craats who could have made the point but doesn't, perhaps because of specialisation (and there is no need to repeat what others have said). In 2015 Van der Plas repeats her comment as something that is rather obvious for didacticians: http://www.wiskundebrief.nl/720.htm#5

[273] VPH might have potential confusions to neglect the Van der Plas (2009) article. None of these confusions are valid but we may list some. (1) It does not explicitly and concretely refer to their work. (2) It does not refer to papers in peer reviewed journals. (3) It does not provide statistics. (4) It might look like a personal opinion. (5) It discusses the link between primary and secondary education, instead of only primary education. (6) It looks at didactics and not student learning. (7) It does not *fully* develop the issue on the effect measure (because it also looks at other issues, like the relation of arithmetic to algebra). These seven possible confusions are invalid, because Van der Plas (2009) remains relevant for the issue of interest, and her analysis implies that the VPH (2009) paper has an invalid approach. If Hickendorff chooses to associate herself strongly with CITO, we may conclude that Van der Plas (2009) actually has *concrete criticism* w.r.t. VPH (2009).

Invalid reasoning and A.D. de Groot's Forum Theory

Based upon their statistical analysis VPH infer that they cannot diagnose a difference in effectiveness in RME and TME at the end of elementary school. Their method is innovative in that they look at the pupils's exam papers rather than final answers to classify which approach each used. However, as clarified above, such classification differs from proper scoring. Teachers of mathematics and test researchers would have some points of doubt:

(1) The official exam rule is that "non-context questions" consist of arithmetic sums only while "context questions" have verbal formulations (narratives), and that only the latter may be done by calculator. This creates a bias towards RME that invented the very notion that "context warrants the calculator" and that relies on context and thus the calculator and trial and error.
(2) VPH don't grade steps and thus have another bias in favour of RME. Categorising students on the methods used in their exam papers cannot replace the basic didactic consideration that one anyhow grades steps to evaluate competence.
(3) A categorisation on techniques cannot discriminate between RME or TME didactics anyhow. By 2009 all textbooks were of the RME kind only, and all kids had been trained in RME fashion.

In October 2013, the then-chairperson of NVvW Marian Kollenveld gave this criticism on the final exam arithmetic test ("Rekentoets"):

> Google Translate 2018: "(...) they are multiple-choice questions and short-answer questions in which the dissolution process is not assessed (only a good answer counts, regardless of the complexity) - in case of a complex question, there are sometimes 4 steps that can all be right or wrong, and can stand for differences in the student's skill, which are not measured now, this also contributes to a minimal score)." [274]

The issues should have been resolved within the setting of A.D. de Groot's "forum theory". [275] [276] The Dutch journal Euclides since 1925 is online now, and there is the obituary of A.D. de Groot, in Euclides, 82 no 3. [277] He got a BSc in mathematics before he switched to psychology, and was a teacher of mathematics for a while. His is a key founder of CITO, where Hickendorff works parttime. I imagine that A.D. de Groot would be aghast to see how psychometricians maltreat the didactics of mathematics. De Groot would also be horrified by their lack of understanding of psychometrics itself. The first thing that a psychometrician should do is to explain that definitions determine the measurements. Thus when you claim to study education, then you define what is involved, and then you also specify what is a success and what is a failure, and you acknowledge the criticism that you also must score the intermediate steps when those are relevant for the strategy of answering a test question. And you should not confuse a technique used (cross-sectional) with didactics (longitudinal).

Forum theory hasn't been much implemented yet, and subsequently we also meet with researchers who refuse to answer to criticism. VPH might think that they are open to criticism, and can refer to discussions in TvO, Psychometrika or presentations at

[274] "(...) het zijn meerkeuzevragen en kort-antwoordvragen waarbij het oplosproces niet wordt beoordeeld (alleen een goed antwoord telt, ongeacht de complexiteit,- bij een complexe vraag zijn er soms wel 4 stappen die allemaal goed of fout kunnen, en ook kunnen staan voor verschillen in vaardigheid van de leerling, die nu niet worden gemeten, ook dit draagt bij aan een minimale score)." http://thomascool.eu/Papers/AardigeGetallen/2016-03-06-NVVW-bestuur-desinformeert-het-parlement-over-het-rekenen.pdf

[275] https://www.knaw.nl/nl/actueel/publicaties/het-forumwaarmerk-van-wetenschap

[276] https://boycottholland.wordpress.com/2015/11/24/a-general-theory-of-knowledge/

[277] https://archief.vakbladeuclides.nl/bestanden/82_2006-07_03.pdf

NVORWO Panama conferences – meetups commonly linked to "realistic" ideology. [278] Obviously I will not deny such communication, but it doesn’t change the present criticism, which they neglect for some years now. I am not aware of other people complaining that VPH do not (adequately) respond to criticism. In fact, I am quite amazed that Liesbeth van der Plas, mathematician Jan van de Craats, mathematician Gerard Verhoef and test-psychologist Ben Wilbrink [279] [280] haven't really deconstructed the KNAW 2009 report, and subsequently Hickendorff's thesis [281] [282] and the discussion on these as well. I suppose that each might have his or her own reasons.

A key difference between these others and me is that I also wrote the books "*Elegance with Substance*" (2009, 2015) and "*Een kind wil aardige en geen gemene getallen*" (2012) and "*A child wants nice and no mean numbers*" (2015, 2018). [283] Thus my position in didactics of mathematics is much stronger compared to these other authors. Obviously also, as an econometrician, I am familiar with the basics of the IRT testing method that VPH employ. Holland is a small country. Also, I am an econometrician and look at these issues also from the viewpoint of (institutional) economics. [284] Thus perhaps it is unavoidable that I might be the only local reseacher who can unravel the knotty problem created by the three groups involved: the ideologues of "realistic" mathematics education, the traditional mathematicians and the psychometricans with their blinders. Yet I do not attend such Panama conferences. Before 2012 I had no real interest in primary education and its research. I never considered myself qualified for didactics in primary school, though a new law in 2016 declares that I am now. [285] The present analysis obviously is sound but still targeted at a very specific issue and point in research. My main point is that I pose questions and would like to hear some answers.

But I must also mention that much of what I say – in this case – really isn't new, see the reference to Van der Plas (2009) and the age-old discussion in statistics and testing about validity. The example on 100 / 4 above is so blatantly obvious, that I cannot see why VPH don't reply to this issue w.r.t. their research. Let me add that before 2016 I only

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[278] http://www.nvorwo.nl/event/panamaconferentie/
[279] http://benwilbrink.nl/projecten/rekenproject.htm
[280] http://thomascool.eu/Papers/Math/2015-09-15-Breach-by-Jan-van-de-Craats-and-Ben-Wilbrink-wrt-scientific-integrity.html
[281] http://benwilbrink.nl/literature/hickendorff_2011.htm Google Translate 2018 of his comment of October 24: “Having read the entire dissertation at least once, it is striking that the research material - the calculations - is limited to what is typically tested in current mathematical education. Note: the current maths education is strongly marked by the ideas of realistic mathematical education. It is for me even the question whether the calculations in Hickendorff's research do belong in the [arithmetic] domain. In any case, they do not belong to the core of this: the 'realistic' mathematical education has been marginalized over the decades into convenient computing, calculating in school contexts that must also be called 'realistic' but that are not, and to mathematical education that is cut off from what later mathematics education presupposes [arithmetic] skills (knowledge of algorithms, basic [arithmetic] facts, can count with fractions). Hickendorff and others have good reasons for limiting the research to the typical calculations as can be found in realistic mathematical education, and unfortunately also in the Cito Eindoets Basisonderwijs and the PPON, but they still run the risk of giving the impression that this research is about calculating skills in the ordinary sense of the word, while in fact it concerns mathematical education as it is deformed under the influence of the not adequately empirically tested [arithmetic] ideas of Hans Freudenthal and his group.”
[282] Wilbrink on Hickendorff’s Chapter 1, Google Translate 2018: “The Lenstra committee also ignored the didactic theory, possibly because no opinion was possible within the committee. But what if a certain didactic theory is demonstrably based on failed psychology? I would like to keep this point at the center of discussion about mathematics education at all times. Marian Hickendorff undoubtedly, too, but she is understandably focused on empirical data about pupils.” Wilbrink allows Hickendorff to get away with “testing without theory” ? What is the meaning of his “center of discussion” ?
[283] http://thomascool.eu/Papers/Math/Index.html
[284] https://boycottholland.wordpress.com/2015/10/31/the-power-void-in-mathematics-education/
[285] https://www.delerarenagenda.nl/blog/weblog/weblog/2016/20160902-vo-docenten-bevoegd-voor-basisschool-%E2%80%98goede-methode-voor-talentvolle-kinderen%E2%80%99

superficially encountered Van Putten and Hickendorff once very briefly, namely at that KNAW 2014 conference, that looked back at the KNAW 2009 report. [286] Thus, with more discussion in person, I could have gathered a better diagnosis of the situation. I can imagine that there might be communication issues between psychometricians and econometricians and didacticians, but I have done my share of looking into psychometrics as well (see the references in VTFD), and it would be no more than rational and scientically warranted if VPH did their share on didactics of mathematics.

The Dutch association of teachers of mathematics NVvW apparently did not debunk the KNAW (2009) report (a Google returns no results). They focused on the "Rekentoets" as applied in secondary education, which is in their own direct interest. The more serious implication however is for primary education: that if you don't properly teach and score the traditional algorithms, then you don't properly prepare students for algebra in secondary education and the rest of their lives. [287] The situation isn't helped much by that NVvW has turned out to be a seriously sick organisation. [288]

The Hickendorff email of 2014 and refusal to correct

A KNAW 2014 conference, looking five years back to 2009, caused me to contact Hickendorff, asking her about didactics and MER and the validity of her research and intended presentation. Hickendorff replied:

> Google Translate: "Dear Thomas Cool, Thank you for your mail. I fear that I can not find the time to view everything you send. In addition, I also wonder if you have come to the right place for me: I am not a didacticist but a psychological researcher, and I also try to stay out of the discussion about didactics as much as possible because I do not believe that that is my expertise. Kind regards, and until a.s Monday, Marian Hickendorff" [289] [290]

Originally, I praised Hickendorff for her modesty that she refrained from a discussion that wasn't her expertise. Hickendorff does not clarify her lack of expertise in the thesis itself, and apparently hasn't told this to the minister of education who might look differently at the KNAW report of 2009 now. I did question her because she involved herself nevertheless, and I observed that she couldn't avoid using an effect measure in her

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[286] http://thomascool.eu/Papers/Math/2014-07-02-KNAW-Rekenen-deugt-niet.html
[287] http://www.wiskundebrief.nl/721.htm#5
[288] http://thomascool.eu/Papers/Math/2016-06-28-Letter-to-NVVW-with-Red-Card.pdf For some Dutch readers, the curious clash in 2016 between the state secretary of education Sander Dekker and the board of NVvW, about a supposed "agreement" on the new highschool test on arithmetic, might be another eye-opener on the disfunctionality of NVvW.
http://www.volkskrant.nl/binnenland/wiskundeleraren-dekker-komt-afspraken-over-betere-rekentoets-niet-na~a4429909
https://boycotholland.wordpress.com/2017/04/08/update-van-bestuur-nvvw-verzint-een-afspraak-met-de-staatssecretaris/
[289] From: "Hickendorff, M." [...]
To: "Thomas Cool / Thomas Colignatus" [...]
Cc: J.A.Bergstra [at] uva.nl, "Craats, Jan van de" [at] uva.nl
Subject: RE: T.b.v. a.s. maandag (KNAW reken-onderwijs)
Date: Fri, 27 Jun 2014 [...]
Beste Thomas Cool, Bedankt voor uw mail. Ik vrees dat ik niet de tijd kan vinden om alles wat u stuurt te bekijken. Daarnaast vraag ik me ook af of u bij mij hiervoor aan het juiste adres bent: ik ben geen didacticus maar psychologisch onderzoeker, en probeer ook zo veel mogelijk buiten de discussie over didactiek te blijven omdat ik niet meen dat dat mijn expertise is. Vriendelijke groeten, en tot a.s. maandag, Marian Hickendorff
[290] I take this statement as it is. If she meant something else, then she should have said something else. Also, I have explained at various locations what her statement implied, and alerted her to this, so she could have corrected me in public since 2014 that she should have been more precise w.r.t. what she actually wanted to express.

research, which effect measure can only be based upon didactic concerns. She did not reply to this, which is a clear breach of integrity of science.

> Google Translate 2018: "6.2 Academic practitioners are honest and loyal about the quality they deliver and they contribute to internal and external assessments of their research." [291]

My questions to Hickendorff amount to an external assessment and she rejected a reply basically by the argument that it was external to her. An analogy: A foot surgeon performing heart surgery rejects answering questions on this by saying, modestly, that he is only a foot surgeon.

Getting clarity on the effectiveness of the didactic approaches of TME versus RME was one of the main objectives of the KNAW report. Curiously, in her thesis Hickendorff claimed such expertise, namely by claiming to do a review. What would be the basis of such claim ? Yet in her email to me she claimed keeping a distance for lack of expertise. This is inconsistent. Clearly my earlier praise for modesty must be withdrawn. She has wrongly informed me, and she should reply to the question on content. She should do so in public. Her disinformative email and the subsequent refusal by her and Van Putten to consider the criticism is in violation of the basic rule in science that researchers must be open to questions and criticism. Perhaps they only follow psychometric convention but keeping a field accountable runs via individual research ethics since one cannot address all at the same time.

Page xvi of Hickendorff's thesis clearly states that *she did report on the effect of didactics on results*.

> "The thesis starts with Chapter 1 reporting a research synthesis of empirical studies that were carried out in the Netherlands into the relation between mathematics education and mathematics proficiency. This chapter is based on work that was done for the KNAW (Royal Dutch Academy of Arts and Sciences) Committee on Primary School Mathematics Teaching [ftnt], whose report came out in 2009. Starting with an overview of results of Dutch national assessments and the position of Dutch students in international assessments, the main body of the chapter is devoted to a systematic review of studies in which the relationship between instructional approach and students' performance outcomes was investigated. The main conclusion that could be drawn was that much is unknown about the relation between mathematics programs and performance outcomes, and that methodologically sound empirical studies comparing different instructional approaches are rare, which may be because they are very difficult to implement. In the remainder of this thesis, the focus is shifted to other determinants of students' mathematics ability related to contemporary mathematics education, such as the strategies students used to solve the problems and characteristics of the mathematics problems. [ftnt: I worked as an associate researcher supporting the Committee. In particular, the Committee requested me to carry out the systematic literature review that formed the basis of chapter 4 in the report. Chapter 1 in the current thesis is based on this work.]"

It is didactics that deals with "the relationship between instructional approach and students' performance outcomes". See also the Dutch translation on p274-275. [292] Her

[291] "6.2. Wetenschapsbeoefenaren laten zich eerlijk en loyaal de maat nemen over de door hen geleverde kwaliteit. Zij werken mee aan in- en externe beoordelingen van hun onderzoek." http://media.leidenuniv.nl/legacy/vsnu-code-wetenschapsbeoefening-2004-%282014%29-def.pdf

[292] Dutch p274-275: "Hoofdstuk 1 van dit proefschrift bevat een onderzoekssynthese van resultaten van Nederlandse empirische studies naar de relatie tussen rekendidactiek en rekenvaardigheid. Dit hoofdstuk is gebaseerd op literatuuronderzoek dat is uitgevoerd voor de adviescommissie Rekenonderwijs op de basisschool [ftnt] ingesteld door de Koninklijke Nederlandse Akademie van

study was a review, but for a review you still must have some qualifications and there are criteria for being critical. In her 2014 email to me she now suggests that she was unqualified to do such a review. Also observe a potential reduction of "empirical studies" to the use of statistics only.

VPH might hold that they only *reviewed* cause-effect research by others, and did not do this kind of research themselves, but this is not relevant here, because in their review they did not criticise the effect measure, as they should have. They might not criticise the effect measures by these other authors because of their own lack of knowledge about didactics of mathematics. When they exclude Van der Plas (2009) for their review study too, then clearly they exclude information about what a valid effect measure would be.

To some extent I can imagine that Hickendorff wants to keep some distance from didactics, since the math war between TME and RME has turned this field into a quagmire indeed. However, the proper response is not neglect but protest and re-engineering. [293] Obviously, this starts from an interest in didactics of mathematics indeed, and an interest in psychology itself might be less encouraging, but the point remains that she started studying arithmetic test scores.

When it becomes an issue of research integrity

My diagnosis is that VPH (i) use selective sources, (ii) use the wrong effect measure so that claimed outcomes are invalid, (iii) have inadequate knowledge about and respect for didactics of mathematics while their topic requires those, (iv) neglect criticism on (i) – (iii). I have documented the case in Dutch [294] and English. [295] Leiden University rejected mediation and thus I submitted the case to the Leiden committee on research integrity. [296]

Wetenschappen (KNAW), wier rapport in 2009 is uitgekomen. Deze systematische kwantitatieve onderzoekssynthese laat geen eenduidige conclusies over het effect van verschillende rekeninstructiemethoden of rekencurricula toe. Enerzijds zijn er weinig methodologisch degelijk opgezette interventiestudies waarin de effecten van verschillende instructieaanpakken vergeleken worden. De wel beschikbare studies zijn bovendien beperkt in verschillende aspecten, zoals steekproefgrootte of inhoudsdomein. Ook zijn didactische kenmerken en instructiekenmerken vaak met elkaar verweven in de programma's die vergeleken zijn, zodat het onmogelijk is de unieke effecten van verschillende kenmerken vast te stellen. Anderzijds zijn de curriculumstudies waarin de uitkomsten van leerlingen die verschillende rekencurricula (rekenmethodes) gevolgd hebben worden vergeleken, beperkt in de mate van controle over de implementatie van het curriculum en in de mogelijk tot het corrigeren voor verstorende variabelen. Hoewel er dus geen algemene hoofdconclusie getrokken kan worden, zijn er wel wat specifieke patronen die uit de bestudeerde onderzoeksresultaten naar voren komen. Ten eerste is het opvallend dat de prestatieverschillen binnen een bepaald type instructieaanpak groter zijn dan tussen verschillende aanpakken. Blijkbaar spelen didactische principes een kleinere rol dan de praktische implementatie door de leerkracht en de interactie tussen de leerkracht en de leerling, bevindingen die in overeenstemming zijn met die van bijvoorbeeld Slavin en Lake (2008) in hun grootschalige internationale onderzoekssynthese."

[293] https://zenodo.org/communities/re-engineering-math-ed/about/

[294] http://www.wiskundebrief.nl/718.htm#7
http://thomascool.eu/Papers/AardigeGetallen/2008-2016-plus-Afgewezen-door-de-WiskundE-brief.html#2016-10-08
http://thomascool.eu/Papers/AardigeGetallen/2016-01-17-Meta-opmerkingen-over-psychologie-en-wiskunde.pdf
http://thomascool.eu/Papers/AardigeGetallen/2016-01-31-Enkele-emails-rekentoets-psychometrie.pdf
http://thomascool.eu/Papers/AardigeGetallen/2016-02-10-Basisprobleem-in-pedagogie-onderwijs-en-didactiek-van-wiskunde.pdf
http://thomascool.eu/Papers/Math/2016-05-25-Email-exchange-with-Kool-Noteboom-Tijdeman.pdf

[295] http://thomascool.eu/Papers/Math/2016-05-09-Letter-to-VOR-and-Trainers-of-teachers.pdf

[296] http://thomascool.eu/Papers/Math/CWI-Leiden/2016-09-30-Letter-to-CWI-anonimised.pdf

It can be observed that procedures on scientific integrity are not well developed yet. [297] Society has shifted from an agricultural to an industrial to a service economy. The conduct of "information workers" becomes ever more important, but regulations on these are lagging. This is awkward especially for specialists, when only a few persons deal with an issue, and when issues of conduct (like also rules of proceedings like these) might have a disproportionate impact. Major concerns w.r.t. breaches of integrity have always been interference with politics or religion or personal advantage for income and status. Such breaches can be seen as coming from external sources. In the present case we have an ivory tower, with tunnel vision, own-language (empirics = science = statistics) and group think. This can be seen as deriving from internal sources in science. Science itself may invite to specialise, but over- and misspecialisation lead astray.

In my view, a professional with personal integrity can still breach the integrity of science. Therefor, I have *specified* what the breaches by VPH have been. The language for such issues is not well developed yet, and one tends to run into confusions because of ambiguous words. (Especially when others start generalising.) For example, a medical doctor might make an error that might even cause the death of a patient. But this doesn't need to be a case of malpractice. It might be a honest mistake. Professionals need freedom and might make mistakes. What can turn this into a breach of integrity (of medicine) is when the doctor neglects criticism and refuses to acknowledge the error. For example, a driver of a car might cause an accident, but still be insured for liabilities. What may turn this in problematic behaviour is when the driver was warned about risky weather conditions, and that he or she took risks that the insurer actually didn't take into account. It becomes a breach of truthful behaviour, for the overall learning process, when the driver doesn't ackowledge the true diagnosis of having taken too much risk.

VPH should have given a reaction to my analysis, in time and in public. This would have been normal scientific procedure: there is criticism on content, and reply on content. Now, there is this discussion on content but in the context of a procedure on integrity, and with a focus on restoring integrity of science.

Originally I had the vague idea that perhaps the Hickendorff (2011) thesis might still be maintained, since the main point of not responding to criticism is from 2014 onwards. However, a thesis should show that the candidate has learned what science is. Clearly Hickendorff hasn't. The thesis is a product of an ivory tower apparently created by Willem Heiser and Kees van Putten. Thus now I put more emphasis on the selective references, i.e. the not-including of Van der Plas (2009) and other didactic considerations. The scientific record better be set straight, so that one could not refer to the present "thesis" as if belonging to the scientific literature. Potentially Hickendorff is the victim of a selective thesis commission, but she also is an apt learner of such selective practices. Thus, my present view is that the thesis should be annulled too. It would be up to another promotor to determine what material can be rewritten in what manner for a revision. This really would be the best decision. Hickendorff is relatively young while Heiser, Van Putten and Tijdeman are retired. Hickendorff potentially has many more years as a potential scientist, and it is better that she learns what science is. Actually, after my original letter to CWI, this should have been the proper response by VPH as well.

[297] https://boycottholland.wordpress.com/2015/11/26/allea-defines-research-integrity-too-narrow

Causal modeling for the basics of didactics

A basic model, mention of psychology, exclusion of didactics

Let us consider the causal modeling behind this. Let me denote *s* for student behaviour (learning, solution strategies), *d* for teacher behaviour (direction, instruction), and *o* for other factors. There will be some feedback when a teacher observes some ineffective learning strategy and adjusts the directions. For now function *f* suffices as a summary what is studied in didactics:

$$s = f[d, o]$$

Different directions *d1* and *d2* would give different outcomes *s1 = f*[*d1,* o] and *s2 = f*[*d2, o*]. Each such relation can be called "a didactic". Could you study *s* while neglecting the functional relationship *s = f*[*d, o*] ? This would be like studying a phenomenon without its causal factors. The differences *s1 – s2* would only be "noise" that cannot be explained. For science this might be a first step but it soon becomes absurd. For example, *A* says to *B:* "You should look at a map because you are driving into the wrong direction." And *B* answers: "No, I am driving. Looking at maps is something else."

This clarification of the definition of didactics shows that the KNAW 2009 committee with its mission quoted above "To survey what is known about the relationship between mathematics education and mathematical proficiency based on existing insights and empirical facts" had a deficient composition, for they lacked didacticians. Arrogantly choosing to reinvent the wheel they came up with "garbage in, garbage out" (GIGO).

In her email of 2014, Hickendorff describes herself as a psychologist. Her thesis also expresses a wish of building a bridge between psychology and psychometrics. We might interprete this as a claim that, in her research frame, psychology was more important than didactics. Teachers get some training on pedagogy but will tend not have a degree in psychology. [298] Thus she would study function *g* that uses factors in psychology:

$$s = g[\psi, o']$$

Instead, the true model is rather that didactics already takes account of student psychology, with a distinction between true ψ for the student and its assumption ψ' by the teacher.

$$s = f[d[\psi'], \psi, o'']$$

If we can assume that there are no crucial errors in judging psychological reactions for most students, then we can assume $\psi = \psi'$, and the latter reduces again to:

$$s = f[d, o]$$

Above we observed that Hickendorff reviewed "the relationship between instructional approach and students' performance outcomes", didn't spot adequate studies, and then looked at alternative explanations like student strategies themselves. My criticism was that Hickendorff incorrectly reduced $d = d[\psi']$ to noise *o*'. She assumes direct causality from ψ on *s* but the main channel is via *d.* While the KNAW study argued for a key role of the teacher, the distinction on TME and RME was rejected, but on invalid grounds.

With this notation in formula's I don't want to suggest exactness. I only think that these schemes help to emphasise the causal presumptions. This should also clarify that psychology is obviously relevant. For Hickendorff it perhaps is a key factor, but didactics might not put much emphasis on this since psychology is only one of the factors.

The following diagrams may clarify Hickendorff's conceptual error. **Figure 13** gives what is likely the "true model" for dominant causality. Potentially there are arrows between all

[298] IGPME looks into psychology and mathematics education. http://www.igpme.org/index.php/home

factors but I now give the hypothesis for the main paths. **Figure 14** gives Hickendorff's position of cutting out the function *f* studied in didactics. Her suggestion is the inference from "students do so" to "students may be competent to do so". Observe that her "try to stay out of the discussion" might still mean that she would respect and include the conclusions from such discussions, like the Van der Plas (2009) paper. However, when her *conduct* is that she *neglects* such discussions, then "try to stay out of" is a misrepresentation of what she actually does.

Figure 13. Likely a good model of dominant causality

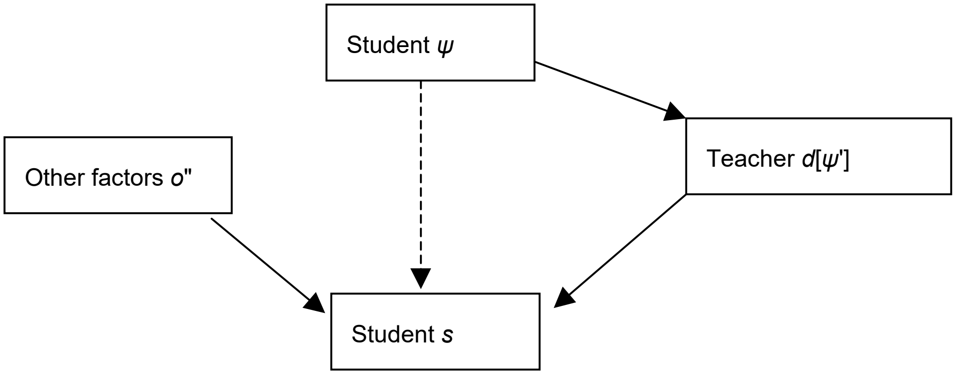
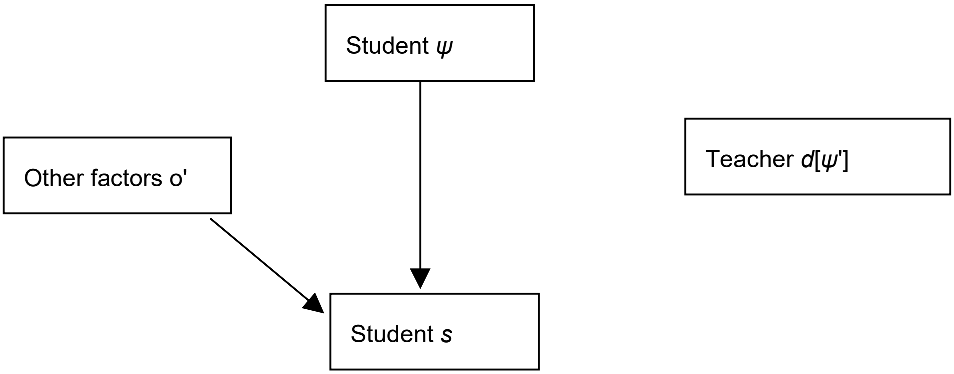


Figure 14. "No didactician" and "try to stay out of the discussions": students invent ("objectively given") algorithms themselves without didactics



Evaluation

There is a simple model in Item Response Theory (IRT) that has questions as items and student answers as responses. This looks at *s* only. I discussed this kind of modeling in my book *Voting Theory for Democracy* (VTFD), Colignatus (2001, 2014). IRT has the nice property that the test tells about both the competence of students and the adequacy of the test itself. However, IRT is only a limited model, and the proper analysis looks wider. Psychometricians focusing on only *s* are at risk of misrepresenting their field of study and the conditionality of their findings.

- Didactics obviously is focused on *affecting* learning behaviour by students.
- It is quite silly to argue (a) that a teacher only does his or her thing, and (b) that what students do is entirely independent.

Secondly, we can only *describe* the very *s* by using information from *d.*

- It are the *d* that define what *a strategy actually is*. The only theory that provides a rationale for what it means "solving correctly" derives from didactics. There is both the algorithm that students use, and possibly an independent *golden standard* provided by a computer programme, but both are designed by didactics.
- If you don't know about *d* then anything that students do is basically random behaviour (with some mean and dispersion).
- Proper didactics also assigns points for the intermediate steps in the algorithm. This valid effect measure would show that students *using the algorithm* would score much better. (The use of the calculator would only give a few points for a right answer.) VPH would argue that this would be irrelevant for their research on *s* ?
- Why would a psychometrician select only *s* and the invalid effect measure of "answering a sum correctly" ? A psychometrician claiming to look at only *s* is at danger of creating his or her own universe of *s,* while neglecting that *s* only is meaningful because of *d.*
- Thus VPH used a statistical exercise to argue against the relevance of *d,* but their exercise was based upon the assumption that *d* was not relevant (for it neglected the discussion on the effect measure).

If a student solves 100 / 4 by means of traditional long division or "realistic" trial and error, then the use of these "strategies" would be random for psychometricians looking at only *s*, because these researchers would not have the didactics *d* that define what the proper algorithms are. Without the use of the algorithm, and only looking at the outcomes, they might determine that 100 / 4 is an “easy” question (with a higher rate of success) and that 57 / 3 is (perhaps) a “hard” question (with a lower rate of success). Without the algorithm such distinction would remain unexplained. Potentially psychometricians might think that "everyone knows what long division is", so that they don't need to check with didacticians of mathematics. In such an ivory tower they might reduce didactics into "ways to teach students about obviously clear techniques, given from heaven". This would be improper research, because it would neglect outcomes from an adjacent field of research (didactics).

This discussion might be contaminated by the context of the Dutch regulations about what is expected from children at the end of primary education. The standard is the CITO test. Hickendorff is associated with CITO. VPH might say that their definition of arithmetic is what CITO has chosen. This might boil down to "testing without theory". Then psychometrics reduces to behaviourism again. However, whatever these test-for-the-test philosophers claim, there is still a distinction between the CITO tests and the didactic objectives that have been selected, as what pupils should be able to do. In this case the objectives w.r.t. algebra in secondary school are clearly important. In that case Hickendorff *as a scientist* might have to criticise CITO instead of embracing it. It is not impossible that CITO has incompetent didacticians of mathematics too.

Clearly, when properly evaluated, the data in the KNAW (2009) report or the Hickendorff (2011) thesis chapter 1, or the evaluations by VPH (2009) in their own (non-review) research on such solution strategies, would generate other conclusions about the mathematical competence of the students (and by implication on the $s = f[d, o]$ relation).

Obviously the other factors *o* can be dominant (Van de Grift), but, in the case of comparing traditional didactics and "realistic" didactics in arithmetic (the present issue of concern), there is a clear dependence:

- Students don't simply invent the traditional algorithms of say long division or solving problems like 1/3 + 1/5. They must be taught via some *d,* and mastery comes from adequate training.
- If you apply the *proper* measure of success (scoring steps in the algorithm) then the difference between *s1* and *s2* will be highly correlated with the difference between *d1* and *d2.* This argument is based upon logic and not in need of a statistical study, and thus cannot be excluded as supposedly being “non-empirical”.
- If you apply an invalid measure of success then you might not see that correlation. In that case you might erroneously conclude that the statistical evidence doesn't support a distinction in effectiveness of either didactic method.

Possible confusions by psychometricians

In itself, when there is a math war between "realistists" and traditionalists, who actually both neglect both empirical research and statistics, and who don't care to design a distinguishing experiment, then I can imagine that psychometricians decide to focus on *s.* It is the kind of research that psychometricians have been creating a tradition in themselves based upon the Item Response Theory (with the risk of tunnel vision). Potentially it might generate results. That said, they still should be open to criticism, that one cannot just focus on *s* while neglecting both *f* and *d.* If the math war is a problem, then the math war should be resolved (and not neglected). Thus, when the psychometricians observe such a math war, then they should protest (too) instead of (only) neglect it. (My advice is an enquiry by parliament. [299])

This neglect of the role of didactics (with the example of long division) links up with the notion that various fields of research are looking into arithmetic: from neuroscientists to psychologists to didacticians. The suggested implication that other fields step in but that didactics might be neglected is a gross generalisation, and quite invalid.

- For example, I have warned neuroscience to beware of conclusions on number sense, when there are some crooked features in current arithmetic. For example, *two and a half* is 2 + ½ but it is written as *two times a half* or 2½ (compare *2a* or 2 km). A conclusion should not be that children have difficulty learning arithmetic, if the cause of learning problems lies in so-called arithmetic itself. See also the issue of pronunciation of the numbers. (New would be a discussion on the errors by Van Putten & Hickendorff and also CITO on 2 + ½, but I have deliberately chosen to first deal with the present conventional points.)
- It requires didactics to grow aware of such issues. Thus multidisciplinary research is welcome and ivory tower research might soon run astray.

Psychometricians should not be so singular as to claim that they can do this research themselves, with only other scientists who they select themselves, while using an invalid generalisation as "others neglect didactics and thus we can do so too". When other scientists join the party on their own initiative and utter criticism, then there is scientific reason to pay attention to the arguments.

NB. Actually, the situation is that the original party had been organised by didactics, and it are the psychometricians who created their own subparty, trying to take over. Let me refer to above quote from chapter 1 of Hickendorff's thesis:

> "(...) a research synthesis of empirical studies that were carried out in the Netherlands into the relation between mathematics education and mathematics proficiency."

Thus the issue is within the realm of didactics of mathematics, and the psychometricians are hired guns to illuminate aspects by their expertise. (They might use the same techniques as for language or other issues.) It can happen that the agent takes over from

[299] https://www.ipetitions.com/petition/tk-onderzoek-wiskundeonderwijs/

the principal, or that the lieutenant ("stadhouder") takes over from the king (William of Orange vs Philip II), but in this case, didactics has a sound position that they aren't fulfilling the contract and doing the job properly.

The causal models and the situation in Holland

Our discussion of these causal models might not be understood without the reference to the developments in Holland.

(1) These insights might be seen as differences in opinion in approaches to research. It might be seen as if VPH (2009) only have a different opinion other than Van der Plas (2009) or me. However, the true problem with VPH are the breaches w.r.t. research integrity w.r.t. the points mentioned above.

(2) VPH might argue that their research would only concern tests on learning. Van der Plas and I provide criticism from didactics, which thus might not apply to their research on learning. The present discussion should clarify that didactics also looks at learning. Thus, if VPH would suggest that criticism from didactics would not apply to their research on learning, then they again would show that they lack in understanding of didactics. Also, such suggestion would be disingenuous since VPH and KNAW (2009:10) point 2 clearly draw conclusions w.r.t. the effectiveness of TME and RME, and thus encroach upon didactics, even while the KNAW committee did not have members with a background in didactics of mathematics.

(3) The causal models are useful for this analysis in institutional economics on the math war. VPH presented an analysis on *s* and the cause *d,* as if there would be no evidence for a relevant difference of effect size between TME and RME, while didactics clearly shows that TME has logic on its side. We also see the problem of the many hands and shared responsibility, when a committee takes over. It were mathematician and chairman Jan Karel Lenstra (without a background in didactics of mathematics) and his full KNAW committee (including Van Putten with assistance by Hickendorff), who supported the invalid analysis. Committee members should respond to criticism *also afterwards,* and not hide behind the committee itself.

Development in 2017-2018

A 2017 study for NRO and IvhO

Hickendorff et al. (2017:24) [300] is a repeat review study commissioned by the Inspectorate of Education (IvhO) with administrative intermediary NRO. The authors qualify their review as “narrative” as opposed to a quantitative meta-analysis. Remarkably, this 2017 “narrative review” still excludes Van der Plas (2009) or my criticism (which one migh qualify as “narrative” too since those don’t rely on statistics but on logic). Hickendorff et al. (2017) finally acknowledge, still confusing “empirical” with “statistical” (p24):

> Google Translate 2018: “Finally, the focus on empirical research limits the scope of the research by not addressing important theories about learning in general and [didactics of arithmetic] in particular.” [301]

Thus, while KNAW (2009) deliberately restricted its attention to statistical findings, Hickendorff at al. (2017) finally agree that such an approach has limited meaning. Yet, not for their own study in 2017 but as recommendation for future research.

However, their comment tends to imply a claim that they are competent to judge upon the importance of didactic theories . Hickendorff already stated her lack of expertise. Co-

[300] https://www.nro.nl/nro-projecten-vinden/?projectid=405-17-920-rekenen%20op%20de%20basisschool

[301] Dutch original: “Ten slotte beperkt de focus op empirische onderzoeken de reikwijdte van het onderzoek doordat niet wordt ingegaan op belangrijke theorieën over leren in het algemeen en rekenwiskundedidactiek in het bijzonder.”(p24)

author T.M.M. Mostert has a MSc degree in "Education and Child Studies", that indeed looks into "factors that effect reading and arithmetic", but this might not be didactics of arithmetic. [302] Co-authors C.J. van Dijk and L.L. van der Zee apparently have no Leiden page. Co-author L.L.M. Jansen [303] has a MSc degree in "Education and Child Studies", and some of her keywords are "mathematics" and "mathematics education" while these do not seem to be covered by her training. Co-author M.F. Fagginger Auer [304] has a background in developmental psychology and a Ph.D. in "methodology and statistics", [305] and its topic appears to be related to the thesis by Hickendorff. My inference is that these authors likely don't have the expertise to really judge that didactics of arithmetic would be relevant. It must be a cheap remark. A symptom is that they did not include such a researcher in their review team.

The subsequent critical question for Hickendorff et al. (2017) would be: who would be the judges on didactics of arithmetic ? If you hire TME then they will reject RME and if you hire RME then they will reject TME. Since the KNAW (2009) word of power there tends to be a new attitude "to take the best of each", without clear criteria what would be "the best", thus with a soup that neglects the discussion before that KNAW (2009) misdirection. Hickendorff et al. do not discuss this moot question who would judge about didactics. Potentially these authors might still think that statistical outcomes would determine which didactics would be "the best" (with some thin air to drop whoever frames the test questions and determines what the proper answers would be).

Overall, Holland has heavily invested in educational degrees such as "Education and Child Studies" and "education management", and Holland suffers a math war, but Holland never got around to set up a decent professorship in the empirical science of didactics of mathematics. KNAW (2009) should have advised to abolish the Freudenthal Head in the Clouds Realistic Mathematics Institute at the University of Utrecht, that pushed RME without proper testing, but the misery continued thanks to the incompetence and arrogance of these psychometricians and child educationalists. [306]

Their claimed result

The Hickendorff et al. (2017) main conclusion is:

> Google Translate 2018: "This means that in the current situation no more than 10 percent of the differences in [arithmetic] performance can be explained by (influenceable and non-influenceable) factors in the educational process." (p95) [307]

They used TIMSS 2015 (Grade 4) and PPON 2011 (Grade 6). [308] We already observed that by 2009 all Dutch textbooks used RME, and thus it should not surprise that these data show less variation in 2011. The TME textbook "Reken Zeker" was started in 2010,

[302] https://www.universiteitleiden.nl/en/staffmembers/terry-mostert#tab-1 and https://www.linkedin.com/in/terry-mostert-617830ba/

[303] https://www.universiteitleiden.nl/en/staffmembers/lisa-jansen#tab-1 and https://www.linkedin.com/in/lisa-jansen-2146ab65/

[304] https://www.linkedin.com/in/marijefaggingerauer/

[305] https://www.narcis.nl/research/RecordID/OND1344773/Language/en

[306] Dutch readers may benefit from criticism by Imelman, Wagenaar and Meijer 2017, http://webwinkel.vangorcum.nl/NL_toonBoek.asp?PublID=5095-0 http://www.beteronderwijsnederland.nl/vakwerk/2018/02/imelman-politiek-pedagogiek/ https://www.beteronderwijsnederland.nl/nieuws/2016/09/in-gesprek-met-prof-dr-imelman/ and also these sources: https://historiek.net/vier-pioniers-van-de-pedagogiek/49844/ https://www.dub.uu.nl/nl/artikel/langeveld-de-tragiek-van-een-befaamd-hoogleraar

[307] Dutch:"Dat betekent dat in de huidige situatie hoogstens 10 procent van de verschillen in rekenprestaties verklaard kan worden door (beïnvloedbare en nietbeïnvloedbare) factoren uit het onderwijsleerproces." (p95)

[308] Scheltens et al. (2013) (with contribution by Hickendorff) "Balans van het rekenwiskundeonderwijs aan het einde van de basisschool 5. Uitkomsten van de vijfde peiling in 2011", PPON-reeks nummer 51, CITO. https://zoek.officielebekendmakingen.nl/blg-219337.pdf

but their students reached Grade 6 only in 2016. Why did Hickendorff et al. (2017) not use my suggestion on using the results of 2016 ? Perhaps though, such would be “original research” and not a “review” study, and if the principal asks for a review then you as an agent might not offer the idea that something better is possible.

Their effect measure is still the outcomes of sums, and they do not explicitly refer to the intended algebra in highschool. Hickendorff et al. (2017) still accept the current tests as valid, though we have seen that they are biased towards RME. These researchers claim to study “education in arithmetic” while in fact they study what RME has created under this false label.

After the KNAW (2009) criticism that adequate studies lacked, the education researchers in Holland and in particular the Freudenthal Head in the Clouds Realistic Mathematics Institute (FHCRMI) in Utrecht in Utrecht did not succeed in setting up an adequate study in 2010-2016, and Hickendorff et al. (2017) still only find Slavin & Lake (208) as the only relevant one. They refer uncritically to the math war in the USA, see our discussion below:

> Google Translate (2018): “The teaching method used is often part of a debate about mathematical education (Slavin & Lake, 2008). [Only one single] review was found of the effects of teaching methods on the [arithmetic] performance of primary school students. Slavin and Lake (2008) concluded on the basis of the median of the effect sizes found that [arithmetic] methods have a negligible to small effect on mathematical performance. Such small positive effects were found for various types of [arithmetic] methods. In general, this review therefore provides little evidence for the proposition that different [arithmetic] methods have different effects on [arithmetic] performance. A comparison with other studies in the review showed that the associated instructional guidance is a more important factor.” (p59) [309]

In their study, TME is called “direct instruction” and RME is called “constructivist instruction”. The didactics are also referred to as “calculation methods”, likely without intending to be denigrating but nevertheless still condescending w.r.t. didactics of mathematics. In Holland, the term “method” is also used for a particular textbook (-series). In the USA the term “curriculum” may be used for a textbook as well. The PPON 2011 study introduces a confusion by using the word “calculation methods” for textbooks too. Its table 9.2 on page 300 compares “calculation methods” but this is erroneous, because this compares textbooks that all use RME. There is no comparison between RME and TME on arithmetic. The conclusion of PPON 2011, that there is hardly difference between the “methods”, should not be seen as a conclusion pertaining to the difference between TME and RME, but only pertains to different RME textbooks. When Hickendort et al. (2017) page 13 & 19 also claim that “calculation methods” hardly differ in results, they might adopt this confusion of PPON 2011 too.

By again excluding Van der Plas (2009) and my criticism, Hickendorff et al. (2017) again manage to conclude that “robust” results would be lacking, while TME has logic on its side:

> Google Translate 2018: “It is striking that there are no robust research results with regard to subject matter or calculation method: neither in the international

[309] Dutch: “De gebruikte lesmethode is vaak onderdeel van debat over het rekenonderwijs (Slavin & Lake, 2008). Naar de effecten van lesmethoden op de rekenprestaties van basisschoolleerlingen is één review gevonden. Slavin en Lake (2008) concludeerden op basis van de mediaan van de gevonden effectgrootten dat rekenmethoden een verwaarloosbaar tot klein effect hebben op rekenprestaties. Dergelijke kleine positieve effecten werden voor diverse soorten rekenmethoden gevonden. Over het algemeen komt uit deze review dus weinig bewijs naar voren voor de stelling dat verschillende rekenmethoden verschillende effecten hebben op rekenprestaties. Uit een vergelijking met andere studies in de review bleek dat de bijbehorende instructiebegeleiding een belangrijkere factor is.” (p59)

> literature nor in the further analyzes of PPON-2011 and TIMSS-2015. Although the importance of these factors is obvious (see also Hiebert & Grouws, 2007; Van Zanten & van den Heuvel-Panhuizen, 2014), it seems difficult to investigate this in a targeted manner. This may be due to the fact that the terms are very broad, the curriculum is strongly related to the legal reference levels and therefore there is little variation in supply because of the used calculation method is related to other school and teacher factors that affect the effects of calculation method can not be determined accurately, or because teachers vary the calculation method use." (p96) [310]

While the authors squeeze in a reference to some didactics, as RME Van Zanten & van den Heuvel-Panhuizen 2014, they still refuse to mention Van der Plas (2009).

In their recommendations for future research they include "calculation methods" – which might mean "didactics of arithmetic" (Dutch "didactiek van de rekenkunde") – but again fail to mention my suggestion to look at the 2016 results on "Reken Zeker".

> Google Translate 2018: "We recommend further research on the following themes: the pedagogical subject knowledge of the teacher, the role of the [arithmetic] coordinator and [arithmetic] policy / vision of the school, and the calculation methods (content and use by teachers)." (p25) [311] (See their p101.)

In 2018, Hickendorff got a "Veni" scholarship for research on pupil arithmetic strategies. [312] I do not have details about her proposal how to study this.

The math war in the USA

Slavin & Lake (2008) is a meta-study that included 87 studies. It is quite possible that these studies do not deal properly with the distinction between TME and RME (and properly re-engineered mathematics education). Slavin currently is director of the Center for Research and Reform at John Hopkins. By training, Slavin is a psychologist and Lake a sociologist. It is not clear to me what their research in didactics of mathematics has been.

S&L p431: "The purpose of this review is to examine the quantitative evidence on elementary mathematics programs to discover how much of a scientific basis there is for competing claims about the effects of various programs. (...) A broad literature search was carried out in an attempt to locate every study that could possibly meet the inclusion requirements." It is important to realise that the USA has still much variety of TME and RME, compared to the dominance of RME in Holland. Thus the USA is better placed to show a difference. My problem is not the use of quantitative methods but the validity of what is measured. For example, KNAW (2009) excluded Van der Plas (2009) perhaps because of lack of statistics but the study was of key importance for validity. We might run into the same problem with the S&L study.

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[310] Dutch:"Opvallend is dat er geen robuuste onderzoeksresultaten zijn met betrekking tot leerstofaanbod of rekenmethode: noch in de internationale literatuur, noch in de nadere analyses van PPON-2011 en TIMSS-2015. Hoewel het belang van deze factoren voor de hand ligt (zie ook Hiebert & Grouws, 2007; Van Zanten & van den Heuvel-Panhuizen, 2014) lijkt het moeilijk deze gericht te onderzoeken. Mogelijk komt dit doordat de begrippen heel breed zijn, het leerstofaanbod sterk samenhangt met de wettelijke referentieniveaus en er daarom weinig variatie in aanbod bestaat, doordat de gebruikte rekenmethode samenhangt met andere school- en leerkrachtfactoren waardoor de effecten van rekenmethode niet zuiver te bepalen zijn, of doordat leerkrachten de rekenmethode verschillend gebruiken." (p96)

[311] Dutch: "Wij bevelen nader onderzoek aan op de volgende thema's: de pedagogisch vakinhoudelijke kennis van de leerkracht, de rol van de rekencoördinator en rekenbeleid/-visie van de school, en de rekenmethoden (inhoud en gebruik door leerkrachten)." (p25)

[312] https://www.universiteitleiden.nl/nieuws/2018/07/veni-beurs-voor-leidse-onderwijswetenschapper-marian-hickendorff

On the other hand, S&L p436 is informative on the math war in the USA. It relates how the NSF funded "reform mathematics" programs but without requiring proper testing: "Yet, experimental control evaluations of these and other curricula that meet the most minimal standards of methodological quality are very few. Only five studies of the NSF programs met the inclusion standards, and all but one of these was a post hoc matched comparison." The post hoc approach suffers the risk of selection bias or censoring, with schools dropping a textbook that doesn't work for them. [313]

S&L show that they are not quite aware of didactics of mathematics and the relation of arithmetic to algebra, when they state (p482): "This is not to say that curriculum is unimportant. There is no point in teaching the wrong mathematics. The research on the NSF supported curricula is at least comforting in showing that reform-oriented curricula are no less effective than traditional curricula on traditional measures, and they may be somewhat more effective, so their contribution to nontraditional outcomes does not detract from traditional ones." Do their studies grade algorithms by steps or do they only look at the outcomes ?

While S&L indicate that RME would give a slightly better median effect size of 0.1, the following indicates that TME could do better with a particular effect size of 0.22.

Namely, my problem now is that Hickendorff et al. (2017) refer to Slavin & Lake (2008) of 9 years earlier. If they had studied the S&L paper more thoroughly, they would have seen that S&L refer to a What Works Clearinghouse 2006 study that wasn't published yet at the time when S&L were writing. In 2017, Hickendorff et al. could have looked. For example, I find this 2013 NCEE Evaluation Brief "After two years, three elementary math curricula outperform a fourth". [314] The outperformed textbook / curriculum is called "*Investigations*" supported by TERC [315] and it is of the RME kind, while the other three are of the TME kind. The Brief p7: "This 0.22 difference (also known as an "effect size") means that a study student at the 50th percentile in math would score 9 percentile points higher as a result of being taught in 1st and 2nd grade with Math Expressions, Saxon, or SFAW/enVision instead of with Investigations." Using a conversion table: [316] with a class of 25 this means 2 more students switching from Fail to Pass. "Even Cohen's 'small' effect of 0.2 would produce an increase from 50% to 58% – a difference that most schools would probably categorise as quite substantial."

NYC Hold is of the TME conviction, and their 2008 review [317] [318] of *Investigations* indicates that the statistical exercise by NCEE / IES was rather superfluous, and needlessly unkind to the pupil guinea pigs, like Ralph Nader testing whether car safety belts really are useful. This only concerned Grade 1 and 2. In itself the 0.22 standard deviation is less than I would expect, but this would also require a look at the Rock & Pollack 2002 ECLS-K test [319] used, getting us further from our present focus on the math war in Holland and getting distracted by the math war in the USA. For due process, let me refer to a remarkably positive EdReport's review [320] of *Investigations* and a reply by the authors on remaining criticism. [321]

[313] S&L p434: "Despite all of these concerns, post hoc studies were reluctantly included in this review for one reason: Without them, there would be no evidence at all concerning most of the commercial textbook series used by the vast majority of elementary schools."
[314] https://ies.ed.gov/ncee/pubs/20134019/pdf/20134019.pdf
[315] https://www.terc.edu
[316] www.leeds.ac.uk/educol/documents/00002182.doc
[317] http://www.nychold.com/terc.html
[318] http://wgquirk.com/TERC.html
[319] https://eric.ed.gov/?id=ED470320
[320] https://www.edreports.org/math/investigations-in-number-data-and-space-3rd-edition-2017/index.html
[321] https://investigations.terc.edu/wp-content/uploads/2017/05/AuthorResponse.pdf?x71805

Conclusions

For this analysis in institutional economics, the causal modeling on didactics and testing on competence in arithmetic and algebra, with a focus on long term memory of pupils, appeared illuminating for understanding the role of formal and informal institutions. Agents in formal institutions on education and its research are most likely influenced by informal institutions that are given by durable ideas and conceptions that do not change easily, in this case on traditional and "realistic" approaches to mathematics education and its research, and on notions what exactly would constitute scientific research and ideas how logic and statistics relate to empirics. The causal modeling provided a framework to understand empirical developments in Holland on mathematics education and its research, also as factors in the overall economy – again see *Elegance with Substance* (2009, 2015).

Psychologists Van Putten & Hickendorff (VPH) and Hickendorff (2011) incorrectly excluded Van der Plas (2009) from their (review) study by confusing empirical science and statistics, while the empirical science of didactics of mathematics would warrant its inclusion. The KNAW (2009) committee had a biased composition without didacticians of arithmetic and algebra and did not correct the error. VPH and KNAW neglect criticism on their conceptual error which is a breach of research integrity. The scientific record must be corrected by removing these "publications" VPH (2009) and KNAW (2009) and Hickendorff (2011) that have been produced with these breaches.

Given that I have no reason to question personal or professional integrity of these psychometricians, my most likely explanation is the ivory tower, in which VPH really adopt these distorted concepts from conventional psychometrics, to insulate themselves from criticism. [322] But this ivory tower or tunnel vision is not science. Science is open minded. It actually doesn't quite matter what confusions VPH have chosen to neglect criticism. Fact is that they breach scientific integrity by selecting their sources and neglecting criticism. That Dutch procedures on research integrity are deficient has not been discussed here.

The KNAW (2009) conclusion that the empirical data in 2009 did not show a difference in effectiveness of TME and RME is false and based upon invalid research and deliberate neglect of information to the contrary. Their position in 2009 can be compared to a position in 1950 that "there is no statistical study that shows that the Moon has another side". With proper tests, that score points for steps in the traditional algorithms in arithmetic, TME should obviously score better than RME that has insufficient training on those algorithms. KNAW (2009) confuses an issue of logic with statistics. Statistics are relevant for effect sizes on particular cases but have limited value for decisions upon principles for curriculum design. Measurements are relevant for student diagnostics which didactics would work for them for particular stages in a curriculum, and such measurements might also be used for statistical reporting, but one should not confuse the purpose of this exercise for something else. Diagnosing students is something else than the KNAW (2009) exercise of trying to stop the social nuisance of a math war between ideologues who misrepresent propaganda as scientific research.

The Freudenthal Head in the Clouds Realistic Mathematics Institute (FHCRMI) at Utrecht University should be abolished as unscientific and comparable to astrology, alchemy or homeopathy. The RME section there has teamed up since 2009 with the STEM researchers so that there is more body to empirical research in education, but this remains a cover up of the unscientific RME core. After being warned by KNAW (2009) they still did not manage in 2010-2016 to set up a distinguishing experiment, as

[322] Maltreat $s = f[d, o]$. Science = statistics. Take the effect measure as outcome only and neglect steps. Arithmetic in elementary school can neglect future algebra. Expertise is a flexible concept.

Hickendorff et al. (2017) observes. Holland better sets up a Simon Stevin Institute for mathematics education and its research.

There remains the statistical question of the unknown effect size of TME over RME in a PPON registration. This likely can be found by looking at the Dutch PPON 2016, and going back to the school archives to recover the data on SES and other variables for the 20 schools that adopted the textbook “Reken Zeker” in 2010, and a control group of normal (RME) students. It must be regretted that this suggestion by Colignatus (2015c) for PPON 2016 was not adopted in time. The VPH neglect of criticism was a factor in the neglect of that suggestion.

Conclusions

The Prologue stated that education is a mess, referring to *Elegance with Substance* (EWS) and *Conquest of the Plane* (COTP) as the evidence. This present look at primary education does not invalidate a similar impression. I am not qualified to judge in this particular field but offer the following conclusions as prospective.

The mathematical structure of arithmetic and geometry is fine, and computing devices and computer algebra programmes are wonders of technical advancement, but something goes seriously wrong between mathematical abstraction on one hand and educational empirics on the other hand.

- Number sense and understanding are hindered and obstructed by taking the English pronunciation of numbers as the norm, while English is a historically grown and clearly confusing dialect of mathematics. Counting with fingers blocks at 10, while it is easy to construct a sign language with place values too. In fractions there is abuse of rank order names and an awkward switch in plus / times, compare 2½ with 2 + ½, while fractions might also be abolished with $2 + 2^H$. Subtraction doesn't use the decimal positional system to its full potential yet, and, by not doing so, creates confusion about it, while enlarging the fear for negative numbers. The latter represent cancellation or making turns, but K1-6 is stuck in “there are no negative apples”.
- Algebraic sense and competence rely upon arithmetic, and thus are hindered and obstructed when arithmetic isn't developed well. Compare current 2½ × 3¼ with proper $(2 + 2^H) \times (3 + 4^H)$.
- Spatial sense and understanding are hindered and obstructed by the absence of the missing link of *named lines,* by not discussing vectors and the Pythagorean Theorem, and by adhering to the Sumerian 360 degrees instead of taking the plane as the unit itself.
- Logical sense and competence in reasoning are hindered and obstructed by above confusions and cumbersomeness, by the witholding of logic and set theory till middle school or later, and by not explicitly developing the notion of proof.

This evidence does not contradict the earlier conclusions of *Elegance with Substance*. To repeat those:

What is seen as mathematics appears to be illogical and/or undidactic. Hence it has to be redesigned. It is no use to improve on the didactics of bad material, it better is replaced. We also considered only a number of topics, a selection of ideas that this author found interesting to develop a bit. More can be found. We should allow for the possibility that teachers have more comments and suggestions themselves (though our critique is that either they don’t have them or don’t follow up on them). The situation is wanting.

This book looks at the result rather than at how this situation could have come about. Still, if the result is inadequate, the conclusion is warranted that some cause is wrong.

One of the most important human characteristics is the preference for what is known and familiar – and mathematicians are only human. They adapt to some degree to new developments and are critical and self-critical, not only with respect to what is discussed but also on how things will change. Nevertheless, key issues got stuck, and the industry as a whole is incapable of freeing itself from grown patterns. New entrants in the industry are conditioned to the blind spots, and pupils and students suffer from them.

The situation is not such that there are no mathematicians to improve on content and that we lack researchers in didactics to improve on that angle. This book will hopefully be read by some in both groups and contribute to improvements. But it would be wrong for governments to think that it would suffice to leave the matter to the industry, and possibly give more subsidies for more of the same. More funds may well mean more outgrowth of awkwardness, cumbersomeness, irrationality. A call for more teaching hours may well mean more hours to mentally torture the students even more. Given this whole industry and the inadequate result the conclusion is rather that the whole industry is to be tackled.

Indeed, it sounds so well. Mathematicians will hold that only they are capable of deciding what is ‘mathematics’. Researchers in the education of ‘mathematics’ will hold that they do the research and nobody else. Will they regard this book as ‘research in the education in mathematics’ ? *Quis custodet custodes ?* It will be a mis-judgement to provide the industry with more funds without serious reorganisation.

In sum, we have considered the work of men and found them to be men. It is a joy to see all these issues that can be improved upon. Let us hope that the field proceeds in this direction indeed. Let economists and the other professions support them.

2015: In Holland the State Secretary on Education Sander Dekker has observed that arithmetic skills are below requirements. He avoids a diagnosis on the Freudenthal "Realistic Mathematics Education" (RME) and thus he doesn't require a reschooling of the 140,000 elementary school teachers. Instead he shifts the burden to the 12,000 teachers of mathematics in secondary education, by requiring an additional arithmetic test for highschool graduation. Apparently he is not aware that creation of arithmetic competence in elementary school is required for later algebraic competence in secondary education. I am sorry to report that there is a breach in the integrity of science in the mathematics education research, so that Mr. Dekker does not get scientifically warranted information. At KNAW there are some abstract thinking mathematicians who think that they know more about mathematics education than empirical scientists, and they don't care about the evidence to the contrary. [323]

2018: This 2nd edition extends on both content and research integrity. Check out the weblog or the book website for developments. [324] Dekker’s successor has a new plan but I haven’t had time to look into this.

Final conclusion

My final conclusion definitely applies to Holland. I tend not to judge about other countries. But the same cumbersome and illogical issues can also be seen internationally. There is a structure to it. It is part of the economics of regulation. Didactics require a mindset sensitive to empirical observation which is not what mathematicians are trained for. Tradition and culture condition mathematicians to see what they are conditioned to see. The industy cannot handle its responsibility. This must hold internationally, country by country. A parliamentary enquiry is advisable, country by country.

Parents are advised to write their representative – and not only those parents who pay for extra private lessons. The professional associations of mathematics and economics are advised to write their parliament in support of that enquiry.

Here ends what I repeated from *Elegance with Substance*.

[323] https://boycottholland.wordpress.com/2014/07/16/integrity-of-science-in-dutch-research-in-didactics-of-mathematics/ [2018: update on figures]
[324] EWS http://thomascool.eu/Papers/Math/Index.html
and CWNN http://thomascool.eu/Papers/NiceNumbers/Index.html

Appendix A. What is new in this book ?

It is generally useful to specify what is new in a book. This overview may repeat some points from my own work that I already presented elsewhere.

(1) Identification of mathematics as the proper language with English as a dialect, so that there is proper perspective and focus on didactics and learning goals (see p17).

(2) Greater awareness that the positional system is under-utilised for its support in counting and arithmetic. Proper use might allow multiplication in First Grade. (This fits Gladwell's comment that "the necessary equation is right there, embedded in the sentence"; but now looks systematically how the positional system can be employed to support education.) (See p18.) Further developed in (2018b).

(3) The latter also concerns the use of the positional system for subtraction (see p97). Further developed in (2018cd).

(4) Pronounciation of numbers with ten, like 19 = *ten & nine* and 23 = *two·ten & three.* (From Gladwell (2008) and Cantonese, but with *ten* instead of *tens* and middle dot instead of hyphen, and for Dutch "tig" instead of "tien".) (See also **Appendix B**.)

(5) The article *"Marcus learns counting and arithmetic with ten"* (p35) that combines these ideas in a draft lesson plan: (a) pronunciation, (b) calling the dialect for what it is, (c) sums that relate to the positional system, (d) tables of addition and multiplication, (e) powers. The tables are further developed in (2018d).

(6) Abolition of fractions by using $x^H = 1 / x$, pronounced as "per *x*", with $H = -1$ the Harremoës operator, pronounced as "eta" (see p98). (Van Hiele (1973) already proposed abolition, and Harremoës (2000) has a symbol for –1 (and much more). New is the choice of pronunciation of "per *x*" instead of the abuse of rank order names (like *"a fifth"*), and of suitable *H* that somewhat looks like "-1" and that gives a half turn on a circle in the complex plane.) See (2018c) too.

(7) Suggestions for gestures or signs that satisfy the positional system, for base 10 (p83) and base 6 (p215). Design principles that fit elementary school.

(8) Identification of the difference in the approaches by Van Hiele (right) and Freudenthal (erroneous), with the distinction between handling abstraction (Van Hiele) and applied mathematical modeling (Freudenthal). Identification of the *missing link* in the standard approaches to geometry: the *named lines.* (This is a copy of §15.2 from *Conquest of the Plane*.) (See p135.)

(9) (a) Identification of Killian's (2006) (2012) treatment of the Pythagorean Theorem in elementary school as a *key supporting step* for Pierre van Hiele's suggestion that vectors can already be presented in elementary school. (b) Presentation of this argument, by sandwiching the topics: (i) presentation of co-ordinates and vectors, (ii) derivation of the theorem, (iii) using the theorem to calculate the lengths of vectors. (c) Demonstration that the presentation of the theorem perfectly fits the Van Hiele didactic approach, with the Van Hiele levels *concrete, sorting, analysis.* (See p113.)

(10) Observation that an earlier analysis on angles and trigonometry is also highly relevant for primary education (see p123).

(11) Explicitation of the relevance for the USA *Common Core* programme, for the implementation given in California (see p141 and p145).

(12) Clarification that proper re-engineering of mathematics education gets bogged down by the math war between the ideologies of traditional TME and “realistic” RME, supported by the ivory tower of the psychometricians who claim expertise on testing but who lack expertise on didactics of mathematics, p145 and following.

Comment w.r.t. Barrow (1993) "Pi in the sky"

I had read Barrow (1993) somewhere before 2009 and referred to him in EWS 2009. Apparently I forgot some details, and in 2012 around my son *M's* 6th birthday it was Gladwell (2008:228) who set me thinking about the number system, as explained above on page 17. Following the hint by Gladwell I wrote the text on *Marcus learns counting and arithmetic with ten,* i.e. fully writing out all pronunciations in the format *two·ten & one* (in 2018 with the ampersand). In 2015 I just completed a 2nd edition of EWS 2015 as well, and this got me trying to remember what Barrow (1993) had to say about these number issues. His book has the subtitle *Counting, thinking and being,* and I did recall that he discussed the history of number systems. Thus, I decided to reread his book.

To my surprise I find a core of my suggestions just stated by Barrow. On his page 35:

> "This method of counting is called the '2-system'. One should compare it to that which we use today which is founded upon the base 10, so that we have distinct words for numbers up to and including ten and then we compose ten-one (which we term 'eleven'), ten-two (twelve), ten-three (thirteen), ten-four (fourteen), up to ten-nine (nineteen) and ten-ten (twenty), before continuing with ten-ten-one which we cal 'twenty-one'."

Barrow indeed uses *ten* rather Galwell's *tens.* Distinctions with my suggestion are:

- to use middle dots rather than hyphens
- to use two·ten & one for 21 and ten·ten for 100 (but Barrow p68 may intend this too).

Barrow also suggests that *eleven* and *twelve* derive from *one left over* and *two left over,* once you have counted to ten. *Twenty* would not derive from two·ten but from *twin of tens.*

Barrow's page 58 mentions a system that uses two hands, structurally the same as my paper *Numbers in base six in First Grade ?* in **Appendix C**, except that I propose to use base six while Barrow describes base five. Observe that base six is better to learn and handle the positional shift. There is also a paradox: The Bombay system works left to right (in Western style) while my proposed system works right to left (as the numbers are supposed to come from India) ... But India is not uniform in its orientation.

> "Elsewhere in India, amongst some traders in the Bombay region, there are still traces of an early base-5 method of counting which uses finger counting in a novel and powerful fashion, enabling much larger number to be dealt with without taxing the memory unduly. The left hand is used in the normal way counting off the fingers from 1 to five, starting with the thumb. But when five is reached this is recorded by raising the thumb of the right hand whereupon counting to the next five begins again with the left hand until ten is reached, then the next finger of the right hand is raised, and so on. This system enables the finger-counter to count to thirty very easily, so that even if he is interrupted or distracted he can determine at a glance where the count has reached."

It is useful that Barrow agrees that the method is powerful. Base 5 might link up easier with base 10 later on. A decision on this however is quickly resolved by the signs in base 10, see page 83 above.

Appendix B. Number sense and sensical numbers

2015 with ampersand 2018

The discussion below was the section on *Number Sense* in EWS 2009. It caused the creation of the draft booklet *A child wants nice and not mean numbers* (2012) – now replaced by this whole book (*not* becoming *no*). The first point is that English is a dialect of mathematics. There is a coherent way to pronounce numbers, to start with in elementary school. The second point is that the positional system is underutilised, and that its proper use would allow a great improvement in arithmetic. Algebra in highschool depends upon arithmetic skills learned in primary education.

Other news is: (i) For negative numbers and subtraction, see p 94. (ii) Later on, I realised that fractions abuse the rank order names: e.g. rank order *fifth* is abused for *a fifth.* There is now the proposal to use $1 / x = x^H$, and pronounce this as "per-*x*". (iii) For an overview of pronunciation, addition, subtraction, multiplication, division, see p 90. The following can be retained as a rough introduction into all of this. It has been edited to fit this book.

Brain, language, sounds and pictures

There is already some remarkable evidence based education (EBE) on arithmetic. Gladwell (2008:228):

> "(…) we store digits in a memory loop that runs for about two seconds."

English numbers are cumbersome to store. Gladwell quotes Stanislas Dehaene:

> "(…) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits."

[PM. Apparently fractions in Chinese are clearer too. Instead of *two-fifths* it would use *two-out of-five*. First creating *fifths* indeed is an additional operation. Perhaps the West is too prim on the distinction between the ratio 2:5 and the number 2/5. Perhaps it does really not make a difference except in terms of pure theory – the verb of considering the ratio and the noun of the result (called "number" when primly formalized in number theory). But fractions are not the topic of present discussion.]

Gladwell on addition:

> "Ask an English-speaking seven-year-old to add *thirty-seven* plus *twenty-two* in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add *three·tens-seven* and *two·tens-two*, and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It's *five·tens-nine*." (Hyphen edited.)

I am not quite convinced by the latter. *Thirty-seven* can be quickly translated into *three·ten & seven,* and *twenty-two* in *two·ten & two*. (Use position *ten* rather than quantity *tens.*) The "thir" and "ty" are liguistic reductions of "three" and "ten". There is no need to create the digital image of the numbers. I can imagine two tracks: pupils who learn to mentally code *thirty* (sound, and mental code too) as *three·ten* (brain meaning) and pupils who follow the longer route via the digits. That said, the Western way is a bit more complicated.

The problem has a quick fix: Use the Cantonese system and sounds for numbers. It would be good EBE to determine whether this would be feasible for an English speaking environment (for starters, perhaps begin in Hong Kong).

Writing left to right, speaking right to left

A deeper issue is that the West reads and writes text from the left to the right while Indian-Arabic or rather Indian numbers are from the right to the left. Thus *fourteen* is 14.

English already adapted a bit, with *twenty-one* and 21. Dutch still has *een-en-twintig* up to *negen-en-negentig*. From hundreds onwards Dutch follows the Indian too, for example *vijf-honderd-een-en-twintig* (521). French of course still has the special *quatre-vingt* for 80 and *quatre-vingt-treize* (80 + 13) for 93.

Sounds and pictures

There is a bit more to it, though. In Gladwell's case the pupils apparently are given a sum via verbal communication. This differs from a written test question. There are two ways to consider a number. 37 can be seen as a series of digits only and pronounced as *three·seven* – like specifying a telephone number – or it can be weighed as *thirty-seven* or *three·ten & seven*. We have to distinguish math from the human mind.

(a) For the mathematical algorithm of addition mentioning only the digits suffices since the order already carries the weights. The mathematically neat way starts with the singles, as indeed Gladwell first mentions 2 plus 7 is 9.

(b) But a human mind tends to have different priorities. It is interested in size. The mind tends to use the weights and to focus on the most important digit. Witness *"nine thousand & four hundred & twenty-six"*. In a written question this tendency is easier to suppress. In a verbal question the tendency is stimulated. Depending upon the circumstances there can be more focus on the size. (The actual algorithm / heuristic that a pupil uses can actually be anything, like first adding up the place values at thousand, then at hundred, ten, one, and then resolve the overflow. The Asian child might indeed start with *three plus two is five*.)

Counting in traditional / verbal manner follows the second approach, and uses the infixes ten, hundred, thousand, ten thousand etcetera to indicate the place and the unit of account. The weight infixes are intended for communicating size and would be redundant for merely transmitting the number, though redundancy can help for checking.

Expressions with weights still can be ambiguous though. With 100 million = 100 *times* 10^6 as the format, it follows that 23 pronounced as *twenty-three* might be understood as 20 *times* 3 giving 60. Dutch has prim *drie-en-twintig* thus with the *plus.* A proper use of weights should fully specify the sum $d \times 1000 + c \times 100 + b \times 10 + a$ for number *dcba.*

Eye, ear, mouth & hand co-ordination

There are two key properties of the Indian order with a Western text direction:

- The mental advantage is that the most important digit is mentioned first.
- The disadvantage is that addition and multiplication work in the opposite direction from reading. It goes against the (over-) flow. For example 17 + 36 = 53 has overflow 7 + 6 = 13 and this has to be processed from the right to the left.

The requirement on eye, ear, mouth & hand co-ordination again shows the importance of Kindergarten – see the work by economist Heckman, e.g. his Tinbergen Lecture, who confirms what Kindergarten teachers have been telling since ages.

History of the decimal system and the zero

Note that the West often speaks about Arabic digits but according to Van der Waerden (1975:58) the Arabs speak about Indian digits so we better follow them:

> "Our digits derive from the Gobar digits which were used in Moorish Spain. The East-Arabic digits are still in use to-day in Turkey, Arabia and Egypt; they are called "Indian digits". It is clear that both were derived from the Brahmi-digits."

A short excursion on the history of the decimal system and zero is useful. Barrow (1993:85) mentions that the Babylonians of 300-200 BC already had a symbol to indicate a blank spot. Possibly Freudenthal 1946 was the first to recover the most likely story on what happened next. It can be observed that Ptolemy in 150 AD wrote whole numbers with Roman numerals but fractions sexagesimally following the Babylonians – and in this positional system he wrote "o" for "ουδεν" ("nothing") when a position was blank. Apparently the Indians became familiar with Greek astronomy from 200 AD onwards. The Indians already had a decimal positional system though of some complexity. They used rhymes and verses to remember long numerical tables, but blank places apparently broke the rhythm and it would have come as an idea that those places could be filled with sounds too. Van der Waerden (1975:57) summarizes:

> "Along with Greek astronomy, the Hindus became acquainted with the sexagesimal system and the zero. They amalgamated this positional system with their own; to their own Brahmin digits 1 – 9, they adjoined the Greek o and they adopted the Greek-Babylonian order. It is quite possible that things went in this way. This detracts in no way from the honor due to the Hindus; it is they who developed the most perfect notation for numbers, known to us."

Clearly, when the zero arrived in Europe again via the Moors in Spain, it helped that astronomers were already used to it. The impact however came from the package deal with the decimal notation in general, that appeared very useful in commerce.

Interestingly, with respect to our discussion of the order of the digits, the Indian system originally had the order from low to high but switched due to the influence of the Greek-Babylonian order. Van der Waerden (1975:55):

> "Bhaskara I, a pupil of Aryabhata, introduced an improved system, which is positional and has zero; it has the further advantage of leaving the poet greater freedom in the choice of syllables and thus enabling him better to meet metrical requirements. According to Datta and Singh, this Bhaskara lived around 520. Like Aryabhata, he begins with the units, followed by the tens, etc., (…) The first to reverse the order (as far as we know) was Jinabhadra Gani, who lived about 537, according to Datta and Singh."

Thus the writing order of Indian numerals may have little to do with the writing order of the Arabic language but rather with the writing order of old Sumer numerals.

[PM. Van der Waerden observes that Sumerian and Chinese results on the Pythagorean Theorem are too similar to be parallel inventions and hence concludes that there must be some common ancestor civilization point where the original invention had been made. We may wonder whether such a point would have to be a very developed civilization. Possibly the basic choice would be to construct houses in rectangular instead of circular form, which is much simpler than what Van der Waerden discusses on celestial events.]

Scope for redesign

The reader might as well skip this subsection on the scope for redesign, since the conclusion will be that we will not quickly change the Indian digits and number order. But some diehards might press on, and it might be relevant for developing more didactics for kids who have problems in co-ordinating eye, ear, mouth and hand.

Overflow

The overflow problem is a bit awkward. It would be interesting – when we are considering changing to Cantonese – to see whether it can be solved at the same time. Thus, can we write numbers in the opposite way ? Let us use the word ***Novel*** when we write <123> for the Indian number 321 (and try not to get confused). To distinguish the Novel from the Indian it will be most useful to write the digits in mirror image (perhaps as they are intended to be read if you change the reading order). Thus 19 becomes 91. It does not take much time to get used to and **Table 15** contains the first practice.

Table 15. Novel versus Indian notation and addition

4321	1234
765	567
98	89
0981	1890

Overflow in Novel is processed neatly in the reading direction. This is straightforward.

Thus, to repeat: the mathematical algorithms for addition and multiplication basically work on the digits and not on how the whole numbers are pronounced. When we work silently on paper, or only pronounce the digits in stated order (with text from left to right) without pronouncing the whole number, and compare Indian and Novel:

- Addition in Indian 17 + 36 = 53 works with the digits as "*one·seven* plus *three·six* gives *five·three*". The order of the digits conflicts with handling overflow.
- Addition in Novel works with digits as "*neves·eno* plus *xis·eerth* is *eerth·evif*". Or *"<seven·one>* plus *<six·three>* is *<three·five>"*. The order of the digits supports handling overflow.

The problem is pronunciation of the whole number

Let us now pronounce the whole number. Something strange happens: the need to size up the number appears to interfere *always* with the reading and writing order.

Consider Indian 5,310,000. The eye traverses first from the left to the right to determine how many digits there are. The pupil deduces that 7 digits are millions, then either calls out the number from memory or the eye goes back, from the right to the left to the beginning, and then the pupil reads it off. Possibly there are parallel processes, as the eye picks out words rather than letters. What remains though is that to say *"five million & three hundred & ten thousand"* is not exactly following the reading order since there is a jump somewhere. The *Jump* is unavoidable since the number of digits has to be counted. As the mind focusses on the most important digit, the speaking order will reflect the order in the mind – which is independent of the reading order.

Thus, where we had the distinction between the *mathematical algorithm* and the *human mind* we now see a parallel distinction between *reading order* and *order of pronunciation*. The tricky issue appears to be pronunciation of the whole number. Digit-wise pronunciation, provided that convention is in place, either Indian or Novel, is feasible. Pronunciation only causes problems when a number is communicated (verbally) with weights. Even a written question may carry this problem if the number is not merely processed in an algorithm but subvocalised. Subvocalisation tends to happen as part of the process of understanding, when the mind wonders what the number means. (The algorithm implicitly assigns weights when the working order implies how overflow is handled. The problem remains in pronunciation: this repeatedly burdens memory with (redundant) information about the weight of the digits.)

The true question is how we would pronounce these Novel numbers as a whole and how pronunciation with size interferes with the neat algorithms. If we follow the Novel reading and writing order (i.e. numbers just like text from left to right), then our mind still wants to pronounce a number starting with the most important digit. In that case the speaking order is opposite to the reading order again.

Table 16 gives the four options: writing Indian / Novel and pronoucing leftward / rightward. The current situation in the West is that the number is written Indian, and spoken differently depending upon size. The cell "India / Arabia" is hypothetical and not relevant since it loses the advantage of pronoucing the most important digit first, without any benefit. Let us consider the two options for Novel.

Table 16. Writing and speaking order of numbers

		Writing order of numbers	
		Left to right - Novel	*Right to left - Indian*
Speaking order of numbers	*Left to right*	Novel-L	West (larger sizes)
	Right to left	Novel-R	India / Arabia English (13-19)

Pronouncing Novel from the left to the right (Novel-L)

In this case 91 is pronounced *nine & one·ten* or simplified *nine & ten*. We stick to the text direction and the linguistic translation of numbers essentially mentions the digits as they appear, while adding the weight. This approach has the drawback that the largest value appears at the end.

There are some epi-phenomena here. People may have a tendency to drop infixes and this may cause ambiguity. <*One·two·three·hundred*> that drops the *ten* might perhaps also be understood as <*one·two·three*> <*hundred*>, which then would be 32100. One option is to first mention the base, as in “million 5.31”.

Pronouncing Novel from the right to the left (Novel-R)

The other possibility is to write 000,013,5 and still say *“five million & three hundred & ten thousand”,* i.e. temporarily speaking (and thus reading) from right to left. (The pronunciation order changes because the number writing order has reversed.) This would combine the Novel notation (so that addition and multiplication follow *text* reading order (reading without speaking)) with starting pronunciation with the biggest position. There would be a small added advantage in that you first count the digits and then have the option to say *“about 5 million”* if that is adequate, without resorting to reading it wholly in reverse direction. Writing from dictation would be more involved, requiring the dictator to either start with the lowest digit or stating the number of places in advance. It seems like a do-able system. Thus: *pronounce the same, write in mirror script.*

Conclusion

We will not quickly drop the Indian digits and number order. But EBE on these aspects will help. The need to size up the number for speaking conflicts with any number order.

Appendix C. Numbers in base six in First Grade ?

2012 with ampersand 2018

Introduction

A sense of number is natural to many mammals and at least humans, see Piazza & Dehaene (2004). We teach children to use their fingers to count to 10. Milikowski (2010): “Kaufmann concludes: a brain doing arithmetic needs the fingers for a long while for support. They apparently help to build a bridge from the concrete to the abstract. In other words: the use of the fingers helps the brain to learn the meaning of the digits.”

There is one *however.* We can wonder: might this not be misplaced concreteness ? Are we perhaps distracted by those ten fingers while mathematical insight can lead to a much better approach ? Might finger counting to 10 not be an archaic simplism without didactic foundation ?

This question causes these subquestions:

(1) Might counting to 10 not be too complex an introduction and might counting with the base of 5 or 6 not be sufficient to achieve insight in the meaning of number and the structure in arithmetic ?

(2) When you use the right hand for units and the left hand to count the number of right hands then don’t you count from 0 to 5 again on the left hand ? Doesn't this satisfy the educational use of the fingers ? And doesn't this mean that we achieve a higher level of abstraction at the same time, since counting hands actually is multiplication ?

(3) Doesn't the complexity of using 10 show from the fact that we use artificial means for the numbers higher than 10, since there are no more fingers, with the well-known difficulty of the positional shift ? Isn't the positional shift easier to grasp when using two hands ?

(4) Isn't the use of the decimal system based upon a misunderstanding, and actually wrong, since with ten fingers we actually should use a system with base 11 (the undecimal system) ?

In a system with base six with two hands, the right hand for the units 0 to 6 and the left hand for the number of right hands, in the order of the Indian-Arabian positional system, we still use the fingers with their great educational value, and (a) we use a limited number of symbols with short calculations for the positional shift, (b) we still have the richness of 36 for serious work, and (c) we use the positional system so that we can achieve elementary insight in the structure of numbers and arithmetic, including multiplication. When we have this foundation in First Grade and lower then the later change to the decimal system seems a repetition of moves, relatively simple and enlightening. If there is insight in the basics of arithmetic then this could make it easier to change to the decimal system with its larger numbers. Perhaps there would be an overall improvement, on balance.

The issue remains tentative because it has not been researched. Few parents will submit their children to such experiments. But we can make the proposition as attractive as possible. The following is targeted at designing the best senary system that could be subjected to research.

A consideration is that students learning to become teachers at elementary school might use the following system to re-experience themselves what steps pupils must learn. This should cause for greater awareness of those steps. This re-experience is best done in a suitably developed system.

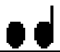
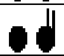
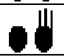
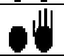
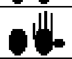
New symbols and names

The senary system is not new, see wikipedia (2012). The examples show that using the same digits in a double role can be confusing. Thus we pick new symbols. We can use the same names 'zero' to 'five' as long as we use systematic pronunciation above six.

The symbols and the first numbers

See Table 1 for the symbols (digits). The number of straight sides in the symbol gives the number (numerical value).

Table 1. Symbols and the first six numbers

Hands	*Symbol*	*Pronunciation*	*Decimal*
	ø	zero	0
	I	one	1
	Γ	two	2
	∆	three	3
	□	four	4
	⌂	five	5

(i) I also considered Λ (capital labda) for 2 but for dyslexia this is too similar to Δ (capital delta). V is already Roman 5. And > (larger than) better is reserved even though it may not be used at this level.

(ii) For arithmetic it is easier to look at your palm and check how the thumb holds down other fingers.

The question "How many fingers do you see ?" starts requiring a distinction between left and right. We might consider gloves or thimbles to indicate the different kinds of counting.

But the simple solution is: Counting the fingers on the back of the hand (with the thumbs in the middle) we use the decimal system, and, counting the fingers on the palm of the hand (with the thumbs sticking out) we use the senary system. [325]

(iii) One advantage of a new system is that we can choose the names systematically. The present decimal system has been stamped by tradition and we write 19 (from left to right, 'ten & nine') but pronounce 'nineteen' (from right to left). It is tempting to write and pronounce the new senary digits from the left to the right. However, the change to the present decimal system later on would become confusing. Hence we maintain the Indian-Arabian numerical order from right to left. Since we are interested in the size of the number we also adopt the order of pronunciation from left to right.

[325] This Appendix C thus opposes page 86. It is natural to start with palms up.

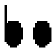
From five to hand

How do we go from five to six ? The best choice is that | on the left hand means, and is called, '(one) (right) hand'. Five fingers mean five when the fingers are spread out while there is one hand when all fingers are held together, see Table 2. The positional shift arises by the equality of a whole right hand with a single finger on the left, see Table 3.

Table 2. From five to hand (value six)

Five fingers, value five	*One (right) hand, value six*
	

Table 3. Positional shift

Equality
 = 

(i) It was an important step in this design to use 'hand' indeed instead of 'six', to signify the new unit of account. Six undoubtedly has a higher level of abstraction but in this phase we intend to support the process towards this abstraction. As a unit of account 'right hand' is too long but the use of 'hand' of course can be supported with the explanation that we are counting right hands.

(ii) It is a research question whether it is better to use base 5, where a right hand with five fingers is equal to a single left finger, and can be replaced by it. The positional shift might be coded as that five right fingers are equal to one left finger. However, this first advantage turns into a later disadvantage. Thinking continually in terms of equality and replacement will slow down the process of counting.

The positional shift can also be given form by letting five fingers be followed by a single finger on the left. In that manner the focus remains on the fingers. When asked what that single finger on the left stands for, one can say: a whole right hand but with fingers closed. The use of language requires accuracy: a hand has five fingers but as a whole it has the value of six fingers – and that whole is represented by holding the fingers together.

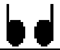
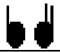
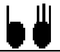
(iii) Advantages of the senary system are the number of divisors and the link to the hours of the day and the number of months. One can imagine adapted clock-faces.

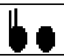
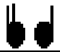
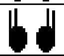
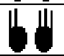
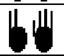
Continued counting

See Table 4 for continued counting from six. Between the words we insert a middle dot that we do not pronounce for multiplication and an ampersand for addition. A dot is better than a hyphen since that can be confusing with minus.

(i) It is a research question whether we can also give the names of the decimal digits and numbers. Thus next to 'hand·one' also 'seven' and up to twelve. It might be confusing to have more names for a number but it might also help pupils to understand the strange words that their parents are speaking. Since children are apt at learning languages these other words need not be confusing, at least when the numbers maintain a system.

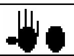
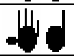
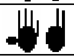
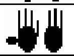
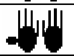
Table 4. Combinations in hand and two·hand

Hands	*Symbol*	*Pronunciation*	*Decimal*
	I ø	hand	6
	I I	hand & one	7
	I Γ	hand & two	8
	I Δ	hand & three	9
	I □	hand & four	10
	I ⌂	hand & five	11

Hands	*Symbol*	*Pronunciation*	*Decimal*
	Γ ø	two·hand	12
	Γ I	two·hand & one	13
	Γ Γ	two·hand & two	14
	Γ Δ	two·hand & three	15
	Γ □	two·hand & four	16
	Γ ⌂	two·hand & five	17

See Table 5 for how the two hands are exhausted at 36 so that we continue counting in Table 8 with lux = hand·hand (from the luxury of a third hand).

Table 5. Combinations in five·hand

Hands	*Symbol*	*Pronunciation*	*Decimal*
	⌂ ø	five·hand	30
	⌂ I	five·hand & one	31
	⌂ Γ	five·hand & two	32
	⌂ Δ	five·hand & three	33
	⌂□	five·hand & four	34
	⌂⌂	five·hand & five	35

Positional shift at hand·hand

The positional shift at hand·hand is a repetition of the positional shift at hand, but it is useful to be explicit about this, so that pupils can verify that it is a repetition. Table 6 gives the step to hand·hand, but the use of a new sign and name as in Table 7 will allow us to count on. The sign for lux assumes a second pupil who uses the fingers of the right hand to start counting in that place value. Counting onwards gives Table 8.

Table 6. From five·hand·five to hand·hand = lux

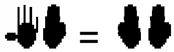
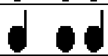
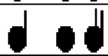
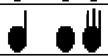
Five·hand & five	*Five·hand + hand = Hand·hand = lux*
	

Table 7. Positional shift

Equality


Table 8. Combinations in lux = hand·hand

Hands	*Symbol*	*Pronunciation*	*Decimal*
	I ø ø	lux	36
	I ø I	lux & one	37
	I ø Γ	lux & two	38
	I ø Δ	lux & three	39
	I ø □	lux & four	40
	I ø ⌂	lux & five	41

Plus and times

The number of possible additions with result ⌂ is limited (I + □ and Γ + Δ, and both in reverse) so that there will quickly be a positional shift, that can be calculated easily as well. Table 9 shows a calculation in columns, that also might be done in larger jumps: Δ + Δ = Δ + Γ + I = I + ⌂ = Iø.

Table 9. Addition in columns

Number	Δ	□	⌂	Iø
Plus	Δ	Γ	I	ø
Is				Iø

The advantage of having both few symbols and numbers till 36 means that we can consider the introduction of multiplication. For these pupils it seems better to speak about ‘times’ and ‘to time’ rather than the long terms ‘multiplication’ and ‘multiply’ (multi-plus).

In Holland, First Grade is limited to addition and subtraction with the numbers to 20 – a bit comparable to the US *Common Core*. This will be related to the positional shift, the illogical pronunciation of the numbers (‘nineteen’ instead of ‘ten & nine’ and ‘twenty’ instead of ‘two·ten’), and the fact that multiplication may quickly give such awkward numbers. When we take a fresh look at the issue then we may agree that learning the numbers to 20 does not have a priority in itself. In the senary system we can count till 36 and this seems doable and clear. Unless research would show that First Grade can only grasp number size but not multiplication.

The curious point is:

> When pupils in First Grade can master above senary system then this itself shows that they can master elementary multiplication. Counting hands namely is multiplication by six. Can they multiply different numbers ?

Standard is Γ + Γ + Γ = Δ × Γ = ıø. There is also Δ + Δ = Γ × Δ = ıø. The discussion of a rectangle and its surface shows that *times* is commutative. Thus, the order of *times* does not matter.

When there are five cats with each two eyes then there are Γ + Γ + Γ + Γ + Γ = ⌂ × Γ = ı□ or *hand & four* eyes in total. With five cats you have five left eyes and five right eyes, thus ⌂ + ⌂ = Γ × ⌂ = ı□. Many pupils of age six could learn this. Would there be a sufficient number of them to introduce the approach in the general curriculum ?

Counting the number of Γs is a higher level of abstraction (the levels identified by Pierre van Hiele). Counting is the ticking-off of the elements of a set. It is a higher level of abstraction to group elements, see a set as a new unit of account, and then tick off the sets.

The following is an important insight with respect to *times*:

> A result like 5 × 2 = 10 is trivial for us but only since we learned this by heart.

Some authors argue that pupils need not learn the *table of times* by heart but must first feel their way. This runs against logic. If you don’t learn the table of *times* by heart then you remain caught in the world of addition. This is very slow and does not contribute to understanding. Remember what *times* is:

(1) Taking a set of sets
(2) To know how you can count single elements but that it is faster to only count the border totals
(3) To know which table to use to look it up (namely × instead of +)
(4) And get your result faster because you know the table by heart
(5) To know all of this.

A calculation like ıΓ × ıı = ıΔΓ contains operations that seem doable at this level. Table 10 uses those higher numbers to make the issue nontrivial. First Grade will use lower numbers. How high can we go ? Nice is ıø × ıø = ıøø but ıı × ıı = ı Γı would remain instructive.

Table 10. Calculating 7 times 8

ıΓ		6 + 2 = 8
ıı	×	6 + 1 = 7
ıΓ		6 + 2 = 8
ıΓø	+	36 + (2 × 6) = 48
ıΔΓ		36 + (3 × 6) + 2 = 56

This calculation shows the advantage of knowing what *times* means. Who knows what it is can understand how the numbers are contructed, and can also understand what arithmetic is (the collection of the weights of the powers of the base number). For this reason it is didactically advantageous to have *times* available as quickly as possible.

Tables

Table 11 gives the table of addition and Table 12 for *times*. Learning by heart is required for the decimal system but inadvisable for the senary system since you would later learn the decimal one. At this stage it suffices to be able to look up the result in the table, and to see how the tables hang together, and how they have some structure (e.g. the diagonals). For example, see in both tables how Δ + Δ + Δ = Δ × Δ.

Table 11. Table of addition for 1 to 12

+	I	Γ	Δ	□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø
I	Γ	Δ	□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø	ΓI
Γ	Δ	□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ
Δ	□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ
□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□
⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂
Iø	II	IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø
II	IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø	ΔI
IΓ	IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø	ΔI	ΔΓ
IΔ	I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø	ΔI	ΔΓ	ΔΔ
I□	I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø	ΔI	ΔΓ	ΔΔ	Δ□
I⌂	Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø	ΔI	ΔΓ	ΔΔ	Δ□	Δ⌂
Γø	ΓI	ΓΓ	ΓΔ	Γ□	Γ⌂	Δø	ΔI	ΔΓ	ΔΔ	Δ□	Δ⌂	□ø

Table 12. Table of *times* for 1 to 12

×	I	Γ	Δ	□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø
I	I	Γ	Δ	□	⌂	Iø	II	IΓ	IΔ	I□	I⌂	Γø
Γ	Γ	□	Iø	IΓ	I□	Γø	ΓΓ	Γ□	Δø	ΔΓ	Δ□	□ø
Δ	Δ	Iø	IΔ	Γø	ΓΔ	Δø	ΔΔ	□ø	□Δ	⌂ø	⌂Δ	Iøø
□	□	IΓ	Γø	Γ□	ΔΓ	□ø	□□	⌂Γ	Iøø	Iø□	IIΓ	IΓø
⌂	⌂	I□	ΓΔ	ΔΓ	□I	⌂ø	⌂⌂	Iø□	IIΔ	IΓΓ	IΔI	I□ø
Iø	Iø	Γø	Δø	□ø	⌂ø	Iøø	IIø	IΓø	IΔø	I□ø	I⌂ø	Γøø
II	II	ΓΓ	ΔΔ	□□	⌂⌂	IIø	IΓI	IΔΓ	I□Δ	I⌂□	Γø⌂	ΓΓø
IΓ	IΓ	Γ□	□ø	⌂Γ	Iø□	IΓø	IΔΓ	I□□	Γøø	ΓIΓ	ΓΓ□	Γ□ø
IΔ	IΔ	Δø	□Δ	Iøø	IIΔ	IΔø	I□Δ	Γøø	ΓIΔ	ΓΔø	Γ□Δ	Δøø
I□	I□	ΔΓ	⌂ø	Iø□	IΓΓ	I□ø	I⌂□	ΓIΓ	ΓΔø	Γ□□	ΔøΓ	ΔΓø
I⌂	I⌂	Δ□	⌂Δ	IIΓ	IΔI	I⌂ø	Γø⌂	ΓΓ□	Γ□Δ	ΔøΓ	ΔΓI	Δ□ø
Γø	Γø	□ø	Iøø	IΓø	I□ø	Γøø	ΓΓø	Γ□ø	Δøø	ΔΓø	Δ□ø	□øø

Advantages and disadvantages

In summary, the advantages of the senary system for First Grade are:

(1) There is a small number of symbols that can be chosen for clarity

(2) The number of calculations for the positional shift is small and clear too

(3) Calculations can be done on two hands, with the right hand for units and the left hand for the number of right hands, in the same order (from right to left) as in the current decimal system

(4) Numbers have the same structure as the decimal system (place value system)

(5) The pronunciation of the numbers is not dictated by tradition but can be chosen systematically

(6) *Times* can be introducted more quickly so that it allows earlier insight in the structure of number and arithmetic (... + *c* × lux + *b* × hand + *a*).

The disadvantages of a senary system are:

(1) It is intended for didactics only and not applied in practice

(2) A question like "How many fingers do you see ?" requires distinction between left and right, back and palm

(3) It may be confusing, on balance, in learning the decimal system later on.

Conclusions

I doubt whether this system with base six will be used in First Grade indeed. Current problems in teaching arithmetic may have to do less with the number system itself, see for comparison the 1950s. In Holland since then there has been a curious move towards not learning the tables by heart, see Milikowski (2004). We may already see a big improvement when misunderstandings like these are resolved. That said, it still is a contribution to think about the number system and its relation to arithmetic.

Libraries have been filled on number and arithmetic but the present discussion seems to includes these useful points:

(1) Above senary system has an attractive form, both by its streamlining and by a minimum of confusion with the decimal system. If you use a senary system, the advice is to use this one. (This would be suitable for students who are learning to become teachers at elementary school.)

(2) This paper gives another perspective on the proposal to revise the names of the decimal numbers (with 11 = ten & one and so on).

(3) Research in both didactics and brains could look with priority whether First Grade can multiply. When pupils can learn above senary system then this already shows their elementary grasp of *times*. Counting hands is *times hand*. Γ × |ø + □ = Γ□ seems doable and shows the structure of numbers. Can pupils also multiply with other numbers ? Five cats with two eyes each gives five times two or hand & four or ten eyes. Seems doable as well. When a range of numbers can be found then this can be exploited to develop arithmetic.

(4) Above discussion may also help to beter target learning aims for Second Grade. Problems like 2 × 10 + 4 = 24 highlight the structure of number as well.

Appendix D. Proposed implementations for English, German, French, Dutch and Danish

English

"&"= "and". The ordinals use -th, e.g. *one-th, two-th, three-th,* .... There is tension between current *three-ten-ths* (3 /10) and mathematical *three·ten-th* (30·*th*), but calculation is done with mathematical name *three per ten.*

zero	0
one	1
two	2
three	3
four	4
five	5
six	6
seven	7
eight	8
nine	9
ten	10

Ten to five·ten

English-M		*Current English*
ten	10	ten
ten & one	11	eleven
ten & two	12	twelve
ten & three	13	thirteen
ten & four	14	fourteen
ten & five	15	fifteen
ten & six	16	sixteen
ten & seven	17	seventeen
ten & eight	18	eighteen
ten & nine	19	nineteen
two·ten	20	twenty

English-M		*Current English*
two·ten	20	twenty
two·ten & one	21	twenty-one
two·ten & two	22	twenty-two
two·ten & three	23	twenty-three
two·ten & four	24	twenty-four
two·ten & five	25	twenty-five
two·ten & six	26	twenty-six
two·ten & seven	27	twenty-seven
two·ten & eight	28	twenty-eight
two·ten & nine	29	twenty-nine
three·ten	30	thirty

English-M		*Current English*
three·ten	30	thirty
three·ten & one	31	thirty-one
three·ten & two	32	thirty-two
three·ten & three	33	thirty-three
three·ten & four	34	thirty-four
three·ten & five	35	thirty-five
three·ten & six	36	thirty-six
three·ten & seven	37	thirty-seven
three·ten & eight	38	thirty-eight
three·ten & nine	39	thirty-nine
four·ten	40	forty

English-M		*Current English*
four·ten	40	forty
four·ten & one	41	forty-one
four·ten & two	42	forty-two
four·ten & three	43	forty-three
four·ten & four	44	forty-four
four·ten & five	45	forty-five
four·ten & six	46	forty-six
four·ten & seven	47	forty-seven
four·ten & eight	48	forty-eight
four·ten & nine	49	forty-nine
five·ten	50	fifty

Numbers of ten

English-M		*Current English*
ten	10	ten
two·ten	20	twenty
three·ten	30	thirty
four·ten	40	forty
five·ten	50	fifty
six·ten	60	sixty
seven·ten	70	seventy
eight·ten	80	eighty
nine·ten	90	ninety
ten·ten, hundred	100	hundred

Ten to million: keep using the current language

			Current English
10^1	ten	10	ten
10^2	ten·ten	100	hundred
10^3	ten·ten·ten	1,000	thousand
10^4	ten·ten·ten·ten	10,000	ten·thousand
10^5	ten·ten·ten·ten·ten	100,000	hundred·thousand
10^6	ten·ten·ten·ten·ten·ten	1,000,000	million

German

The choice of *zig* instead of *zehn* cannot be avoided because of the confusion between *neunzehn* (zig & neun) and *neunzig* (neun·zig) if zehn were used. It remains an option to use English *ten* or scientific *deca,* but this seems unnecessary and unlikely.

“&”= “und”. The choices of *ein* instead of *eins* and *sieb* instead of *sieben* are optional. Given that *ein* and *sieb* already are used, as in *ein-und-siebzig,* I have opted to use them universally.

The ordinals would use -te, e.g. *ein-te, zwei·zig & ein-te.*

null	0
ein, eins	1
zwei	2
drei	3
vier	4
fünf	5
sechs	6
sieb, sieben	7
acht	8
neun	9
zig, zehn	10

Zig zu fünf·zig

Deutsch-M		*Deutsch heute* (current German)
zig	10	zehn
zig & ein	11	elf
zig & zwei	12	zwölf
zig & drei	13	dreizehn
zig & vier	14	vierzehn
zig & fünf	15	fünfzehn
zig & sechs	16	sechzehn
zig & sieb	17	siebzehn
zig & acht	18	achtzehn
zig & neun	19	neunzehn
zwei·zig	20	zwanzig

Deutsch-M		*Deutsch heute*
zwei·zig	20	zwanzig
zwei·zig & ein	21	ein-und-zwanzig
zwei·zig & zwei	22	zwei-und-zwanzig
zwei·zig & drei	23	drei-und-zwanzig
zwei·zig & vier	24	vier-und-zwanzig
zwei·zig & fünf	25	fünf-und-zwanzig
zwei·zig & sechs	26	sechs-und-zwanzig
zwei·zig & sieb	27	sieben-und-zwanzig
zwei·zig & acht	28	acht-und-zwanzig
zwei·zig & neun	29	neun-und-zwanzig
drei·zig	30	dreißig

Deutsch-M		*Deutsch heute*
drei·zig	30	dreißig
drei·zig & ein	31	ein-und-dreißig
drei·zig & zwei	32	zwei-und-dreißig
drei·zig & drei	33	drei-und-dreißig
drei·zig & vier	34	vier-und-dreißig
drei·zig & fünf	35	fünf-und-dreißig
drei·zig & sechs	36	sechs-und-dreißig
drei·zig & sieb	37	sieben-und-dreißig
drei·zig & acht	38	acht-und-dreißig
drei·zig & neun	39	neun-und-dreißig
vier·zig	40	vierzig

Deutsch-M		*Deutsch heute*
vier·zig	40	vierzig
vier·zig & ein	41	ein-und-vierzig
vier·zig & zwei	42	zwei-und-vierzig
vier·zig & drei	43	drei-und-vierzig
vier·zig & vier	44	vier-und-vierzig
vier·zig & fünf	45	fünf-und-vierzig
vier·zig & sechs	46	sechs-und-vierzig
vier·zig & sieb	47	sieben-und-vierzig
vier·zig & acht	48	acht-und-vierzig
vier·zig & neun	49	neun-und-vierzig
fünf·zig	50	fünfzig

The numbers of zig

Deutsch-M		*Deutsch heute*
zig	10	zig
zwei·zig	20	zwanzig
drei·zig	30	dreißig
vier·zig	40	vierzig
fünf·zig	50	fünfzig
sechs·zig	60	sechzig
sieb·zig	70	siebzig
acht·zig	80	achtzig
neun·zig	90	neunzig
zig·zig, hundert	100	hundert

Ten to million: keep using the current language above zig

			Deutsch heute
10^1	zig	10	zehn
10^2	zig·zig	100	hundert
10^3	zig·zig·zig	1,000	tausend
10^4	zig·zig·zig·zig	10,000	zig·tausend
10^5	zig·zig·zig·zig·zig	100,000	hundert·tausend
10^6	zig·zig·zig·zig·zig·zig	1,000,000	Million

French

In French there is no problem in taking *dix* as the base for the numbers of ten.

The numbers of 70-100 are fully written out because of the complex French originals.

“&”= “et”. The ordinals would be *-ième: un-ième,deux-ième, ...*

zéro	0
un	1
deux	2
trois	3
quatre	4
cinq	5
six	6
sept	7
huit	8
neuf	9
dix	10

Dix to cinq·dix

Français-M		*Français aujourd'hui*
dix	10	dix
dix & un	11	onze
dix & deux	12	douze
dix & trois	13	treize
dix & quatre	14	quatorze
dix & cinq	15	quinze
dix & six	16	seize
dix & sept	17	dix-sept
dix & huit	18	dix-huit
dix & neuf	19	dix-neuf
deux·dix	20	vingt

Français-M		*Français aujourd'hui*
deux·dix	20	vingt
deux·dix & un	21	vingt et un
deux·dix & deux	22	vingt-deux
deux·dix & trois	23	vingt-trois
deux·dix & quatre	24	vingt-quatre
deux·dix & cinq	25	vingt-cinq
deux·dix & six	26	vingt-six
deux·dix & sept	27	vingt-sept
deux·dix & huit	28	vingt-huit
deux·dix & neuf	29	vingt-neuf
trois·dix	30	trente

Français-M		*Français aujourd'hui*
trois·dix	30	trente
trois·dix & un	31	trente et un
trois·dix & deux	32	trente-deux
trois·dix & trois	33	trente-trois
trois·dix & quatre	34	trente-quatre
trois·dix & cinq	35	trente-cinq
trois·dix & six	36	trente-six
trois·dix & sept	37	trente-sept
trois·dix & huit	38	trente-huit
trois·dix & neuf	39	trente-neuf
quatre·dix	40	quarante

Français-M		*Français aujourd'hui*
quatre·dix	40	quarante
quatre·dix & un	41	quarante et un
quatre·dix & deux	42	quarante-deux
quatre·dix & trois	43	quarante-trois
quatre·dix & quatre	44	quarante-quatre
quatre·dix & cinq	45	quarante-cinq
quatre·dix & six	46	quarante-six
quatre·dix & sept	47	quarante-sept
quatre·dix & huit	48	quarante-huit
quatre·dix & neuf	49	quarante-neuf
cinq·dix	50	cinquante

Français-M		*Français aujourd'hui*
sept·dix	70	soixante-dix
sept·dix & un	71	soixante et onze
sept·dix & deux	72	soixante-douze
sept·dix & trois	73	soixante-treize
sept·dix & quatre	74	soixante-quatorze
sept·dix & cinq	75	soixante-quinze
sept·dix & six	76	soixante-seize
sept·dix & sept	77	soixante-dix-sept
sept·dix & huit	78	soixante-dix-huit
sept·dix & neuf	79	soixante-dix-neuf
huit·dix	80	quatre-vingts

huit·dix	80	quatre-vingts
huit·dix & un	81	quatre-vingt-un
huit·dix & deux	82	quatre-vingt-deux
huit·dix & trois	83	quatre-vingt-trois
huit·dix & quatre	84	quatre-vingt-quatre
huit·dix & cinq	85	quatre-vingt-cinq
huit·dix & six	86	quatre-vingt-six
huit·dix & sept	87	quatre-vingt-sept
huit·dix & huit	88	quatre-vingt-huit
huit·dix & neuf	89	quatre-vingt-neuf
neuf·dix	90	quatre-vingt-dix

neuf·dix	90	quatre-vingt-dix
neuf·dix & un	91	quatre-vingt et onze
neuf·dix & deux	92	quatre-vingt-douze
neuf·dix & trois	93	quatre-vingt-treize
neuf·dix & quatre	94	quatre-vingt-quatorze
neuf·dix & cinq	95	quatre-vingt-quize
neuf·dix & six	96	quatre-vingt-seize
neuf·dix & sept	97	quatre-vingt-dix-sept
neuf·dix & huit	98	quatre-vingt-dix-huit
neuf·dix & neuf	99	quatre-vingt-dix-neuf
dix·dix, cent	100	cent

The numbers of dix

Français-M		*Français aujourd'hui*
dix	10	dix
deux·dix	20	vingt
trois·dix	30	trente
quatre·dix	40	quarante
cinq·dix	50	cinquante
six·dix	60	soixante
sept·dix	70	soixante-dix
huit·dix	80	quatre-vingts
neuf·dix	90	quatre-vingt-dix
dix·dix	100	cent

Ten to million: keep using the current language

			Français aujourd'hui
10^1	dix	10	dix
10^2	dix·dix	100	cent
10^3	dix·dix·dix	1,000	mille
10^4	dix·dix·dix·dix	10,000	dix·mille
10^5	dix·dix·dix·dix·dix	100,000	cent·mille
10^6	dix·dix·dix·dix·dix·dix	1,000,000	million

Dutch

The choice of *tig* instead of *tien* cannot be avoided because of the confusion between *negentien* (tig & negen) and *negentig* (negen·tig) if *tien* were used. It remains an option to use English *ten,* but this seems unnecessary and unlikely. “&”= “en”.

Ordinals use -de: *een-de, twee-de, drie-de,* ..., *tig-de,* .....

nul	0
een	1
twee	2
drie	3
vier	4
vijf	5
zes	6
zeven	7
acht	8
negen	9
tig, tien	10

From ten to fifty

Nederlands-M		*Huidig Nederlands*
tig	10	tien
tig & een	11	elf
tig & twee	12	twaalf
tig & drie	13	dertien
tig & vier	14	veertien
tig & vijf	15	vijftien
tig & zes	16	zestien
tig & zeven	17	zeventien
tig & acht	18	achttien
tig & negen	19	negentien
twee·tig	20	twintig

Nederlands-M		*Huidig Nederlands*
twee·tig	20	twintig
twee·tig & een	21	een-en-twintig
twee·tig & twee	22	twee-en-twintig
twee·tig & drie	23	drie-en-twintig
twee·tig & vier	24	vier-en-twintig
twee·tig & vijf	25	vijf-en-twintig
twee·tig & zes	26	zes-en-twintig
twee·tig & zeven	27	zeven-en-twintig
twee·tig & acht	28	acht-en-twintig
twee·tig & negen	29	negen-en-twintig
drie·tig	30	dertig

Nederlands-M		*Huidig Nederlands*
drie·tig	30	dertig
drie·tig & een	31	een-en-dertig
drie·tig & twee	32	twee-en-dertig
drie·tig & drie	33	drie-en-dertig
drie·tig & vier	34	vier-en-dertig
drie·tig & vijf	35	vijf-en-dertig
drie·tig & zes	36	zes-en-dertig
drie·tig & zeven	37	zeven-en-dertig
drie·tig & acht	38	acht-en-dertig
drie·tig & negen	39	negen-en-dertig
vier·tig	40	veertig

Nederlands-M		*Huidig Nederlands*
vier·tig	40	veertig
vier·tig & een	41	een-en-veertig
vier·tig & twee	42	twee-en-veertig
vier·tig & drie	43	drie-en-veertig
vier·tig & vier	44	vier-en-veertig
vier·tig & vijf	45	vijf-en-veertig
vier·tig & zes	46	zes-en-veertig
vier·tig & zeven	47	zeven-en-veertig
vier·tig & acht	48	acht-en-veertig
vier·tig & negen	49	negen-en-veertig
vijf·tig	50	vijftig

The numbers of tig

Nederlands-M		*Huidig Nederlands*
tig	10	tien
twee·tig	20	twintig
drie·tig	30	dertig
vier·tig	40	veertig
vijf·tig	50	vijftig
zes·tig	60	zestig
zeven·tig	70	zeventig
acht·tig	80	tachtig
negen·tig	90	negentig
tig·tig, honderd	100	honderd

Ten to million: keep using the current language above tig

			Huidig Nederlands
10^1	tig	10	tien
10^2	tig·tig	100	honderd
10^3	tig·tig·tig	1,000	duizend
10^4	tig·tig·tig·tig	10,000	tig·duizend
10^5	tig·tig·tig·tig·tig	100,000	honderd·duizend
10^6	tig·tig·tig·tig·tig·tig	1,000,000	miljoen

Danish

Danish can use current *ti* as below, but also has the option to use English *ten.*

“&”= “og”. For the ordinals a suggestion would be to use *-de* like English *-th.*

nul	0
en	1
to	2
tre	3
fire	4
fem	5
seks	6
syv	7
otte	8
ni	9
ti	10

From ten to fifty

Dansk-M		*Dansk i dag*
ti	10	ti
ti & en	11	elleve
ti & to	12	tolv
ti & tre	13	tretten
ti & fire	14	fjorten
ti & fem	15	femten
ti & seks	16	seksten
ti & syv	17	sytten
ti & otte	18	atten
ti & ni	19	nitten
to·ti	20	tyve

Dansk-M		*Dansk i dag*
to·ti	20	tyve
to·ti & en	21	en-og-tyve
to·ti & to	22	to-og-tyve
to·ti & tre	23	tre-og-tyve
to·ti & fire	24	fire-og-tyve
to·ti & fem	25	fem-og-tyve
to·ti & seks	26	seks-og-tyve
to·ti & syv	27	syv-og-tyve
to·ti & otte	28	otte-og-tyve
to·ti & ni	29	ni-og-tyve
tre·ti	30	tredive

Dansk-M		*Dansk i dag*
tre·ti	30	tredive
tre·ti & en	31	en-og-tredive
tre·ti & to	32	to-og-tredive
tre·ti & tre	33	tre-og-tredive
tre·ti & fire	34	fire-og-tredive
tre·ti & fem	35	fem-og-tredive
tre·ti & seks	36	seks-og-tredive
tre·ti & syv	37	syv-og-tredive
tre·ti & otte	38	otte-og-tredive
tre·ti & ni	39	ni-og-tredive
fire·ti	40	fyrre

Dansk-M		*Dansk i dag*
fire·ti	40	fyrre
fire·ti & en	41	en-og-fyrre
fire·ti & to	42	to-og-fyrre
fire·ti & tre	43	tre-og-fyrre
fire·ti & fire	44	fire-og-fyrre
fire·ti & fem	45	fem-og-fyrre
fire·ti & seks	46	seks-og-fyrre
fire·ti & syv	47	syv-og-fyrre
fire·ti & otte	48	otte-og-fyrre
fire·ti & ni	49	ni-og-fyrre
fem·ti	50	halvtreds

The numbers of ti

Dansk-M		*Dansk i dag*
ti	10	ti
to·ti	20	tyve
tre·ti	30	tredive
fire·ti	40	fyrre
fem·ti	50	halvtreds
seks·ti	60	tres
syv·ti	70	halvfjerds
otte·ti	80	firs
ni·ti	90	halvfems
ti·ti, hundrede	100	hundrede

Ten to million: keep using the current language

			Dansk i dag
10^1	ti	10	ti
10^2	ti·ti	100	hundrede
10^3	ti·ti·ti	1,000	tusind
10^4	ti·ti·ti·ti	10,000	ti·tusind
10^5	ti·ti·ti·ti·ti	100,000	hundrede·tusind
10^6	ti·ti·ti·ti·ti·ti	1,000,000	million

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PM 1. Colignatus is the name of Thomas Cool in science. See http://thomascool.eu.
PM 2. References in footnotes need not be repeated here.
PM 3. Commentary by anonymous researchers is acknowledged and has contributed to the quality of the present analysis.

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