NOTE ON A DIOPHANTINE APPROXIMATION

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IT has long been known that, if a is an irrational number and b any number, integers x and y can be found so that

$$ay-x-b \tag{I}$$

is arbitrarily small.

Tchebychef proved that there is an infinite number of integer y's such that

$$|ay-x-b| < \frac{1}{2\,|y|}, \tag{II}$$

Hermite that the same is true if $\frac{1}{2}$ is replaced by the smaller number $\sqrt{\frac{2}{27}}$, and Minkowski* that

$$|ay-x-b| < \frac{1}{4\,|y|} \tag{III}$$

holds for an infinite number of integer values of y.

It is explicitly stated by Minkowski that for the truth of (III) b must not be an integer, and, in fact, it had previously been practically established by Markoff and Hurwitz† that, when b is an integer, the strongest general inequality of type (II) is

$$|ay-x-b| < \frac{1}{\sqrt{5}\,|y|}$$

Inasmuch as $\sqrt{\frac{2}{27}}$ is less than $\sqrt{\frac{1}{5}}$, and Hermite's argument does not depend on b being non-integer, that argument needs modification before

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* See Minkowski, *Werke*, Vol. 1, p. 320. Hermite's paper occurs in his *Œuvres*, Vol. 3, and in *Crelle's Journal* for 1879.

† Hurwitz, *Math. Ann.*, Vol. 39, p. 279. Cf. Klein's lithographed lectures on the Theory of Numbers.

it can be accepted. It seems probable that once the method is perfected it can be extended to two or more linear forms of the type (I).*

The object of this note, however, is to prove that Minkowski's result is final, *i.e.* that, if $k < \frac{1}{4}$, it is possible to choose a and b so that there is not an infinite number of integers y for which

$$|ay - x - b| < \frac{k}{|y|}.$$

The argument is quite simple, but it does not establish the result of Minkowski, only that no stronger result can be universally true. I take $b = \frac{1}{2}$, and remark that if,

$$|ay - x - \tfrac{1}{2}| < \frac{k}{|y|},$$

for a negative value of y, the same is true for the corresponding positive value, though not for the same value of x.

Thus, if the result is not final, we should have

$$|ay - x - \tfrac{1}{2}| < \frac{k}{y}$$

for an infinite number of positive values of y and $k < \frac{1}{4}$. That is to say

$$\left| a - \frac{2x+1}{2y} \right| < \frac{k}{y^2},$$

or

$$\left| a - \frac{2x+1}{2y} \right| < \frac{h}{(2y)^2} \quad (h < 1).$$

There would thus be an infinite number of approximations ξ/η to a in which ξ is odd, η even, and

$$\left| a - \frac{\xi}{\eta} \right| < \frac{h}{\eta^2} \quad (h < 1). \tag{IV}$$

It may be assumed that ξ and η are co-prime, for, if they were not, we could deduce a similar inequality in which they were, with a smaller value of h.

Now any approximation satisfying the last named condition must (as can be easily proved) be either a principal, or an auxiliary, convergent of the continued fraction for a. I first arrange that the denominators of all the principal convergents are odd, so that ξ/η is none of these.

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* And so to establish a theorem, the truth of which is conjectured by Hardy and Littlewood, *Acta Math.*, Vol. 37, p. 178, under Theorem 1.341.

This is secured by writing

$$a = \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{a_n +} \cdots = (a_1, a_2, \ldots, a_n, \ldots),$$

in which a_1 is odd and the other a's all even; the approximation ξ/η obtained above must then be an auxiliary convergent.

Suppose that p_n/q_n is the convergent corresponding to a_n, and that p/q is an auxiliary convergent between p_n/q_n and p_{n+1}/q_{n+1}; then we have

$$\frac{p}{q} = \frac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}},$$

where α is an integer less than a_{n+1}. A simple calculation gives

$$\left| \frac{p}{q} - a \right| = \frac{1}{\lambda q^2},$$

where
$$\lambda = (\alpha, a_n, a_{n-1}, \ldots) + (\beta, a_{n+2}, a_{n+3}, \ldots), \tag{V}$$

and
$$\alpha + \beta = a_{n+1}.$$

If p/q satisfies the condition (IV) we must have

$$\lambda > \frac{1}{h} > 1:$$

hence either $\alpha = 1$ or $\beta = 1$. Taking the two cases we find

$$\lambda < 1 + \frac{1}{a_{n+1} - 1}$$

in each.

It is now quite clear that, if the a's steadily increase without limit, there will not be an infinite number of such λ's greater than $1/h$, if h is less than unity. This is the result required. Briefly, if we take $b = \frac{1}{2}$ and

$$a = (a_1, a_2, \ldots, a_n, \ldots),$$

when a_1 is odd, the other a's even and steadily increasing, there is not an infinite number of values of y for which

$$|ay - x - b| < \frac{1}{k|y|} \quad \left(k < \frac{1}{4}\right).$$

It may be noted that, even in this case, there is an infinite number of values of y for which

$$| ay-x-b | < \frac{1}{4 | y |}.$$

In fact, with either $\alpha = 1$ or $\beta = 1$ above, p is odd and q is even, and from (V)

$$\lambda > \frac{1}{1+1/a_n} + \frac{1}{a_{n+1}} > 1 - \frac{1}{a_n} + \frac{1}{a_{n+1}},$$

and

$$\lambda > 1 - \frac{1}{a_{n+2}} + \frac{1}{a_{n+1}},$$

in the two cases. With our conditions the second value of λ exceeds unity; so, taking a p/q for which $\beta = 1$, we get the result, which is, of course, in accordance with the general one of Minkowski.