THE THEORY OF CAUCHY'S PRINCIPAL VALUES. (FOURTH PAPER: *The Integration of Principal Values*—CONTINUED—*with Applications to the Inversion of Definite Integrals*)

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I.

Introductory.

1. This paper is designed to supplement and complete three papers published under the same title in 1901–2.* In these three papers I considered in detail a number of questions connected with principal values, and in particular the questions of the continuity, differentiation, and integration of principal values which involve a continuous parameter. In the last of them I began to consider the equation

$$(1) \qquad P\int_a^A dx\, P\int_b^B f(x, y)\, dy = P\int_b^B dy\, P\int_a^A f(x, y)\, dx,$$

where $f(x, y)$ is a function affected with singularities of a special form. It will probably be convenient for me to repeat the principal results at which I arrived. I shall confine myself at present to the case in which all the limits are finite, and I shall simplify the statement of the results by the introduction of certain definitions.

I shall call a straight line parallel to either axis *a standard curve of the first kind*, or, more shortly, *a line* C_1. A curve of continuous curvature, whose tangent is nowhere parallel to either axis, I shall call *a standard curve of the second kind*, or *a line* C_2. Such a curve has the property that its equation may be expressed in either of the forms

$$y = X(x), \qquad x = Y(y),$$

where X and Y are functions whose first two derivatives are continuous and whose first derivatives do not vanish for any value of x or y in question. The simplest example of such a curve is the line $x = y$.

It generally happens, in cases which lead to applications of any interest,

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that the discontinuities of f lie on a finite number of lines C_1 or C_2. Occasionally a line of discontinuities presents itself which violates the conditions for a line C_2 in that its tangent is parallel to an axis at a finite number of points. This case is not considered in the general theorems which follow, and additional investigation is necessary when it occurs.

2. In my third paper I dealt with the case in which no two singular curves intersect. I first proved that—

(i.) *If f is a continuous function of both variables, except along a finite number of lines C_1 parallel to the axis of y, and if*

$$P\int_a^A f\,dx$$

is uniformly convergent in (b, B), then

$$P\int_a^A dx\int_b^B f\,dy = \int_b^B dy\,P\int_a^A f\,dx. \tag{2}$$

In other words, the equation (1) holds, but two of the symbols of the principal value contained in it are unnecessary.

The simplest case of this theorem is that in which

$$f = \frac{\Theta(x, y)}{x-\alpha},$$

where $a < \alpha < A$, and Θ is a function continuous without exception, together with its first derivative Θ_x. If all the first and second derivatives of Θ are continuous I shall say that f has a *standard discontinuity* along the line $x = \alpha$. Similarly I shall say that f has a standard discontinuity along the line C_2 represented by the equations

$$y = X(x), \qquad x = Y(y),$$

if it is capable of expression in the forms

$$\frac{\Theta(x, y)}{y-X}, \qquad \frac{\Phi(x, y)}{x-Y},$$

where Θ and Φ, together with all their first and second derivatives, are continuous without exception.

With this notation, my second result was

(ii.) *If f, together with all its first and second derivatives, is continuous throughout (a, A, b, B), except that it has standard discontinuities* *along a finite number of non-intersecting lines C_2, then*

$$(3)\qquad \int_a^A dx\, P\int_b^B f\,dy = \int_b^B dy\, P\int_a^A f\,dx.$$

Here, again, it will be noticed, only two symbols of the principal value are necessary, but these are not the same two as in equation (2).*

3. By a combination of Theorems (i.) and (ii.), and the theorem which results from (i.) when x and y are interchanged, we can deal with all cases of interest in which no two curves of discontinuity intersect. The really interesting case, however, is that in which there are such intersections. It is clear that, by dividing up the rectangle of integration, we can reduce this case to that in which there is only *one* intersection; and this case can be subdivided into three, according as the intersecting curves are (*a*) two lines C_1, (*b*) two lines C_2, or (*c*) a line C_1 and a line C_2.

With case (*a*) I dealt in my third paper, by means of the theorem

(iii.) *If* $$f = \Theta / \{(x-\alpha)(y-\beta)\},$$
where $a < \alpha < A$, $b < \beta < B$, and Θ, together with all its first and second derivatives, is continuous without exception, then

$$(4)\qquad P\int_a^A dx\, P\int_b^B f\,dy = P\int_b^B dy\, P\int_a^A f\,dx.$$

In other words, (1) holds provided *all* the signs of the principal value are retained.

4. I shall now proceed to consider cases (*b*) and (*c*). As regards the former, I indicated my result in general terms at the end of my third paper. If
$$f = \psi/\lambda\mu,$$
$\lambda = 0$ and $\mu = 0$ being the curves of discontinuity, then (1) may be true, or it may be untrue, the difference, in the latter case, between the two sides of the equation being
$$\frac{2\pi^2\psi(\alpha, \beta)}{\dfrac{\partial(\lambda, \mu)}{\partial(\alpha, \beta)}},$$

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* I considered before the case in which f is of the form
$$\frac{\Theta}{y-X}\left\{\log\left(\frac{1}{y-X}\right)\right\}^{a_1}\left\{\log\log\left(\frac{1}{y-X}\right)\right\}^{a_2}\dots;$$
but this more general case will not concern us in this paper.

where α, β are the coordinates of the point of intersection, supposed to lie *inside* (a, A, b, B).* This result I shall now proceed to define more precisely and to prove. I have had the outlines of the proof in my hands for years, but have never published it, as it was only recently that I realised what interesting applications the result has to the problem of the inversion of a definite integral, which has been so prominent in recent mathematical literature.

II.

Proofs of the General Theorems.

5. Let us suppose that

$$a = b = -1, \qquad A = B = 1,$$

and that the point of intersection of the singular curves is (0, 0). It is clear that these hypotheses do not involve any real loss of generality.

If ϵ and ϵ' are any positive numbers, however small, we have, by Theorem 2 above, the equations

$$\int_{-1}^{1} dx\, P \int_{\epsilon}^{1} f\, dy = \int_{\epsilon}^{1} dy\, P \int_{-1}^{1} f\, dx, \qquad \int_{-1}^{1} dx\, P \int_{-1}^{-\epsilon'} f\, dy = \int_{-1}^{-\epsilon'} dy\, P \int_{-1}^{1} f\, dx.$$

Hence

(1) $$\int_{-1}^{1} dx\, P \int_{-1}^{1} f\, dy = \left(\int_{-1}^{-\epsilon'} + \int_{\epsilon}^{1}\right) dy\, P \int_{-1}^{1} f\, dx + \Delta(\epsilon, \epsilon'),$$

where

(2) $$\Delta(\epsilon, \epsilon') = \int_{-1}^{1} dx\, P \int_{-\epsilon'}^{\epsilon} f\, dy,$$

provided only that this last expression has a meaning. If this is so, and if we can prove that when ϵ and ϵ' tend independently to zero, $\Delta(\epsilon, \epsilon')$ tends to a limit Δ, we shall arrive at the equation

(3) $$\int_{-1}^{1} dx\, P \int_{-1}^{1} f\, dy = \int_{-1}^{1} dy\, P \int_{-1}^{1} f\, dx + \Delta.$$

Evaluation of Δ.

6. Let ρ and ρ' be small positive numbers. Then

$$\Delta(\epsilon, \epsilon') = J + J' + \Delta(\rho, \rho', \epsilon, \epsilon'),$$

* We shall consider later on the case of an intersection *on the boundary* of the rectangle.

where $$J = \int_\rho^1 dx\, P\int_{-\epsilon'}^{\epsilon} f\,dy, \quad J' = \int_{-1}^{-\rho'} dx\, P\int_{-\epsilon'}^{\epsilon} f\,dy,$$

and $$\Delta(\rho, \rho', \epsilon, \epsilon') = \int_{-\rho'}^{\rho} dx\, P\int_{-\epsilon'}^{\epsilon} f\,dy,$$

provided only that this last expression has a meaning.

Now, by Theorem 2, we have

$$J = \int_{-\epsilon'}^{\epsilon} dy\, P\int_\rho^1 f\,dx;$$

and therefore, when ρ is fixed, we can, given any positive number σ, so choose η that

$$|J| < \sigma,$$

for $0 < \epsilon \leqslant \eta$, $0 < \epsilon' \leqslant \eta$. Similarly, we can ensure that $|J'| < \sigma$.

I shall call a function $\phi(\rho, \rho', \epsilon, \epsilon')$ *negligible* if it tends to the limit zero when ρ, ρ', ϵ, and ϵ' tend independently to zero; that is to say if, given any positive number σ, we can choose ζ and η, so that $|\phi| < \sigma$ for $0 < \rho \leqslant \zeta$, $0 < \rho' \leqslant \zeta$, $0 < \epsilon \leqslant \eta$, $0 < \epsilon' \leqslant \eta$. In particular ϕ is negligible if, when ρ and ρ' have any fixed values, $\phi \to 0$ with ϵ and ϵ', uniformly for all pairs of sufficiently small values of ρ and ρ'. For example, J and J' are negligible.

Now let us assume for a moment that we have proved that

$$\Delta(\rho, \rho', \epsilon, \epsilon') = \Delta + \delta_1 + \delta_2 + \ldots + \delta_k, \tag{4}$$

where Δ is a constant and δ_s a negligible function. Then

$$|\Delta(\epsilon, \epsilon') - \Delta| \leqslant |J| + |J'| + \sum_{s=1}^{k} |\delta_s|.$$

Given σ we can choose ζ and η, so that

$$|J| < \frac{\sigma}{k+2}, \quad |J'| < \frac{\sigma}{k+2}, \quad |\delta_s| < \frac{\sigma}{k+2},$$

for $0 < \rho \leqslant \zeta$, $0 < \rho' \leqslant \zeta$, $0 < \epsilon \leqslant \eta$, $0 < \epsilon' \leqslant \eta$; and then

$$|\Delta(\epsilon, \epsilon') - \Delta| < \sigma,$$

so that $\Delta(\epsilon, \epsilon') \to \Delta$ as $\epsilon \to 0$, $\epsilon' \to 0$.

Thus, in order to establish the truth of the equation (3), all that is necessary is to express $\Delta(\rho, \rho', \epsilon, \epsilon')$ in the form (4).

7. We are supposing that $\lambda = 0$, $\mu = 0$ are the curves of discontinuity, and that they intersect without touching at the origin; further, that

$$f = \frac{\psi}{\lambda\mu} = \frac{\phi(x, y)}{\{y - X_1(x)\}\{y - X_2(x)\}},$$

where ψ, ϕ, λ, μ are functions of x and y, continuous, with all their first and second derivatives, throughout the rectangle of integration. Also X_1', X_2' are of constant sign, and $X_1'(0) \neq X_2'(0)$; and X_1'', X_2'' are continuous.

We shall write $\phi(x, y) = \phi(x, 0) + y\phi_1(x, y)$,

so that $$\phi_1(x, y) = \{\phi(x, y) - \phi(x, 0)\}/y = \frac{\partial\phi(x, \theta y)}{\partial y} \quad (0 < \theta < 1),$$

and
$$\frac{\partial\phi_1}{\partial y} = -\frac{1}{y^2}\left\{\phi(x, y) - \phi(x, 0) - y\frac{\partial\phi(x, y)}{\partial y}\right\}$$
$$= \frac{\partial^2\phi(x, \theta y)}{\partial y^2} - \tfrac{1}{2}\frac{\partial^2(x, \theta' y)}{\partial y^2},$$

where $0 < \theta < 1$, $0 < \theta' < 1$. From these equations it follows that $\partial\phi_1/\partial y$ is continuous, and
$$|\phi_1| < K, \quad \left|\frac{\partial\phi_1}{\partial y}\right| < K.$$

8. Let $$\Delta(\rho, \rho', \epsilon, \epsilon') = \Delta_0 + \Delta_1,$$

where Δ_0 and Δ_1 are obtained from Δ by replacing ϕ by $\phi(x, 0)$ and by $y\phi_1(x, y)$ respectively. We shall prove first that Δ_1 is well defined and negligible. We have
$$\Delta_1 = \int_{-\rho'}^{\rho} dx\, P\int_{-\epsilon'}^{\epsilon} \frac{y\phi_1(x, y)\, dy}{(y - X_1)(y - X_2)} = \int_{-\rho'}^{\rho} \chi(x, \epsilon, \epsilon')\, dx,$$

say. Since $$\frac{\partial}{\partial y}\{y\phi_1(x, y)\} = \frac{\partial\phi(x, y)}{\partial y}$$

is continuous, $\chi(x, \epsilon, \epsilon')$ is defined for all values of ϵ and ϵ'. Also, in virtue of what was proved in the last section concerning $\partial\phi_1/\partial y$, we may integrate by parts, and obtain
$$\chi(x, \epsilon, \epsilon') = \frac{1}{2(X_1 - X_2)}\Big[\phi_1(x, \epsilon)\{X_1 \log(\epsilon - X_1)^2 - X_2 \log(\epsilon - X_2)^2\}$$
$$- \phi_1(x, -\epsilon')\{X_1 \log(\epsilon' + X_1)^2 - X_2 \log(\epsilon' + X_2)^2\}$$
$$- \int_{-\epsilon'}^{\epsilon} \frac{\partial\phi_1}{\partial y}\{X_1 \log(y - X_1)^2 - X_2 \log(y - X_2)^2\}\, dy\Big].$$

Now $$\left|\frac{X_1}{X_1 - X_2}\right| < K, \quad \left|\frac{X_2}{X_1 - X_2}\right| < K,$$

for $-\rho' \leqslant x \leqslant \rho$. Hence the integrated part of $\chi(x, \epsilon, \epsilon')$ is in absolute value less than KM, where

$$M = |\log(\epsilon - X_1)^2| + |\log(\epsilon - X_2)^2| + |\log(\epsilon' + X_1)^2| + |\log(\epsilon' + X_2)^2|.$$

The remaining part of $\chi(x, \epsilon, \epsilon')$ is in absolute value less than

$$K \int_{-\epsilon'}^{\epsilon} \{|\log(y - X_1)^2| + |\log(y - X_2)^2|\}\, dy.$$

But
$$\int_{-\epsilon'}^{\epsilon} |\log(y - X_1)^2|\, dy = -\int_{-\epsilon'}^{\epsilon} \log(y - X_1)^2\, dy$$
$$= -(\epsilon - X_1)\log(\epsilon - X_1)^2 - (\epsilon' + X_1)\log(\epsilon' + X_1)^2 + 2(\epsilon + \epsilon'),$$

which is certainly less than KM. Similarly

$$\int_{-\epsilon'}^{\epsilon} |\log(y - X_2)^2|\, dy < KM.$$

Hence, finally,
$$|\chi(\epsilon, \epsilon')| < KM.$$

From this it follows that Δ_1 is well defined. Also

$$|\Delta_1| < K \int_{-\rho'}^{\rho} M\, dx.$$

But the last integral consists of four parts of which one is

$$\int_{-\rho'}^{\rho} |\log(\epsilon - X_1)^2|\, dx = -\int_{-\rho'}^{\rho} \log(\epsilon - X_1)^2\, dx.$$

In this last integral make the substitution $X_1(x) = u$. We know that X_1' is of constant sign, say positive; then X_1' lies between certain positive limits, and

$$-\int_{-\rho'}^{\rho} \log(\epsilon - X_1)^2\, dx < -K \int_{-\varpi'}^{\varpi} \log(\epsilon - u)^2\, du,$$

where ϖ and ϖ' are functions of ρ and ρ' respectively, such that

$$\lim_{\rho \to 0} \frac{\varpi}{\rho} = \lim_{\rho' \to 0} \frac{\varpi'}{\rho'} = X_1'(0).$$

Hence

$$\int_{-\rho'}^{\rho} |\log(\epsilon - X_1)^2|\, dx$$
$$< -K\{(\varpi - \epsilon)\log(\varpi - \epsilon)^2 + (\varpi' + \epsilon)\log(\varpi' + \epsilon)^2 - 2\varpi - 2\varpi'\},$$

and this last function is plainly negligible. Similarly for the other three integrals which also form part of

$$\int_{-\rho'}^{\rho} M\,dx\,;$$

and so this integral is negligible, and therefore $\Delta_1(\rho, \rho', \epsilon, \epsilon')$ is negligible.

9. We may therefore consider, instead of Δ, the function Δ_0 defined by the equation

$$\Delta_0(\rho, \rho', \epsilon, \epsilon') = \int_{-\rho'}^{\rho} \phi(x, 0)\,dx\, P\int_{-\epsilon'}^{\epsilon} \frac{dy}{(y-X_1)(y-X_2)}$$

$$= \tfrac{1}{2}\int_{-\rho'}^{\rho} \log\left(\frac{\epsilon-X_1}{\epsilon-X_2}\,\frac{\epsilon'+X_2}{\epsilon'+X_1}\right)^2 \frac{\phi(x, 0)\,dx}{X_1-X_2}.$$

The last integral is plainly convergent; and so Δ_0 is well defined. This remark completes the proof that $\Delta(\rho, \rho', \epsilon, \epsilon')$ is well defined.

Now let

$$\phi(x, 0) = \phi(0, 0)+\phi_2(x, 0)$$

and let

$$\Delta_0 = \Delta_{0,0}+\Delta_2,$$

where $\Delta_{0,0}$ and Δ_2 denote the functions deduced from Δ_0 by replacing $\phi(x, 0)$ by $\phi(0, 0)$ and $\phi_2(x, 0)$ respectively. Then

$$|\phi_2| < K|x|, \qquad \left|\frac{\phi_2}{X_1-X_2}\right| < K;$$

and so

$$|\Delta_2| < K\int_{-\rho'}^{\rho} M\,dx.$$

It therefore follows, from the work of the last section, that Δ_2, like Δ_1, is negligible.

10. It remains to consider

$$\Delta_{0,0}(\rho, \rho', \epsilon, \epsilon') = \tfrac{1}{2}\phi(0, 0)\int_{-\rho'}^{\rho} \log\left(\frac{\epsilon-X_1}{\epsilon-X_2}\,\frac{\epsilon'+X_2}{\epsilon'+X_1}\right)^2 \frac{dx}{X_1-X_2}.$$

We write

$$\gamma_1 = X_1'(0), \quad \gamma_2 = X_2'(0), \quad \frac{1}{X_1-X_2} = \frac{1}{(\gamma_1-\gamma_2)x}+\Xi(x).$$

Then

$$|\Xi(x)| < K;$$

and if

$$\Delta_{0,0} = \bar{\Delta}+\Delta',$$

$\bar{\Delta}$ and Δ' being formed from $\Delta_{0,0}$ by replacing $1/(X_1-X_2)$ by $1/\{(\gamma_1-\gamma_2)x\}$ and $\Xi(x)$ respectively, we can prove that Δ' is negligible by the same method that we used for Δ_1 and Δ_2.

11. We remain with

$$\overline{\Delta}(\rho, \rho', \epsilon, \epsilon') = \frac{\phi(0,0)}{2(\gamma_1-\gamma_2)} \int_{-\rho'}^{\rho} \log\left(\frac{\epsilon - X_1}{\epsilon - X_2}\,\frac{\epsilon' + X_2}{\epsilon' + X_1}\right)^2 \frac{dx}{x} = \frac{\phi(0,0)}{2(\gamma_1-\gamma_2)}(j_1 - j_2),$$

where $j_1 = P\int_{-\rho'}^{\rho} \log\left(\frac{\epsilon - X_1}{\epsilon' + X_1}\right)^2 \frac{dx}{x}, \qquad j_2 = P\int_{-\rho'}^{\rho} \log\left(\frac{\epsilon - X_2}{\epsilon' + X_2}\right)^2 \frac{dx}{x}.$

It is to be observed that neither j_1 nor j_2 is convergent if the sign of the principal value is removed, except in the particular case in which $\epsilon = \epsilon'$.

To j_1 let us supply the substitution $X_1(x) = u$.* We obtain

$$j_1 = P\int_{-\varpi'}^{\varpi} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 \frac{du}{xX_1'(x)} = j_1' + j_1'',$$

where $j_1' = P\int_{-\varpi'}^{\varpi} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 \frac{du}{u}, \qquad j_1'' = \int_{-\varpi'}^{\varpi} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 R\,du,$

and $$R = \frac{1}{xX_1'(x)} - \frac{1}{u},$$

so that $|R| < K$. The integral j_1'' may now be shown to be negligible by a mere repetition of some of our previous arguments. And, of course, j_2 may be treated in the same way as j_1.

12. We have now only to consider

$$\overline{\Delta}_0(\rho, \rho', \epsilon, \epsilon') = \frac{\phi(0,0)}{2(\gamma_1-\gamma_2)}(j_1' - j_2'),$$

where $j_1' = P\int_{-\varpi'}^{\varpi} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 \frac{du}{u}, \qquad j_2' = P\int_{-\overline{\varpi}'}^{\overline{\varpi}} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 \frac{du}{u},$

ϖ, ϖ', $\overline{\varpi}$, $\overline{\varpi}'$ being numbers such that

$$\varpi/\rho \to \gamma_1, \quad \varpi'/\rho' \to \gamma_1, \quad \overline{\varpi}/\rho \to \gamma_2, \quad \overline{\varpi}'/\rho' \to \gamma_2,$$

as ρ and ρ' tend to zero.

Let τ be a positive number less than the least of the moduli of ϖ, ϖ', $\overline{\varpi}$, and $\overline{\varpi}'$. If a denotes any one of the four latter numbers that is positive, and β any one of them that is negative, it is clear that, when once ρ, ρ', and τ have been determined, the integrals

$$\int_{\tau}^{a} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 \frac{du}{u}, \qquad \int_{\beta}^{-\tau} \log\left(\frac{\epsilon - u}{\epsilon' + u}\right)^2 \frac{du}{u}$$

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* The ordinary process of substitution may be employed, since X_1 and its first two derivatives are continuous. See my first paper of this series, pp. 33 *et seq.*

tend to zero with ϵ and ϵ', and are therefore negligible. Hence we may replace $\bar{\Delta}_0$ by

$$\bar{\Delta} = \frac{\phi(0,0)}{2(\gamma_1-\gamma_2)}(i_1-i_2),$$

where

$$i_1 = P\int_{-\tau\,\mathrm{sgn}\,\gamma_1}^{\tau\,\mathrm{sgn}\,\gamma_1} \log\left(\frac{\epsilon-u}{\epsilon'+u}\right)^2 \frac{du}{u}, \qquad i_2 = P\int_{-\tau\,\mathrm{sgn}\,\gamma_2}^{\tau\,\mathrm{sgn}\,\gamma_2} \log\left(\frac{\epsilon-u}{\epsilon'+u}\right)^2 \frac{du}{u}.$$

13. If γ_1 and γ_2 have the same sign

$$i_1 = i_2, \quad \bar{\Delta} = 0, \quad \text{and} \quad \Delta(\epsilon, \epsilon') \to \Delta = 0.$$

If they have opposite signs, let γ_1, say, be positive. Then

$$\begin{aligned} i_1 - i_2 &= 2P\int_{-\tau}^{\tau} \log\left(\frac{\epsilon-u}{\epsilon'+u}\right)^2 \frac{du}{u} \\ &= 2\int_0^{\tau} \log\left(\frac{\epsilon-u}{\epsilon+u}\right)^2 \frac{du}{u} + 2\int_0^{\tau} \log\left(\frac{\epsilon'-u}{\epsilon'+u}\right)^2 \frac{du}{u} \\ &= 2\int_0^{\tau/\epsilon} \log\left(\frac{1-u}{1+u}\right)^2 \frac{du}{u} + 2\int_0^{\tau/\epsilon'} \log\left(\frac{1-u}{1+u}\right)^2 \frac{du}{u} \\ &\to 2\int_0^{\infty} \log\left(\frac{1-u}{1+u}\right)^2 \frac{du}{u} \to -2\pi^2. \end{aligned}$$

Thus, in this case $\quad \Delta(\epsilon, \epsilon') \to \Delta = -\dfrac{2\pi^2\phi(0,0)}{\gamma_1-\gamma_2}.$

14. But

$$\gamma_1 = \left(\frac{dX_1}{dx}\right)_0 = -\left(\frac{\partial\lambda}{\partial x}\right)_0 \Big/ \left(\frac{\partial\lambda}{\partial y}\right)_0, \qquad \gamma_2 = -\left(\frac{\partial\mu}{\partial x}\right)_0 \Big/ \left(\frac{\partial\mu}{\partial y}\right)_0,$$

$$\phi(0,0) = \psi(0,0)\lim\left(\frac{y-X_1}{\lambda}\,\frac{y-X_2}{\mu}\right) = \psi(0,0)\Big/\left\{\left(\frac{\partial\lambda}{\partial y}\right)_0\left(\frac{\partial\mu}{\partial y}\right)_0\right\}.$$

Hence, finally, Δ is equal to

$$-\{2\pi^2\psi(0,0)\}\Big/\left[\frac{\partial(\lambda,\mu)}{\partial(x,y)}\right]_0$$

or to zero. In forming the Jacobian, it is to be observed that $\lambda = 0$ is the curve which makes a positive acute angle with the axis of x.

We can therefore state

THEOREM A.—*If $f(x,y)$, together with all its first and second derivatives, is continuous throughout (a, A, b, B), except that it has standard discontinuities along two standard curves of the second type, $\lambda(x,y)=0$ and $\mu(x,y)=0$, which intersect once only, and simply, at the point*

(α, β); *if, further, we express* $f(x, y)$ *in the form* $\psi(x, y)/\lambda(x, y)\,\mu(x, y)$, *then will*

$$\int_a^A dx\, P\int_b^B f\,dy = \int_b^B dy\, P\int_a^A f\,dx + \Delta,$$

where

$$\Delta = \frac{2\pi^2\psi(\alpha, \beta)}{\dfrac{\partial(\lambda, \mu)}{\partial(\alpha, \beta)}},\ 0$$

according as, at the point (α, β), *the tangents of the angles made by the curves* $\lambda = 0$, $\mu = 0$ *with the axis of* x *have not or have the same sign. In the former case it is to be understood that* $\lambda = 0$ *corresponds to the positive sign.*

15. Exceptional cases of this theorem arise when the point (α, β) falls on a side or at a corner of the rectangle (a, A, b, B). It will, however, be convenient to postpone the consideration of these until we have dealt with the case in which a line C_1 and a line C_2 intersect within the rectangle.

We shall suppose that $a = b = -1$, $A = B = 1$, as before, that the line C_1 is $y = 0$, and that $\lambda(x, y) = 0$, the line C_2, passes through the origin, so that $\alpha = 0$, $\beta = 0$, as before. And we shall suppose $f(x, y)$ to be expressed in the form

$$f(x, y) = \frac{\psi(x, y)}{y\lambda(x, y)}.$$

It is clear that, in this case, the sign of the principal value will be required in at least *three* places, viz., before each sign of integration with respect to y and the inner sign of integration with respect to x. We accordingly take $\epsilon = \epsilon'$ (§ 5), and our final equation will be

$$\int_{-1}^{1} dx\, P\int_{-1}^{1} f\,dy = P\int_{-1}^{1} dy\, P\int_{-1}^{1} f\,dx + \Delta, \tag{5}$$

where

$$\Delta = \lim_{\epsilon \to 0} \Delta(\epsilon) = \lim_{\epsilon \to 0} \int_{-1}^{1} dx\, P\int_{-\epsilon}^{\epsilon} f\,dy, \tag{6}$$

provided this limit exists.

16. Our argument now follows very closely the lines of §§ 7–13. We replace $\Delta(\epsilon)$ by

$$\Delta(\rho, \rho', \epsilon) = \int_{-\rho'}^{\rho} dx\, P\int_{-\epsilon}^{\epsilon} f\,dy,$$

and $f = \psi/y\lambda$ by
$$\frac{\phi(x, y)}{y(y-X_1)},$$

and write
$$\phi(x, y) = \phi(x, 0) + y\phi_1(x, y),$$

as in § 7. Then $\Delta(\rho, \rho'; \epsilon)$ may be replaced by $\Delta_0 + \Delta_1$, where

$$\Delta_0 = \int_{-\rho'}^{\rho} \phi(x, 0)\, dx\, P\int_{-\epsilon}^{\epsilon} \frac{dy}{y(y-X_1)}$$

and
$$\Delta_1 = \int_{-\rho'}^{\rho} dx\, P\int_{-\epsilon}^{\epsilon} \frac{\phi_1(x, y)\,dy}{y-X_1},$$

and the last integral is easily proved to be negligible by a slight modification of the argument of § 8.

Again*
$$\Delta_0 = \tfrac{1}{2}\int_{-\rho'}^{\rho} \log\left(\frac{\epsilon - X_1}{\epsilon + X_1}\right)^2 \phi(x, 0)\, \frac{dx}{X_1},$$

which we replace by $\Delta_{0,0} + \Delta_2$, where

$$\Delta_{0,0} = \tfrac{1}{2}\phi(0, 0) \int_{-\rho'}^{\rho} \log\left(\frac{\epsilon - X_1}{\epsilon + X_1}\right)^2 \frac{dx}{X_1},$$

$$\Delta_2 = \tfrac{1}{2}\int_{-\rho'}^{\rho} \log\left(\frac{\epsilon - X_1}{\epsilon + X_1}\right)^2 \phi_2(x)\, \frac{dx}{X_1},$$

$$\phi_2(x) = \phi(x, 0) - \phi(0, 0).$$

Since $|\phi_2/X_1| < K$, the integral Δ_2 may be shown to be negligible.

Finally, we transform $\Delta_{0,0}$ by the substitution $X_1(x) = u$, and we find (cf. § 12) that it may be replaced by

$$\frac{\phi(0, 0)}{2\gamma_1} \int_{-\tau \operatorname{sgn} \gamma_2}^{\tau \operatorname{sgn} \gamma_1} \log\left(\frac{\epsilon - u}{\epsilon + u}\right)^2 \frac{du}{u},$$

which tends, as $\epsilon \to 0$, to the limit

$$\frac{\phi(0, 0)}{2\gamma_1 \operatorname{sgn} \gamma_1} \int_{-\infty}^{\infty} \log\left(\frac{1-u}{1+u}\right)^2 \frac{du}{u} = -\frac{\pi^2 \phi(0, 0)}{|\gamma_1|}.$$

Since
$$\gamma_1 = -\left(\frac{\partial\lambda}{\partial x}\right)_0 \Big/ \left(\frac{\partial\lambda}{\partial y}\right)_0, \qquad \phi(0, 0) = \psi(0, 0) \Big/ \left(\frac{\partial\lambda}{\partial y}\right)_0,$$

we find ultimately that
$$\Delta = -\frac{\pi^2 \psi(0, 0)}{\left|\dfrac{\partial\lambda}{\partial x}\right|_0 \operatorname{sgn}\left(\dfrac{\partial\lambda}{\partial y}\right)_0}.$$

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* It is here that the importance of having $\epsilon = \epsilon'$ appears. We have

$$P\int_{-\epsilon'}^{\epsilon} \frac{dy}{y(y-X_1)} = \frac{1}{2X_1} \log\left(\frac{\epsilon - X_1}{\epsilon' + X_1} \cdot \frac{\epsilon'}{\epsilon}\right)^2,$$

and the terms involving $\log \epsilon$ and $\log \epsilon'$ would prove intractable.

Hence we obtain

THEOREM B.—*If the conditions of Theorem A are satisfied, except that the curve* $\mu(x, y) = 0$ *is replaced by the straight line* $y = \beta$, *and if*

$$f(x, y) = \frac{\psi(x, y)}{(y-\beta)\,\lambda(x, y)},$$

then will

$$\int_a^A dx\, P\int_b^B f\,dy = P\int_b^B dy\, P\int_a^A f\,dx + \Delta,$$

where

$$\Delta = -\frac{\pi^2\,\psi(\alpha, \beta)}{\left\{\left|\frac{\partial\lambda}{\partial x}\right| \operatorname{sgn}\left(\frac{\partial\lambda}{\partial y}\right)\right\}_{x=\alpha,\, y=\beta}}.$$

It will be seen that

$$\Delta = \pm\frac{\pi^2\,\psi(\alpha, \beta)}{\left(\frac{\partial\lambda}{\partial x}\right)_{x=\alpha,\, y=\beta}},$$

where the ambiguous sign is that of γ_1. If $\gamma_1 > 0$,

$$\Delta = \pi^2\,\psi(\alpha, \beta) \Big/ \left(\frac{\partial\lambda}{\partial x}\right)_{\alpha,\,\beta};$$

and, since

$$\frac{\partial(\lambda,\, y-\beta)}{\partial(x, y)} = \frac{\partial\lambda}{\partial x},$$

we can obtain Δ by the same rule as is prescribed by Theorem A, provided we halve the result. If $\gamma_1 < 0$, we can apply the same rule, but then we must take the two singular curves to be

$$\lambda \equiv y-\beta = 0, \qquad \mu(x, y) = 0.$$

There is, of course, a corresponding theorem for the case in which the line C_1 is $x-\alpha = 0$.

17. The exceptional cases of Theorem A, referred to in § 15, are as follows:—

THEOREM A*a*.—*If the conditions of Theorem A are satisfied, except that* (α, β) *falls on a side, though not at a corner, of the rectangle, then the result of the theorem must be modified by dividing* Δ *by* 2, *and by inserting an additional sign of the principal value—before the outer sign of integration with respect to* x, *if* $\beta = b$ *or* B, *before the outer sign of integration with respect to* y, *if* $\alpha = a$ *or* A.

THEOREM A*b*.—*If, however,* (α, β) *falls at a corner of the rectangle, the repeated integrals cease to be convergent, except in the special case in* *which the singular curves make equal and opposite angles with the axes.* In this special case the result of Theorem A still holds if Δ is divided by 4.*

18. *A verification.*—It will, I think, tend to clearness if, before proceeding to applications of these theorems, I verify them on a simple example.

Let
$$f(x, y) = 1/\{(y-mx)(y-nx)\},$$
where $m \neq 0$, $n \neq 0$, $m \neq n$. First take $a = b = -1$, $A = B = 1$. Then
$$\int_{-1}^{1} dx\, P\int_{-1}^{1} f\,dy = \frac{1}{2(m-n)}\int_{-1}^{1} \log\left(\frac{1-mx}{1+mx}\cdot\frac{1+nx}{1-nx}\right)^2 \frac{dx}{x},$$
$$\int_{-1}^{1} dy\, P\int_{-1}^{1} f\,dx = \frac{1}{2(m-n)}\int_{-1}^{1} \log\left(\frac{y+m}{y-m}\cdot\frac{y-n}{y+n}\right)^2 \frac{dy}{y}.$$
The substitution $x = 1/y$ transforms the first of these integrals into
$$-\left(\int_{1}^{\infty} + \int_{-\infty}^{-1}\right) \log\left(\frac{y+m}{y-m}\cdot\frac{y-n}{y+n}\right)^2 \frac{dy}{y};$$
and the difference of the two repeated integrals is therefore
$$-\frac{1}{2(m-n)}\int_{-\infty}^{\infty} \log\left(\frac{y+m}{y-m}\cdot\frac{y-n}{y+n}\right)^2 dy = -\frac{2\pi^2}{m-n},\ 0,\ \frac{2\pi^2}{m-n},$$
according as (*a*) $m > 0 > n$, (*b*) m and n have the same sign, or (*c*) $m < 0 < n$. These results agree with Theorem A.

If we take $a = 0$, $A = 1$, $b = -1$, $B = 1$, we find
$$\int_{0}^{1} dx\, P\int_{-1}^{1} f\,dy = \frac{1}{2(m-n)}\int_{0}^{1}\log\left(\frac{1-mx}{1+mx}\cdot\frac{1+nx}{1-nx}\right)^2 \frac{dx}{x},$$
$$P\int_{-1}^{1} dy\, P\int_{0}^{1} f\,dx = \frac{1}{2(m-n)}\, P\int_{-1}^{1} \log\left(\frac{y-n}{y-m}\right)^2 \frac{dy}{y}.$$
The last integral is not convergent if the sign of the principal value is removed. It may, however, be transformed into
$$\frac{1}{2(m-n)}\int_{0}^{1} \log\left(\frac{y+m}{y-m}\cdot\frac{y-n}{y+n}\right)^2 \frac{dy}{y},$$
and we find, as before, that the difference of the repeated integrals is $-\pi^2/(m-n)$, 0, or $\pi^2/(m-n)$. This agrees with Theorem A*a*.

If we take $a = b = 0$, $A = B = 1$, the repeated integrals
$$\int_{0}^{1} dx\, P\int_{0}^{1} f\,dy = \frac{1}{2(m-n)}\int_{0}^{1}\log\left(\frac{1-mx}{1-nx}\right)^2 \frac{dx}{x}, \qquad \int_{0}^{1} dy\, P\int_{0}^{1} f\,dx = \frac{1}{2(m-n)}\int_{0}^{1}\log\left(\frac{y-n}{y-m}\right)^2 \frac{dy}{y}$$
are not convergent unless $m+n = 0$. If this condition is satisfied their difference is easily found to be $-\pi^2/4m$ or $\pi^2/4m$, according as m is positive or negative. This agrees with Theorem A*b*.

If we take $a < 0 < A$, $b = -1$, $B = 1$, and $n = 0$, so that
$$f(x, y) = 1/\{y(y-mx)\},$$

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* In this case, all of one singular curve, except its point of intersection with the other, lies outside the rectangle.

we obtain an illustration of Theorem B. Then

$$\int_a^A dx\, P\int_{-1}^1 f\,dy = \frac{1}{2m}\int_a^A \log\left(\frac{1-mx}{1+mx}\right)^2\frac{dx}{x}, \qquad P\int_{-1}^1 dy\, P\int_a^A f\,dx = \frac{1}{2m}P\int_{-1}^1 \log\left(\frac{y-ma}{y-mA}\right)^2\frac{dy}{y}.$$

The last integral, which is no longer convergent if the sign of the principal value is removed, is equal to

$$\frac{1}{2m}\int_0^1 \log\left(\frac{y-ma}{y+ma}\right)^2\frac{dy}{y} - \frac{1}{2m}\int_0^1 \log\left(\frac{y-mA}{y+mA}\right)^2\frac{dy}{y}$$

$$= -\frac{1}{2m}\int_{-\infty}^a \log\left(\frac{1-mx}{1+mx}\right)^2\frac{dx}{x} - \frac{1}{2m}\int_A^\infty \log\left(\frac{1-mx}{1+mx}\right)^2\frac{dx}{x};$$

and so the difference between the repeated integrals is

$$\frac{1}{2m}\int_{-\infty}^\infty \log\left(\frac{1-mx}{1+mx}\right)^2\frac{dx}{x} = -\frac{\pi^2}{|m|};$$

a result which agrees with Theorem B.

III.

Applications to the Inversion of Definite Integrals.

19. The preceding results have many interesting applications to the problem of the "inversion of a definite integral," which forms the starting point for the modern theory of "integral equations."

(i.) It is easy to see that

$$P\int_0^1 \cot\pi(y-\beta)\,dy = \tfrac{1}{2}\left[\log\sin^2\pi(y-\beta)\right]_0^1 = 0 \qquad (0<\beta<1).$$

Hence, if $0<x<1,\ 0<\beta<1,$ we have

$$P\int_0^1 \operatorname{cosec}\pi(x-y)\operatorname{cosec}\pi(y-\beta)\,dy$$

$$= \operatorname{cosec}\pi(x-\beta)\,P\int_0^1\{\cot\pi(x-y)+\cot\pi(y-\beta)\}\,dy = 0.$$

Now, take $a=b=0,\ A=B=1,$ and

$$f(x, y) = \operatorname{cosec}\pi(x-y)\operatorname{cosec}\pi(y-\beta)\,\phi(x).$$

The singular curves are $x-y=0$ and $y-\beta=0$, which intersect in the one point (β, β). Also $\lambda = \sin\pi(x-y)$, $(\partial\lambda/\partial x)_{\beta,\beta} = \pi$, and

$$\psi(\beta, \beta) = \lim_{x\to\beta,\, y\to\beta}\left\{\frac{y-\beta}{\sin\pi(y-\beta)}\phi(x)\right\} = \phi(\beta)/\pi.$$

Hence, by Theorem B, we obtain $\Delta = \phi(\beta)$. Thus

(1) $$P\int_0^1 \operatorname{cosec}\pi(y-\beta)\,dy\,P\int_0^1 \operatorname{cosec}\pi(x-y)\,\phi(x)\,dx = -\phi(\beta).$$

This result may be stated by means of the two formulæ

$$(2)\ P\int_0^1 \operatorname{cosec}\pi(x-y)\,\phi(x)\,dx = \chi(y), \quad P\int_0^1 \operatorname{cosec}\pi(x-y)\,\chi(x)\,dx = -\phi(y),$$

which is a typical "inversion formula." An example is given by the equations

$$P\int_0^1 \frac{\cos\pi(1-\lambda)x}{(\sin\pi x)^\lambda}\,\frac{dx}{\sin\pi(x-y)} = -\frac{\sin\pi(1-\lambda)y}{(\sin\pi y)^\lambda},$$

$$P\int_0^1 \frac{\sin\pi(1-\lambda)x}{(\sin\pi x)^\lambda}\,\frac{dx}{\sin\pi(x-y)} = \frac{\cos\pi(1-\lambda)y}{(\sin\pi y)^\lambda},$$

which I proved in a paper in the *Quarterly Journal* for 1901.* Here $\lambda < 1$. It will be observed that, if $\lambda > 0$, the subject of integration has further infinities along the lines $x = 0, 1$, such that its integral up to $x = 0$ or $x = 1$ is absolutely convergent. It is easy, and in no way relevant to our present purpose, to prove that our conclusions are not affected by this circumstance.

It is clear that the value of

$$P\int_0^1 \frac{dy}{\sin\pi(y-\beta)}\int_a^A \frac{\phi(x)}{\sin\pi(x-y)}\,dx$$

is $-\phi(\beta)$, provided only $a < \beta < A$. If $a < A < \beta$ or $\beta < a < A$, it is zero; if a or A is equal to β it is $-\frac{1}{2}\phi(\beta)$. But to take $a = 0$, $A = 1$ clearly leads to the most elegant result.

If $\phi(x) = 1$, we find that

$$\chi(y) = P\int_0^1 \operatorname{cosec}\pi(x-y)\,dx = \left[\frac{1}{2\pi}\log\tan^2\tfrac{1}{2}\pi(x-y)\right]_0^1 = \frac{1}{\pi}\log\cot^2\tfrac{1}{2}\pi y.$$

Thus, if $0 < y < 1$, we have

$$P\int_0^1 \frac{\log\cot^2\frac{1}{2}\pi x}{\sin\pi(x-y)}\,dx = -\pi.$$

We can, of course, obtain the values of any number of special integrals in this manner.

(ii.) Since

$$\cot\pi(x-y)\cot\pi(y-\beta) = \cot\pi(x-\beta)\{\cot\pi(x-y)+\cot\pi(y-\beta)\}+1,$$

we have $$P\int_0^1 \cot\pi(x-y)\cot\pi(y-\beta)\,dy = 1 \quad (0<x<1,\ 0<\beta<1).$$

Hence we obtain the formula

$$(3)\quad P\int_0^1 \cot\pi(y-\beta)\,dy\,P\int_0^1 \cot\pi(x-y)\,\phi(x)\,dx = \int_0^1 \phi(x)\,dx - \phi(\beta),$$

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* *Quart. Jour. of Math.*, Vol. XXXII., p. 383.

$\phi(x)$ being any function with continuous first and second differential coefficients.

In this case again we may replace the limits of integration with respect to x by any numbers a and A, such that $a < \beta < A$.

An equivalent manner of stating this result is by means of the pair of formulæ

$$(4)\qquad \begin{cases} P\int_0^1 \cot\pi(x-y)\,\phi(x)\,dx = \quad \chi(y) - \int_0^1 \chi(x)\,dx, \\ P\int_0^1 \cot\pi(x-y)\,\chi(x)\,dx = -\phi(y) + \int_0^1 \phi(x)\,dx. \end{cases}$$

These formulæ appear to have been given by Hilbert in his lectures: a proof, based on quite different ideas, has been given by Kellogg.*

(iii.) A third example is obtained by taking

$$f(x, y) = \frac{\phi(x)}{(\cos\pi x - \cos\pi y)(\cos\pi y - \cos\pi\beta)}.$$

In this case there are *three* corrections to be applied, viz., for the points (β, β), $(0, 0)$, and $(1, 1)$; and Theorems B and A*b* have both to be used. Using the equation

$$P\int_0^1 \frac{dy}{\cos\pi y - \cos\pi\beta} = 0 \quad (0 < \beta < 1),$$

we easily obtain

$$(5)\quad P\int_0^1 \frac{dy}{\cos\pi y - \cos\pi\beta}\, P\int_0^1 \frac{\phi(x)\,dx}{\cos\pi x - \cos\pi y}$$
$$= \frac{1}{\sin^2\pi\beta}\left\{\cos^2\tfrac{1}{2}\pi\beta\,\phi(0) + \sin^2\tfrac{1}{2}\pi\beta\,\phi(1) - \phi(\beta)\right\}.$$

This formula may be expressed in a particularly elegant form if $\phi(x) = \sin\pi x\,\psi(x)$, so that $\phi(0) = 0$, $\phi(1) = 1$. We then obtain the formulæ

$$(6)\qquad P\int_0^1 \frac{\sin\pi x\,\psi(x)\,dx}{\cos\pi x - \cos\pi y} = \chi(y), \quad P\int_0^1 \frac{\sin\pi y\,\chi(x)\,dx}{\cos\pi x - \cos\pi y} = -\psi(y).$$

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* *Math. Annalen*, Bd. LVIII., p. 442. It is to be observed that the term

$$\int_0^1 \chi(x)\,dx$$

in the first line of (4) is inserted merely for the sake of formal parallelism with the second line; it may be removed, or any other constant substituted, without affecting the result.

For example, if $\phi(x) = \cos n\pi x$, and if we notice that

$$P\int_0^1 \frac{\cos n\pi x}{\cos \pi x - \cos \pi y}\,dx = \frac{\sin n\pi y}{\sin \pi y},$$

we obtain the formula

$$P\int_0^1 \frac{\sin n\pi y}{\sin \pi y}\,\frac{dy}{\cos \pi y - \cos \pi\beta} = \frac{1}{\sin^2 \pi\beta}\left\{\cos^2 \tfrac{1}{2}\pi\beta + (-1)^n \sin^2 \tfrac{1}{2}\pi\beta - \cos n\pi\beta\right\}.$$

With this formula may be associated

$$\int_0^1 \frac{\sin n\pi y}{\sin \pi y}\,\frac{dy}{\cosh \pi\beta - \cos \pi y} = \frac{1}{\sinh^2 \pi\beta}\left\{\cosh^2 \tfrac{1}{2}\pi\beta - (-1)^n \sinh^2 \tfrac{1}{2}\pi\beta - e^{-n\pi\beta}\right\},$$

which may be obtained by applying Theorem A*b* to the function

$$f(x, y) = \frac{\cos n\pi x}{(\cos \pi x - \cos \pi y)(\cosh \pi\beta - \cos \pi y)};$$

or otherwise deduced from well known results.

20. (iv.) The formulæ (1)–(6) are capable of various interesting transformations. Thus, if in (2), we write

$$\tan \pi x = t, \quad \tan \pi y = \tau, \quad \frac{\phi(\arctan t/\pi)}{\sqrt{(1+t^2)}} = \lambda(t), \quad \frac{\chi(\arctan \tau/\pi)}{\sqrt{(1+\tau^2)}} = \mu(\tau),$$

we are led to the formulæ

$$(7) \qquad P\int_{-\infty}^{\infty} \frac{\lambda(t)}{t-\tau}\,dt = \pi\mu(\tau), \qquad P\int_{-\infty}^{\infty} \frac{\mu(t)}{t-\tau}\,dt = -\pi\lambda(\tau).$$

This is a most interesting pair of formulæ, and it is worth while to attempt to determine as precisely as possible sufficient conditions to be satisfied by $\lambda(t)$ in order that the formulæ shall certainly be valid. But we shall defer this for a moment, until we arrive at the formulæ more directly (see §§ 22–24).

Similarly, the transformation $\tan \frac{1}{2}\pi x = t$, $\tan \frac{1}{2}\pi y = \tau$ leads us to the equations

$$(8) \qquad P\int_0^{\infty} \frac{1+\tau^2}{(t-\tau)(1+t\tau)}\,\lambda(t)\,dt = \pi\mu(\tau), \qquad P\int_0^{\infty} \frac{1+\tau^2}{(t-\tau)(1+t\tau)}\,\mu(t)\,dt = -\pi\lambda(\tau).$$

Similarly, too, we obtain from (4) the formulæ

$$P\int_{-\infty}^{\infty} \frac{1+t\tau}{t-\tau}\,\frac{\lambda(t)\,dt}{1+t^2} = \pi\mu(\tau) - \int_{-\infty}^{\infty} \frac{\mu(\tau)\,d\tau}{1+\tau^2},$$

$$P\int_{-\infty}^{\infty} \frac{1+t\tau}{t-\tau}\,\frac{\mu(t)\,dt}{1+t^2} = -\pi\lambda(\tau) + \int_{-\infty}^{\infty} \frac{\lambda(\tau)\,d\tau}{1+\tau^2},$$

also given by Kellogg.*

Finally, from (6), we obtain

$$(9) \qquad P\int_0^{\infty} \frac{t}{t^2-\tau^2}\,\frac{\lambda(t)}{1+t^2}\,dt = -\tfrac{1}{2}\pi\,\frac{\mu(\tau)}{1+\tau^2}, \qquad P\int_0^{\infty} \frac{\tau}{t^2-\tau^2}\,\mu(t)\,dt = \tfrac{1}{2}\pi\lambda(\tau).$$

21. (v.) If we take $a = b = -1$, $A = B = 1$,

$$f(x, y) = \frac{xy\,\phi(x)}{(x-y)(1-xy)(y-\beta)(1-\beta y)}, \quad \frac{(1+x^2)(1+y^2)\,\phi(x)}{(x-y)(1-xy)(y-\beta)(1-\beta y)},$$

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* *L.c.*, p. 451.

and observe that

$$P\int_{-1}^{1} \frac{y\,dy}{(x-y)(1-xy)(y-\beta)(1-\beta y)} = P\int_{-1}^{1} \frac{(1+y^2)\,dy}{(x-y)(1-xy)(y-\beta)(1-\beta y)} = 0,$$

if $-1 < x < 1$, $-1 < \beta < 1$, we are led to the equations

$$(10)\quad \int_{-1}^{1} \frac{y\,dy}{(y-\beta)(1-\beta y)}\, P\int_{-1}^{1} \frac{x\phi(x)\,dx}{(x-y)(1-xy)}$$
$$= \left\{\frac{\pi}{2(1-\beta^2)}\right\}^2 \left\{-4\beta^2\phi(\beta) + (1+\beta)^2\phi(1) + (1-\beta)^2\phi(-1)\right\},$$

$$(11)\quad P\int_{-1}^{1} \frac{(1+y^2)\,dy}{(y-\beta)(1-\beta y)}\, P\int_{-1}^{1} \frac{(1+x^2)\,\phi(x)\,dx}{(x-y)(1-xy)}$$
$$= \left(\frac{\pi}{1-\beta^2}\right)^2 \left\{-(1+\beta^2)^2\phi(\beta) + (1+\beta)^2\phi(1) + (1-\beta)^2\phi(-1)\right\}.$$

Each of these results may evidently be stated as an inversion formula; and either of them, or any of the preceding formulæ, may be applied to give the values of any number of particular integrals. Thus, to put $\phi(x) = 1$ in (10) leads to the formula

$$P\int_{-1}^{1} \log\left(\frac{1+y}{1-y}\right) \frac{dy}{(y-\beta)(1-\beta y)} = \frac{\pi^2}{2(1-\beta^2)} \quad (0 < \beta < 1).$$

22. *Examples with Infinite Limits.*—I shall not attempt to formulate any general theorems as extensions of Theorems A, &c., to the cases in which some or all of the limits are infinite. Such cases are better considered individually; and I shall consider one case, perhaps the most interesting of all, in detail.

(vi.) Let us suppose first that $b = -\infty$, $B = \infty$, and

$$f(x, y) = \frac{\phi(x)}{(x-y)(y-\beta)}.$$

Since

$$P\int_{-\infty}^{\infty} \frac{dy}{(x-y)(y-\beta)} = 0,$$

the result to which we are led is

$$(12)\qquad P\int_{-\infty}^{\infty} \frac{dy}{y-\beta}\, P\int_a^A \frac{\phi(x)\,dx}{x-y} = -\Delta,$$

where $\Delta = 0$, if β falls outside the interval (a, A), $\Delta = \pi^2\phi(\beta)$, if $a < \beta < A$, and $\Delta = \frac{1}{2}\pi^2\phi(\beta)$, if $\beta = a$ or A. Let us suppose $a < \beta < A$.

Theorem B assures us of the truth of the equation

$$(13)\quad P\int_b^B dy\, P\int_a^A \frac{\phi(x)\,dx}{x-y} = \int_a^A \phi(x)\,dx\, P\int_b^B \frac{dy}{(x-y)(y-\beta)} - \pi^2\phi(\beta),$$

if $b < \beta < B$. It is clear that the conditions that we may replace b and

B by $-\infty$ and ∞ in this equation are that

$$\int_a^A \phi(x)\,dx \int_B^\infty \frac{dy}{(x-y)(y-\beta)} \to 0, \quad \int_a^A \phi(x)\,dx \int_{-\infty}^b \frac{dy}{(x-y)(y-\beta)} \to 0,$$

as $B \to \infty$ and $b \to -\infty$. But these conditions are certainly satisfied. For the first equation is

$$\lim_{B\to\infty} \int_a^A \phi(x) \log\left(\frac{B-x}{B-\beta}\right) \frac{dx}{x-\beta} = 0.$$

Now $\log\left(\frac{B-x}{B-\beta}\right) = \log\left(1-\frac{x-\beta}{B-\beta}\right) = -\left(\frac{x-\beta}{B-\beta}\right)\Big/\left(1-\theta\frac{x-\beta}{B-\beta}\right),$

where $0 < \theta < 1$: thus, for large values of B, we have

$$\left|\log\left(\frac{B-x}{B-\beta}\right)\right| < \frac{K}{B}|x-\beta|;$$

and from this the result follows at once. Hence

$$(14) \qquad P\int_{-\infty}^\infty \frac{dy}{y-\beta}\, P\int_a^A \frac{\phi(x)\,dx}{x-y} = -\Delta.$$

23. The question as to whether we can replace a and A by $-\infty$ and ∞ in this equation is decidedly more difficult. It is most convenient to begin by first establishing the equation

$$(15) \quad P\int_b^B \frac{dy}{y-\beta}\, P\int_{-\infty}^\infty \frac{\phi(x)\,dx}{x-y} = \int_{-\infty}^\infty \phi(x)\,dx\, P\int_b^B \frac{dy}{(x-y)(y-\beta)} - \pi^2\phi(\beta)$$

$$(b < \beta < B).$$

I shall prove that this equation holds if only

$$\int^\infty \frac{\phi(x)}{x}\,dx, \qquad \int_{-\infty} \frac{\phi(x)}{x}\,dx$$

are convergent.

It is clear that (13) will pass over into (15) as $a \to -\infty$ and $A \to \infty$, if only

$$(16) \qquad \lim_{A\to\infty} P\int_b^B \frac{dy}{y-\beta} \int_A^\infty \frac{\phi(x)}{x-y}\,dx = 0,$$

with a similar equation involving a. Now the integral

$$\chi(A, y) = \int_A^\infty \frac{\phi(x)}{x-y}\,dx$$

is convergent, since $x/(x-y)$ is monotonic, and uniformly convergent* for

* See Bromwich, *Infinite Series*, pp. 433 *et seq.*

$b \leqslant y \leqslant B$, or, indeed, throughout any finite interval of values of y. For a similar reason, we have

$$\frac{\partial \chi(A, y)}{\partial y} = \int_A^\infty \frac{\phi(x)}{(x-y)^2}\, dx.$$

Also $\chi(A, y) = \dfrac{A}{A-y}\displaystyle\int_A^{A'} \frac{\phi(x)}{x}\, dx, \quad \dfrac{\partial \chi(A, y)}{\partial y} = \dfrac{A}{(A-y)^2}\displaystyle\int_A^{A''} \frac{\phi(x)}{x}\, dx,$

where $A' > A$, $A'' > A$. Hence, given ϵ, we can so choose A_0 that

$$|\chi| < \epsilon, \qquad \left|\frac{\partial \chi}{\partial y}\right| < \epsilon / A,$$

for $A \geqslant A_0$ and throughout any finite interval of values of y. Now

$$P\int_b^B \chi(A, y)\frac{dy}{y-\beta} = \left(P\int_b^{2\beta-b} + \int_{2\beta-b}^B\right)\chi(A, y)\frac{dy}{y-\beta} = J_1 + J_2,$$

where
$$J_1 = \int_0^{\beta-b} \{\chi(A, \beta+u) - \chi(A, \beta-u)\}\frac{du}{u}$$
$$= 2\int_0^{\beta-b} \frac{\partial \chi(A, t)}{\partial t}\, du$$

(t being a function of u which certainly lies within the larger of the two intervals $b < t < 2\beta - b$, $b < t < B$); and

$$J_2 = \int_{2\beta-b}^B \chi(A, y)\frac{dy}{y-\beta} = \chi(A, w)\int_{2\beta-b}^B \frac{dy}{y-\beta},$$

w lying between $2\beta - b$ and B. From these equations, and the inequalities for χ and $\partial\chi/\partial y$ proved above, the truth of (16), and therefore of (15), follows immediately.

The question which remains is whether we can replace b and B by $-\infty$ and ∞ in (15). And this will be so if only

$$(17) \quad \int_{-\infty}^\infty \phi(x)\,dx\, P\int_B^\infty \frac{dy}{(x-y)(y-\beta)} = \tfrac{1}{2}\int_{-\infty}^\infty \phi(x)\log\left(\frac{B-x}{B-\beta}\right)^2 \frac{dx}{x-\beta} \to 0,$$

with a similar condition in b. That this is so when the limits are a and A was proved in § 22; hence it is enough for us to prove that

$$(18) \qquad \lim_{B\to\infty} \tfrac{1}{2}\int_A^\infty \phi(x)\log\left(\frac{B-x}{B-\beta}\right)^2 \frac{dx}{x-\beta} = 0 \quad (A > \beta).$$

Moreover we may without real loss of generality take $\beta = 0$, and consider the equation

$$(18') \qquad \lim_{B\to\infty} \tfrac{1}{2}\int_A^\infty \phi(x)\log\left(1 - \frac{x}{B}\right)^2 \frac{dx}{x} = 0.$$

24. I shall now prove that the conditions (i.) that $|\phi(x)| < K$, and (ii.) that

$$\int^{\infty} \phi(x) \frac{\log x}{x} dx$$

is convergent, are sufficient to ensure the truth of (18′). Since $1/(\log x)$ is monotonic, the second of these ensures that $\int^{\infty} \frac{\phi(x)}{x}$ is convergent, as has already been assumed.

Now, let

$$\int_A^{\infty} = \int_A^{A_1} + \int_{A_1}^{(1-\lambda)B} + \int_{(1-\lambda)B}^{(1+\lambda)B} + \int_{(1+\lambda)B}^{\infty} = J_1 + J_2 + J_3 + J_4,$$

say, where $0 < \lambda < 1$. Choose A_1 so that

$$(19) \qquad \left| \int_{A'}^{A''} \frac{\phi(x)}{x} dx \right| < \sigma,$$

for $A'' > A' \geqslant A_1$. When A_1 has been chosen we can choose B_1, so that

$$(20) \qquad |J_1| < \sigma \quad (B \geqslant B_1).$$

Also the function $\log(1-x/B)^2$ is negative for $A_1 < x < (1-\lambda)B$, and decreases algebraically as x increases; hence we may apply Bonnet's form of the second mean-value theorem* to the integral J_2, and so obtain

$$J_2 = \log \lambda \int_{\xi}^{(1-\lambda)B} \frac{\phi(x)}{x} dx,$$

where $A_1 < \xi < (1-\lambda)B$; and so we can choose B_2 so that

$$(21) \qquad |J_2| < \sigma \log\left(\frac{1}{\lambda}\right) \quad (B \geqslant B_2).$$

Again, $$J_3 = \tfrac{1}{2} \int_{1-\lambda}^{1+\lambda} \phi(\xi B) \log(1-\xi)^2 \frac{d\xi}{\xi},$$

and so

$$(22) \quad J_3 < K \int_{1-\lambda}^{1+\lambda} \log\left(\frac{1}{1-\xi}\right)^2 \frac{d\xi}{\xi} < \frac{K}{1-\lambda} \int_{1-\lambda}^{1+\lambda} \log\left(\frac{1}{1-\xi}\right)^2 d\xi$$

$$= \frac{2K\lambda}{1-\lambda}\left\{1 + \log\left(\frac{1}{\lambda}\right)\right\} < K\lambda \left\{1 + \log\left(\frac{1}{\lambda}\right)\right\},$$

provided $1/(1-\lambda) < K$, a condition which will certainly be satisfied.

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* See, for example, Bromwich's *Infinite Series*, pp. 426 *et seq.*, or my *Course of Pure Mathematics*, p. 286.

Finally, $$J_4 = J_{4,1}+J_{4,2}+J_{4,3},$$

where $$J_{4,1} = \int_{(1+\lambda)B}^{\infty} \frac{\phi(x)\log x}{x}\,dx, \quad J_{4,2} = -\log B \int_{(1+\lambda)B}^{\infty} \frac{\phi(x)}{x}\,dx,$$

$$J_{4,3} = \int_{(1+\lambda)B}^{\infty} \frac{\phi(x)}{x} \log\left(1-\frac{B}{x}\right) dx.$$

We can choose $B_{4,1}$, so that

(23) $$|J_{4,1}| < \sigma \quad (B \geqslant B_{4,1});$$

and, since $$J_{4,2} = -\log B \int_{(1+\lambda)B}^{\infty} \frac{\phi(x)\log x}{x}\,\frac{dx}{\log x}$$

$$= -\frac{\log B}{\log\{(1+\lambda)B\}} \int_{(1+\lambda)B}^{\xi} \frac{\phi(x)\log x}{x}\,dx,$$

where $\xi > (1+\lambda)B$, we can choose $B_{4,2}$, so that

(24) $$|J_{4,2}| < \sigma \quad (B \geqslant B_{4,2}).$$

And $$J_{4,3} = \log\left(\frac{\lambda}{1+\lambda}\right) \int_{(1+\lambda)B}^{\xi} \frac{\phi(x)}{x}\,dx,$$

so that we can choose $B_{4,3}$ so that

(25) $$|J_{4,3}| < \sigma \log(1/\lambda) \quad (B \geqslant B_{4,3}).$$

Let B_0 be the greatest of B_1, B_2, $B_{4,1}$, $B_{4,2}$, $B_{4,3}$. Then

$$\left|\tfrac{1}{2}\int_A^{\infty} \phi(x)\log\left(1-\frac{x}{B}\right)^2 \frac{dx}{x}\right| \leqslant |J_1|+|J_2|+|J_3|+|J_{4,1}|+|J_{4,2}|+|J_{4,3}|$$

$$< \sigma\left\{3+2\log\left(\frac{1}{\lambda}\right)\right\}+K\lambda\left\{1+\log\left(\frac{1}{\lambda}\right)\right\} \quad (B \geqslant B_0).$$

Given δ we can choose λ, so that $K\lambda\left\{1+\log\left(\frac{1}{\lambda}\right)\right\} < \frac{1}{2}\delta$, and then σ, so that $\sigma\{3+2\log(1/\lambda)\} < \frac{1}{2}\delta$. It follows that the integral tends to zero as $B \to \infty$, and we may therefore replace B by ∞ in (15). The result at which we thus arrive seems of sufficient interest to be stated as a theorem.

THEOREM C.—*If* (i.) $\phi(x)$ *and its first two derivatives are continuous,* (ii.) $|\phi(x)| < K$ *for all values of* x, *and* (iii.) *the integrals*

$$\int^{\infty} \frac{\phi(x)\log x}{x}\,dx, \quad \int_{-\infty} \frac{\phi(x)\log(-x)}{x}\,dx$$

are convergent, then will

$$P\int_{-\infty}^{\infty} \frac{dy}{y-\beta} P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x-y} = -\pi^2\phi(\beta)\,;$$

or, in other words, if $$P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x-y} = \pi\chi(y),$$

then will $$P\int_{-\infty}^{\infty} \frac{\chi(x)\,dx}{x-y} = -\pi\phi(y).$$

The form of this result has been already arrived at otherwise (§ 20), but without an accurate investigation of sufficient conditions for its truth. A familiar example is obtained by taking

$$\phi(x) = \cos mx, \qquad \chi(x) = -\sin mx.$$

The inversion formula itself has a familiar look, but I am not aware that I have ever seen it before, much less any accurate discussion of it proceeding by methods in any way resembling those of the preceding sections.

25. (vii.) The following formulæ are of a similar character :—

(a) $$\int_0^{\infty} \log\left(\frac{x+\eta}{x-\eta}\right)^2 \phi(x)\frac{dx}{x} = 4\int_0^{\eta} dy\, P\int_0^{\infty} \frac{\phi(x)\,dx}{x^2-y^2} + \pi^2\phi(0),$$

$$\int_0^{\infty} dy\, P\int_0^{\infty} \frac{\phi(x)\,dx}{x^2-y^2} = -\tfrac{1}{4}\pi^2\phi(0)\,;$$

(b) $$P\int_0^{\infty} \frac{dy}{y^2-\beta^2} P\int_0^{\infty} \frac{\phi(x)\,dx}{x^2-y^2} = \left(\frac{\pi}{2\beta}\right)^2 \{\phi(0)-\phi(\beta)\},$$

$$P\int_0^{\infty} \frac{x\phi(x)}{x^2-y^2}\,dx = \tfrac{1}{2}\pi\chi(y), \qquad P\int_0^{\infty} \frac{y\chi(x)}{x^2-y^2}\,dx = -\tfrac{1}{2}\pi\phi(y)\,;$$

(c) $$P\int_0^{\infty} \frac{y^2\,dy}{y^2-\beta^2} P\int_0^{\infty} \frac{\phi(x)\,dx}{x^2-y^2} = -\tfrac{1}{4}\pi^2\phi(\beta).$$

The last of these formulæ, which also leads to the inversion formula written just above it, was given by Schlömilch (*Analytische Studien*, Bd. II., p. 156), but without any accurate definition or discussion. Interesting formulæ may be obtained from any of the above equations by particular choices of $\phi(x)$, *e.g.*, $\frac{1}{x^2+a^2}$, $\frac{\cos mx}{x^2+a^2}$, ...

26. (viii.) The following example, which is of a somewhat different character, leads to a very interesting generalisation of Theorem C.

Take $a = b = -\infty$, $A = B = \infty$, and

$$f(x, y) = \frac{\phi(x)}{(y-\beta)\{x-R(y)\}},$$

where $R(y)$ is a rational function of y whose numerator is of higher degree than its denominator, and which is such that the equation $R(y) = x$ has, for all values of x, *real* roots only. Then

$$P\int_{-\infty}^{\infty} \frac{dy}{(y-\beta)\{x-R(y)\}} = 0;$$

and so we are led to the formulæ

(26)
$$P\int_{-\infty}^{\infty} \frac{dy}{y-\beta} P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x-R(y)} = \mp\pi^2\phi\{R(\beta)\},$$

(27)
$$\begin{cases} P\int_{-\infty}^{\infty} \dfrac{\phi(x)\,dx}{x-y} = \pi\chi(y), \\ P\int_{-\infty}^{\infty} \dfrac{\chi\{R(x)\}\,dx}{x-y} = \mp\pi\phi\{R(y)\}, \end{cases}$$

where the upper or lower sign is to be chosen according as $R'(\beta) \gtrless 0$. The last formulæ embody a generalisation of Theorem C. In particular we may take

$$R(y) = ay - \frac{b}{y-B} - \frac{c}{y-C} - \dots - \frac{k}{y-K} \quad (a, b, c, \dots, k > 0).$$

Suppose, for example, that

$$R(y) = ay - \frac{b}{y} \quad (a, b > 0), \qquad \phi(x) = e^{ix}, \qquad \chi(x) = ie^{ix}.$$

We obtain*
$$P\int_{-\infty}^{\infty} e^{i(ax-b/x)} \frac{dx}{x-y} = \pi i e^{i(ay-b/y)},$$

$$P\int_0^{\infty} \cos\left(ax - \frac{b}{x}\right) \frac{dx}{x^2-y^2} = -\tfrac{1}{2}\pi \sin\left(ay - \frac{b}{y}\right), \quad P\int_0^{\infty} \sin\left(ax - \frac{b}{x}\right) \frac{x\,dx}{x^2-y^2} = \tfrac{1}{2}\pi\cos\left(ay - \frac{b}{y}\right).$$

With these formulæ may be associated those obtained from the formula

$$\int_{-\infty}^{\infty} \frac{dy}{y^2+\beta^2} P\int_{-\infty}^{\infty} \frac{\phi(x)}{x-R(y)}\,dx = \int_{-\infty}^{\infty} \phi(x)\,dx\, P\int_{-\infty}^{\infty} \frac{dy}{(y^2+\beta^2)\{x-R(y)\}} = \frac{\pi}{\beta}\mathbf{R}\left[\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x-R(\beta i)}\right]$$

[where $\phi(x)$ is real]. For example, taking $\phi(x) = \sin x$, $R(y) = ay - b/y$, we obtain

$$\int_0^{\infty} \cos\left(ay - \frac{b}{y}\right) \frac{dy}{y^2+\beta^2} = \frac{1}{\beta}\int_0^{\infty} \frac{x \sin x\, dx}{x^2 + \left(a\beta + \dfrac{b}{\beta}\right)^2} = \frac{\pi}{2\beta} e^{-a\beta - b/\beta}.$$

27. It is instructive to consider also the case in which the roots of $x = R(y)$ are not always real. Suppose, *e.g.*,

$$R(y) = ay + \frac{b}{y} \quad (a, b > 0).$$

The roots are real only if $x^2 \geqslant 4ab$, and

$$P\int_{-\infty}^{\infty} \frac{dy}{(y-\beta)\left(x - ay - \dfrac{b}{y}\right)} = -\frac{x-(2b/\beta)}{x-a\beta-(a/\beta)} \frac{\pi}{\sqrt{(4ab-x^2)}},\ 0,$$

* See *Quarterly Journal*, Vol. XXXII., p. 374; Bromwich, *Infinite Series*, p. 496.

according as $x^2 < 4ab$ or $x^2 > 4ab$. Let us, for simplicity, take $\beta = 1$. Then we obtain

$$-\pi \int_{-2\sqrt{(ab)}}^{2\sqrt{(ab)}} \frac{x-2b}{x-a-b} \frac{\phi(x)\,dx}{\sqrt{(4ab-x^2)}} = P\int_{-\infty}^{\infty} \frac{dy}{y-1}\, P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x-ay-(b/y)} + \pi^2\phi(a+b)\,\mathrm{sgn}\,(a-b).$$

Suppose $b < a$, and let $\tau = \sqrt{(b/a)}$, $m = 2\sqrt{(ab)}$. Then, making the substitution $x = mt$, we obtain, after a little reduction, the formulæ

$$P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x-y} = \pi\chi(y),$$

$$P\int_{-\infty}^{\infty} \frac{\chi\left(ay+\frac{b}{y}\right)}{y-1}\,dy = -\pi\phi(a+b) + 2\tau\int_0^{\pi} \frac{\cos\theta-\tau}{1-2\tau\cos\theta+\tau^2}\,\phi(m\cos\theta)\,d\theta.$$

In particular, the assumption $\phi(x) = e^{ix}$ leads to the formulæ

$$P\int_0^{\infty} \cos\left(ax+\frac{b}{x}\right)\frac{dx}{1-x^2} = \tfrac{1}{2}\pi\sin(a+b) - \tfrac{1}{2}\int_0^{\pi} \frac{1-\tau^2}{1-2\tau\cos\theta+\tau^2}\sin(m\cos\theta)\,d\theta$$

$$= \tfrac{1}{2}\pi\sin(a+b) - \pi(\tau J_1 - \tau^3 J_3 + \tau^5 J_5 - \ldots),$$

$$P\int_0^{\infty} \sin\left(ax+\frac{b}{x}\right)\frac{dx}{1-x^2} = -\tfrac{1}{2}\pi\cos(a+b) + \tfrac{1}{2}\int_0^{\pi} \frac{1-\tau^2}{1-2\tau\cos\theta+\tau^2}\cos(m\cos\theta)\,d\theta$$

$$= -\tfrac{1}{2}\pi\cos(a+b) - \pi(\tau^2 J_2 - \tau^4 J_4 + \tau^6 J_6 - \ldots),$$

where $0 < b < a$, $\tau = \sqrt{(b/a)}$, $m = 2\sqrt{(ab)}$, and the J's denote Bessel's functions with argument m. With these should be associated the further formulæ*

$$\int_0^{\infty} \cos\left(ax+\frac{b}{x}\right)\frac{dx}{1+x^2} = \tfrac{1}{2}\pi e^{-(a-b)} - \pi(\tau J_1 + \tau^3 J_3 + \tau^5 J_5 + \ldots),$$

$$\int_0^{\infty} \sin\left(ax+\frac{b}{x}\right)\frac{dx}{1+x^2} = \tfrac{1}{2}\pi e^{-(a-b)} - \pi(\tau^2 J_2 + \tau^4 J_4 + \tau^6 J_6 + \ldots).$$

28. It is unnecessary to multiply examples of the very large number of classes of formulæ that can be obtained in this kind of way. There is, however, one kind of case that we have not yet considered. In all the cases that have been considered so far, the number of intersections of the singular curves has been finite. I shall conclude by considering one example in which this is not the case. It is, of course, possible to construct any number of such examples, and the justification of any result thus obtained involves theoretical difficulties additional to those which have been dealt with already. With these, however, I do not propose to deal; for the applications of such results as I have obtained do not seem to be of great interest. I shall, therefore, merely give a single example of the kind of formulæ that can be obtained.

It is easy to see that, if x, β, and b are real and no two of them differ by an integer, then

$$P\int_{-\infty}^{\infty} \frac{1}{\sin\pi(x-y)\sin\pi(y-\beta)}\frac{dy}{y-b} = 0.$$

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* *Messenger of Mathematics*, Vol. XXXVIII., p. 129.

The result follows, in fact, at once, when Cauchy's theorem is applied to the integral

$$\int \frac{1}{\sin \pi(x-y) \sin \pi(y-\beta)} \frac{dy}{y-b},$$

the contour of integration being the ordinary "infinite semicircle" described on and above the real axis, and the poles of the subject of integration, which are all real, being avoided by small semicircles in the usual manner.

Suppose, then, that

$$f(x, y) = \frac{\phi(x)}{(x-a)(y-b) \sin \pi(x-y) \sin \pi(y-\beta)}.$$

We obtain

$$0 = P\int_{-\infty}^{\infty} \frac{dy}{(y-b) \sin \pi(y-\beta)} P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{(x-a) \sin \pi(x-y)} + \Sigma\Delta,$$

where the sign of summation applies to all points of intersection of singular curves.

These points are (i.) $x = a$, $y = b$; (ii.) $x = a$, $y = a+n$; (iii.) $x = b+m$, $y = b$; (iv.) $x = a$, $y = \beta+n$; (v.) $x = \beta+m+n$, $y = \beta+n$; where m and n have any integral values, positive or negative. In order to ensure that *three* curves shall in no case intersect in one point it is necessary to suppose that no two of a, b, β differ by an integer.

We easily find

$$\Delta_{a,b} = 0,$$

$$\Delta_{a,a+n} = \frac{\pi\phi(a)}{\sin \pi(a-\beta)} \frac{1}{a-b+n},$$

$$\Delta_{b+m,b} = \frac{\pi}{\sin \pi(b-\beta)} \frac{(-1)^m \phi(b+m)}{b-a+m},$$

$$\Delta_{a,\beta+n} = 0,$$

$$\Delta_{\beta+m+n,\beta+n} = \frac{(-1)^{m+n} \phi(\beta+m+n)}{(\beta+m+n-a)(\beta+n-b)};$$

and, after a little reduction, arrive at the formulæ

$$P\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{(x-a) \sin \pi(x-y)} = \pi\chi(y),$$

$$P\int_{-\infty}^{\infty} \frac{\chi(x)\,dx}{(x-b)\sin\pi(x-y)}$$

$$= -\pi\phi(a)\cot\pi(b-a)\operatorname{cosec}\pi(y-a) - \cot\pi(y-b)\sum_{-\infty}^{\infty}\frac{(-1)^m\phi(y+m)}{y-a+m}$$

$$+\operatorname{cosec}\pi(y-b)\sum_{-\infty}^{\infty}\frac{(-1)^m\phi(b+m)}{b-a+m}.$$

For instance, if $\phi\equiv 1$, then $\chi\equiv 0$. Hence we should have

$$0 = -\pi\cot\pi(b-a)\operatorname{cosec}\pi(y-a)$$

$$-\cot\pi(y-b)\sum_{-\infty}^{\infty}\frac{(-1)^m}{y-a+m} + \operatorname{cosec}\pi(y-b)\sum_{-\infty}^{\infty}\frac{(-1)^m}{b-a+m},$$

an equation the truth of which may easily be verified.