THE APPLICATION OF BASIC NUMBERS TO BESSEL'S AND LEGENDRE'S FUNCTIONS

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CONTENTS.

PART I.

1.

Let $[n]$ denote $\dfrac{p^n-1}{p-1}$; then, since $[1] = 1$,

$$[2] = 1+p, \qquad [3] = 1+p+p^2,$$

and $$[-1] = -p^{-1}, \qquad [-2] = -p^{-1}-p^{-2}, \qquad \dots .$$

The number $[n]$ is analogous to the natural number n. Various functions analogous to the functions x^n, $\exp(x)$, $\Gamma(x)$, $J_n(x)$, $P_n(x)$ may be formed, in which these numbers $[1]$, $[2]$, ... occupy the place taken by the natural numbers in the ordinary functions of analysis. Such generalized functions may be obtained as solutions of differential equations analogous to the equations satisfied by the simpler functions. In any number $[n]$ a base p is implied, and, if the base $p = 1$, the number reduces, in the limit, to the natural number n. It may be convenient to call $[n]$ the *basic num-*

* Four papers, two communicated in March and two in April, have been condensed into a single paper.

ber n, and the functions formed in this manner *basic functions*. In this paper the application of the numbers will be directed towards obtaining the following extension of Neumann's addition theorem for the function J_0.

$$\mathrm{J}_0(be^{\iota\beta}, ce^{\iota\gamma}) = J_{[0]}(b)\,\mathfrak{J}_{[0]}\left(\frac{c}{p}\right) - 2p\cos a\, J_{[1]}(b)\,\mathfrak{J}_{[1]}\left(\frac{c}{p}\right)$$

$$+2p^4\cos 2a\, J_{[2]}(b)\,\mathfrak{J}_{[2]}\left(\frac{c}{p}\right) - \ldots + (p-1)\,\phi(bc\beta\gamma)$$

$$(\beta-\gamma = a).$$

This is only one of an infinite number of independent addition theorems, all of which reduce, when $p=1$, to Neumann's addition theorem for $J_0(R)$. The form of the general series which contains the above and a similar series symmetrical in b and c as particular cases will be indicated. The second part of the paper treats of the series corresponding to the various transformations of Legendre's functions, especially the expansions of $P_n(\cos\theta)$. Incidentally it is shown that

$$(y-x)^{-1} = \Sigma\,[2n+1]\,P_{[n]}(x)\,Q_{[n]}(y),$$

and the summation of a case of $F([a][\beta][\gamma][\delta][\epsilon])$ is effected.

2.

The results numbered (1)–(7) in this article are such as will be required in subsequent work. The theorems corresponding to the binomial and exponential theorems are well known in other forms, but it seems convenient to collect them together and express them in the form most useful for reference. If n be a positive integer, we take

$$(1-x)_n = (1-x)(1-px)(1-p^2x)\ldots(1-p^{n-1}x).$$

In general $$(1-x)_n = \prod_{r=0}^{\infty}\frac{(1-p^{n-r-1}x)}{(1-p^{-r-1}x)} \quad (p>1). \tag{a}$$

If, however, $p<1$, the proper infinite product expression for $(1-x)_n$ is

$$(1-x)_n = \mathop{\mathrm{L}}_{\kappa=\infty}\frac{(1-x)(1-px)\ldots(1-p^{\kappa}x)}{(1-p^{n}x)(1-p^{n+1}x)\ldots(1-p^{n+\kappa}x)}. \tag{β}$$

The expansion of these products in infinite series has been considered by many writers, and we may express the analogue of the binomial theorem as

$$(1-x)_n = 1-\frac{[n]}{[1]}x+p\frac{[n][n-1]}{[2]!}x^2-\ldots$$

$$+(-1)^r p^{\frac{1}{2}r(r-1)}\frac{[n][n-1]\ldots[n-r+1]}{[r]!}x^r+\ldots. \tag{1}$$

The series is convergent if $p>1$ and $x<p$, if $p<1$ and $x<p^{-n}$, if $p=1$ and $x<1$.

From (a) and (β) we obtain without difficulty

$$\frac{(1-x)_{-n}}{(1-p^{-n-1}x)}=(1-x)_{-n-1}, \qquad \frac{(1-x)_{-n}}{(1-p^{-n-1}x)(1-p^{-n-2}x)}=(1-x)_{-n-2}, \tag{2}$$

which will be required in subsequent work.

The Function E_p :—If in series (1) we replace x by $\lambda(1-p)/p^n$, and make n infinite, we obtain

$$\underset{n=\infty}{\mathrm{L}}\left(1+\frac{\lambda(p-1)}{p^n}\right)_n=1+\frac{\lambda}{[1]!}+\frac{\lambda^2}{[2]!}+\dots. \tag{γ}$$

The series is absolutely convergent if $p>1$, and is convergent for all other values of p, subject to obvious limitations of λ. The infinite product is convergent in form (a) when $p>1$, and in form (β) when $p<1$. We therefore write the function (γ) as $E_p(\lambda)$, the analogue of the exponential function.

If we invert the base p, the number $[1]$ is unchanged, but the basic number $[r]$ is transformed into $p^{1-r}[r]$, and the function $E_p(x)$ becomes

$$E_{p^{-1}}(x)=1+\frac{x}{[1]}+p\frac{x^2}{[2]!}+\dots+p^{\frac{1}{2}r(r-1)}\frac{x^r}{[r]!}+\dots.$$

The following properties may be obtained without difficulty :—

$$E_p(x)\,E_{p^{-1}}(y)=1+\frac{(x+y)}{[1]}+\frac{(x+y)(x+py)}{[2]!}+\frac{(x+y)(x+py)(x+p^2y)}{[3]!}+\dots, \tag{3}$$

$$E_p(a)\,E_{p^{-1}}(-a)=1=E_p(-a)\,E_{p^{-1}}(a), \tag{4}$$

which is the analogue of $\exp(-a)=1/\exp(a)$,

$$E_{p^2}\left(\frac{\lambda}{[2]}\right)=1+\frac{\lambda}{[2]}+\frac{\lambda^2}{[2][4]}+\dots+\frac{\lambda^r}{\{2r\}!}+\dots. \tag{5}$$

The use of the symbol $\{2r\}!$ to denote $[2][4]\dots[2r]$ will be convenient in lengthy expressions.

It is well known that

$$\begin{aligned}F([a][\beta][\gamma]p^{\gamma-a-\beta})&=1+\frac{[a][\beta]}{[1][\gamma]}p^{\gamma-a-\beta}+\dots\\&=\prod_{n=0}^{n=\infty}\frac{[\gamma-a+n][\gamma-\beta+n]}{[\gamma-a-\beta+n][\gamma+n]}.\end{aligned} \tag{6}$$

If we invert the base p, the series becomes $F([a][\beta][\gamma]p)$, while the form of the infinite product remains unchanged. Both series are, however, required to fully represent the infinite product, because, while the product is absolutely convergent for all values of the base p, the series $F([a][\beta][\gamma]p^{\gamma-a-\beta})$ is absolutely convergent if $p>1$ and the series $F([a][\beta][\gamma]p)$ is absolutely convergent if $p<1$. Both series are, however, convergent for all other values of p including unity, subject to the condition $\gamma-a-\beta>1$.

The infinite product, when expressed in terms of the generalized gamma function, takes the form

$$\frac{1}{p^{a\beta}}\frac{\Gamma_p([\gamma-a-\beta])\,\Gamma_p([\gamma])}{\Gamma_p([\gamma-a])\,\Gamma_p([\gamma-\beta])},$$

and this is equal to $F([a][\beta][\gamma]p)$ or $F([a][\beta][\gamma]p^{\gamma-a-\beta})$, subject to the conditions stated above. The function Γ_p is defined, p. 361, Ser. 2, Vol. 1. Some detailed properties of this function and its derivatives will be found in a paper in *Proc. R.S. London*, Vol. LXXIV. (1904).

3. *The Functions* $J_{[n]}$ *and* $\mathfrak{J}_{[n]}$.

We define $J_{[n]}(\lambda, x)$ as $\displaystyle\sum_{r=0}^{r=\infty}(-1)^r\frac{\lambda^{n+2r}x^{[n+2r]}}{[n+r]!\,[r]!\,(2)_n(2)_{n+r}}$,

in which $[n+r]! = \Gamma_p([n+r+1])$ and $(2)_n$ satisfies the relation

$$(2)_n\Gamma_p([n+1]) = [2]^n\,\Gamma_{p^2}([n+1]).$$

The function $(2)_n$ reduces to 2^n if the base $p=1$. It will be convenient to denote $(2)_n\Gamma_p([n+1])$ by $\{2n\}!$, a function which has the following difference equation :—

$$\{2n\}! = [2n]\,\{2n-2\}!.$$

The complete solution of the differential equation satisfied by $J_{[n]}$ is given in *Proc. R.S. Edin.*, Session 1903–1904. If we invert the base p in the function

$$J_{[n]}(\lambda) = \Sigma(-1)^r\frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!},$$

we obtain the series

$$p^{n^2}\Sigma(-1)^r\frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}p^{2r(n+r)}$$

and denote this by $p^{n^2}\mathfrak{J}_{[n]}(\lambda)$.

It has been shown in result (5) that

$$E_{p^2}(x) = \Sigma \frac{x^r [2]^r}{\{2r\}!};$$

therefore $$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = 1 + \frac{\lambda t}{\{2\}!} + \frac{\lambda^2 t^2}{\{4\}!} + \dots$$

and $$E_{p^2}\left(-\frac{\lambda t}{[2]}\right) = 1 - \frac{\lambda t^{-1}}{\{2\}!} + \frac{\lambda^2 t^{-2}}{\{4\}!} - \dots .$$

Taking the product of these, we obtain, on arranging according to powers of t,

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]t}\right) = J_{[0]}(\lambda) + tJ_{[1]}(\lambda) + t^2 J_{[2]}(\lambda) + \dots$$
$$- t^{-1} J_{[1]}(\lambda) + t^{-2} J_{[2]}(\lambda) - \dots,$$

which, since* $$J_{[n]} = (-1)^n J_{[-n]},$$

may be written $$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda t}{[2]}\right) = \sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\lambda).$$

Similarly we can obtain

$$E_{p^{-2}}\left(\frac{\lambda t}{[2]}\right) E_{p^{-2}}\left(-\frac{\lambda t}{[2]}\right) = \sum_{n=-\infty}^{+\infty} p^{n^2} t^n \mathfrak{J}_{[n]}\left(\frac{\lambda}{p}\right).$$

Both of these are analogous to

$$\exp\left\{\frac{\lambda}{2}\left(t - \frac{1}{t}\right)\right\} = \sum_{-\infty}^{+\infty} t^n J_n(\lambda).$$

Writing at length

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]t}\right) = J_{[0]}(\lambda) + tJ_{[1]}(\lambda) + \dots + t^r J_{[r]}(\lambda) + \dots$$
$$- \frac{1}{t} J_{[1]}(\lambda) + \frac{1}{t^2} J_{[2]}(\lambda) - \dots, \quad (8)$$

$$E_{p^{-2}}\left(-\frac{\lambda t}{[2]}\right) E_{p^{-2}}\left(\frac{\lambda}{[2]t}\right) = \mathfrak{J}_{[0]}\left(\frac{\lambda}{p}\right) - pt\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right) + \dots$$
$$+ (-)^r p^{r^2} t^r \mathfrak{J}_{[r]}\left(\frac{\lambda}{p}\right) - \dots + \frac{p}{t}\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right) + \frac{p^4}{t^2}\mathfrak{J}_{[2]}\left(\frac{\lambda}{p}\right) + \dots, \quad (9)$$

we see that the product of the left-hand sides of (8) and (9) is unity, since $E_p(a) E_{p^{-1}}(-a) = 1$, as was shown in result (14). This gives rise to various interesting results, by equating to zero the coefficients of various

* *Trans. R.S. Edin.*, Vol. XLI., Part 1, 1904, pp. 112, 113.

powers of t in the product of the series on the right-hand sides of the above expressions. By considering the terms independent of t we obtain

$$1 = J_{[0]}(\lambda)\,\mathfrak{J}_{[0]}\left(\frac{\lambda}{p}\right) + 2p\,J_{[1]}(\lambda)\,\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right) + \ldots + 2p^{r^2}J_{[r]}(\lambda)\,\mathfrak{J}_{[r]}\left(\frac{\lambda}{p}\right) + \ldots .$$

In a paper (*Trans. R.S. Edin.*, Vol. XLI.) a result

$$1 = J_{[0]}(\lambda)\,\mathfrak{J}_{[0]}(\lambda) + \frac{[4]}{[2]}J_{[1]}(\lambda)\,\mathfrak{J}_{[1]}(\lambda) + \ldots + p^{r(r-1)}\frac{[4r]}{[2r]}J_{[r]}(\lambda)\,\mathfrak{J}_{[r]}(\lambda) \quad (9\text{A})$$

is obtained, and subsequently the general form including an infinite number of theorems similar to the above is given. This is of interest as showing that an infinite number of forms of the general addition theorem exist.

4.

By (8), we have

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = \frac{1}{E_{p^2}\left(-\frac{\lambda}{[2]\,t}\right)}\left\{J_{[0]} + tJ_{[1]} + t^2J_{[2]} + \ldots - \frac{1}{t}J_{[1]} + \frac{1}{t^2}J_{[2]} - \ldots\right\},$$

but

$$E_{p^2}\left(-\frac{\lambda}{[2]\,t}\right)E_{p^{-2}}\left(\frac{\lambda}{[2]\,t}\right) = 1\,;$$

therefore

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = E_{p^{-2}}\left(\frac{\lambda}{[2]\,t}\right)\left\{J_{[0]} + tJ_{[1]} + \ldots - \frac{1}{t}J_{[1]} + \ldots\right\}. \quad (10)$$

Expanding the exponential basic functions, and equating coefficients of the various powers of t, from the terms independent of t, we get

$$1 = J_{[0]}(\lambda) + \frac{\lambda}{[2]}J_{[1]}(\lambda) + p^2\frac{\lambda^2}{[2][4]}J_{[2]}(\lambda) + \ldots + p^{r(r-1)}\frac{\lambda^r}{\{2r\}!}J_{[r]}(\lambda) + \ldots .$$

This theorem is a particular case of a more general theorem for $J_{[n]}(\lambda, x)$. (*Trans. R.S. Edin.*, Vol. XLI., Part 1, Art. 8, p. 25.) By equating the coefficients of t^n in (10), we obtain

$$\frac{\lambda^n}{\{2n\}!} = J_{[n]} + \frac{\lambda}{\{2\}!}J_{[n+1]} + p^2\frac{\lambda^2}{\{4\}!}J_{[n+2]} + \ldots + p^{r(r-1)}\frac{\lambda^r}{\{2r\}!}J_{[n+r]}. \quad (11)$$

Equate the coefficients of t^{-n}; then we obtain

$$p^{n(n-1)}\frac{\lambda^n}{\{2n\}!}J_{[0]} + p^{(n+1)n}\frac{\lambda^{n+1}}{\{2n+2\}!}J_{[1]} + \ldots \text{ ad inf.}$$

$$= p^{(n-1)(n-2)}\frac{\lambda^{n-1}}{\{2n-2\}!}J_{[1]} - p^{(n-2)(n-3)}\frac{\lambda^{n-2}}{\{2n-4\}!}J_{[2]} + \ldots . \quad (12)$$

Putting n in succession as 1, 2, 3, ..., we obtain from the latter formula

$$J_{[1]} = \frac{\lambda}{[2]} J_{[0]} + p^2 \frac{\lambda^2}{[2][4]} J_{[1]} + p^6 \frac{\lambda^3}{[2][4][6]} J_{[2]} + \dots$$

and
$$\frac{\lambda}{[2]} J_{[1]} - J_{[2]} = p^2 \frac{\lambda^2}{[2][4]} J_{[0]} + p^6 \frac{\lambda^3}{[2][4][6]} J_{[1]} + \dots,$$

and so on. From the two expressions just given, we find

$$J_{[2]} = \frac{\lambda^2}{[2][4]} J_{[0]} + p^2 \frac{\lambda^3}{[2][4][6]} \frac{[4]}{[2]} J_{[1]}(\lambda) + \dots,$$

and, finally,

$$J_{[n]}(\lambda) = \frac{\lambda^n}{[n]!\,(2)_n} J_{[0]}(\lambda) + p^2 \frac{\lambda^{n+1}}{[n+1]!\,(2)_{n+1}} J_{[1]}(\lambda) + \dots$$
$$+ p^{r(r-1)} \frac{\lambda^{n+r-1}}{[n+r-1]!\,(2)_{r-1}} J_{[r-1]}(\lambda) + \dots. \qquad (13)$$

5.

In this article it will be shown that

$$J_{[n]}(\kappa\lambda) = \kappa^n J_{[n]}(\lambda) - \kappa^{n+1} \frac{\lambda}{[2]} \left(\kappa - \frac{1}{\kappa}\right) J_{[n+1]}(\lambda)$$
$$+ \kappa^{n+2} \frac{\lambda^2}{[2][4]} \left(\kappa - \frac{1}{\kappa}\right) \left(\kappa - \frac{p^2}{\kappa}\right) J_{[n+2]}(\lambda) - \dots, \qquad (14)$$

the coefficient of $J_{[n+r]}(\lambda)$ being

$$(-1)^r \kappa^{n+r} \frac{\lambda^r}{[2][4]\dots[2r]} \left(\kappa - \frac{1}{\kappa}\right) \left(\kappa - \frac{p^2}{\kappa}\right) \dots \left(\kappa - \frac{p^{2r-2}}{\kappa}\right).$$

When $p = 1$ this theorem reduces to

$$J_n(\kappa\lambda) = \kappa^n J_n(\lambda) - \lambda\kappa^{n+1} \left(\frac{\mu}{2}\right) J_{n+1}(\lambda) + \lambda^2 \frac{\kappa^{n+2}}{2!} \left(\frac{\mu}{2}\right)^2 J_{n+2}(\lambda) - \dots,$$

in which $\mu = \kappa - \kappa^{-1}$: see Todhunter, *Functions of Laplace, Lamé, and Bessel*, p. 326. A theorem which may be derived from the above is

$$p^{-mn} J_{[n]}(p^m \lambda) = J_{[n]}(\lambda) - \lambda \frac{[2m]}{[2]} (p-1) J_{[n+1]}(\lambda)$$
$$+ \lambda^2 \frac{[2m][2m-2]}{[2][4]} (p-1)^2 p^2 J_{[n+2]}(\lambda) - \dots. \qquad (15)$$

As before, we have

$$E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\kappa\lambda}{[2]\,t}\right)$$
$$= J_{[0]}(\kappa\lambda)+tJ_{[1]}(\kappa\lambda)+\ldots+t^rJ_{[r]}(\kappa\lambda)+\ldots-t^{-1}J_{[1]}(\kappa\lambda)+t^{-2}J_{[2]}(\kappa\lambda)-\ldots,$$

$$E_{p^2}\left(\frac{\lambda\kappa t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]\,\kappa t}\right)$$
$$= J_{[0]}(\lambda)+\kappa tJ_{[1]}(\lambda)+\ldots+\kappa^r t^r J_{[r]}(\lambda)+\ldots$$
$$-\kappa^{-1}t^{-1}J_{[1]}(\lambda)+\ldots+(-1)^r\kappa^{-r}t^{-r}J_{[r]}(\lambda)-\ldots.$$

Now, by means of results (3) and (4), we can show that

$$\frac{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\kappa\lambda}{[2]\,t}\right)}{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]\,\kappa t}\right)} = \frac{E_{p^2}\left(-\frac{\kappa\lambda}{[2]\,t}\right)}{E_{p^2}\left(-\frac{\lambda}{[2]\,\kappa t}\right)} = E_{p^2}\left(-\frac{\kappa\lambda}{[2]\,t}\right) E_{p^{-2}}\left(\frac{\lambda}{[2]\,\kappa t}\right)$$
$$= 1-\frac{\lambda}{t}\,\frac{\left(\kappa-\frac{1}{\kappa}\right)}{[2]}+\frac{\lambda^2}{t^2}\,\frac{\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)}{[2][4]}-\ldots; \tag{16}$$

so that

$$J_{[0]}(\kappa\lambda)+tJ_{[1]}(\kappa\lambda)+\ldots \text{ ad inf.}-t^{-1}J_{[1]}(\kappa\lambda)+t^{-2}J_{[2]}(\kappa\lambda)-\ldots \text{ ad inf.}$$
$$=\left\{1-\frac{\lambda}{t}\,\frac{\left(\kappa-\frac{1}{\kappa}\right)}{[2]}+\ldots\right\}$$
$$\times\left\{J_{[0]}(\lambda)+\kappa tJ_{[1]}(\lambda)+\ldots-\kappa^{-1}t^{-1}J_{[1]}(\lambda)+\kappa^{-2}t^{-2}J_{[2]}(\lambda)-\ldots\right\}.$$

Equating the coefficients of t^n in these products, we obtain

$$J_{[n]}(\kappa\lambda) = \kappa^n J_{[n]}(\lambda)-\kappa^{n+1}\frac{\lambda}{[2]}\left(\kappa-\frac{1}{\kappa}\right)J_{[n+1]}(\lambda)$$
$$+\kappa^{n+2}\frac{\lambda^2}{[2][4]}\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)J_{[n+2]}(\lambda)-\ldots$$

In a similar manner we can obtain

$$\kappa^n J_{[n]}(\lambda) = J_{[n]}(\kappa\lambda)+\frac{\lambda}{[2]}\left(\kappa-\frac{1}{\kappa}\right)J_{[n+1]}(\kappa\lambda)$$
$$+\frac{\lambda^2}{[2][4]}\left(\kappa-\frac{1}{\kappa}\right)\left(p^2\kappa-\frac{1}{\kappa}\right)J_{[n+2]}(\lambda)+\ldots. \tag{17}$$

Interesting particular cases of these are ($\kappa = \sqrt{2}$)

$$J_{[n]}(\sqrt{2}\lambda) = 2^{\frac{1}{2}n}\left\{J_{[n]}(\lambda) - \frac{\lambda}{[2]}J_{[n+1]}(\lambda) + \frac{\lambda^2}{[2][4]}(2-p^2)J_{[n+2]}(\lambda) - \dots\right\} \tag{18}$$

and ($\kappa = p^m$)

$$J_{[n]}(p^m\lambda) = p^{mn}\left\{J_{[n]}(\lambda) - \lambda\frac{[2m]}{[2]}(p-1)J_{[n+1]}(\lambda) + \dots \right.$$
$$\left. + (-1)^r\lambda^r\frac{[2m]\dots[2m-2r+2]}{[2][4]\dots[2r]}p^{r(r-1)}(p-1)^r J_{[n+r]}(\lambda) + \dots\right\}, \tag{19}$$

$$p^{mn}J_{[n]}(\lambda) = J_{[n]}(p^m\lambda) + \lambda\frac{[2m]}{[2]}(p-1)p^{-m}J_{[n+1]}(p^m\lambda)$$
$$+\lambda^2\frac{[2m][2m-2]}{[2][4]}(p-1)^2p^{-2m}J_{[n+2]}(p^m\lambda) - \dots. \tag{20}$$

The equations (8) and (9) of Art. 3 easily lead to various theorems respecting the functions $J_{[n]}$ and $\mathfrak{J}_{[n]}$ when n is a positive integer. The theorems obtained as above in this way are, however, valid when n is not restricted to positive integral values; this is evident at once if we compare the coefficients of the powers of λ in the series on the right side of (14) with the coefficients of like powers of λ in $J_{[n]}(\kappa\lambda)$. We have from the right side the following terms involving λ^{n+2r}:—

$$(-1)^r\left[\frac{\kappa^n\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!} + \frac{\kappa^{n+1}\left(\kappa - \frac{1}{\kappa}\right)\lambda^{n+2r}}{\{2\}!\,\{2n+2r\}!\,\{2r-2\}!}\right.$$
$$\left. + \frac{\kappa^{n+2}\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)\lambda^{n+2r}}{\{4\}!\,\{2n+2r\}!\,\{2r-4\}!} + \dots\right],$$

which may be expressed as

$$(-1)^r\frac{\kappa^n\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}\left\{1 + \frac{[2r]}{[2]}(\kappa^2-1) + \frac{[2r][2r-2]}{[2][4]}(\kappa^2-1)(\kappa^2-p^2) + \dots\right\}$$
$$\equiv \frac{\kappa^{n+2r}\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}.$$

This holds for all values of n, subject to the interpretation of $\{2n+2r\}!$ as

$$\{2n+2r\}! = (2)_{n+r}\Gamma_p([n+r+1]) = [2]^{n+r}\Gamma_{p^2}([n+r+1]).$$

6.

We have so far introduced two generalized forms of Bessel's function. The function E_p gives us also a third form, which we denote $\mathrm{J}_{[n]}$, as distinguished from $J_{[n]}$ and $\mathfrak{J}_{[n]}$. $\mathrm{J}_{[n]}$ is perhaps worth noticing on account of its connexion with an expression analogous to cos (x cos ϕ).

Since $$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = 1+\frac{\lambda t}{[2]}+\frac{\lambda^2 t^2}{[2][4]}+\ldots+\frac{\lambda^r t^r}{\{2r\}!}+\ldots$$

and $$E_{p^{-2}}\left(-\frac{\lambda}{[2]\,t}\right) = 1-\frac{\lambda t^{-1}}{[2]}+p^2\frac{\lambda^2 t^{-2}}{[2][4]}-\ldots+(1)^r p^{r(r-1)}\frac{\lambda^r t^{-r}}{\{2r\}!}+\ldots,$$

forming the product of these, we obtain, by (3),

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{1/p^2}\left(-\frac{\lambda}{[2]\,t}\right) = 1+\frac{\lambda}{[2]}\left(t-\frac{1}{t}\right)+\frac{\lambda^2}{[2][4]}\left(t-\frac{1}{t}\right)\left(t-\frac{p^2}{t}\right)$$
$$+\frac{\lambda^2}{[2][4][6]}\left(t-\frac{1}{t}\right)\left(t-\frac{p^2}{t}\right)\left(t-\frac{p^4}{t}\right)+\ldots. \quad (21)$$

If, however, the product be formed in a series of ascending and descending powers of t, the expression for the product is

$$\mathrm{J}_{[0]}(\lambda)+t\mathrm{J}_{[1]}(\lambda)+t^2\mathrm{J}_{[2]}(\lambda)+\ldots-pt^{-1}\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right)+p^4t^{-2}\mathfrak{J}_{[2]}\left(\frac{\lambda}{p}\right)-\ldots, \quad (22)$$

in which $$\mathrm{J}_{[n]}(\lambda) = \Sigma\,(-1)^r p^{r(r-1)}\frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}.$$

If $p = 1$ and $t = e^{i\phi}$, the series (21) becomes $e^{i\lambda \sin\phi}$.

Since the expansion in ascending and descending powers of t is a Laurent series, we have

$$\mathrm{J}_{[n]}(\lambda) = \frac{1}{2\pi i}\int_C E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^{-2}}\left(-\frac{\lambda}{[2]\,t}\right)\frac{dt}{t^{n+1}},$$

$$p^{n^2}\mathfrak{J}_{[n]}(\lambda) = \frac{1}{2\pi i}\int_C E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^{-2}}\left(-\frac{\lambda}{[2]\,t}\right)\frac{dt}{t^{1-n}}.$$

Taking C to be a circle of unit radius, so that $t = e^{i\theta}$,

$$\mathrm{J}_{[n]}(\lambda) = \frac{1}{2\pi}\int_0^{2\pi}\left(1+\frac{\lambda}{[2]}(e^{i\theta}-e^{-i\theta})\right.$$
$$\left.+\frac{\lambda^2}{[2][4]}(e^{i\theta}-e^{-i\theta})(e^{i\theta}-p^2e^{-i\theta})+\ldots\right)e^{-ni\theta}d\theta, \quad (23)$$

subject to the uniform convergence of the series. Also

$$p^{n^2}\mathfrak{J}_{[n]}(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} E_{p^2}\left(\frac{\lambda e^{i\theta}}{[2]}\right) E_{p^{-2}}\left(-\frac{\lambda e^{-i\theta}}{[2]}\right)e^{ni\theta}d\theta. \quad (24)$$

7. *Addition Theorems.*

Many of the expressions used in the following analysis are rather complicated: it will be useful, therefore, to compare the expressions with those which correspond to them in the case of the ordinary Bessel function; the work will, therefore, be on the same lines as the analysis on pp. 25, 26, 27 of Gray and Mathews' *Treatise on Bessel Functions*, and an analogous notation will be used. References will be made in the form [*G.M.*, p. 25, (62)].

We know that

$$E_p(a)\,E_{p^{-1}}(b) = 1+\frac{(a+b)}{[1]!}+\frac{(a+b)(a+pb)}{[2]!}+\dots.$$

From this we deduce that

$$E_{p^2}\left(\frac{xt}{[2]}\right)E_{p^{-2}}\left(\frac{yt}{[2]}\right) = 1+\frac{(x+y)\,t}{\{2\}!}+\frac{(x+y)(x+p^2y)\,t^2}{\{4\}!}+\dots = \mathfrak{E}(x,\,y,\,t) \quad (25)$$

and

$$E_{p^2}\left(-\frac{x}{[2]\,t}\right)E_{p^{-2}}\left(-\frac{y}{[2]\,t}\right) = 1-\frac{(x+y)\,t}{\{2\}!}+\frac{(x+y)(x+p^2y)\,t^2}{\{4\}!}-\dots$$

$$= \mathfrak{E}(-x,\,-y,\,t). \quad (26)$$

Forming the product of these, we have

$$E_{p^2}\left(\frac{xt}{[2]}\right)E_{p^2}\left(-\frac{x}{[2]\,t}\right)E_{p^{-2}}\left(\frac{yt}{[2]}\right)E_{p^{-2}}\left(-\frac{y}{[2]\,t}\right) = \mathfrak{E}(x,\,y,\,t)\,\mathfrak{E}(-x,\,-y,\,t). \quad (27)$$

By means of results (8) and (9) the left side of expression (27) may be written in the form

$$\sum_{m=-\infty}^{+\infty} t^m J_{[m]}(x)\times\sum_{r=-\infty}^{+\infty} t^r p^{r^2}\mathfrak{J}_{[r]}\left(\frac{y}{p}\right) = \mathfrak{E}(x,\,y,\,t)\,\mathfrak{E}(-x,\,-y,\,t). \quad (28)$$

Equating the coefficients of t^n on both sides of this, we obtain

$$\sum_{m=-\infty}^{m=+\infty} p^{(n-m)^2} J_{[m]}(x)\,\mathfrak{J}_{[n-m]}\left(\frac{y}{p}\right) = \frac{(x+y)(x+yp^2)\dots(x+yp^{2n-2})}{\{2n\}!}$$

$$\left\{1-\frac{(x+y)(x+yp^{2n})}{[2n+2][2]}+\frac{(x+y)(x+p^2y)(x+yp^{2n})(x+yp^{2n+2})}{[2n+2][2n+4][2][4]}+\dots\right\}$$

$$= \mathrm{J}_n(x,\,y). \quad (29)$$

The series on the right side of the above reduces to $J_n(x+y)$, in case $p=1$. A particular case of this theorem is

$$J_{[0]}(x)\,\mathfrak{J}_{[0]}\left(\frac{y}{p}\right)-2pJ_{[1]}(x)\,\mathfrak{J}_{[1]}\left(\frac{y}{p}\right)+\dots+(-1)^n 2p^{n^2}J_{[n]}(x)\,\mathfrak{J}_{[n]}\left(\frac{y}{p}\right)+\dots$$

$$= 1-\frac{(x+y)^2}{\{2\}!\,\{2\}!}+\frac{(x+y)^2(x+p^2y)^2}{\{4\}!\,\{4\}!}-\dots. \quad (30)$$

8.

Neumann has shown that

$$J_0(\sqrt{b^2+2bc\cos a+c^2}) = J_0(b)J_0(c)+2\Sigma(-1)^s J_s(b)J_s(c)\cos(sa).$$

We proceed to obtain the analogous theorem for the function $J_{[n]}$.

We have $$E_{p^2}\left(\frac{\kappa x t}{[2]}\right)E_{p^2}\left(-\frac{x}{[2]\kappa t}\right) = \sum_{-\infty}^{+\infty}\kappa^n t^n J_{[n]}(x).$$

Now

$$E_{p^2}\left(\frac{\kappa x t}{[2]}\right)E_{p^2}\left(-\frac{x}{[2]\kappa t}\right) = E_{p^2}\left(\frac{\kappa x t}{[2]}\right)E_{p^2}\left(-\frac{\kappa x}{[2]t}\right)E_{p^{-2}}\left(\frac{\kappa x}{[2]t}\right)E_{p^2}\left(-\frac{x}{[2]\kappa t}\right),$$

because the product of the two middle E functions in the expression on the right side of the equation is unity, by theorem (4). The product of the four E functions may also be written

$$\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x)E_{p^{-2}}\left(\frac{\kappa x}{[2]t}\right)E_{p^2}\left(-\frac{x}{[2]\kappa t}\right);$$

so that $$\sum_{n=-\infty}^{+\infty}\kappa^n t^n J_{[n]}(x) = E_{p^{-2}}\left(\frac{\kappa x}{[2]t}\right)E_{p^2}\left(-\frac{x}{[2]\kappa t}\right)\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x).$$

[Cf. *G.M.*, p. 25, (62).]

Transforming the E functions in this equation by means of (4), we have

$$\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x) = E_{p^2}\left(-\frac{\kappa x}{[2]t}\right)E_{p^{-2}}\left(\frac{x}{[2]\kappa t}\right)\sum_{n=-\infty}^{+\infty}\kappa^n t^n J_{[n]}(x). \qquad (31)$$

The product of the E functions in this equation is the series

$$1-\left(\kappa-\frac{1}{\kappa}\right)\frac{x}{[2]t}+\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)\frac{x^2}{[2][4]t^2}-\dots,$$

the analogue of $$\exp\left\{\frac{x}{2t}\left(\kappa-\frac{1}{\kappa}\right)\right\};$$

therefore $$\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x) = \left\{1-\left(\kappa-\frac{1}{\kappa}\right)\frac{x}{[2]t}+\dots\right\}\sum_{n=-\infty}^{+\infty}\kappa^n t^n J_{[n]}(x).$$

Put $x = r$, $\kappa = e^{\iota\theta}$, and let

$$i^n \sin_n\theta = \frac{(e^{\iota\theta}-e^{-\iota\theta})(e^{\iota\theta}-p^2e^{-\iota\theta})(e^{\iota\theta}-p^4e^{-\iota\theta})\dots}{(1+p)(1+p^2)(1+p^3)\dots} \text{ to } n \text{ factors};$$

then

$$\sum_{-\infty}^{+\infty} J_{[n]}(re^{i\theta})t^n = \left\{1-\frac{ir\sin\theta}{[1]t}+\frac{i^2r^2\sin_2\theta}{[2]!\,t^2}-\frac{i^3r^3\sin_3\theta}{[3]!\,t^3}+\dots\right\}\sum_{-\infty}^{+\infty} e^{n\iota\theta}J_{[n]}(r)t^n. \qquad (32)$$

Equating coefficients of t^n, we have

$$J_{[n]}(re^{i\theta}) = e^{ni\theta}\left\{J_{[n]}(r) - \frac{ir\sin\theta}{[1]}e^{i\theta}J_{[n+1]}(r) + \frac{i^2r^2\sin_2\theta}{[2]!}e^{2i\theta}J_{[n+2]}(r) - \dots\right\}, \quad (33)$$

analogous to *G.M.*, p. 26, (64), ed. 1895.

If, in this equation, we put $e^{i\theta} = i$, then we obtain

$$J_{[n]}(ir) = i^n\left\{J_{[n]}(r) + \frac{2r}{[2]}J_{[n+1]}(r) + \frac{2(1+p^2)r^2}{[2][4]}J_{[n+2]}(r) + \dots\right\}. \quad (34)$$

9.

In this article we shall obtain briefly the theorems for the function $\mathfrak{J}_{[n]}$ corresponding to those obtained in the last article for $J_{[n]}$.

We have shown in result (9) that

$$E_{p^{-2}}\left(\frac{x\kappa t}{[2]}\right)E_{p^{-2}}\left(-\frac{x}{[2]\kappa t}\right) = \sum_{n=-\infty}^{n=\infty} p^{n^2}\kappa^n\mathfrak{J}_{[n]}\left(\frac{x}{p}\right)t^n.$$

Now

$$E_{p^{-2}}\left(\frac{x\kappa t}{[2]}\right)E_{p^{-2}}\left(-\frac{x}{[2]\kappa t}\right)$$
$$= E_{p^{-2}}\left(\frac{x\kappa t}{[2]}\right)E_{p^{-2}}\left(-\frac{\kappa x}{[2]t}\right)E_{p^2}\left(\frac{\kappa x}{[2]t}\right)E_{p^{-2}}\left(-\frac{x}{[2]\kappa t}\right),$$

and from this we deduce, as in the last article, that

$$\sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{\kappa x}{p}\right)t^n = E_{p^2}\left(\frac{x}{[2]\kappa t}\right)E_{p^{-2}}\left(-\frac{\kappa x}{[2]t}\right)\sum_{n=-\infty}^{+\infty} p^{n^2}\kappa^n\mathfrak{J}_{[n]}\left(\frac{x}{p}\right).$$

The product of the two E functions is, by theorem (3), after some obvious reductions,

$$1 - \left(\kappa - \frac{1}{\kappa}\right)\frac{x}{[2]t} + \left(\kappa - \frac{1}{\kappa}\right)\left(p^2\kappa - \frac{1}{\kappa}\right)\frac{x^2}{[2][4]t^2} - \dots.$$

This is analogous to $\exp\left\{\frac{x}{2t}\left(\kappa - \frac{1}{\kappa}\right)\right\}$,

but differs from the corresponding series obtained previously in connection with $J_{[n]}$. We have now

$$\sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{\kappa x}{p}\right)t^n = \left\{1 - \left(\kappa - \frac{1}{\kappa}\right)\frac{x}{[2]t}\right.$$
$$\left. + \left(\kappa - \frac{1}{\kappa}\right)\left(p^2\kappa - \frac{1}{\kappa}\right)\frac{x^2}{[2][4]t^2} - \dots\right\}\sum_{-\infty}^{+\infty} p^{n^2}\kappa^n\mathfrak{J}_{[n]}\left(\frac{x}{p}\right). \quad (35)$$

Put $x = r$, $\kappa = e^{\iota\theta}$, and let $i^n \sin_n(-\theta)$ denote, as before,

$$\frac{(e^{-\iota\theta}-e^{\iota\theta})(e^{-\iota\theta}-p^2e^{\iota\theta})\dots(e^{-\iota\theta}-p^{2n-2}e^{\iota\theta})}{(1+p)(1+p^2)\dots(1+p^n)};$$

then

$$\sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{re^{i\theta}}{p}\right)t^n$$
$$= \left\{1+\frac{ir\sin(-\theta)}{[1]t}+\frac{i^2r^2\sin_2(-\theta)}{[2]!\,t^2}+\dots\right\}\sum_{n=-\infty}^{+\infty} p^{n^2}e^{n\iota\theta}\mathfrak{J}_{[n]}\left(\frac{r}{p}\right). \quad (36)$$

Equating coefficients of t^n, we obtain

$$\mathfrak{J}_{[n]}\left(\frac{re^{i\theta}}{p}\right) = e^{n\iota\theta}\mathfrak{J}_{[n]}\left(\frac{r}{p}\right)+\sum_0^\infty \frac{i^sr^s\sin_s(-\theta)}{[s]!}e^{(n+s)\iota\theta}p^{s(s+2n)}\mathfrak{J}_{[n+s]}\left(\frac{r}{p}\right), \quad (37)$$

analogous to result (33) of the preceding article, and corresponding to *G.M.*, p. 26, (64), ed. 1895.

If in (37) we put $e^{\iota\theta} = i$, we obtain

$$\mathfrak{J}_{[n]}\left(\frac{ir}{p}\right) = \sum_{s=0}^{s=\infty} i^n \frac{2^s r^s}{\{2s\}!} p^{s(s+2n)}\mathfrak{J}_{[n+s]}\left(\frac{r}{p}\right). \quad (38)$$

10.

In result (32) put $r = b$, $\theta = \beta$, and in (36) put $r = c$, $\theta = \gamma$; then multiply the results together: we obtain

$$\sum_{n=-\infty}^{+\infty} J_{[n]}(be^{\iota\beta})t^n \sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{ce^{\iota\gamma}}{p}\right)t^n$$
$$= \left\{1-if'(b,c,\beta,\gamma)\frac{t^{-1}}{[1]}+i^2f''(b,c,\beta,\gamma)\frac{t^{-2}}{[2]!}-\dots\right\}$$
$$\times\sum_{-\infty}^{+\infty} e^{n\iota\beta}J_{[n]}(b)t^n\sum_{-\infty}^{+\infty} e^{n\iota\gamma}p^{n^2}\mathfrak{J}_{[n]}\left(\frac{c}{p}\right)t^n, \quad (39)$$

in which

$$f^{(n)}(b,c,\beta,\gamma) = b^n\sin_n\beta-[n]b^{n-1}c\sin_{n-1}\beta\sin(-\gamma)$$
$$+\frac{[n][n-1]}{[2]!}b^{n-2}c^2\sin_{n-2}\beta\sin_2(-\gamma)-\dots+(-)^nc^n\sin_n(-\gamma). \quad (40)$$

This expression (40) reduces, when $p = 1$, to $(b\sin\beta+c\sin\gamma)^n$. If $\beta = \gamma = 0$, then $f^{(n)}(b,c,\beta,\gamma) = 0$. Equation (39) is analogous to *G.M.*, p. 26, (66).

11.

Consider now the expression on the left side of (39), which we wil write for convenience

$$\Sigma J_{(n)}(b\kappa)t^n\,\Sigma p^{n^2}\mathfrak{J}_{[n]}\left(\frac{c\kappa_1}{p}\right)t^n \quad (\kappa = e^{\iota\beta},\ \kappa_1 = e^{\iota\gamma})$$

Expressed in terms of the E functions, this is

$$E_{p^2}\left(\frac{b\kappa t}{[2]}\right) E_{p^2}\left(-\frac{b\kappa}{[2]t}\right) E_{p^{-2}}\left(\frac{c\kappa_1 t}{[2]}\right) E_{p^{-2}}\left(-\frac{c\kappa_1}{[2]t}\right).$$

Taking now the product of the first and third E functions, also the product of the second and fourth, we have by (3), after some obvious reductions,

$$\left\{1+\frac{(b\kappa+c\kappa_1)}{[2]}t+\frac{(b\kappa+c\kappa_1)(b\kappa+p^2c\kappa_1)}{[2][4]}t^2+\dots\right\}$$

$$\times\left\{1-\frac{(b\kappa+c\kappa_1)}{[2]}t^{-1}+\frac{(b\kappa+c\kappa_1)(b\kappa+p^2c\kappa_1)}{[2][4]}t^{-2}-\dots\right\}, \tag{41}$$

analogous to $$\exp\left\{\frac{b\kappa+c\kappa_1}{2}\left(t-\frac{1}{t}\right)\right\}.$$

The product (41) gives us a Laurent series, in which the coefficients of the powers of t are functions analogous to $J_n(b\kappa+c\kappa_1)$, which we will denote by the symbol $\mathrm{J}_n(b\kappa, c\kappa_1)$. We see that

$$\mathrm{J}_0(b\kappa, c\kappa_1) = 1-\frac{(b\kappa+c\kappa_1)^2}{[2]^2}+\frac{(b\kappa+c\kappa_1)^2(b\kappa+p^2c\kappa_1)^2}{[2]^2[4]^2}-\dots, \tag{42}$$

and in general

$$\mathrm{J}_n(b\kappa, c\kappa_1) = \frac{(b\kappa+c\kappa_1)(b\kappa+p^2c\kappa_1)\dots(b\kappa+p^{2n-2}c\kappa_1)}{[2][4]\dots[2n]}$$

$$\times\left\{1-\frac{(b\kappa+c\kappa_1)(b\kappa+p^{2n}c\kappa_1)}{[2][2n+2]}+\dots\right\}. \tag{43}$$

When n is not integral the finite product in (43) must be replaced by a suitable infinite product, as

$$\frac{1}{\{2n\}!}\prod_{m=0}^{m=\infty}\frac{(b\kappa+p^{2m}c\kappa_1)}{(b\kappa+p^{2n+2m}c\kappa_1)}b^n\kappa^n \qquad (p<1).$$

We have now

$$\sum_{n=0}^{n=\infty}\mathrm{J}_n(be^{\iota\beta}, ce^{\iota\gamma})t^n+\sum_{n=1}^{n=\infty}(-1)^n\mathrm{J}_n(be^{\iota\beta}, ce^{\iota\gamma})t^{-n}$$

$$=\left\{1-if'(b, c, \beta, \gamma)\frac{t^{-1}}{[1]}+i^2f''(b, c, \beta, \gamma)\frac{t^{-2}}{[2]!}-\dots\right\}$$

$$\times\sum_{n=-\infty}^{+\infty}e^{n\iota\beta}J_{[n]}(b)t^n\sum_{-\infty}^{+\infty}p^{n^2}e^{ni\gamma}\mathfrak{J}_{[n]}\left(\frac{c}{p}\right)t^n. \tag{44}$$

Equating the coefficients of the various powers of t, we obtain

$$\mathrm{J}_n(be^{\iota\beta}, ce^{\iota\gamma}) = C_0-C_1if'(b, c, \beta, \gamma)+C_2i^2f''(b, c, \beta, \gamma)-\dots, \tag{45}$$

which is the analogue of *G. M.*, p. 26, (67).

12.

In this article we shall write down the expressions for C_0, C_1, ... Since (*Trans. R.S. Edin.*, Vol. XLI., Part 1, pp. 110–113)

$$J_{[n]}(a) = (-1)^n J_{[-n]}(a) \quad \text{and} \quad \mathfrak{J}_{[n]}\left(\frac{b}{p}\right) = (-1)^n \mathfrak{J}_{[-n]}\left(\frac{b}{p}\right),$$

we can write by means of these equations

$$\begin{aligned} C_0 = \; & e^{n\iota\beta} J_{[n]}(b)\, \mathfrak{J}_{[0]}\left(\frac{c}{p}\right) + e^{(n-1)i\beta+i\gamma} p J_{[n-1]}(b)\, \mathfrak{J}_{[1]}\left(\frac{c}{p}\right) + \dots \\ & \qquad + e^{n\iota\gamma} p^{n^2} J_{[0]}(b)\, \mathfrak{J}_{[n]}\left(\frac{c}{p}\right) \\ & - e^{(n+1)\iota\beta-\iota\gamma} p J_{[n+1]} \mathfrak{J}_{[1]} + e^{(n+2)\iota\beta-2\iota\gamma} p^4 J_{[n+2]} \mathfrak{J}_{[2]} - \dots \text{ ad inf.} \\ & - e^{(n+1)i\gamma-i\beta} p^{(n+1)^2} J_{[1]} \mathfrak{J}_{[n+1]} + e^{(n+2)\iota\gamma-2\iota\beta} p^{(n+2)^2} J_{[2]} \mathfrak{J}_{[n+2]} - \dots \text{ ad inf.,} \end{aligned} \tag{46}$$

and in general

$$\begin{aligned} C_s = \; & e^{(n+s)\iota\beta} J_{[n+s]} \mathfrak{J}_{[0]} + e^{(n+s-1)\iota\beta+\iota\gamma} p J_{[n+s-1]} \mathfrak{J}_{[1]} + \dots \\ & \qquad + e^{(n+s)\iota\gamma} p^{(n+s)^2} J_{[0]} \mathfrak{J}_{[n+s]} \\ & - e^{(n+s+1)\iota\beta-\iota\gamma} p J_{[n+s+1]} \mathfrak{J}_{[1]} + e^{(n+s+2)\iota\beta-2\iota\gamma} p^4 J_{[n+s+2]} \mathfrak{J}_{[2]} - \dots \text{ ad inf.} \\ & - p^{(n+s+1)^2} e^{(n+s+1)i\gamma-i\beta} J_{[1]} \mathfrak{J}_{n+s+1]} \\ & + p^{(n+s+2)^2} e^{(n+s+2)\iota\gamma-2\iota\beta} J_{[2]} \mathfrak{J}_{[n+s+2]} - \dots \text{ ad inf.} \end{aligned} \tag{47}$$

This formula is very complicated, but simplifies in two cases, viz., if $\beta = \gamma = 0$; then $f^{(n)}(\beta, \gamma, b, c) = 0$. We have in this case

(i.)

$$\begin{aligned} \text{J}_n(b, c) = J_{[n]}(b)\, \mathfrak{J}_{[0]}\left(\frac{c}{p}\right) + p J_{[n-1]}(b)\, \mathfrak{J}_{[1]}\left(\frac{c}{p}\right) + \dots + p^{n^2} J_{[0]}(b)\, \mathfrak{J}_{[n]}\left(\frac{c}{p}\right) \\ - [p^{(n+1)^2} J_{[1]} \mathfrak{J}_{[n+1]} + p J_{[n+1]} \mathfrak{J}_{[1]}] \\ + [p^{(n+2)^2} J_{[2]} \mathfrak{J}_{[n+2]} + p^4 J_{[n+2]} \mathfrak{J}_{[2]}] - \dots \\ + (-1)^r [p^{(n+r)^2} J_{[r]} \mathfrak{J}_{[n+r]} + p^{r^2} J_{[n+r]} \mathfrak{J}_{(r]}] + \dots \end{aligned} \tag{48}$$

A more general form of this result will be found in *Trans. R.S. Edin.*, Vol. XLI., Part 1, 1904, p. 117.

(ii.) If $p = 1$ and $be^{i\beta} + ce^{i\gamma}$ be a real quantity, then

$$f^{(n)}(b, c, \beta, \gamma) = (b \sin\beta + c \sin\gamma)^n = 0,$$

Putting $\beta - \gamma = a$, we have the addition theorem for Bessel functions.

(iii.) If we put $be^{\iota\beta}+ce^{\iota\gamma}=a$ (a real quantity),

$$a^2 = b^2+c^2+2bc\cos(\beta-\gamma);$$

also

$$b\sin\beta+c\sin\gamma = 0;$$

therefore $if'(b, c, \beta, \gamma) = \frac{bi}{p+1}(e^{\iota\beta}-e^{-\iota\beta})-\frac{ci}{p+1}(e^{-i\gamma}-e^{i\gamma}) = 0$

and

$$i^2f''(b, c, \beta, \gamma) = i^2\left\{b^2\frac{e^{\iota\beta}-e^{-\iota\beta})(e^{\iota\beta}-p^2e^{-\iota\beta})}{(1+p)(1+p^2)}-bc(1+p)\frac{(e^{\iota\beta}-e^{-\iota\beta})(e^{-\iota\gamma}-e^{\iota\gamma})}{(1+p)(a+p)}\right.$$
$$\left.+c^2\frac{(e^{-\iota\gamma}-e^{\iota\gamma})(e^{-\iota\gamma}-p^2e^{\iota\gamma})}{(1+p)(1+p^2)}\right\}.$$

This expression reduces, since $be^{\iota\beta}-be^{-\iota\beta}+ce^{\iota\gamma}-ce^{-\iota\gamma}=0$, to

$$\frac{2i^3(1-p^2)\,bc\sin(\beta-\gamma)}{(1+p)(1+p^2)}. \qquad (49)$$

The expressions $f^{(2)}, f^{(4)}, \ldots$ are capable of similar reductions, but I defer this, merely remarking that the problem of reduction is to find the simplest form of expressions homogeneous in x and y derived from the product of the two series

$$\left\{1-\frac{x}{[2]t}\left(\kappa-\frac{1}{\kappa}\right)+\frac{x^2}{[2][4]t^2}\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)-\ldots\right\}$$
$$\times\left\{1-\frac{y}{[2]t}\left(\kappa_1-\frac{1}{\kappa_1}\right)+\frac{y^2}{[2][4]t^2}\left(\kappa_1-\frac{1}{\kappa_1}\right)\left(p^2\kappa_1-\frac{1}{\kappa_1}\right)-\ldots\right\}.$$

subject to the condition

$$x\left(\kappa-\frac{1}{\kappa}\right)+y\left(\kappa_1-\frac{1}{\kappa_1}\right)=0.$$

13.

Referring to the expressions given in Art. 12 for $C_0, C_1, C_2, \ldots$, we see hat, if $n=0$ and $be^{\iota\beta}+ce^{\iota\gamma}$ be a real quantity, then, since $f'(b, c, \beta, \gamma)=0$,

$$J_0(be^{\iota\beta}, ce^{\iota\gamma}) = C_0-C_2f''(b, c, \beta, \gamma)+\ldots, \qquad (50)$$

nd in this expression

$$C_0 = J_{[0]}(b)\,\mathfrak{J}_{[0]}\left(\frac{c}{p}\right)-2p\cos(\beta-\gamma)J_{[1]}(b)\,\mathfrak{J}_{[1]}\left(\frac{c}{p}\right)+\ldots$$
$$+(-1)^n\,2p^{n^2}\cos n(\beta-\gamma)\,J_{[n]}(b)\,\mathfrak{J}_{[n]}\left(\frac{c}{p}\right)+\ldots,$$

$$C_2 f'' = -\frac{2i(1-p^2)\,bc\sin(\beta-\gamma)}{(1+p)(1+p^2)}$$

$$\times \{[e^{2\iota\beta}J_{[2]}\mathfrak{J}_{[0]}+e^{\iota(\beta+\gamma)}pJ_{[1]}\mathfrak{J}_{[1]}+e^{2\iota\gamma}p^4J_{[0]}\mathfrak{J}_{[2]}$$

$$-p\,(e^{4\iota\beta}J_{[3]}\mathfrak{J}_{[1]}+p^8e^{4i\gamma}J_{[1]}\mathfrak{J}_{[3]})\,e^{-\iota(\beta+\gamma)}$$

$$+p^4\,(e^{6i\gamma}J_{[4]}\mathfrak{J}_{[2]}+p^{12}e^{6\iota\gamma}J_{[2]}\mathfrak{J}_{[4]})\,e^{-2\iota(\beta+\gamma)}-\ldots\}+\ldots$$

$$+i^r f^{(r)}C_{(r)}+\ldots, \qquad (51)$$

which is the extension of Neumann's addition theorem, we may write the theorem

$$J_0(be^{\iota\beta}, ce^{\iota\gamma}) = J_{[0]}(b)\,\mathfrak{J}_{[0]}\left(\frac{c}{p}\right)-2p\cos a\,J_{[1]}(b)\,\mathfrak{J}_{[1]}\left(\frac{c}{p}\right)$$

$$+2p^4\cos 2a\,J_{[2]}(b)\,\mathfrak{J}_{[2]}\left(\frac{c}{p}\right)-\ldots+(p-1)\,\phi(b, c, \beta, \gamma)$$

$$(\beta-\gamma = a). \qquad (52)$$

Finally we may remark that the addition theorem investigated above is one of an infinite number of such addition theorems. The terms in the general theorem being of the form

$$p^{n(n-\nu)}\frac{[4n\nu]}{[2n\nu]}\cos n(\beta-\gamma)\,J_{[n]}(b)\,\mathfrak{J}_{[n]}\left(\frac{c}{p^{1-\nu}}\right), \qquad (52\text{A})$$

ν being an arbitrary integer, if $\nu = 0$, we have the set of theorems which we have been discussing.

PART II.

14. *Transformations of the Function* $P_{[n]}$.

It is well known that

$$F(a, \beta, \gamma, x) = (1-x)^{\gamma-a-\beta}F(\gamma-a, \gamma-\beta, \gamma, x)$$

$$= (1-x)^{-a}F\left(a, \gamma-\beta, \gamma, \frac{x}{x-1}\right)$$

$$= (1-x)^{-\beta}F\left(\beta, \gamma-a, \gamma, \frac{x}{x-1}\right). \qquad (53)$$

These expressions are a set of four equal particular integrals of the differential equation of the hypergeometric series denoted*

$$y_1 = y_2 = y_{17} = y_{18}.$$

* Forsyth, *Treatise on Differential Equations*, 2nd ed., pp. 192, 194.

Let $F([a][\beta][\gamma]x)$ denote the series

$$1+\frac{[a][\beta]}{[1][\gamma]}x+\frac{[a][a+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}x^2+\dots;$$

then we shall show that

$$1+\frac{[a][\beta]}{[1][\gamma]}p^{\gamma-a-\beta}x+\frac{[a][a+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}p^{2(\gamma-a-\beta)}x^2+\dots$$

$$=(1-x)_{\gamma-a-\beta}\left\{1+\frac{[\gamma-a][\gamma-\beta]}{[1][\gamma]}x+\frac{[\gamma-a][\gamma-a+1][\gamma-\beta][\gamma-\beta+1]}{[2]![\gamma][\gamma+1]}x^2+\dots\right\} \tag{54}$$

$$=(1-x)_{-a}\left\{1-\frac{[a][\gamma-\beta]}{[1][\gamma]}\frac{px}{(p^{a+1}-x)}+\frac{[a][a+1][\gamma-\beta][\gamma-\beta+1]}{[2]![\gamma][\gamma+1]}\frac{p^2x^2}{(p^{a+1}-x)(p^{a+2}-x)}-\dots\right\} \tag{55}$$

$$=(1-x)_{-\beta}\left\{1-\frac{[\beta][\gamma-a]}{[1][\gamma]}\frac{px}{(p^{\beta+1}-x)}+\frac{[\beta][[\beta+1][\gamma-a][\gamma-a+1]}{[2]![\gamma][\gamma+1]}\frac{p^2x^2}{(p^{\beta+1}-x)(p^{\beta+2}-x)}-\dots\right\}, \tag{56}$$

If $p=1$, these series reduce to the series in expression (53).

Consider series (55): we may write this series

$$(1-x)_{-a}\left\{1+\frac{[-a][\gamma-\beta]}{[1][\gamma]}\frac{x}{1-p^{-a-1}x}+\dots\right\},$$

and again, by means of result (2), we are able to write this in the form

$$(1-x)_{-a}+\frac{[-a][\gamma-\beta]}{[1][\gamma]}x(1-x)_{-a-1}+\frac{[-a][-a-1][\gamma-\beta][\gamma-\beta-1]}{[1][2][\gamma][\gamma+1]}x^2(1-x)_{-a-2}+\dots. \tag{57}$$

Replacing $(1-x)_{-a-r}$ by expansions in convergent series (1), we obtain the double series

$$\left\{1-\frac{[-a]}{[1]}x'+\ldots+(-1)^r p^{\frac{1}{2}r(r-1)}\frac{[-a][-a-1]\ldots[-a-r+1]}{[r]!}x^r+\ldots\right\}$$

$$+\frac{[-a][\gamma-\beta]}{[1][\gamma]}x\left\{1-\frac{[-a-1]}{[1]}x+\ldots\right.$$

$$\left.+(-1)^{r-1}p^{\frac{1}{2}(r-1)(r-2)}\frac{[-a-1][-a-2]\ldots[-a-r+1]}{[r]!}x^{r-1}+\ldots\right\}$$

$$+\frac{[-a][-a-1][\gamma-\beta][\gamma-\beta+1]}{[2]![\gamma][\gamma+1]}x^2\left\{1-\frac{[-a-2]}{[1]}x+\ldots\right.$$

$$\left.+(-1)^{r-2}p^{\frac{1}{2}(r-2)(r-3)}\frac{[-a-2][-a-3]\ldots[-a-r+1]}{[r]!}x^{r-2}+\ldots\right\}$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\,. \tag{58}$$

Collect the terms according to powers of x. We obtain the series

$$1+p^{\gamma-a-\beta}\frac{[a][\beta]}{[1][\gamma]}x+\ldots,$$

because the coefficient of x^r, after some obvious reduction, may be written

$$p^{-ra}\frac{[a][a+1]\ldots[a+r-1]}{[r]!}$$

$$\times\left\{1+p\frac{[-r][\gamma-\beta]}{[1][\gamma]}+p^2\frac{[-r][-r+1][\gamma-\beta][\gamma-\beta+1]}{[1][2][\gamma][\gamma+1]}+\ldots\right\}, \tag{59}$$

and the series within the large brackets is, for all values of r, by theorem (7),

$$p^{r(\gamma-\beta)}\frac{\Gamma_p([\gamma])\,\Gamma_p([\beta+r])}{\Gamma_p([\gamma+r])\,\Gamma_p([\beta])};$$

so that when r is a positive integer the expression (59), which is the coefficient of x^r, becomes

$$p^{r(\gamma-a-\beta)}\frac{[a][a+1]\ldots[a+r-1][\beta]\ldots[\beta+r-1]}{[r]![\gamma][\gamma+1]\ldots[\gamma+r-1]}.$$

This establishes the theorem

$$F([a][\beta][\gamma]p^{\gamma-a-\beta}x)=(1-x)_{-a}\left\{1+\frac{[a][\gamma-\beta]}{[1][\gamma]}\frac{px}{(x-p^{a+1})}+\ldots\right\}, \tag{60}$$

which is the analogue of $y_1=y_{17}$, among the twenty-four particular integrals of the hypergeometric differential equation.

Since the series F is symmetrical in α and β, we have, by an interchange of α and β,

$$F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}x) = (1-x)_{-\beta}\left\{1+\frac{[\beta][\gamma-\alpha]}{[1][\gamma]}\frac{px}{(x-p^{\beta+1})}+\dots\right\},$$

which corresponds to $y_1 = y_{18}$.

The transformation analogous to

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma, x)$$

is due to Heine, being easily deduced from his theorem,

$$\phi[a, b, c, q, x] = \prod_{n=0}^{n=\infty}\frac{\left(1-\frac{abx}{c}q^n\right)}{(1-q^n)}\,\phi\left[\frac{c}{a}, \frac{c}{b}, c, q, \frac{abx}{c}\right],$$

ϕ denoting Heine's series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\dots,$$

if we put $a = p^{\alpha}$, $b = p^{\beta}$, $c = p^{\gamma}$, $q = p$, $x = p^{\gamma-\alpha-\beta}x$.

The result is

$$F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}x) = (1-x)_{\gamma-\alpha-\beta}F([\gamma-\alpha][\gamma-\beta][\gamma]x). \tag{61}$$

We now proceed to apply these and other special transformations to the functions $P_{[n]}$ and $Q_{[n]}$.

15. *The Function* $P_{[n]}$.

If in theorem (60) of the last article we make

$$\alpha = -\tfrac{1}{2}n, \qquad \beta = \tfrac{1}{2}(n+1), \qquad \gamma = \tfrac{1}{2},$$

and change p into p^2, also making x into x^2/p, we have, after some obvious reductions,

$$1-p^{1-n}\frac{[n][n+1]}{[2]!}x^2+p^{-n}\frac{[n-2][n][n+1][n+3]}{[4]!}x^4-\dots$$

$$= \prod_{r=0}^{r=\infty}\frac{(1-p^{2r-1}x^2)}{(1-p^{n+2r-1}x^2)}\left\{1+p^{-n}\frac{[n]^2}{[2]!}\frac{x^2}{(p^{n-3}x^2-1)}\right.$$

$$\left.+p^{-2n}\frac{[n]^2[n-2]^2}{[4]!}\frac{x^4}{(p^{n-3}x^2-1)(p^{n-5}x^2-1)}-\dots\right\}. \tag{62}$$

We compare the series on the left side of the above expression with the series

$$1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)\,n\,(n+1)(n+3)}{4!}x^4-\dots,$$

which is a particular solution of Legendre's differential equation; I have discussed this series in the more general form in connection with the differential equation in (*Trans. R.S. Edin.*, "Generalized Functions of Legendre and Bessel," Vol. XLI., Art. 5, p. 20). In the case when n is an even positive integer it is easily shown that the series on the left side of (62) is $C.P_{[n]}(x)$,

$$P_{[n]}(x) = \frac{[2n]!}{[n]!\,[n]!\,(2)_n} \left\{ x^n - p^3 \frac{[n][n-1]}{[2][2n-1]} x^{n-2} + \dots \right\}.$$

If in result (60) we put $\alpha = -\frac{1}{2}n$, $\beta = -\frac{1}{2}(n-1)$, $\gamma = \frac{1}{2}-n$, and change p into p^2, also x into p^3/x^2, we obtain

$$1-p^3 \frac{[n][n-1]}{[n][2n-1]} x^{-2} + p^8 \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]} x^{-4} - \dots$$

$$= \prod_{r=0}^{r=\infty} \frac{(x^2-p^{2r+3})}{(x^2-p^{2r-n+3})} \left\{ 1 - \frac{[n]^2}{[2][2n-1]} \frac{p^{n+2}}{(p^{n+1}-x^2)} \right.$$

$$\left. + \frac{[n][n-2]^2}{[2][4][2n-1][2n-3]} \frac{p^{2n+4}}{(p^{n+1}-x^2)(p^{n-1}-x^2)} - \dots \right\}. \quad (63)$$

Multiplying both sides of this equation by $\dfrac{\Gamma_p([2n+1])}{\Gamma_p([n+1])\,\Gamma_p([n+1])\,(2)_n}$, we obtain an interesting transformation of $P_{[n]}(x)$. For the differential equation of the functions $P_{[n]}$, $Q_{[n]}$, and its properties, I refer to *Trans. R.S. Edin.*, Vol. XLI. Two other similar theorems may be obtained in the same way, if we substitute the particular values of α, β, γ, given above, in equation (56). Theorem (61) reduces to an identity if we substitute special values, so as to obtain $P_{[n]}(x)$, both sides of the equation becoming $CP_{[n]}(x)$.

16.

In this article the transformations corresponding to

$P_n(\cos\theta)$

$$= \cos^n\theta \left\{ 1 - \frac{n(n-1)}{2^2} \tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{2^2.4^2} \tan^4\theta - \dots \right\} \quad (64)$$

$$= \frac{1.3\dots(2n-1)}{2^n.n!} \left\{ \cos n\theta + \frac{1.n}{1.(2n-1)} \cos(n-2)\theta \right.$$

$$\left. + \frac{1.3.n(n-1)}{1.2(2n-1)(2n-3)} \cos(n-4)\theta + \dots \right\} \quad (65)$$

will be considered.

Take a series

$$x^n\left\{1+\frac{[n][n-1]}{[2][2]}\left(p^2-\frac{p^3}{x^2}\right)\right.$$
$$\left.+\frac{[n][n-1][n-2][n-3]}{[2][4][2][4]}\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)+\ldots\right\}, \quad (66)$$

of which the general term is

$$x^n\frac{[n][n-1][n-2]\ldots[n-2r+1]}{\{[2][4][6]\ldots[2r]\}^2}\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)\ldots\left(p^{4r-2}-\frac{p^{2r+1}}{x^2}\right).$$

If now we expand* the products which involve x in each term, we obtain that the coefficient of x^n in (66) is

$$1+p^2\frac{[n][n-1]}{[2][2]}+p^8\frac{[n][n-1][n-2][n-3]}{[2][4][2][4]}+\ldots$$
$$+p^{2r^2}\frac{[n]\ldots[n-2r+1]}{\{[2]\ldots[2r]\}^2}+\ldots,$$

and generally that the coefficient of x^{n-2r} is

$$(-1)^r p^{r(r+2)}\frac{[n]\ldots[n-2r+1]}{\{[2][4]\ldots[2r]\}^2}$$
$$\times\left\{1+p^{2(r+2)}\frac{[n-2r][n-2r-1]}{[2r+2][2r+2]}\frac{[2r+2]}{[2]}+\ldots\right\}. \quad (67)$$

The general term of the series within the large brackets in (67) is

$$p^{2s(r+s)}\frac{[n-2r]\ldots[n-2r-2s+1]}{\{[2r+2]\ldots[2r+2s]\}^2}\frac{[2r+2]\ldots[2r+2s]}{[2]\ldots[2s]}.$$

When n is positive and integral these series are finite, but for all values of n they are particular cases of series $F([a][\beta][\gamma]x)$. The application of

$$F([a][\beta][\gamma]p^{\gamma-a-\beta}) = \prod_0^\infty\frac{[\gamma-\beta+n][\gamma-a+n]}{[\gamma-a-\beta+n][\gamma+n]}$$

in the case $a = r-\frac{1}{2}n$, $\beta = s-\frac{1}{2}(n-1)$, $\gamma = r+1$, the base p being changed into p^2, gives us

$$1+p^{2(r+2)}\frac{[n-2r][n-2r-1]}{[2][2r+2]}+\ldots$$
$$=\frac{[2][4]\ldots[2r]}{[2n-1][2n-3]\ldots[2n-2r+1]}\frac{[1][3]\ldots[2n-1]}{[n]!} \quad (68)$$

or an equivalent expression in infinite products, when n is not integral.

* "Series connected with the Enumeration of Partitions," p. 69, (11), (12).

The series (67), which is the coefficient of x^{n-2r}, is then, by (68) and (67),

$$(-1)^r p^{r(r+2)} \frac{[n]\dots[n-2r+1]}{\{[2][4]\dots[2r]\}^2} \frac{[2]\dots[2r]}{[2n-1]\dots[2n-2r+1]} \frac{[1][3]\dots[2n-1]}{[n]!},$$

which is

$$(-1)^r p^{r(r+2)} \frac{[2n]!}{[n]!\,[n]!\,(2)_n} \frac{[n][n-1]\dots[n-2r+1]}{[2]\dots[2r][2n-1]\dots[2n-2r+1]}, \qquad (69)$$

namely, the coefficient of x^{n-2r} in the standard form of $P_{[n]}(x)$. When n is not positive and integral we must replace $[2n]!$ and $(2)_n$ by infinite products defined in Art. 3. We have finally

$$P_{[n]}(x) = \sum_{r=0}^{r=\infty} x^n \frac{[n][n-1][n-2]\dots[n-2r+1]}{\{[2][4][6]\dots[2r]\}^2}$$
$$\times\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)\dots\left(p^{4r-2}-\frac{p^{2r+1}}{x^2}\right). \qquad (70)$$

The extension of (64) is thus determined; for, putting $x = \cos\theta$, $p = 1$, the theorem reduces to

$$P_n(\cos\theta) = \cos^n\theta\left\{1-\frac{n(n-1)}{2^2}\tan^2\theta+\dots\right\}.$$

17.

Consider the series

$$\frac{[2n]!}{[n]!\,[n]!\,(2)_n}\left[\frac{1}{(2)_n}\left\{x+\frac{1}{x}\right\}_n - \frac{p}{(2)_{n-2}}\frac{[n][n-1]}{[n][2n-1]}\left\{x+\frac{p^2}{x}\right\}_{n-2}+\dots\right], \qquad (71)$$

which, if $p = 1$, reduces term by term to $P_n\left\{\frac{1}{2}\left(x+\frac{1}{x}\right)\right\}$.

The general term of the series within the brackets is

$$(-1)^r p^{r(2r+1)} \frac{1}{(2)_{n-2r}} \frac{[n][n-1]\dots[n-2r+1]}{[2][4]\dots[2r][2n-1]\dots[2n-2r+1]}\left\{x+\frac{p^{2r}}{x}\right\}_{n-2r}$$

If n be a positive integer,

$$\left\{x+\frac{1}{x}\right\}_n = \left(x+\frac{1}{x}\right)\left(x+\frac{p^2}{x}\right)\left(x+\frac{p^4}{x}\right)\dots \text{ to } n \text{ factors},$$

$$\left\{x+\frac{p^2}{x}\right\}_n = \left(x+\frac{p^2}{x}\right)\left(x+\frac{p^4}{x}\right)\left(x+\frac{p^6}{x}\right)\dots \text{ to } n \text{ factors}.$$

In case n be not a positive integer,

$$\left\{x+\frac{1}{x}\right\}_n = \prod_{r=1}^{\infty} \frac{\left(x+\frac{p^{2n-2r}}{x}\right)}{\left(x+\frac{p^{-2r}}{x}\right)} x^n \quad (p>1) \quad \text{or} \quad \prod_{r=0}^{\infty} \frac{\left(x+\frac{p^{2r}}{x}\right)}{\left(x+\frac{p^{2n+2r}}{x}\right)} x^n \quad (p<1).$$

For all values of n we have shown* that

$$\left\{x+\frac{1}{x}\right\}_n$$

$$= x^n + \frac{[2n]}{[2]} x^{n-2} + p^2 \frac{[2n][2n-2]}{[2][4]} x^{n-4} + p^6 \frac{[2n][2n-2][2n-4]}{[2][4][6]} x^{n-6} + \ldots.$$

If, now, we expand $\left\{x+\frac{1}{x}\right\}_n$, &c., in expression (71), and collect the terms according to powers of x, we find that x^n arises only from the first term of (71), and the term which involves it is $x^n/(2)_n$, and in general the terms involving x^{n-2r} form the series

$$p^{r(r-1)} \frac{[2n]\ldots[2n-2r+2]}{[2]\ldots[2r].(2)_n} \left\{1 - p\frac{[2n-2r][2r]}{[2][2n-1]} \right.$$

$$\left. + p^4 \frac{[2n-2r][2n-2r-2][2r][2r-2]}{[2][4][2n-1][2n-3]} - \ldots \right\} x^{n-2r}. \quad (72)$$

The series within the large brackets is a particular case of the general hypergeometric series (7).

When n is positive and integral the sum of (72) is

$$p^{r(r-1)} \frac{\{[2n]\ldots[2n-2r+2]\}\{[1][3]\ldots[2r+1]\} x^{n-2r}}{\{[2][4]\ldots[2r]\}\{[2n-1]\ldots[2n-2r+1]\}(2)_n}.$$

If n be not a positive integer, the finite products in this expression will be replaced by appropriate expressions in terms of Γ_p functions, which, for n integral and positive, will reduce to the expression given above. We have then transformed (71) into

$$\frac{[2n]!}{[n]![n]!(2)_n(2)_n} \left\{x^n + \frac{[2n]}{[2][2n-1]} x^{n-2} + p^2 \frac{[1][3][2n][2n-2]}{[2][4][2n-1][2n-3]} x^{n-4} + \ldots\right\}, \quad (73)$$

* *Supra*, Ser. 2, Vol. 1, p. 69, (11).

which is the extension of

$$P_n\left\{\tfrac{1}{2}\left(x+\frac{1}{x}\right)\right\} = \frac{2n!}{n!\,n!\,2^{2n}}\left\{x^n + \frac{1.2n}{2(2n-1)}x^{n-2} + \dots\right\}.$$

$[n]! = \Gamma_p([n+1])$.—If we make $p = 1$ and $x+x^{-1} = 2\cos\theta$, we obtain

$$P_n(\cos\theta) = \frac{2n!}{n!\,n!\,2^{2n}}\left\{\cos n\theta + \frac{1.2n}{2(2n-1)}\cos(n-2)\theta + \dots\right\}.$$

18.

It is easily established that

$$\lambda^n = \frac{(2)_n[n]!\,[n]!}{[2n+1]!}\left\{[2n+1]\,P_{[n]}(\lambda) + p^3[2n-3]\frac{[2n+1]}{[2]}P_{[n-2]}(\lambda)\right.$$
$$\left. + p^6[2n-7]\frac{[2n+1][2n-1]}{[2][4]}P_{[n-4]}(\lambda) + \dots\right\}, \quad (74)$$

for, replacing the functions P by their series expansions, we find that the terms involving λ^{n-2r} give the series

$$p^{r(r+2)}\frac{[2n-2r]!}{\{2n-2r\}!\,\{2r\}!\,[n-2r]!}[2n+1]\,\lambda^{n-2r}\left[1 + \sum_{s=1}^{\infty}(-)^s p^{s(s+1)-2rs}\right.$$
$$\left. \times \frac{\{2r\}!}{\{2r-2s\}!\,\{2s\}!}\,\frac{[2n-1]\dots[2n-2s+3]}{[2n-2r-1]\dots[2n-2r-2s+1]}[2n-4s+1]\right].$$

This series reduces identically to zero, as may be seen if we sum the series term by term; for we find that the sum of r terms is for all values of r a factor of the $(r+1)$-th term, and the expression for the sum will vanish by reason of the presence of a zero factor: the above reasoning establishes a more general theorem (cf. *Trans. R.S. Edin.*, *loc. cit.*).

$$\lambda^n \underset{\dots}{x^{[n]}} = \frac{(2)_n[n]!\,[n]!}{[2n+1]!}\left\{[2n+1]\,P_{[n]}(\lambda, x)\right.$$
$$\left. + p^3[2n-3]\frac{[2n+1]}{[2]}P_{[n-2]}(\lambda, x) + \dots\right\}. \quad (75)$$

We compare this with the theorem in Art. 36, p. 20, Todhunter, *Functions of Laplace, Lamé, and Bessel*, and proceed to show that

$$\frac{1}{y-x} = \sum_{0}^{\infty}(2n+1)\,Q_n(y)\,P_n(x)$$

may be extended in the form

$$\frac{1}{\mu-\lambda}=\sum_{n=0}^{n=\infty}[2n+1]\,Q_{[n]}(\mu)\,P_{[n]}(\lambda). \tag{76}$$

If $\mu>\lambda$, then we have

$$\frac{1}{\mu-\lambda}=\frac{1}{\mu}+\frac{\lambda}{\mu^2}+\frac{\lambda^2}{\mu^3}+\dots.$$

Now express each power of λ in a series of $P_{[n]}$ functions by means of the theorem (75), and then collect all the terms which involve the same coefficient. Thus $P_{[n]}(\lambda)$ will arise from $\frac{\lambda^n}{\mu^{n+1}}$, $\frac{\lambda^{n+2}}{\mu^{n+3}}$, ..., and, for the multiplier of it,

from $\frac{\lambda^n}{\mu^{n+1}}$ we get $\frac{1}{\mu^{n+1}}\frac{[n]!\,[n]!\,(2)_n}{[2n+1]!}[2n+1]$,

from $\frac{\lambda^{n+2}}{\mu^{n+3}}$ we get $\frac{1}{\mu^{n+3}}\frac{[n+2]!\,[n+2]!\,(2)_{n+2}}{[2n+5]!}[2n+1]\,p^3\frac{[2n+5]}{[2]}$,

from $\frac{\lambda^{n+4}}{\mu^{n+5}}$ we get $\frac{1}{\mu^{n+5}}\frac{[n+4]!\,[n+4]!\,(2)_{n+4}}{[2n+9]!}[2n+1]\,p^6\frac{[2n+9][2n+7]}{[2][4]}$.

From this we see that the multiplier of $P_{[n]}(\lambda)$ is

$$[2n+1]\frac{[n]!\,[n]!\,(2)_n}{[2n]!}\left\{\mu^{-n-1}+p^3\frac{[n+1][n+2]}{[2][2n+3]}\mu^{-n-3}+\dots\right\}$$
$$=[2n+1]\,Q_{[n]}[\mu].$$

The series $Q_{[n]}$ reduces term by term to Legendre's series Q_n if we make the base p unity. The series

$$Q_{[n]}(\mu,x)$$
$$=\frac{[n]!\,[n]!\,(2)_n}{[2n+1]}\left\{\mu^{-n-1}x^{[-n-1]}+p^3\frac{[n+1][n+2]}{[2][2n+3]}\mu^{-n-3}x^{[-n-3]}+\dots\right\}$$

will be found discussed as a solution along with $P_{[n]}(\lambda, x)$ of a certain differential equation analogous to Legendre's in *Trans. R.S. Edin.*, Vol. XLI., 1904. We write now

$$\frac{1}{\mu-\lambda}=\sum_{n=0}^{n=\infty}[2n+1]\,P_{[n]}(\lambda)\,Q_{[n]}(\mu)\quad(\mu>\lambda,\ \mu>1). \tag{77}$$

Subject to convergence of the series, we may also write down a more general theorem, viz.,

$$\sum_{n=0}^{n=\infty}[2n+1]\,P_{[n]}(\lambda, x)\,Q_{[n]}(\mu, y)=\frac{1}{\mu y^{[1]}}+\frac{\lambda x^{[1]}}{\mu^2 y^{[2]}}+\dots. \tag{78}$$

19. *A Case of Summation of* $F([a][\beta][\gamma][\delta][\epsilon]x)$.

In Art. 14 we have shown that

$$F([a][\beta][\gamma]p^{\gamma-a-\beta}x)=(1-x)_{\gamma-a-\beta}F([\gamma-a][\gamma-\beta][\gamma]x). \tag{79}$$

Now $(1-x)_{\gamma-a-\beta}=1-\dfrac{[\gamma-a-\beta]}{[1]}x+\dots$

$$+(-1)^r p^{\frac{1}{2}r(r-1)}\frac{[\gamma-a-\beta]\dots[\gamma-a-\beta-r+1]}{[r]!}x^r-\dots.$$

Replacing the functions in (79) by their expansions in series of powers of x, and equating the coefficients of equal powers of x, we obtain from the terms involving x^n

$$\frac{[\gamma-a]_n[\gamma-\beta]_n}{[n]!\,[\gamma]_n}+\frac{[\gamma-a]_{n-1}[\gamma-\beta]_{n-1}}{[n-1]!\,[\gamma]_{n-1}}\frac{[a+\beta-\gamma]}{[1]}p^{\gamma-a-\beta}+\dots+\frac{[a+\beta-\gamma]_n}{[n]!}$$
$$=\frac{[a]_n[\beta]_n}{[n]!\,[\gamma]_n}p^{n(\gamma-a-\beta)},$$

in which $\qquad [\gamma]_n=[\gamma][\gamma+1][\gamma+2]\dots[\gamma+n-1]$.

Change $\gamma-a$ to x, $\gamma-\beta$ to y, γ to z; then

$$\frac{[x]_n[y]_n}{[z]_n}+\Sigma\frac{[n]!}{[r]!\,[n-r]!}\frac{[x]_{n-r}[y]_{n-r}}{[z]_{n-r}}[z-x-y]_r\,p^{r(x+y-z)}$$
$$=\frac{[z-x]_n[z-y]_n}{[z]_n}p^{n(x+y-z)}. \tag{80}$$

Divide throughout by $[x]_n[y]_n/[z]_n$, and put

$$x=a-\delta+1,\qquad y=a-\epsilon+1,\qquad z=a-\beta+1,\qquad n=-a;$$

the series (80) now becomes

$$1+p\frac{[a][\beta][\delta+\epsilon-a-\beta-1]}{[1][\delta][\epsilon]}$$
$$+p^2\frac{[a][a+1][\beta][\beta+1][\delta+\epsilon-a-\beta-1][\delta+\epsilon-a-\beta]}{[2][\delta][\delta+1][\epsilon][\epsilon+1]}+\dots$$
$$=p^{a\beta}\frac{\Pi_p([a-\delta])\,\Pi_p([\beta-\delta])\,\Pi_p([a-\epsilon])\,\Pi_p([\beta-\epsilon])}{\Pi_p([a+\beta-\delta])\,\Pi_p([a+\beta-\epsilon])\,\Pi_p([-\delta])\,\Pi_p([-\epsilon])}. \tag{81}$$

In this equation α is a negative integer owing to the manner in which we obtained the identity (80), but both the product and the series are symmetrical in α and β, and β is not restricted to integral values; hence we can write (81) as valid in general, subject to convergence conditions.

The following are two examples of summation :—

$$p^{-n^2}\left\{\frac{\Gamma_p([2n+1])}{\Gamma_p([n+1])\,\Gamma_p([n+1])}\right\}^2$$
$$=1+\sum_{r=1}^{\infty} p^{r(r-2n)}\left\{\frac{[n][n-1]\dots[n-r+1]}{[r]!}\right\}^2\frac{[2n+1][2n+2]\dots[2n+r]}{[r]!}, \tag{82}$$

$$J_{[1]}(a)\,\mathfrak{J}_{[1]}(a)=\frac{[4]}{[2]}J_{[0]}(a)\,\mathfrak{J}_{[2]}(a)+p^2\frac{[8]}{[4]}J_{[1]}(a)\,\mathfrak{J}_{[3]}(a)+\dots$$
$$+p^{r(r-1)}\frac{[4r]}{[2r]}J_{[r-1]}(a)\,\mathfrak{J}_{[r+1]}(a)+\dots. \tag{83}$$