THE CLASSIFICATION OF RATIONAL APPROXIMATIONS

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Introduction.

It was proved by Hermite* that, if θ is any incommensurable number, there exists an infinite number of rational approximations x/y such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{y^2\sqrt{3}},$$

and that for two consecutive approximations x/y, x'/y' of this class

$$xy' - x'y = \pm 1.$$

The same theorem has been proved by Minkowski† when $\sqrt{3}$ is replaced by 2.

The main object of this paper is to prove that, *if*

$$1\tfrac{1}{2} \leqslant k \leqslant 2,$$

then for two consecutive approximations that satisfy the condition

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2},$$

we have
$$xy' - x'y = \pm 1. \tag{A}$$

If k is outside the limits stated, this last relation need not be satisfied by two consecutive approximations (§ 7).

What is alluded to as classification means that, when θ is given, a class of approximations consists of those satisfying

$$| x/y - \theta | < 1/ky^2,$$

* *Œuvres*, Vol. 1, p. 168. Cf. also the letters to Jacobi.

† *Ges. Abhandlungen*, Vol. 1, p. 320; *Math. Ann.*, Vol. 54, p. 91.

where k is fixed. The class is infinite for all θ's when $k \leqslant \sqrt{5}$.* When approximations are said to be arranged in order, it is order of increasing complexity, *i.e.* increasing values of y, that is meant.

1. We shall first prove that, if

$$|x/y-\theta| < 1/y^2,$$

and the fraction x/y is in its lowest terms, then it is a convergent to the continued fraction for θ.

In fact, express x/y as a continued fraction, and arrange so that the penultimate convergent x'/y' errs from x/y in the same direction that θ differs from x/y.

Since $$|x/y-x'/y'| = 1/yy',$$

it is clear that θ lies between the two fractions; we can therefore write

$$\theta = \frac{px+x'}{py+y'} \quad (p > 0);$$

and hence, if $$\frac{x}{y} = c_0 + \frac{1}{c_1} + \cdots \frac{1}{c_n},$$

or $$(c_0;\ c_1, c_2, \ldots, c_n),$$

we have $$\theta = (c_0;\ c_1, c_2, \ldots, c_n, p).\dagger$$

If $p > 1$, the C.F. for θ agrees with that for x/y as far as c_n; if $p < 1$, the C.F. for θ only differs in that c_n is to be increased by the integral part of $1/p$. In the first case, x/y is a principal convergent to the C.F. for θ, and in the second it is an auxiliary convergent.

2. Let the C.F. for θ be

$$(a_0;\ a_1, a_2, \ldots, a_n, \ldots),$$

* Hurwitz, *Math. Ann.*, Vol. 39, p. 279, and Markoff, *ibid.*, Vol. 15, p. 384. The approximations of Hermite do not include all that satisfy the condition indicated. What they are and what they are not has been elegantly determined by Humbert, *Comptes Rendus*, Dec. 1915. Cf. Minkowski, *Ges. Abhandlungen*, Vol. 1, p. 279. The case $\Omega = 1$ is here referred to.

† We also write

$$(c_0;\ c_1, c_2, c_3, \ldots) \quad \text{for} \quad c_0 + \frac{1}{c_1 +} \frac{1}{c_2 +} \cdots$$

and $$(c_1, c_2, c_3, \ldots) \quad \text{for} \quad \frac{1}{c_1 +} \frac{1}{c_2 +} \frac{1}{c_3 +} \cdots.$$

and denote the convergent corresponding to a_n by p_n/q_n; and write

$$\omega_n = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots).$$

Then
$$\theta = \frac{\omega_n p_n + p_{n-1}}{\omega_n q_n + q_{n-1}},$$

and
$$\left| \frac{p_n}{q_n} - \theta \right| = \frac{1}{\lambda_n q_n^2},$$

where $\lambda_n = \omega_n + \dfrac{q_{n-1}}{q_n} = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots) + (a_n,\ a_{n-1},\ a_{n-2},\ \ldots).$ (2 . 1)

Next consider an auxiliary (or intermediate) convergent p/q, where

$$p = a p_{n-1} + p_{n-2}, \qquad q = a q_{n-1} + q_{n-2},$$

a being an integer less than a_n. It is easy to see that

$$\left| \frac{p}{q} - \theta \right| = \frac{1}{\lambda q^2},$$

where
$$\lambda = (\beta,\ a_{n+1},\ a_{n+2},\ \ldots) + (a,\ a_{n-1},\ a_{n-2}), \tag{2 . 2}$$

and
$$a + \beta = a_n.$$

In fact, we have
$$\theta = \frac{\eta p + p_{n-1}}{\eta q + q_{n-1}},$$

where
$$\eta = (\beta,\ a_{n+1},\ a_{n+2},\ \ldots),$$

and so
$$\frac{p}{q} - \theta = \frac{1}{q(\eta q + q_{n-1})} = \frac{1}{q^2(\eta + q_{n-1}/q)};$$

and since
$$\frac{q_{n-1}}{q} = (a,\ a_{n-1},\ a_{n-2},\ \ldots),$$

the result follows.

If p/q is really an auxiliary (and not a principal) convergent, a, β are both positive. Hence λ is in all cases less than 2, and less than 1 unless either a or β is unity. Thus :—

The approximations to θ which belong to the class defined by

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{y^2},$$

are all included among the principal convergents for θ and the auxiliary *convergents adjacent to them. The former must satisfy the condition and the latter may do.*

The formulæ (2 . 1) and (2 . 2) are fundamental.

3. If a convergent intermediate between p_{n-1}/q_{n-1} and p_n/q_n satisfies the condition

$$\left|\frac{p}{q}-\theta\right| < \frac{1}{kq^2},$$

then p_n/q_n also satisfies the condition. For

$$\left|\frac{p}{q}-\theta\right| = \frac{1}{\lambda q^2},$$

$$\left|\frac{p_n}{q_n}-\theta\right| = \frac{1}{\lambda_n q_n^2},$$

and we have only to see that $\lambda_n > \lambda$.

We have
$$\lambda = \frac{1}{\beta+x}+\frac{1}{a+y} \quad (x,\, y < 1),$$

$$\lambda_n = \frac{1}{x}+\frac{1}{a_n+y},$$

$$\lambda_n-\lambda = \frac{\beta}{x(x+\beta)}-\frac{\beta}{(a+y)(a_n+y)}.$$

The first term here is greater than

$$\frac{\beta}{\beta+1},$$

and the other less than
$$\frac{\beta}{aa_n}.$$

The result $\lambda_n > \lambda$ follows at once.

4. As regards principal convergents, we have

$$\lambda_n = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots)+(a_n,\ a_{n-1},\ a_{n-2},\ \ldots),$$

or say
$$\lambda_n = \xi_n+\eta_n.$$

It is clear that
$$\lambda_{n-1} = \xi_{n-1}+\eta_{n-}\ ,$$

where
$$\xi_{n-1} = a_n + 1/\xi_n,$$
$$1/\eta_n = a_n + \eta_{n-1},$$
so that
$$\xi_{n-1} + \eta_{n-1} = 1/\xi_n + 1/\eta_n.$$
It follows that
$$\lambda_n + \lambda_{n-1} = \xi_n + 1/\xi_n + \eta_n + 1/\eta_n;$$
and since
$$\xi_n > a_{n+1}, \quad 1/\eta_n > a_n,$$
we deduce
$$\lambda_n + \lambda_{n-1} > a_{n+1} + 1/a_{n+1} + a_n + 1/a_n.$$
In all cases therefore
$$\lambda_n + \lambda_{n-1} > 4,$$
and, unless
$$a_n = a_{n+1} = 1,$$
$$\lambda_n + \lambda_{n-1} > 4\tfrac{1}{2}.$$

It follows that except when, in the C.F. for θ, a_n is ultimately always unity, there is an infinite number of values of n for which $\lambda_n > 2\tfrac{1}{4}$.

In the exceptional case it is clear that, when $n \to \infty$.
$$\lambda_n \to \sqrt{5}.$$

To be more explicit, suppose a_r is the last a that is greater than unity, then
$$\lambda_n > \sqrt{5} \quad \text{for} \quad n-r \text{ odd},$$
and
$$\lambda_n < \sqrt{5} \quad \text{for} \quad n-r \text{ even}.$$

Hence we have the result, first explicitly stated by Hurwitz, that for any irrational θ there is an infinite number of approximations x/y such that
$$\left| \frac{x}{y} - \theta \right| < \frac{1}{y^2\sqrt{5}},$$
while, if $k > \sqrt{5}$, there are irrationals θ having only a finite number of approximations such that
$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2}.$$

We have only to see, in fact, that, for all irrationals θ, $\lambda_n > \sqrt{5}$ for an infinite number of n's.

The class corresponding to a value of k is always infinite if $k \leqslant \sqrt{5}$, but if $k > \sqrt{5}$ it is finite for some θ's.

The simplest exceptional θ is
$$\theta = \tfrac{1}{2}(\sqrt{5}-1) = (1, 1, 1, \ldots).$$

5. Proceeding now to the proof of the Theorem (A), which is my main thesis, it may be remarked that, when an approximation x/y is assigned, there are only two fractions x'/y' ($x' < x$, $y' < y$) that satisfy the condition

$$xy'-x'y = \pm 1,$$

viz. one giving the upper and one the lower sign. Thus we have only to examine whether one of these two fractions is the member of the class that next precedes x/y.

If the condition for the class be too severe (*i.e.*, if the value of k be too large), the preceding member may be further away from x/y than either fraction x'/y', while if the condition be too lax the preceding member of the class may be nearer to x/y than either. This rough argument indicates the nature of the theorem and the manner of reasoning to be used in proof.

6. Another remark will be more directly useful. Suppose $x = p_n$, $y = q_n$, p_n/q_n being, as usual, the convergent to θ corresponding to a_n. Then, unless $a_n = 1$, the fractions x'/y' for which

$$xy'-x'y = \pm 1$$

are clearly the convergent p_{n-1}/q_{n-1} and the auxiliary one just preceding p_n/q_n.

If $a_n = 1$, the latter does not exist, and is replaced by the convergent p_{n-2}/q_{n-2}.

If x/y is the auxiliary convergent just before p_n/q_n, and $a_n > 2$, the two fractions x'/y' are p_{n-1}/q_{n-1} and the auxiliary convergent just before x/y.

If $a_n = 2$, the latter is absent and is replaced by p_{n-2}/q_{n-2}.

Finally, when x/y is the auxiliary convergent just after p_{n-1}/q_{n-1}, the two fractions in question are p_{n-1}/q_{n-1} and p_{n-2}/q_{n-2}.

These are all the cases that need be mentioned.

7. PROOF OF THEOREM (A).

I. If $k > 2$, all the approximations x/y must be principal convergents. We can readily construct a case in which, while p_{n+1}/q_{n+1} satisfies the condition, we have $a_{n+1} = 2$, but p_n/q_n does not satisfy it; for here

$$\lambda_n < 2+\frac{1}{a_{n+2}}+\frac{1}{a_n},$$

and by taking a_n and a_{n-2} large enough we can ensure $\lambda_n < k$.

It follows (§ 6) that the member x/y next preceding p_{n+1}/q_{n+1} does not satisfy

$$p_{n+1}y-q_{n+1}x = \pm 1.$$

II. If $k=2$, all the members are again principal convergents. If p_{n+1}/q_{n+1} satisfies the condition, then, if $a_{n+1}>1$, p_n/q_n does also (2.1), while if $a_{n+1}=1$, p_n/q_n may not. Yet since

$$\lambda_{n-1}+\lambda_n > 4,$$

either p_n/q_n or p_{n-1}/q_{n-1} must satisfy the condition; so that in either case (see § 6) the preceding member x/y does satisfy

$$p_{n+1}y-q_{n+1}x = \pm 1,$$

and the theorem is proved.

III. If $k<1\frac{1}{2}$, we have only to indicate an exception to the relation

$$xy'-x'y = \pm 1.$$

We take $a_n=3$, so that between p_{n-1}/q_{n-1} and p_n/q_n there are two auxiliary convergents, viz.,

$$p/q \text{ corresponding to } \alpha=1,\ \beta=2,$$

$$p'/q' \quad\quad ,, \quad\quad \alpha=2,\ \beta=1;$$

and we are going to arrange that p/q satisfies the class condition

$$\left|\frac{p}{q}-\theta\right| < \frac{1}{kq^2},$$

while p'/q' does not. In this case p_n/q_n must satisfy the condition (§ 3), and the preceding member p/q does not obey the formula

$$p_nq-q_np = \pm 1.$$

A simple calculation is needed.

We can write
$$\lambda = \frac{1}{2+\xi}+\frac{1}{1+\eta},$$

$$\lambda' = \frac{1}{1+\xi}+\frac{1}{2+\eta},$$

$$\xi = (a_{n+1},\ a_{n+2},\ \ldots),$$

$$\eta = (a_{n-1},\ a_{n-2},\ \ldots),$$

and we have $$\lambda-\lambda' = \frac{1}{(1+\eta)(2+\eta)} - \frac{1}{(1+\xi)(2+\xi}, $$

so that $\lambda > \lambda'$ if only $\xi > \eta$.

It will be clear now that, k being less than $1\frac{1}{2}$ but ever so near it, we can so arrange a case that

$$\lambda > k \quad \text{and} \quad \lambda' < k;$$

this completes the argument for $k < 1\frac{1}{2}$.

What has been stated refers to cases where k is nearly $1\frac{1}{2}$, but to deal with other cases we need only take $a_{n+1} = 4, 5,$ &c., instead of 3. The *crux* of the matter is that $\lambda > \lambda'$.

IV. The final case is $$1\tfrac{1}{2} \leqslant k < 2.$$

If an auxiliary convergent between p_{n-1}/q_{n-1} and p_n/q_n belongs to the class, we must have $$a_n = 2, \quad \alpha = \beta = 1 \quad (\S\ 2).$$

Suppose now that p_n/q_n satisfies the condition

$$\left| \frac{p_n}{q_n} - \theta \right| < \frac{1}{kq_n^2}.$$

The preceding member of the class may be the intermediate convergent if $a_n = 2$, but if $a_n > 2$ it must be p_{n-1}/q_{n-1}; and both these satisfy

$$p_n x - q_n y = \pm 1 \quad (\S\ 6).$$

There remains the case $a_n = 1$. Clearly p_{n-1}/q_{n-1} may not belong to the class; if it does not, no auxiliary convergent between p_{n-2}/q_{n-2} and p_{n-1}/q_{n-1} can possibly belong (§ 3), while, since $\lambda_{n-2}+\lambda_{n-1} > 4$, p_{n-2}/q_{n-2} must belong.

This completes the proof as far as a principal convergent p_{n-1}/q_{n-1} is concerned.

It remains to discuss the member immediately before an auxiliary convergent that is like the one above. As indicated $a_n = 2$, $\lambda_{n-1} > 2$, and p_{n-1}/q_{n-1} is the preceding member.

Since it satisfies the condition, the proof is now finished.

8. In cases where successive approximations of the class satisfy the relation $$xy' - x'y = \pm 1,$$

the whole class will differ from the class of principal convergents to θ in two possible ways :—

(i) if $a_{n+1} = 1$, the convergent p_n/q_n may have to be rejected ;

(ii) if $a_n = 2$, the auxiliary convergent between p_{n-1}/q_{n-1} and p_n/q_n may have to be inserted (§ 7, IV).

In these cases the C.F. for θ can be modified* (by the introduction of some minus signs, if necessary), so that the approximations of the class and no others occur as principal convergents.

Suppose, in fact, that x/y, x'/y', x''/y'' are three consecutive members of the class, so that

$$xy'-x'y = \pm 1, \quad x'y''-x''y' = \pm 1.$$

We can write
$$x = \lambda x' + \mu x'',$$
$$y = \lambda y' + \mu y'',$$

and it readily follows that λ, μ are integers of which μ is ± 1 ; since λ must be positive this is the continued fraction algorithm. If $\mu = +1$, the partial quotient λ is preceded by a positive sign, otherwise by a negative sign.

A simple example will best indicate the method of modification.

Take
$$\theta = (4,\ 2,\ 4,\ 1,\ 4,\ \dots):$$

then the principal convergents are

$$\tfrac{1}{4},\ \tfrac{2}{9},\ \tfrac{9}{40},\ \tfrac{11}{49},\ \tfrac{53}{236},\ \dots,$$

and the auxiliary convergent between $\frac{1}{4}$ and $\frac{2}{9}$, viz. $\frac{1}{5}$, may belong, while the principal convergent $\frac{9}{40}$ may not. In fact, taking

$$\left| \frac{x}{y} - \theta \right| = \frac{1}{\lambda y^2},$$

we have for the first
$$\lambda > 1{\cdot}6,$$

* Cf. Hermite, *l.c.*, who deals with $k = \sqrt{3}$, and Minkowski (*Math. Ann.*, Vol. 54, p. 91 ; *Ges. Abhandlungen*, Vol. 1, p. 320), who deals with $k = 2$. The process is a definite one, as there stated, but not a direct one.

and for the second $\lambda < 1{\cdot}5$,

so that, with $k = 1{\cdot}6$, the first must be inserted and the second rejected.

First, change from $(4,\ 2,\ 4,\ 1,\ 4,\ \ldots)$,

to $(4,\ 2,\ 5,\ -5,\ \ldots)$:

the new convergents are $\frac{1}{4},\ \frac{2}{9},\ \frac{11}{49},\ \frac{53}{236},\ \ldots,$

and $\frac{9}{40}$ has disappeared.

Secondly, change to $(4, 1, 1, -6, -5, \ldots)$, and the convergents are

$$\tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{2}{9},\ \tfrac{11}{49},\ \tfrac{53}{236},\ \ldots,$$

as desired.

The rules will be obvious and are easily established.

It may be mentioned that, since

$$\lambda_n + \lambda_{n+1} > 4,$$

two consecutive principal convergents have never to be rejected.

9. We have seen (§ 4) that, if

$$\theta = \tfrac{1}{2}(\sqrt{5}-1) = (1,\ 1,\ 1,\ \ldots),$$

and $k > \sqrt{5}$, there is only a finite number of rational approximations x/y to θ, such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2}.$$

A more general result has been established by Markoff,* viz. that, if θ has only a finite number of rational approximations such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{3y^2},$$

then θ must be a quadratic surd.

The first result, and some idea of Markoff's method, can be alternatively obtained by following the line of argument used by Liouville† in his classical proof of the existence of transcendent numbers.

* *Math. Ann.*, Vol. 15, p. 381. It will, I think, be clear to any reader that the results of Hurwitz quoted earlier all really occur in this paper.

† *Liouville's Journal*, Vol. 16, p. 133. The argument has been often reproduced: see, *e.g.*, Young, *The Theory of Sets of Points*, p. 7.

In fact, with the above value of θ, we have

$$f(\theta) \equiv \theta^2+\theta-1 = 0,$$

whence

$$f\left(\frac{x}{y}\right)-f(\theta) = f'(\xi)\left(\frac{x}{y}-\theta\right),$$

where ξ is between x/y and θ. It follows that

$$\left|\frac{x}{y}-\theta\right| = \left|\frac{f(x/y)}{f'(x)}\right|.$$

Now, x and y being integers,

$$\left|f\left(\frac{x}{y}\right)\right| = \frac{\lambda}{y^2},$$

where λ is a positive integer, and so

$$f'(\xi) \to f'(\theta) = \sqrt{5},$$

when $x/y \to \infty$. We deduce that for any rational approximation

$$\left|\frac{x}{y}-\theta\right| \to \frac{\lambda}{y^2\sqrt{5}},$$

when $y \to \infty$; since $\lambda \geqslant 1$, the result follows at once, and it also follows that $\lambda = 1$ for the best approximations.

In general, when θ is a quadratic surd, λ will be the minimum of a certain indefinite quadratic binary form, and it was in finding λ that Markoff succeeded.

10. I shall now attempt to supplement the just quoted theorem of Markoff.

Suppose that, to make up the digits in the continued fraction for θ, we have a cycle of m 1's and another of m 2's, it being understood that each cycle can be used any number of times in succession. It is quite clear that the set of θ's so constructed is non-enumerable, and therefore that there are non-algebraic—*i.e.* transcendent—numbers among them; it is easy to see that the value of λ_n corresponding to a partial quotient a_n is distinctly less than 3, except when a_{n+1} is a 2 with a 1 and a 2 adjacent to it. In this case, when $m \to \infty$ and $n \to \infty$, $\lambda_n \to \frac{\sqrt{5}-1}{2}+\sqrt{2}+1$, say ϕ, and it follows that, if $k > \phi$, there are transcendent numbers with only a finite number of approximations such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2}.$$

The value of ϕ is $3{\cdot}0322 < 3\frac{1}{30}$.

It seems likely that, by a different choice of the two cycles, we could reduce the critical value of ϕ to 3, but I do not see at present how to do this. I may add that Markoff's result relating to the quadratic surds is by no means easy to establish; it probably does contain the key to the supplementary theorem here indicated.