# THE HARMONIC FUNCTIONS ASSOCIATED WITH THE PARABOLIC CYLINDER

(SECOND PAPER.)

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1. It has been shewn by Weber* that the functions associated with the parabolic cylinder in harmonic analysis are solutions of a differential equation of the form

$$y''+Zy = 0,$$

where $Z$ is a quadratic function of the independent variable $z$. There is no loss of real generality in taking the equation to be

$$y''+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y = 0;$$

and, when $n$ is a positive integer, a solution of the equation is

$$(-)^n e^{\frac{1}{4}z^2}\frac{d^n}{dz^n}(e^{-\frac{1}{2}z^2}).$$

This function is intimately connected with Hermite's† polynomial

$$U_n(z) = e^{z^2}\frac{d^n}{dz^n}(e^{-z^2}).$$

References to other investigations of the solutions of the differential equations are to be found in papers‡ published in the *Proceedings* in 1903 and 1910. It should be stated here that the standard solution of the

* *Math. Ann.*, Vol. 1 (1869), pp. 1-36.

† *Comptes rendus*, t. 58 (1864), pp. 93-97, 266-273.

‡ Whittaker, *Proceedings*, Ser. 1, Vol. 35 (1903), pp. 417-427; Watson, *Proceedings*, Ser. 2, Vol. 8 (1910), pp. 393-421. See *Encyclopédie des Sciences Math.*, Vol. 2, t. 5, p. 325. Reference may also be made to the work of Markoff, *Bulletin de l'Académie des Sciences de St. Pétersbourg*, Sér. 5, t. 9 (1898), pp. 435-446, and A. Milne, *Proc. Edinburgh Math. Soc.*, Vol. 33 (1915), pp. 53-59, concerning the zeros of $D_n(z)$.

equation, when $n$ is unrestricted, is taken by Whittaker to be

$$D_n(z) = -\frac{\Gamma(n+1)}{2\pi i} e^{-\frac{1}{4}z^2} z^n \int_{\infty z}^{(0+)} e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} \, dt,$$

and other solutions are $D_n(-z)$, $D_{-n-1}(\pm iz)$.

When $n$ is a positive integer, then

$$D_n(z) = (-)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n} (e^{-\frac{1}{2}z^2}) = (-)^n 2^{-\frac{1}{2}n} e^{-\frac{1}{4}z^2} U_n(z/\sqrt{2}).$$

The most remarkable discovery which has been made concerning the function $D_n(z)$ is contained in Adamoff's* approximation, namely that, when $n$ is a large integer and $x$ is positive,

$$D_n(x) = 2^{\frac{1}{2}} n^{\frac{1}{2}n} e^{-\frac{1}{2}n} \left[\cos(xn^{\frac{1}{2}} - \tfrac{1}{2}n\pi) + n^{-\frac{1}{2}} w_n(x)\right],$$

where

$$w_n(x) = e^{\frac{1}{4}x^2} \vartheta_n(x)/(x\sqrt{\pi}),$$

and $|\vartheta_n(x)| < 3{\cdot}35\ldots$, when $n > 2$; and $w_n(0) = 0$ or $\frac{1}{6}\vartheta/n$ (where $-1 < \vartheta < 1$), according as $n$ is odd or even.

In my former paper, I shewed that Adamoff's investigation would be extended to $D_n(z)$, when $z$ is complex, provided that $|I(z)|$ is not too large: and this result made it easy to give a formal proof of a theorem enunciated by Hermite and Whittaker, concerning the expansibility of an arbitrary analytic function in the form $\Sigma\, a_n D_n(z)$:—a result analogous to Neumann's expansions in series of Legendre polynomials and Bessel functions.

In this paper, I investigate various types of asymptotic formulæ and expansions of $D_n(z)$, by the method of steepest descents; the results contained in the paper are therefore to be associated with those previously obtained by Debye† and by myself‡ for Bessel functions.

It should be stated here that the method of steepest descents§ is a mode of discussing the integral $\int e^{nf(s)} \phi(s)\, ds$ by choosing a contour which passes through a zero of $f'(s)$, and which

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* *Ann. de l'Inst. Polytechnique de St. Pétersbourg*, t. 5 (1906), pp. 127–143. More recently, Perron, *Archiv der Math. u. Phys.*, Ser. 3, Vol. 22 (1914), pp. 329–340, has given some approximate formulæ for the confluent hypergeometric function of which $D_n(z)$ is a special form, but the limits of error in the formulæ are not investigated with the preciseness of the earlier work of Adamoff.

† *Math. Ann.*, Vol. 67 (1909), pp. 535–558; *Münchener Sitzungsberichte* [5], 1910.

‡ *Proceedings*, Ser. 2, Vol. 16 (1917), pp. 150–174. The present investigation is much more closely connected with this paper than with the paper of which it is nominally a sequel.

§ The method is to be traced to a posthumous paper by Riemann, *Werke*, 1876, p. 405. See also Brillouin, *Ann. de l'École Norm. Sup.*, Sér. 3, t. 33 (1916), pp. 17–69. Riemann's method was a combination of the method of steepest descents and the method of stationary phase.

is such that $If(s)$ is constant on the contour; so that $Rf(s)$ falls away from its maximum as rapidly as possible, $f(s)$ being monogenic. The name is derived (like the German "Methode der Sattelpunkte" and the French "Méthode du Col") from a consideration of the surface on which the coordinates of any point are $R(s)$, $I(s)$, $Rf(s)$; the contour is the plan of a curve drawn on the surface so as to pass through a saddlepoint and at all other points on it to be as steep as possible.

2. *Whittaker's integral for $D_n(2\zeta\sqrt{n})$.*

It is convenient to take a new variable in place of $z$, the argument of $D_n(z)$. We shall write throughout $z = 2\zeta\sqrt{n}$, $x = 2\xi\sqrt{n}$, where $\xi$ is real; it will be supposed that $n$ is positive (except in §§ 10–14, 16, 17, in which it is assumed merely that $R(n) > 0$ and $|\arg n| < \frac{1}{2}\pi$), so that, with these exceptions, $x$ is also real.

In Whittaker's integral, already quoted, we take a new parametric variable $s$, defined by the equation

$$t = -zs\sqrt{n},$$

where $|\arg s| \leqslant \pi$ on the path of integration. This gives

$$D_n(z) = \frac{\Gamma(n+1)}{2\pi i\, n^{\frac{1}{2}n}} \int_{-\infty}^{(0+)} s^{-n-1} \exp\{-\tfrac{1}{4}z^2 + n^{\frac{1}{2}}zs - \tfrac{1}{2}ns^2\}\, ds$$

$$= \pi^{-1}(e/n)^{\frac{1}{2}n}\,\Gamma(n+1)\,\Delta_n(\zeta),$$

if $$\Delta_n(\zeta) = \frac{1}{2i}\int_{-\infty}^{(0+)} \exp\{-n(\zeta^2 + \tfrac{1}{2} + \log s - 2\zeta s + \tfrac{1}{2}s^2)\}\, s^{-1} ds$$

$$= \frac{1}{2i}\int_{-\infty}^{(0+)} e^{-n\tau} s^{-1}\, ds,$$

where $$\tau \equiv \zeta^2 + \tfrac{1}{2} + \log s - 2\zeta s + \tfrac{1}{2}s^2.$$

We proceed to discuss the properties of $D_n(z)$ by means of the slightly simpler function $\Delta_n(\zeta)$.

3. *Inequalities satisfied by $\Delta_n(\xi)$ when $\xi \geqslant 1$.*

We shall now apply the principles of the method of steepest descents to the contour integral for $\Delta_n(\xi)$, introduced in § 2, in the special case in which $\xi \geqslant 1$.

Write $\xi = \cosh\alpha$, where $\alpha \geqslant 0$; then the value of $\tau$ becomes

$$\cosh^2\alpha + \tfrac{1}{2} + \log s - 2s\cosh\alpha + \tfrac{1}{2}s^2.$$

Evidently $\tau$, *qua* function of $s$, is stationary when

$$(1/s)-2\cosh\alpha+s=0,$$

*i.e.*, when
$$s=e^{\pm\alpha}.$$

At these points $\tau$ is real, and so we choose (if possible) a contour on which $I(\tau)=0$.

Writing $s=re^{i\theta}$, we see that the contour must be given by the whole or part of the curve whose equation in polar coordinates is

$$\theta-2r\sin\theta\cosh\alpha+r^2\sin\theta\cos\theta=0.$$

This equation is satisfied, either if $\theta=0$, or if

$$r=\sec\theta\,\{\cosh\alpha\pm\sqrt{(\cosh^2\alpha-\theta\cot\theta)}\}.$$

If the lower sign be taken for the radical $r$ is *positive* when $-\pi\leqslant\theta\leqslant\pi$, and $r$ is an even function of $\theta$; as $\theta$ increases from 0 to $\pi$, $r$ varies* from $e^{-\alpha}$ to $+\infty$.

If the upper sign were taken for the radical, $r$ would vary from $e^{\alpha}$ to $+\infty$ as $\theta$ increases from 0 to $\frac{1}{2}\pi$; and when $\frac{1}{2}\pi<\theta<\pi$, $r$ would be negative.

The curves on which $\theta-2r\sin\theta\cosh\alpha+r^2\sin\theta\cos\theta$ vanishes are shewn in Figs. 1 and 2, the latter shewing the special case $\alpha=0$; the dotted portions correspond to the negative values of $r$.

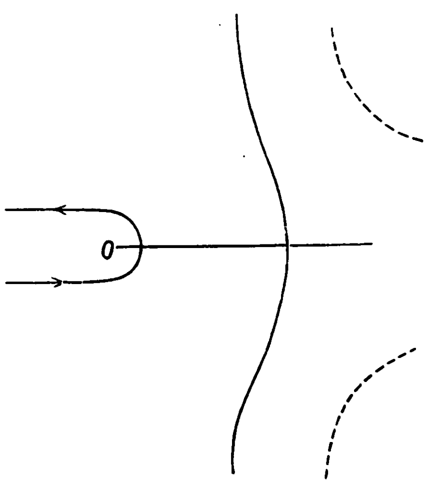


FIG. 1.

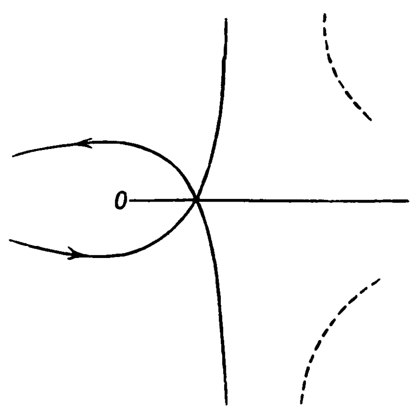


FIG. 2.

Consequently the curve $r=\sec\theta\,\{\cosh\alpha-\sqrt{(\cosh^2\alpha-\theta\cot\theta)}\}$ is a

* It appears that $r$ does not necessarily increase steadily; for, when $\theta=0$, $r$ is stationary and its second differential coefficient with regard to $\theta$ is negative if $\cosh\alpha<2/\sqrt{3}$.

contour of the appropriate type; and it gives rise to the definite integral

$$\Delta_n(\xi) = \frac{1}{2i}\int_{-\pi}^{\pi} e^{-nT}\left(\frac{1}{r}\frac{dr}{d\theta}+i\right)d\theta,$$

where $r$ is the stated function of $\theta$ and $T$ is defined as a function* of $\theta$ by the equation

$$T = \cosh^2 a+\tfrac{1}{2}+\log r-2r\cos\theta\cosh a+\tfrac{1}{2}r^2\cos 2\theta.$$

To indicate the dependence of $T$ on $\theta$ and $a$, we shall frequently write $T(\theta, \cosh a)$ in place of $T$; we thus have

$$T(\theta, \cosh a) = \log\left[\sec\theta\left\{\cosh a-\sqrt{(\cosh^2 a-\theta\cot\theta)}\right\}\right]+\cosh^2 a+\tfrac{1}{2}-\theta\cot 2\theta$$
$$-\cosh a\sec^2\theta\left\{\cosh a-\sqrt{(\cosh^2 a-\theta\cot\theta)}\right\}.$$

Since $r$ and $T$ are *even* functions of $\theta$, we see at once that

$$\Delta_n(\xi) = \int_0^{\pi} e^{-nT}d\theta.$$

From the equations

$$dT/d\theta = \{(1/r)-2\cos\theta\cosh a+r\cos 2\theta\}(dr/d\theta)+2r\sin\theta(\cosh a-r\cos\theta),$$
$$0 = 2\sin\theta(r\cos\theta-\cosh a)(dr/d\theta)+(1-2r\cos\theta\cosh a+r^2\cos 2\theta)$$

(the latter being derived from the fact that $I(\tau)$ is zero on the contour), we see that

$$dT/d\theta = r[2\sin\theta(\cosh a-r\cos\theta)]^{-1}\left[\{1/r-2\cos\theta\cosh a+r\cos 2\theta\}^2\right.$$
$$\left.+4\sin^2\theta(\cosh a-r\cos\theta)^2\right]$$
$$= r[2\sin\theta(\cosh a-r\cos\theta)]^{-1}\left[\{1/r-2\cos\theta\cosh a+r\}^2\right.$$
$$\left.+4\sinh^2 a\sin^2\theta\right].$$

Since $$\cosh a-r\cos\theta = +\sqrt{(\cosh^2 a-\theta\cot\theta)} \geqslant 0,$$

we deduce that, when $0\leqslant\theta\leqslant\pi$, $T$ is a non-decreasing function of $\theta$, and so
$$T \geqslant T(0, \cosh a) = \sinh a\cosh a-a \geqslant 0.$$

It now follows immediately from the last integral given for $\Delta_n(\xi)$ that

$$0 \leqslant \Delta_n(\cosh a) \leqslant \pi e^{-nT(0,\,\cosh a)}.$$

An inequality for $\Delta_n(\cosh a)$, which is stronger than that just obtained

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* In fact $T = R(\tau)$; although $\tau$ reduces to $T$ on the contour considered, it avoids confusion to make a distinction between $\tau$, a function of the complex variable $s$, and $T$, a function of the real variable $\theta$.

(except when $a$ is small) can be derived from the former of the pair of expressions given for $dT/d\theta$; for we have

$$dT/d\theta \geqslant 2r \sin\theta\,(\cosh a - r\cos\theta)$$

$$= 2\theta \left\{ 1 - \frac{\cosh a}{\cosh a + \surd(\cosh^2 a - \theta\cot\theta)} \right\}$$

$$\geqslant 2\theta \{1 - e^{-a}\cosh a\},$$

since $\theta\cot\theta \leqslant 1$. Hence, on integration, $T - T(0, \cosh a) \geqslant \tfrac{1}{2}\theta^2(1-e^{-2a})$, and so

$$\Delta_n(\cosh a) \leqslant \int_0^{\pi} \exp\{-nT(0, \cosh a) - \tfrac{1}{2}n\theta^2(1-e^{-2a})\}\, d\theta$$

$$\leqslant (\tfrac{1}{2}\pi/n)^{\frac{1}{2}} (1-e^{-2a})^{-\frac{1}{2}} \exp\{-nT(0, \cosh a)\};$$

and this is the inequality to which reference was made.

The results of this section may be summed up in the following theorem :—

*When $x \geqslant 2\surd n$, $D_n(x)$ is positive and does not exceed the smaller of the functions*

$$\Gamma(n+1)\, e^{\frac{1}{2}n} (2n)^{-n} \{x + \surd(x^2 - 4n)\}^n \exp\{-\tfrac{1}{4}x\surd(x^2-4n)\},$$

$$(2\pi)^{-\frac{1}{2}}\Gamma(n+1)\, e^{\frac{1}{2}n}(2n)^{-n-\frac{1}{2}}(x^2-4n)^{-\frac{1}{4}}$$
$$\times \{x + \surd(x^2-4n)\}^{n+\frac{1}{2}} \exp\{-\tfrac{1}{4}x\surd(x^2-4n)\}.$$

4. *The behaviour of $\Delta_n(\xi)$ and $\Delta_n'(\xi)$ when $\xi \geqslant 1$.*

It is easy to see that

$$\frac{dT(\theta, \xi)}{d\xi} = \frac{\sec^2\theta}{\surd(\cosh^2 a - \theta\cot\theta)} [\{\cosh a \sin^2\theta - \surd(\cosh^2 a - \theta\cot\theta)\}^2$$
$$+ \cos^2\theta\{\cosh^2 a \sin^2\theta + \sinh^2 a\}] \geqslant 0;$$

and consequently

$$\frac{dT(\theta, \xi)}{d\xi} = \frac{d}{d\theta}[2\tan\theta\surd(\cosh^2 a - \theta\cot\theta) + 2\cosh a\,(\theta - \tan\theta)].$$

From the integrals

$$\Delta_n(\xi) = \int_0^{\pi} e^{-nT} d\theta, \qquad \Delta_n'(\xi) = -n\int_0^{\pi} \frac{dT(\theta, \xi)}{d\xi} e^{-nT} d\theta,$$

it is now evident (i) that, when $x \geqslant 2\surd n$, $D_n(x)$ is a positive decreasing function of $x$ when $n$ is fixed, and (ii) that, when $\xi$ is fixed, and not less than 1, $\Delta_n(\xi)$ and $-n^{-1}\Delta_n'(\xi)$ are positive decreasing functions of $n$, since

$$T \geqslant T(0, \xi) = \sinh\alpha \cosh\alpha - \alpha \geqslant 0.$$

5. *Simple properties of* $\Delta_n(1)$ *and* $\Delta_n'(1)$.

The functions $\Delta_n(1)$, $\Delta_n'(1)$ are not so amenable to analysis as $J_n(n)$, $J_n'(n)$. It is easily verified that, when $\theta$ is small, $27T(\theta, 1)/\theta^3 \sim 8\sqrt{3} - 12\theta$, and so $T(\theta, 1)/\theta^3$ initially *decreases*, though it ultimately *increases*, since it tends to infinity as $\theta \to \pi$.

From a theorem of Bromwich (see his *Infinite Series*, p. 444) it follows [since $T(\theta, 1)$ is monotonic] that

$$\lim_{n\to\infty} [n^{\frac{1}{3}}\Delta_n(1)] = \lim_{n\to\infty}\left[n^{\frac{1}{3}}\int_0^\infty \exp(-8n\sqrt{3}\,.\,\theta^3/27)\,d\theta\right] = \tfrac{1}{2}\Gamma(\tfrac{1}{3})/3^{\frac{1}{6}}.$$

Also, it is easy to shew that

$$\frac{d[n^{\frac{1}{3}}\Delta_n(1)]}{dn} = \tfrac{1}{3}n^{\frac{1}{3}}\int_0^\pi \left[\theta\frac{dT}{d\theta} - 3T\right]e^{-nT}\,d\theta;$$

but, since the integrand is not one-signed, we cannot infer that $n^{\frac{1}{3}}\Delta_n(1)$ is monotonic. On the other hand, since the integrand is negative for sufficiently small values of $\theta$, we *can* infer that $n^{\frac{1}{3}}\Delta_n(1)$ is a decreasing function of $n$ *when $n$ is sufficiently large*.

Similarly, if we put

$$2\tan\theta\sqrt{(\cosh^2\alpha - \theta\cot\theta)} + 2\cosh\alpha(\theta - \tan\theta) \equiv G(\theta, \cosh\alpha),$$

it follows from § 4 that
$$\frac{d[n^{-\frac{1}{3}}\Delta_n'(1)]}{dn} = n^{\frac{1}{3}}\int_0^\pi (TG' - \tfrac{2}{3}GT')\,e^{-nT}\,d\theta,$$

where $G$ is written in place of $G(\theta, 1)$. Since, when $\theta$ is small,

$$TG' - \tfrac{2}{3}GT' \sim -8\theta^6/135,$$

it follows that $-n^{-\frac{1}{3}}\Delta_n'(1)$ is a positive increasing function of $n$, *when $n$ is sufficiently large*.

[*Added, December 15th*, 1917.—A proof will be given at the end of § 7*a* that $TG' - \tfrac{2}{3}GT'$ is negative throughout the range $0 < \theta < \pi$. This result obviously shews that $-n^{-\frac{1}{3}}\Delta_n'(1)$ is a positive increasing function of $n$ *for all positive values of $n$*.]

6. *Properties of the contour introduced in* § 3.

We shall now make a rather elaborate investigation with a view to proving ultimately that $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$ is a non-increasing function of $n$ when $\xi$ is fixed and not less than 1, provided that $\xi - 1$ is not too large. The range of values of $\xi$ for which we shall prove the theorem is, in fact, $1 \leqslant \xi \leqslant 1{\cdot}2$ approximately. The theorem may be true for greater values of $\xi$, but it is then comparatively unimportant. The fact that it is true for the range of values just specified will be found later (§ 18) to be of vital importance in dealing with the convergence of series of Kapteyn's type when $\xi \geqslant 1$.

We shall retain the notation of § 3 and consider values of $\theta$ between 0 and $\pi$ only.

Since $$d(r\sin\theta)/d\theta = \sin\theta\,(dr/d\theta)+r\cos\theta$$
$$= \tfrac{1}{2}(1-r^2)/(\cosh a - r\cos\theta)$$
$$= \tfrac{1}{2}(1-r^2)(\cosh^2 a-\theta\cot\theta)^{-\frac{1}{2}},$$

we see that the ordinate of the contour increases with $\theta$ so long as $r<1$, and decreases when $r>1$.

Now $r=1$ only when $\theta - 2\cosh a\sin\theta + \sin\theta\cos\theta = 0$.

Since* $\theta\operatorname{cosec}\theta+\cos\theta$ decreases steadily from 2 to $\frac{1}{2}\pi$ as $\theta$ increases from 0 to $\frac{1}{2}\pi$, and then increases steadily from $\frac{1}{2}\pi$ to $+\infty$ as $\theta$ increases from $\frac{1}{2}\pi$ to $\pi$, we see that there is *one* value only of $\theta$ (other than $\theta=0$), say $\theta=\theta_a$, for which $\theta\operatorname{cosec}\theta+\cos\theta-2\cosh a$ vanishes; and $\theta_a$ is an obtuse angle exceeding $\theta_0$, where $\theta_0$ is the value of $\theta$ for which $\theta\operatorname{cosec}\theta+\cos\theta-2$ vanishes.

[The angle $\theta_0$ is 2·139 radians; the sexagesimal measure of this angle is 122° 32′.]

It is evident that† $r<1$ when $0<\theta<\theta_a$, and $r>1$ when $\theta_a<\theta<\pi$; and so $r\sin\theta$ increases steadily as $\theta$ increases from 0 to $\theta_a$, and then decreases steadily as $\theta$ increases from $\theta_a$ to $\pi$.

We shall subsequently (§ 7) require an upper bound to the value of $f(a)$, where $f(a)$ is the value of $r\sin\theta/(\cosh a-r\cos\theta)$ when $\theta=\theta_a$.

We have

$$f(a)=\sin\theta_a/(\cosh a-\cos\theta_a) = 2\sin^2\theta_a/(\theta_a-\sin\theta_a\cos\theta_a).$$

But $\sin^2\theta_a \leqslant \sin^2\theta_0$, $\theta_a-\sin\theta_a\cos\theta_a \geqslant \theta_0-\sin\theta_0\cos\theta_0$, since $\theta_a$ and $\theta_0$ are obtuse angles of which the former is the greater.

Hence $f(a)\leqslant f(0)=\sin\theta_0/(1-\cos\theta_0)=\cot\frac{1}{2}\theta_0 = 0·548$.

That is to say, 0·548 is the upper bound of $f(a)$. The fact which will be found essential in § 7 is that this is less than unity.

We shall next shew that, when $0<\theta<\theta_a$, the contour is *concave* to the initial line.

It is easily seen that

$$\frac{d(r\sin\theta)}{d(r\cos\theta)} = \frac{(1-r^2)\sin\theta}{\cos\theta-2r\cosh a+r^2\cos\theta};$$

and the contour will be *concave*, when $0<\theta<\theta_a$, if this expression

* The differential coefficient of this function is $-\cos\theta\operatorname{cosec}^2\theta\,(\theta-\sin\theta\cos\theta)$.

† If $a=0$, $r=1$ when $\theta=0$; for positive values of $a$, $r<1$ when $\theta=0$

has a positive differential coefficient with regard to $\theta$ for these values of $\theta$.

We shall need the following preliminary results (valid when $\theta$ lies between 0 and $\pi$) to prove this theorem :—

(i) $1-2r\cos\theta\cosh\alpha+r^2\cos^2\theta = 1-\theta\cot\theta \geqslant 0,$

(ii) $\cos\theta-2r\cosh\alpha+r^2\cos\theta = \cos\theta-\theta\operatorname{cosec}\theta \leqslant 0,$

(iii) $\cosh\alpha-2r\cos\theta+r^2\cosh\alpha \geqslant 1-2r\cos\theta\cosh\alpha+r^2,$

(iv) $2(\sin\theta-\theta\cos\theta) \geqslant \theta-\sin\theta\cos\theta.$

Proofs of (i) and (ii) are left to the reader. To prove (iii), we see that the difference of the expressions is $(\cosh\alpha-1)(1+2r\cos\theta+r^2)$, and this is never negative. To prove (iv), we observe that

$$2(\sin\theta-\theta\cos\theta)-(\theta-\sin\theta\cos\theta)$$

vanishes when $\theta=0$, while for values of $\theta$ between 0 and $\pi$ it has the positive differential coefficient $2\sin\theta(\theta-\sin\theta)$, and so is not negative.

Now it is easy to shew that

$$(\cosh\alpha-r\cos\theta)(\cos\theta-2r\cosh\alpha+r^2\cos\theta)^2\frac{d}{d\theta}\left[\frac{d(r\sin\theta)}{d\theta}\bigg/\frac{d(r\cos\theta)}{d\theta}\right]$$

$$= 2(\cosh\alpha-2r\cos\theta+r^2\cosh\alpha)(1-2r\cosh\alpha\cos\theta+r^2\cos^2\theta)$$
$$+r(\cos\theta-2r\cosh\alpha+r^2\cos\theta)(1-2r\cos\theta\cosh\alpha+r^2)$$

$$\geqslant 2(1-\theta\cot\theta)(1-2r\cos\theta\cosh\alpha+r^2)$$
$$-r(\theta\operatorname{cosec}\theta-\cos\theta)(1-2r\cos\theta\cosh\alpha+r^2)$$

$$= \{2(1-\theta\cot\theta)-r(\theta\operatorname{cosec}\theta-\cos\theta)\}(1-2r\cos\theta\cosh\alpha+r^2)$$

$$\geqslant (1-r)(\theta\operatorname{cosec}\theta-\cos\theta)(1-2r\cos\theta\cosh\alpha+r^2) \geqslant 0,$$

provided that $r<1$. Hence the differential coefficient of

$$\{d(r\sin\theta)/d(r\cos\theta)\}$$

is positive when $0<\theta<\theta_\alpha$, and so *the contour is concave to the initial line for this range of values of* $\theta$.

7. *A family of bicircular quartics associated with the contour.*

It will next be shewn that

$$\frac{dT(\theta,\cosh\alpha)}{d\theta}\bigg/\frac{dG(\theta,\cosh\alpha)}{d\theta}$$

*is an increasing function of* $\theta$, when $0 < \theta < \pi$; where, with the notation of § 5,

$$G(\theta, \cosh \alpha) \equiv 2 \tan \theta \sqrt{(\cosh^2 \alpha - \theta \cot \theta)} + 2 \cosh \alpha \, (\theta - \tan \theta).$$

From the formulæ of §§ 3, 4 it is evident that

$$\frac{dT(\theta, \cosh \alpha)}{d\theta} \Big/ \frac{dG(\theta, \cosh \alpha)}{d\theta} = \frac{(1 - 2r \cos \theta \cosh \alpha + r^2)^2 + 4r^2 \sinh^2 \alpha \sin^2 \theta}{2r \sin \theta \, (r^2 - 2r \cos \theta \cosh \alpha + \cosh 2\alpha)}.$$

Now write*

$$X = a\,(\cosh \alpha - r \cos \theta)/\sinh \alpha, \qquad Y = ar \sin \theta/\sinh \alpha,$$

where $a$ is any positive constant, and we have

$$\frac{dT(\theta, \cosh \alpha)}{d\theta} \Big/ \frac{dG(\theta, \cosh \alpha)}{d\theta} = \frac{\sinh \alpha}{2a} \, \frac{(X^2 + Y^2 - a^2)^2 + 4a^2 Y^2}{Y(X^2 + Y^2 + a^2)}.$$

Hence, if we write

$$(X^2 + Y^2 - a^2)^2 + 4a^2 Y^2 = 4a\lambda Y(X^2 + Y^2 + a^2),$$

we have to shew that $\lambda$ increases steadily as the point $(X, Y)$ traces out the curve $C$ defined by

$$X = a\,(\cosh \alpha - r \cos \theta)/\sinh \alpha, \qquad Y = ar \sin \theta/\sinh \alpha;$$

where $\theta$ is a parameter increasing from 0 to $\pi$, and $r$ is, of course, written for brevity in place of $\sec \theta \, \{\cosh \alpha - \sqrt{(\cosh^2 \alpha - \theta \cot \theta)}\}$.

We shall consequently study the family of bicircular quartics

$$(X^2 + Y^2 - a^2)^2 + 4a^2 Y^2 = 4a\lambda Y(X^2 + Y^2 + a^2),$$

$\lambda$ being the parameter of the family. Only one quartic can be drawn through any real point, other than the points for which both $Y = 0$ and $X^2 + Y^2 = a^2$; and all the quartics pass through these points, which are $(\pm a, 0)$. We shall shew that, as a point $P$ describes the curve $C$ from $\theta = +0$ to $\theta = \pi$, the value of $\lambda$ for the quartic through $P$ steadily increases from 0 to $\infty$; and this will establish the theorem to be proved.

Each quartic is symmetrical with regard to the $Y$-axis, and it is wholly above the $X$-axis if† $\lambda \geqslant 0$; it meets the $Y$-axis at $(0, \pm ai)$, $[0, 2\lambda a \pm a(4\lambda^2 - 1)^{\frac{1}{2}}]$, and the two latter points are real only when $\lambda \geqslant \frac{1}{2}$.

* The analysis breaks down when $\alpha$ is actually zero, but it can then be replaced by the simpler analysis of § 7*a*.

† When $\lambda < 0$ the quartic lies below the $X$-axis, and therefore we may confine our attention to *positive* (including zero) values of $\lambda$.

When $\lambda = 0$, the quartic consists of the two points $(\pm a, 0)$; all the quartics touch the axis of $X$ at these points, while the shape of the quartics near them is that of the parabolas $2a\lambda Y = (X \mp a)^2$; these parabolas increase in size with $\lambda$.

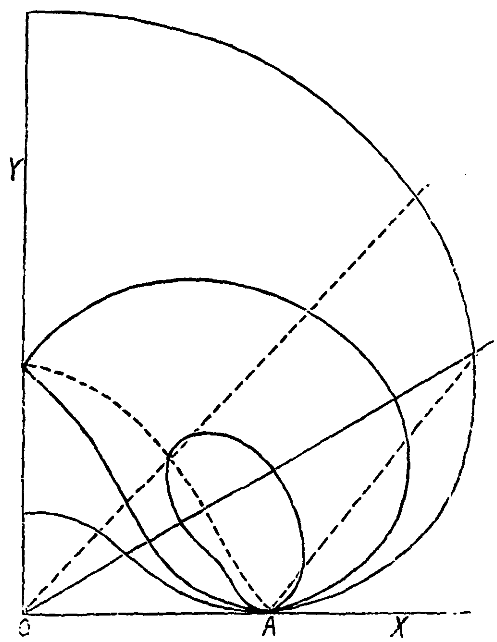


FIG. 3.

Each quartic consists of a number of closed branches,* and this number cannot exceed 2; for if there were 3, a circle drawn through a point on each branch would cut each branch a second time and would meet the curve twice at each circular point, so that a quartic and a conic would have 10 intersections, which is impossible. When $\lambda > \frac{1}{2}$, there cannot even be two branches, for, by symmetry, they would have 0 or 4 intersections with the axis of $Y$.

Hence, when $\lambda < \frac{1}{2}$, each quartic is bipartite, and, when $\lambda > \frac{1}{2}$, each quartic is unipartite; and by a consideration of the curves near $(\pm a, 0)$, we see that the quartic $(\lambda)$ is wholly enclosed by the part or parts of the quartic $(\lambda')$ when $\lambda' > \lambda$.

* The curves shewn in the figure are those for which $\lambda = 1/\sqrt{8}, \frac{1}{2}, 1/\sqrt{2}$; the dotted curve and line are the sextic and its asymptote which will be introduced presently: the line $Y = X \cot \frac{1}{2}\theta_0$ is also shewn.

When $\lambda = \frac{1}{2}$, the quartic has a double point at $(0, a)$, and we have a transition from the bipartite to the unipartite type.

We shall next consider the inflexions of each of the quartics; and we shall discuss them by means of the obvious result that a closed curve cannot have one inflexion without having a second, and if it has two (or more) inflexions, a bitangent can be drawn such that two inflexions lie between its points of contact.

Now the quartic has the following* eight bitangents:—

(i) $Y = 0,\ Y = 2a\lambda/(1-\lambda^2)$.

(ii) $\lambda^2 Y^2 + X^2 = 0$.

(iii) $Y = mX + c$, where

$$c = a\{1 \pm \sqrt{(1-4\lambda^2)}\}/(2\lambda), \qquad m^2 = \{c(1-\lambda^2)-a\lambda\}/(a\lambda^3).$$

The pair (ii) are always imaginary. The set (iii) are real if, and only if, $\lambda \leqslant \frac{1}{2}$, *i.e.*, if the quartic is bipartite; since

$$a^2 - (c/m)^2 = ac\lambda > 0,$$

the set (iii) all pass between the two branches of the curve. Also $Y = 0$ touches at $(\pm a, 0)$, and $Y = 2a\lambda/(1-\lambda^2)$ touches at real points only if $\lambda \leqslant 1/\sqrt{3}$; these points of contact are given by

$$a^2 - X^2 = 4a^2\lambda^4/(1-\lambda^2)^2,$$

and so they lie between the lines $X = \pm a$.

Now consider the area which forms the greatest possible region bounded by arcs of any particular quartic, and the segment of contact of the bitangent $Y = 0$ and the segment of contact of the bitangent

$$Y = 2a\lambda/(1-\lambda^2),$$

the latter being omitted if it does not touch at real points. The boundary of this area is a *simple closed curve* with no real bitangents, except one (or two) which touches at every point of a segment of a straight line. This curve consequently has no inflexions, and since the whole of the portion of the quartic lying on the right of $X = a$ forms part of it, it follows that *no quartic has any inflexions on the right of the line* $X = a$.

---

* Salmon, *Higher Plane Curves*, pp. 231–237, shews that a bicircular quartic has not more than 8 bitangents, and gives a method for deducing them in pairs from four canonical forms of the equation of the quartic; as it does not seem easy to obtain all of the four forms, I derived the bitangents by the direct process of substituting $mX+c$ for $Y$ in the equation, and writing down the conditions that the result should be a perfect square in $X$.

Hence, if a point starts from $(a, 0)$ to the right and travels along the quartic until it again reaches the line $X = a$, the tangent to the quartic at the point has a continually increasing slope, which increases from 0 to $+\infty$ and then (as the point moves to the left on the upper part of the quartic) from $-\infty$ to a finite negative value.

Moreover, the locus of points of contact with the quartics of tangents parallel to the $Y$-axis is the sextic

$$Y^6 + Y^4(X^2+a^2) - Y^2(X^4 - 10X^2a^2 + a^4) - (X^2-a^2)^2(X^2+a^2) = 0.$$

Since the product of the roots of this cubic in $Y^2$ is positive, and the sum of the roots is negative, the cubic has one positive root and cannot have more than one; and hence, corresponding to any positive value of $X$, there is one and only one point on the sextic in the *positive quadrant*. The ordinate of this point vanishes when, and only when, $X = a$.

The portion of the sextic in the region in which $Y \geqslant 0$, $X \geqslant a$ consequently consists of a single branch starting from $(a, 0)$ and terminating at infinity* on $Y = X$; and, at points between this branch and the $X$-axis, the quartics obviously have positive slopes, and are consequently *convex* to the $X$-axis.

Further, when $X$ is sufficiently large (in fact when $X \geqslant a \times 1{\cdot}9 \ldots$ approximately), the sextic lies *above* the line $Y = X \cot \frac{1}{2}\theta_0$, since

$$\cot \tfrac{1}{2}\theta_0 = 0{\cdot}548 < 1,$$

and the asymptote of the sextic is $Y = X$.

Next consider the curve $C$ (introduced on p. 125) traced out by the point whose coordinates are

$$X = a(\cosh \alpha - r\cos\theta)/\sinh \alpha, \qquad Y = ar\sin\theta/\sinh\alpha,$$

as $\theta$ increases from 0 to $\pi$.

As $\theta$ increases from 0 to $\theta_\alpha$ the point moves to the right from $(a, 0)$ with an increasing ordinate. This portion of the curve traced out by the point has a decreasing slope, and consequently it cannot meet any particular quartic more than once, since each quartic has an increasing slope on the right of $X = a$ and an increasing ordinate.

The portion of $C$ for which $\theta_\alpha < \theta < \pi$ has a negative slope, and it lies on the right of the line $X = a(\cosh\alpha - \cos\theta_\alpha)/\sinh\alpha$ and below the line $Y = X\cot\frac{1}{2}\theta_0$, by § 6; and so it lies entirely in a region in which the quartics have positive slopes, if we choose $\alpha$ so that

$$a(\cosh\alpha - \cos\theta_\alpha)/\sinh\alpha \geqslant a \times 1{\cdot}9 \ldots$$

* The lines $Y = \pm X$ are the only real asymptotes of the sextic.

Since $\theta_a$ is obtuse, this condition is certainly satisfied when $\coth a \geqslant 1{\cdot}9 \ldots$, *i.e.*, when $\cosh a \leqslant 1{\cdot}2 \ldots$.

When this condition is satisfied, the portion of $C$ under consideration obviously cannot meet any particular quartic more than once.

Hence, if a point $P$ travels along $C$ from $\theta = +0$, the parameter $\lambda_P$ of the quartic through $P$ must be monotonic as $\theta$ increases from 0 to $\theta_a$, and it must also continue to be monotonic as $\theta$ increases from $\theta_a$ to $\pi$; for, if $\lambda_P$ had any maxima or minima, we could find two points on the curve (one on each side of the critical point) at which $\lambda_P$ had the same value, and these two points would lie on the same quartic.

Since $\lambda_P \to 0$ as $\theta \to +0$ and $\lambda_P \to \infty$ as $\theta \to \pi$, while $\lambda_P$ is positive when $\theta = \theta_a$, it is obvious that $\lambda_P$ must be a steadily increasing function of $\theta$; and so we have established the required result that, when $a$ has any fixed value such that $1 < \cosh a \leqslant 1{\cdot}2 \ldots$,

$$\frac{dT(\theta, \cosh a)}{d\theta} \bigg/ \frac{dG(\theta, \cosh a)}{d\theta}$$

is an increasing function of $\theta$.

*7a. The simplification produced by making $a = 0$.*

In the special case in which $a = 0$, the analysis of § 7 breaks down; we then have to prove that $(1-2r\cos\theta+r^2)/(2r\sin\theta)$ is an increasing function of $\theta$.

Write $X = 1-r\cos\theta$, $Y = r\sin\theta$, and we see that we have to prove that $(X^2+Y^2)/(2Y)$ is an increasing function of $\theta$.

We therefore study the family of circles $X^2+Y^2 = 2\lambda Y$. These touch the axis of $X$ at the origin, and have positive slope when $\lambda > 0$, $X > 0$, $Y \leqslant X$. The curve traced out by $(1-r\cos\theta, r\sin\theta)$, as $\theta$ increases from $+0$ to $\theta_0$, is concave to the axis of $X$, and the ordinate and abscissa of a point on it are positive and increase as $\theta$, the parameter of the point, increases. This part of the curve consequently meets each circle only once. The portion of the curve for which $\theta_0 < \theta < \pi$ has a negative slope, and lies below $Y = X\cot\frac{1}{2}\theta_0$, and therefore in a region in which the circles have positive slope. The proof that $(X^2+Y^2)/(2Y)$ increases as the point $(X, Y)$ moves along the curve now follows exactly as in § 7.

It may be remarked that, even in this particular case, a direct algebraical proof, that

$$(1-2r\cos\theta+r^2)/(2r\sin\theta)$$

is an increasing function of $\theta$ on the curve under consideration, seems to be so arduous as to be impracticable.

[*Added, December* 15*th*, 1917.—The result stated at the end of § 5, that $TG'-\frac{2}{3}GT'$ is negative when $0 < \theta < \pi$, may be proved by similar graphical considerations: here $T$, $G$ stand for $T(\theta, 1)$, $G(\theta, 1)$ respectively. For we have

$$\frac{GG'^2}{T'^2} = \frac{8XY^3}{(X^2+Y^2)^2} = 8\sin^3\phi\cos\phi,$$

where $\tan\phi = Y/X$. Now, since $X = 1-r\cos\theta$, $Y = r\sin\theta$, $\phi$ steadily decreases as $\theta$ increases from 0 to $\pi$; for when $\theta$ is less than $\theta_0$ the curve traced out by $(X, Y)$ is concave to the $X$-axis, and when $\theta > \theta_0$ the curve has an ordinate which decreases as $\theta$ increases. Hence

$\phi$ steadily decreases from $\frac{1}{3}\pi$ to 0 as $\theta$ increases from 0 to $\pi$. But, when $0 < \phi < \frac{1}{3}\pi$,

$$\frac{d}{d\phi}(\sin^3\phi\cos\phi) = \sin^4\phi\,(3\cot^2\phi - 1) > 0,$$

and so $GG''/T'^2$ is a steadily decreasing function of $\theta$; hence

$$T'(2GG'' + G'^2) - 2GG'T'' < 0,$$

that is to say,

$$\frac{d}{d\theta}[3T - 2(GT'/G')] < 0.$$

Therefore $3T - 2(GT'/G')$ is a decreasing function of $\theta$ which vanishes when $\theta = 0$, and so is negative when $\theta > 0$; and from this result it is obvious that $3TG' - 2GT'$ is negative when $\theta > 0$.]

8. *The cardinal theorem concerning* $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$.

We are now in a position to prove the important theorem that, when $\xi$ is fixed (and $1 \leqslant \xi \leqslant 1\cdot2\ldots$), $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$ is a non-increasing function of $n$.

For the values of $\xi$ under consideration, we have, from the results of §§ 7 and 7*a*,

$$T''G' - T'G'' \geqslant 0.$$

It follows that

$$\frac{d(T'G/G')}{d\theta} - \frac{dT}{d\theta} = \frac{(T''G' - T'G'')\,G}{G'^2} \geqslant 0,$$

since $G$ is not negative; for $G$ is bounded* when $0 \leqslant \theta \leqslant \pi$, while $G$ vanishes when $\theta = 0$ and $G' = dT/d\xi \geqslant 0$, by § 4, when $0 \leqslant \theta \leqslant \pi$.

Hence $(T'G/G') - T$ is a non-decreasing function of $\theta$, and it is equal to $-T(0, \cosh\alpha)$ when $\theta = 0$; and hence we at once get

$$G(\theta, \cosh\alpha)\,\frac{dT(\theta, \cosh\alpha)}{d\theta} \geqslant \{T(\theta, \cosh\alpha) - T(0, \cosh\alpha)\}\,\frac{dG(\theta, \cosh\alpha)}{d\theta}.$$

Next, we shall shew that

$$\Delta_n(\xi)\,\frac{\partial^2\Delta_n(\xi)}{\partial n\,\partial\xi} - \frac{\partial\Delta_n(\xi)}{\partial\xi}\,\frac{\partial\Delta_n(\xi)}{\partial n} \leqslant 0.$$

We have

$$\Delta_n(\xi) = \int_0^\pi e^{-nT(\psi,\xi)}\,d\psi, \qquad \frac{\partial\Delta_n(\xi)}{\partial\xi} = -n\int_0^\pi \frac{dG(\theta,\xi)}{d\theta}\,e^{-nT(\theta,\xi)}\,d\theta,$$

and, differentiating under the integral sign, and then integrating by parts,†

* It is easy to see that $G(\theta, \cosh\alpha) = 2(\theta\cosh\alpha - r\sin\theta)$.

† Compare the corresponding analysis in the *Proceedings*, Ser. 2, Vol. 16, pp. 164, 165.

we easily find that

$$\Delta_n(\xi)\frac{\partial^2\Delta_n(\xi)}{\partial n\,\partial\xi}-\frac{\partial\Delta_n(\xi)}{\partial\xi}\,\frac{\partial\Delta_n(\xi)}{\partial n}=-n\int_0^\pi\int_0^\pi\Omega(\theta,\psi)e^{-n\{T(\theta,\xi)+T(\psi,\xi)\}}\,d\theta\,d\psi,$$

where $\Omega(\theta,\psi)=G(\theta,\xi)\dfrac{dT(\theta,\xi)}{d\theta}+\{T(\psi,\xi)-T(\theta,\xi)\}\dfrac{dG(\theta,\xi)}{d\theta}$

$$\geqslant G(\theta,\xi)\frac{dT(\theta,\xi)}{d\theta}+\{T(0,\xi)-T(\theta,\xi)\}\frac{dG(\theta,\xi)}{d\theta}$$

$$\geqslant 0,$$

by the result just proved; and so we have

$$\Delta_n(\xi)\,\frac{\partial^2\Delta_n(\xi)}{\partial n\,\partial\xi}-\frac{\partial\Delta_n(\xi)}{\partial\xi}\,\frac{\partial\Delta_n(\xi)}{\partial n}\leqslant 0,$$

when $1\leqslant\xi\leqslant 1\cdot 2\,\ldots$.

It follows from this inequality, exactly as in *Proceedings*, Ser. 2, Vol. 16, pp. 164, 165, that $\Delta_n(\xi)/\Delta_n(1)$ is a non-increasing function of $n$ when $\xi$ is fixed and $1\leqslant\xi\leqslant 1\cdot 2\,\ldots$. Changing to parabolic cylinder functions, we see at once that $D_n(2\xi\surd n)/D_n(2\surd n)$ is a non-increasing function of $n$ when $\xi$ is fixed and $1\leqslant\xi\leqslant 1\cdot 2\,\ldots$.

9. *An upper bound to* $|D_n(z)|$ *when the order is a positive integer and* $z$ *is unrestricted.*

It will now be shewn that, when $n$ is an integer,

$$|D_n(z)|<\Gamma(n+1)\,e^{\frac{1}{2}n}\,(2n)^{-n}\,\left|\,\{z+\surd(z^2-4n)\}^n\exp\{-\tfrac{1}{4}z\surd(z^2-4n)\}\right|,$$

where that value of $\surd(z^2-4n)$ is taken which makes

$$|z-\surd(z^2-4n)|\leqslant 2\surd n.$$

In the formula $$\Delta_n(\zeta)=\frac{1}{2i}\int_{-\infty}^{(0+)}e^{-n\tau}s^{-1}\,ds,$$

obtained in § 2, take the contour to be a circle, with centre at the origin, terminated at the points at which $\arg s=\pm\pi$, together with the lines joining the end points of this arc to $-\infty$. The integrals along the lines cancel, since $n$ is an integer.

To obtain the best possible inequality, take the circle to pass through a stationary point of $\tau$, namely, the point at which $s=e^{-(\alpha+i\beta)}\equiv s_0$, say, where $\zeta=\cosh(\alpha+i\beta)$ and $\alpha\geqslant 0$, $-\pi\leqslant\beta\leqslant\pi$.

Writing $s=e^{-\alpha+i\theta}$ on the circle and putting $\tau=\tau_0$ when $s=s_0$, we see that, when $s$ is on the circle,

$$R(\tau-\tau_0) = 2e^{-2\alpha}\sin^2\tfrac{1}{2}(\theta+\beta)\{e^{2\alpha}-\cos(\theta-\beta)\} \geqslant 0.$$

Hence the modulus of $\Delta_n(\zeta)$ cannot exceed $\pi\,|\exp(-n\tau_0)|$.

Now $$s_0 = \cosh(\alpha+i\beta)-\sinh(\alpha+i\beta) = \zeta-\sqrt{(\zeta^2-1)},$$

where that sign is given to the radical which makes $|s_0|<1$; and on substituting for $\tau_0$ in terms of $\zeta$, and thence of $z$, we at once obtain the inequality stated.

Since $$R(\tau_0) = \tfrac{1}{2}\sinh 2\alpha\cos 2\beta-\alpha \geqslant 0,$$

if $\alpha=0$ or if $\beta=0$, we see that, if $z$ is positive and equal to $x$ (whether greater than, less than, or equal to $2\sqrt{n}$) then $\Delta_n(\xi)<\pi$, and so

$$|D_n(x)| < \Gamma(n+1)\,e^{\frac{1}{2}n}\,n^{-\frac{1}{2}n}.$$

This inequality shews that Adamoff's function $w_n(x)$ is, in reality, $O(n)$ at most, where the constant involved in the symbol $O$ is independent of $x$.

It is easy to construct a very similar inequality when $n$ is not restricted to be an integer by obtaining a suitable upper bound for the integrand on the straight lines which pass from the circle to infinity.

10. *The expression of a definite integral in terms of parabolic cylinder functions.*

We shall subsequently need the result

$$\int_{-\infty}^{\infty} s^{-n-1}\exp(-\tfrac{1}{4}z^2+n^{\frac{1}{2}}zs-\tfrac{1}{2}ns^2)\,ds = (2\pi n^n)^{\frac{1}{2}}\,e^{\mp\frac{1}{2}(n+1)\pi i}D_{-n-1}(\mp iz).$$

The upper or lower sign is to be taken according as the path of integration passes above or below the origin.

This is most simply proved by taking $R(n)<0$, deforming the contour into the rays $\arg s=0$, $\arg s=\pm\pi$, expanding $\exp(n^{\frac{1}{2}}zs)$ in powers of $z$ and integrating term by term; the result follows for all values of $n$ by the general theory of analytic continuation.

11. *Asymptotic expansions of $D_n(z)$ when $z$ and $n$ are large.*

We shall now investigate asymptotic expansions corresponding to those obtained by Debye for $J_n(z)$. The analysis is similar to that given by Debye, and will be stated very briefly, except in the discussion of the range of validity of the expansions; the portions of Debye's work which correspond to the last mentioned investigation seem to be somewhat inadequate.

The expansions will be deduced from the following Lemma:—

LEMMA.—*If* (i) $f(t)$ *is analytic when* $|t| \leqslant a+\delta$, *where* $a>0$, $\delta>0$, *except at a branch-point at the origin, and*

$$f(t) = \sum_{m=1}^{\infty} a_m t^{(m/r)-1}$$

*when* $|t| \leqslant a$; (ii) $|f(t)| < Ke^{bt}$, *where* $K$ *and* $b$ *are independent of* $t$, *when* $t$ *is positive and* $t \geqslant a$; (iii) $|\arg n| \leqslant \frac{1}{2}\pi - \Delta$, *where* $\Delta > 0$; (iv) $|n|$ *is sufficiently large: then there exists a complete asymptotic expansion given by the formula*

$$\int_0^\infty e^{-nt} f(t)\,dt \sim \sum_{m=1}^{\infty} a_m \Gamma(m/r)\, n^{-(m/r)}.$$

The proof of the lemma is easy; for, if $M$ be any fixed integer, we have

$$\left| f(t) - \sum_{m=1}^{M-1} a_m t^{(m/r)-1} \right| < K_1 t^{(M/r)-1} e^{bt}$$

throughout the range of integration, where $K_1$ is some number independent of $t$.

Hence $$\int_0^\infty e^{-nt} f(t)\,dt = \sum_{m=0}^{M-1} \int_0^\infty e^{-nt} a_n t^{(m/r)-1}\,dt + R_M,$$

where $$|R_M| < \int_0^\infty |e^{-nt}|\, K_1 t^{(M/r)-1} e^{bt}\,dt$$

$$< K_1 \Gamma(M/r) / \{R(n) - b\}^{M/r},$$

provided that $R(n) > b$, which is the case when $|n|$ is sufficiently large; and since $\{R(n)-b\}^{-1} = O(1/n)$ for the range of values of $n$ under consideration, we have

$$\int_0^\infty e^{-nt} f(t)\,dt = \sum_{m=1}^{M-1} a_m \Gamma(m/r)\, n^{-m/r} + O(n^{-M/r}),$$

and so the integral possesses the complete asymptotic expansion, which is of Poincaré's type.

12. *The application of the method of steepest descents to Whittaker's integral.*

We take the form of Whittaker's integral obtained in § 2, namely,

$$\Delta_n(\zeta) = \frac{1}{2i}\int_{-\infty}^{(0+)} e^{-n\tau} s^{-1}\,ds,$$

where
$$\tau = \zeta^2+\tfrac{1}{2}+\log s-2\zeta s+\tfrac{1}{2}s^2;$$

this formula is obviously valid when $|\arg n| \leqslant \frac{1}{2}\pi-\Delta$, where $\Delta$ is any positive number. As in § 9, we write $\zeta = \cosh(a+i\beta)$, where $a \geqslant 0$, $-\pi \leqslant \beta \leqslant \pi$, and also $s = re^{i\theta}$; further, let $a+i\beta = \gamma$, $e^{-\gamma} = s_0$.

It is evident that $\tau$ is stationary when $s = s_0$; let the value of $\tau$ at this point be $\tau_0$.

Then
$$\begin{aligned}\tau-\tau_0 = \log r+a-r\{&e^a\cos(\theta+\beta)+e^{-a}\cos(\theta-\beta)\}\\ &+\tfrac{1}{2}r^2\cos 2\theta+1+\tfrac{1}{2}e^{-2a}\cos 2\beta\\ &+i[\theta+\beta-r\{e^a\sin(\theta+\beta)+e^{-a}\sin(\theta-\beta)\}\\ &\qquad+\tfrac{1}{2}r^2\sin 2\theta-\tfrac{1}{2}e^{-2a}\sin 2\beta].\end{aligned}$$

It is necessary, before we apply the method of steepest descents, to study the form of the curve on which $\tau-\tau_0$ is real, namely

$$r^2\sin\theta\cos\theta-r\{e^a\sin(\theta+\beta)+e^{-a}\sin(\theta-\beta)\}+\theta+\beta-e^{-2a}\sin\beta\cos\beta = 0.$$

The curve has already been considered (§§ 3, 6) when $\beta = 0$; and since its equation is unaffected by changing the sign of $\beta$, provided that the sign of $\theta$ be also changed (so that the curve is reflected in the real axis), it is sufficient to consider in detail values of $\beta$ such that $0 < \beta \leqslant \pi$; further, since the effect of increasing $\beta$ by $\pi$ leaves the equation unaffected, provided that $\theta$ be diminished by $\pi$ (*i.e.*, provided the curve is turned through two right angles), it is really sufficient to consider values of $\beta$ between 0 and $\frac{1}{2}\pi$.

Suppose therefore that $0 < \beta \leqslant \frac{1}{2}\pi$, and consider the values of $r$ corresponding to various given values of $\theta$.

We have $\theta+\beta-e^{-2a}\sin\beta\cos\beta$ positive when $0 \leqslant \theta \leqslant \pi$, and $e^a\sin(\theta+\beta)+e^{-a}\sin(\theta-\beta)$ changes sign once (from + to −) at a value of $\theta$ greater than $\frac{1}{2}\pi$.

Hence, when $\frac{1}{2}\pi < \theta < \pi$, there is one positive root of the quadratic in $r$; this root is a continuous function of $\theta$ and tends to a limit when $\theta \to \frac{1}{2}\pi$; but it tends to $+\infty$ as $\theta \to \pi$ (in the latter case, the negative root tends to a finite limit).

When $0 < \theta < \frac{1}{2}\pi$, there are two distinct positive roots so long as

$$\{e^{a}\sin(\theta+\beta)+e^{-a}\sin(\theta-\beta)\}^2-4\sin\theta\cos\theta\{\theta+\beta-e^{-2a}\sin\beta\cos\beta\}>0.$$

The last expression may be written in the form

$$4\sin\theta\cos\theta[\{\sinh^2 a\sin^2(\theta+\beta)/(\sin\theta\cos\theta)\}$$
$$+\{\sin(\theta+\beta)\cos\beta\sec\theta\}-\theta-\beta].$$

The differential coefficient of the expression in square brackets is

$$\sin(\theta+\beta)\sin(\theta-\beta)\sec^2\theta\{\sinh^2 a\operatorname{cosec}^2\theta+1\},$$

and so the expression has one minimum in the range of values of $\theta$ under consideration, namely $\theta=\beta$; its value is then $\cosh 2a\sin 2\beta-2\beta$.

[The expression $\cosh 2a-2\beta\operatorname{cosec}2\beta$ is positive* when $\zeta$ ($\equiv \xi+i\eta$) is on the right of the curve shewn in Fig. 4 by a continuous line on the right of the $\eta$-axis; the curve has a *point saillant* at $\xi=1$, $\eta=0$.]

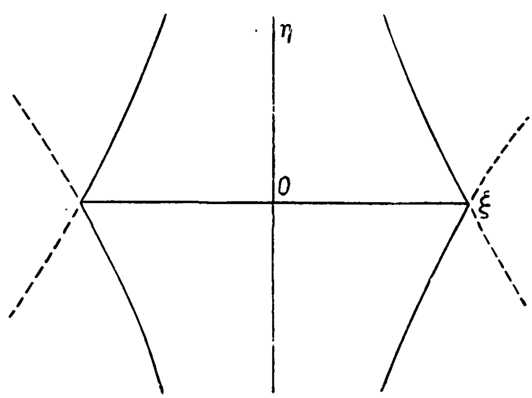


FIG. 4.

There are thus three cases:—(i) If $\zeta$ is on the right of the curve, we

* The expression is written in this form to cover the case when $\beta$ is negative. The reader should note that it did not seem practicable to draw the curves in Figs. 5–11 accurately; and the sketches given indicate merely the general form of the curves.

The dotted curve in Fig. 4 (which is drawn accurately) is $\sinh 2a\cos 2\beta=2a$; see § 18. The other continuous curve is the image in the $\eta$-axis of the first curve; see § 16.

get two distinct values of $r$ for each value of $\theta$ between 0 and $\frac{1}{2}\pi$, as in Fig. 5. (ii) If $\zeta$ is on the curve, the contour has a second double point

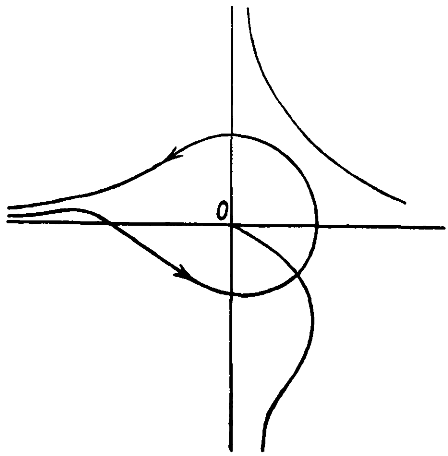


FIG. 5.

where $\theta = \beta$, as in Fig. 6. (iii) If $\zeta$ is on the left of the curve, there are two values of $\theta$ such that between them $r$ is imaginary, and the curve is as in Fig. 7.

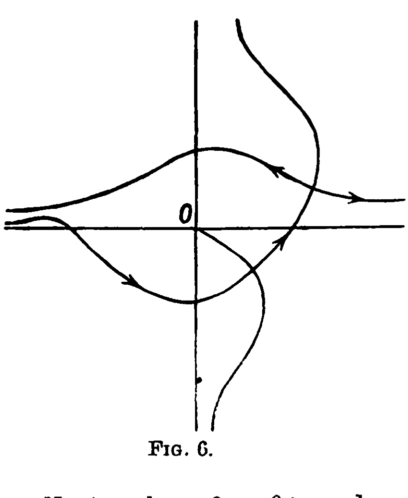


FIG. 6.

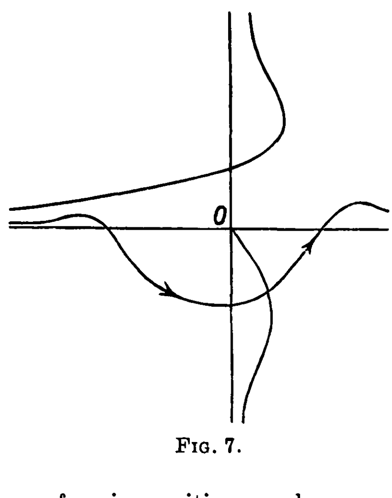


FIG. 7.

Next, when $0 > \theta > -\frac{1}{2}\pi$, one value of $r$ is positive so long as $\theta > -\beta + e^{-2\alpha}\sin\beta\cos\beta$, and when $\theta$ is less than this value, both are positive, provided that $e^{\alpha}\sin(\theta+\beta)+e^{-\alpha}\sin(\theta-\beta)$ vanishes for a greater value of $\theta$ than $-\beta+e^{-2\alpha}\sin\beta\cos\beta$, *i.e.*, provided that

$$\tan^{-1}(\tanh\alpha\tan\beta)-\beta+e^{-2\alpha}\sin\beta\cos\beta < 0;$$

and this inequality is always true.

[To establish the inequality, we call the expression on the left $f(\alpha, \beta)$; then

$$\frac{\partial f(\alpha, \beta)}{\partial \alpha} = \frac{e^{-2\alpha} \sin \beta \cos \beta (\sinh 2\alpha - \cos 2\beta)}{\cosh^2 \alpha \cos^2 \beta + \sinh^2 \alpha \sin^2 \beta}.$$

Hence, if $\cos 2\beta < 0$, $f(\alpha, \beta) \leqslant f(\infty, \beta) = 0$; and if $\cos 2\beta > 0$, $f(\alpha, \beta)$ decreases as $\alpha$ increases from 0 to $\frac{1}{2}\sinh^{-1}(\cos 2\beta)$, and subsequently increases, so that $f(\alpha, \beta)$ does not exceed the greater of $f(0, \beta)$ and $f(\infty, \beta)$. Therefore, always, $f(\alpha, \beta) \leqslant 0$.]

Consequently, when $-\beta + e^{-2\alpha} \sin \beta \cos \beta > \theta > -\frac{1}{2}\pi$, there are two positive values of $r$ which coincide only when $\theta = -\beta$ [the maximum of $\sinh^2 \alpha \sin^2 (\theta+\beta)/(\sin \theta \cos \theta) + \{\sin(\theta+\beta) \cos \beta \sec \theta\} - \theta - \beta$].

The larger root tends to $+\infty$ as $\theta \to -\frac{1}{2}\pi$, and when $\theta < -\frac{1}{2}\pi$ it becomes negative; the other root remains positive and tends to a finite limit as $\theta \to -\pi$; when $\theta < -\pi$, there are two positive roots (the negative root becoming positive through infinity when $\theta = -\pi$) so long as $\{\sinh^2 \alpha \sin^2 (\theta+\beta)/(\sin \theta \cos \theta)\} + \{\sin(\theta+\beta) \cos \beta \sec \theta\} - \theta - \beta$ is negative; since the expression has become positive when $\theta$ has decreased to $-\pi-\beta$, the curve must lie as shewn in all three figures.

13. *Asymptotic expansions derived from Whittaker's integral.*

We now consider
$$\frac{1}{2i} \int e^{-n\tau} s^{-1} ds,$$

taken along the contour marked in that one of Figs. 5, 6, 7 which is appropriate to the particular value of $\zeta$ under consideration.

On each contour, $\tau$ decreases from $+\infty$ to $\tau_0$, and then increases to $+\infty$. We accordingly write the integral in the form

$$\frac{1}{2i} \int_{\tau_0}^{\infty} e^{-n\tau} \left\{ \frac{d \log (s_1/s_0)}{d\tau} - \frac{d \log (s_2/s_0)}{d\tau} \right\} d\tau,$$

where $s_1$, $s_2$ are the two values of $s$ corresponding to a specified value of $\tau$, so chosen that $\arg s_1 \geqslant -\beta \geqslant \arg s_2$.

Now, if we write $v \equiv \log (s/s_0)$, we have

$$\tau - \tau_0 = \sum_{m=0}^{\infty} c_m v^{m+2},$$

where
$$c_m = (2^{m+1} s_0^2 - s_0^2 - 1)/(m+2)!.$$

For sufficiently small values of $|\tau - \tau_0|$ this series can be reverted, giving an expansion of the form

$$v = \sum_{m=0}^{\infty} a_m (\tau - \tau_0)^{\frac{1}{2}(m+1)}/(m+1),$$

provided that $c_0 \neq 0$, *i.e.*, provided that $s_0 \neq \pm 1$.

Hence we get $$\frac{dv}{d\tau} = \sum_{m=0}^{\infty} \tfrac{1}{2}a_m(\tau-\tau_0)^{\frac{1}{2}(m-1)}.$$

To determine the radius of convergence of this series, we observe that

$$dv/d\tau = (1-2\zeta s+s^2)^{-1}.$$

and so $v$ is a monogenic function of $\tau$ except when $s = e^{\pm\gamma}$; hence the only possible singularities are at the values of $\tau$ given by the formulæ $\tau_0-2r\pi i$, $-\tau_0+2r\pi i$. Of the latter system the only points which are singularities of the particular branch of $v$ under consideration (defined by the power series and its continuations for positive values of $\tau-\tau_0$) are the three points $-\tau_0$, $-\tau_0 \pm 2\pi i$. For, the singularities being given by $\tau-\tau_0 = 2r\pi i$, $-2\tau_0+2r\pi i$, we see that if $\tau_0$ be varied so as to make $-2\tau_0+2r\pi i$ tend to zero, then the radius of convergence will tend to zero, and so $c_0$ must tend to zero; for, if $c_0$ did not tend to zero, the series would have a non-zero radius of convergence.

Now consider the zeros of $-2\tau_0+2r\pi i$, *i.e.*, of $2\gamma-\sinh 2\gamma+2r\pi i$.

If $$2\gamma-\sinh 2\gamma+2r\pi i = 0,$$

then $$2r\pi-2i\gamma = \sin(2r\pi-2i\gamma).$$

This, regarded as an equation in $\gamma$, has a triple zero* when

$$2i\gamma-2r\pi = 0,$$

and it has simple zeros, two in the strip $\pi < R(2r\pi-2i\gamma) < \frac{5}{4}\pi$, two in the strip $2\pi < R(2r\pi-2i\gamma) < \frac{9}{4}\pi$, and so on; none of these latter zeros are on the boundaries of the strips and none are such that

$$I(2r\pi-2i\gamma) = 0.$$

Hence the only zeros which make $c_0$ vanish are those for which

$$2i\gamma-2r\pi = 0;$$

and $\gamma = 0$, $\pm\pi$ are the only ones for which $-\pi \leqslant \beta \leqslant \pi$; these correspond to the values $0$, $\mp 1$ of $r$.

Hence the radius of convergence of the series for $v$ is the smallest of $2\pi$, $|2\tau_0|$, $|2\tau_0 \pm 2\pi i|$; and none of these are small except when $\gamma$ is nearly equal to $0$ or $\pm\pi$.

*Hence the Lemma of* § 11 *is applicable except near the points* $\zeta = \pm 1$, *which will require special consideration.*

---

* These results are due to Hardy, *Messenger of Mathematics*, Vol. 31, p. 163.

Following the methods of Debye,* we find (by using what is practically Lagrange's expansion) that $a_m$ is the coefficient† of $v^m$ in the expansion of $v^{m+1}(\tau-\tau_0)^{-\frac{1}{2}(m+1)}$ in ascending powers of $v$.

We shall write $$a_{2m}c_0^{m+\frac{1}{2}} \equiv A_m(\gamma),$$

so that

$$A_0(\gamma) = 1,$$

$$A_1(\gamma) = \tfrac{1}{3} - \tfrac{11}{24}\coth\gamma + \tfrac{5}{24}\coth^2\gamma,$$

$$A_2(\gamma) = \tfrac{1}{54} - \tfrac{67}{432}\coth\gamma + \tfrac{427}{1152}\coth^2\gamma - \tfrac{595}{1728}\coth^3\gamma + \tfrac{385}{3456}\coth^4\gamma.$$

In the case of $s_1$, it is easy to see from Fig. 1 that we must take

$$c_0^{-\frac{1}{2}} = +i/\sqrt{(e^{-\gamma}\sinh\gamma)},$$

where $\sqrt{(\sinh\gamma)}$ is positive when $\gamma$ is real and positive; for $s_2$ we must take the other value of $c_0^{-\frac{1}{2}}$, and hence we find that

$$\frac{1}{2i}\int_{\tau_0}^{\infty} e^{-n\tau}\left\{\frac{d\log(s_1/s_0)}{d\tau} - \frac{d\log(s_2/s_0)}{d\tau}\right\}d\tau$$

$$\sim \frac{e^{-n\tau_0}}{2\sqrt{(e^{-\gamma}\sinh\gamma)}}\sum_{m=0}^{\infty}\frac{A_m(\gamma)}{c_0^m}\int_{\tau_0}^{\infty} e^{-n(\tau-\tau_0)}(\tau-\tau_0)^{m-\frac{1}{2}}\,d\tau$$

$$\sim \frac{e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{2\sqrt{(ne^{-\gamma}\sinh\gamma)}}\sum_{m=0}^{\infty}\frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m}.$$

*Hence, if* $\gamma$ *be such that* $\cosh 2\alpha - 2\beta\operatorname{cosec}2\beta > 0$, *i.e., if* $\zeta$ *be on the right of the curve shewn in* Fig. 4, *then the asymptotic expansion of* $D_n(z)$ *is given by the formula*

$$D_n(z) \sim \frac{\Gamma(n+1)\,e^{\frac{1}{2}n}\,e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{2\sqrt{(\pi n^{n+1}e^{-\gamma}\sinh\gamma)}}\sum_{m=0}^{\infty}\frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m},$$

where $$\gamma = \cosh^{-1}(\tfrac{1}{2}z/\sqrt{n}).$$

[The dominant term in this expansion, namely,

$$\tfrac{1}{2}\Gamma(n+1)\exp\{\tfrac{1}{2}n - n\sinh\gamma\cosh\gamma + n\gamma\}/\sqrt{(\pi n^{n+1}e^{-\gamma}\sinh\gamma)},$$

is the second of the two approximations given at the end of § 3 in the special case when $z$ is positive and exceeds $2\sqrt{n}$.]

* In Debye's expression (*Math. Ann.*, Vol. 67, p. 545) a factor 3 is omitted from the coefficient of $c_1^2c_2/c_0^3$ in $a_4$. The effect of this correction is to change the expression $\frac{7}{576}\cot^2\tau_0$ which occurs in his subsequent work into $\frac{77}{576}\cot^2\tau_0$.

† Expressions for the coefficients $a_m$ as determinants, whose elements are multiples of the coefficients $c_r$, are given in the *Messenger of Mathematics*, Vol. 46, pp. 97–101.

If, however, $\zeta$ is on the left of the curve given by

$$\cosh 2\alpha - 2\beta \operatorname{cosec} 2\beta = 0,$$

then the integral

$$\frac{1}{2i}\int_{\tau_0}^{\infty} e^{-n\tau}\left\{\frac{d\log(s_1/s_0)}{d\tau} - \frac{d\log(s_2/s_0)}{d\tau}\right\} d\tau$$

still has the same asymptotic expansion, *though it represents a different function of z*. For the contour in the $s$-plane (Fig. 7) is no longer one which is symbolised by $(-\infty; 0+)$; it is a contour going from $\infty e^{-\pi i}$ to $+\infty$ below the origin, or from $+\infty$ to $\infty e^{\pi i}$ above the origin, according* as $I(\zeta) \geqslant 0$ or $I(\zeta) \leqslant 0$, *provided that $\zeta$ is on the right of the curve*†

$$\cosh 2\alpha - 2(\beta \pm \pi) \operatorname{cosec}(2\beta \pm 2\pi) = 0.$$

Hence, by the result of § 10, the integral is equal to

$$\pm \frac{1}{2i} e^{-\frac{1}{2}n}(2\pi n^n)^{\frac{1}{2}} e^{\pm\frac{1}{2}(n-1)\pi i} D_{-n-1}(\pm iz),$$

and so we obtain the asymptotic expansion‡

$$D_{-n-1}(\pm iz) \sim \frac{e^{\mp\frac{1}{2}n\pi i} e^{\frac{1}{2}n} e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{\sqrt{(2n^{n+1}e^{-\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m},$$

where the upper or lower sign is taken according as $I(\zeta) \geqslant 0$ or $I(\zeta) \leqslant 0$, and it is supposed that $\zeta$ lies between the curve

$$\cosh 2\alpha - 2\beta \operatorname{cosec} 2\beta = 0,$$

and the curve $\quad \cosh 2\alpha - 2(\beta \pm \pi) \operatorname{cosec}(2\beta \pm 2\pi) = 0.$

---

* If $I(\zeta) = 0$ and $-1 < \zeta < 1$, we may take $\beta$ positive or negative at pleasure $(-\pi < \beta < \pi)$.

† If $\zeta$ were on the left of this curve we should get a contour *starting at* $+\infty$, *encircling the origin, and returning to* $+\infty$. This curve is the image in the imaginary axis of the curve $\cosh 2\alpha = 2\beta \operatorname{cosec} 2\beta$ (see Fig. 4).

‡ When $\zeta$ is actually on the curve given by $\cosh 2\alpha = 2\beta \operatorname{cosec} 2\beta$, there is a second stationary point on the contour in the $s$-plane at which $\tau = -\tau_0$; near this point, $d(\log s)/d\tau$ is $O\{(\tau+\tau_0)^{-\frac{1}{2}}\}$; this pole of the integrand raises the order of the integral along the part of the contour near the point by $n^{\frac{1}{2}}$, which is $o\{\exp(-2n\tau_0)\}$, because $\tau_0$ is not small and $R(\tau_0) < 0$ (since $\zeta$ is on the left of the dotted curve in Fig. 4); and so the asymptotic expansion is unaffected when $n$ is positive. If $n$ is complex we may get an additional series to be inserted; this series is easily written down, and it seems unnecessary to give it here. In this case we may go along the contour either to $+\infty$ or to $-\infty$; and so the two functions $D_n(z)$ and $e^{\pm\frac{1}{2}n\pi i}\Gamma(n+1)(2\pi)^{-\frac{1}{2}} D_{-n-1}(\pm iz)$ have the same asymptotic expansion when $\zeta$ is on the curve and is not close to the point $\zeta = 1$.

14. *The asymptotic expansion of* $D_{-n-1}(\mp iz)$.

In the preceding section we obtained the asymptotic expansion of $D_{-n-1}(\pm iz)$ when $z$ lay in a certain region of the plane and the upper or lower sign was taken so as to make $R(\pm iz) \leqslant 0$. We shall next consider $D_{-n-1}(\mp iz)$, where $R(\pm iz) \leqslant 0$.

To do this we make use of the second stationary point of

$$\tau \equiv \tfrac{1}{4}\zeta^2+\tfrac{1}{2}+\log s-2\zeta s+\tfrac{1}{2}s^2,$$

namely, $s_0' = e^{+(\alpha+i\beta)}$; at this point $\tau = -\tau_0$, and the contour given by the method of steepest descents has for its equation in polar coordinates

$$r^2\sin\theta\cos\theta - r\left\{e^{\alpha}\sin(\theta+\beta)+e^{-\alpha}\sin(\theta-\beta)\right\}+\theta-\beta+e^{2\alpha}\sin\beta\cos\beta = 0,$$

with a double point where $r = e^{\alpha}$, $\theta = \beta$.

When $\beta$ is positive and acute, the curve is as shewn in Figs. 8, 9, 10, and when $\beta$ is positive and obtuse, it is as shewn in Fig. 11. The effect of changing the sign of $\beta$ is to reflect the curve in the real axis. When $\beta = 0$, the figure degenerates into Fig. 1, and when $\beta = \pm\pi$ the figure

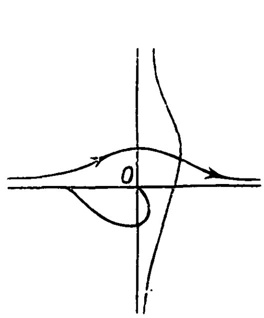


FIG. 8.

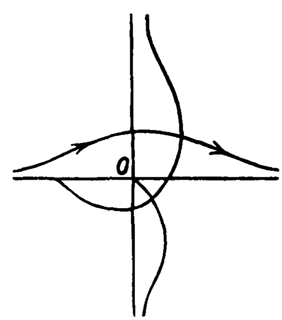


FIG. 9.

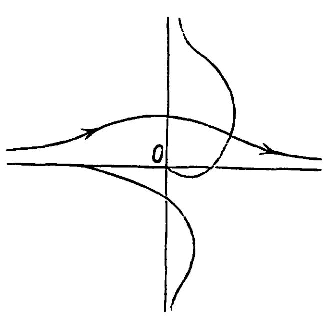


FIG. 10.

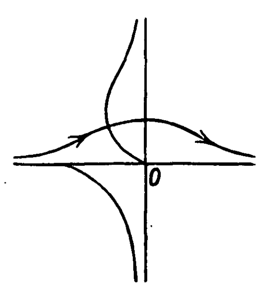


FIG. 11.

also degenerates; in the special case $\beta = 0$, the contour may be taken to go either from $\infty e^{\pi i}$ or from $\infty e^{-\pi i}$ to $+\infty$, the important stationary point (cf. footnote ‡, p. 140) being that at $-\tau_0$; the cases $\beta = \pm \pi$ are derivable from $\beta = 0$, since $-i\cosh(\alpha \pm i\pi) = +i\cosh\alpha$. The contour going from $\infty e^{\pm\pi i}$ to $+\infty$ gives, in general, the asymptotic expansion of $D_{-n-1}(\mp iz)$.

Writing $w = \log(s/s_0')$, we have

$$\tau + \tau_0 = \sum_{m=0}^{\infty} c_m' w^{m+2},$$

where $$c_n' = \{(2^{m+1}-1)s_0^{-2}-1\}/(m+2)!,$$

and hence $$d(\log s)/d\tau = \Sigma\, \tfrac{1}{2} a_m' (\tau+\tau_0)^{m-\frac{1}{2}},$$

where $a_0' = c_0'^{-\frac{1}{2}}$, $a_1' = -c_1'/c_0'^2$, ..., $(c_0' = e^\gamma \sinh\gamma)$, and, generally,

$$a_{2m}' = A_m(-\gamma)/(c_0')^{m+\frac{1}{2}}.$$

To fix the sign of $\sqrt{c_0'}$, we observe that, for the branch of the contour going from $s_0'$ to $+\infty$, $d(\log s)/d\tau$ is *positive* in the limiting case when $\gamma$ is positive; and so we get, in general,

$$\int_{\infty\exp(\pm\pi i)}^{\infty} e^{-n\tau} s^{-1} ds \sim \frac{e^{n\tau_0}}{+\sqrt{(ne^\gamma \sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\,\Gamma(m+\frac{1}{2})}{(ne^\gamma \sinh\gamma)^m},$$

where $\arg(\sinh\gamma)$ is zero when $\gamma$ is positive.

From this result* we at once get (except when $\beta = \pm\pi$)

$$D_{-n-1}(\mp iz) \sim \frac{e^{\pm\frac{1}{2}(n+1)\pi i} e^{\frac{1}{2}n} e^{-n(\gamma-\sinh\gamma\cosh\gamma)}}{\sqrt{(2n^{n+1}e^\gamma \sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\,\Gamma(m+\frac{1}{2})}{(ne^\gamma\sinh\gamma)^m}.$$

[The asymptotic expansion of $D_{-n-1}(\pm iz)$ on the right of the curve $\cosh 2\alpha = 2\beta\operatorname{cosec} 2\beta$ is now at once derivable from the formula

$$D_n(z) = (2\pi)^{-\frac{1}{2}}\Gamma(n+1)\left\{e^{\frac{1}{2}n\pi i} D_{-n-1}(iz) + e^{-\frac{1}{2}n\pi i} D_{-n-1}(-iz)\right\},$$

by a use of the result of § 13.]

Next, write $A_m(\gamma) \equiv A_{m,1}(\gamma) + A_{m,2}(\gamma)$, where $A_{m,1}(\gamma)$, $A_{m,2}(\gamma)$ are respectively even and odd functions of $\gamma$. From the formula just quoted, which connects $D_n(z)$, $D_{-n-1}(iz)$, $D_{-n-1}(-iz)$, we see that *between the curves whose equations are*

$$\cosh 2\alpha = 2\beta\operatorname{cosec} 2\beta \quad \text{and} \quad \cosh 2\alpha = 2(\beta \pm \pi)\operatorname{cosec}(2\beta \pm 2\pi),$$

* When $\beta = 0$, there is a second double point on the contour, but this does not affect the result (cf. footnote ‡, p. 140).

*$D_n(z)$ possesses the asymptotic expansion*

$$D_n(z) \sim \frac{\Gamma(n+1)\, e^{\frac{1}{2}n}\, e^{\pm\frac{1}{4}\pi i}}{\sqrt{(\pi n^{n+1} \sinh\gamma)}} \left[ \sum_{m=0}^{\infty} \frac{A_{m,1}(\gamma)\, e^{\frac{1}{2}m\pi i} \{\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})\} \cos\phi_m}{(n \sinh\gamma)^m} + \sum_{m=1}^{\infty} \frac{A_{m,2}(\gamma)\, e^{\frac{1}{2}(m+1)\pi i} \{\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})\} \sin\phi_m}{(n\sinh\gamma)^m} \right],$$

*where* $$i\phi_m = \tfrac{1}{2}\gamma \mp \tfrac{1}{4}\pi i - n\tau_0 + m\gamma + \tfrac{1}{2}m\pi i,$$

*provided that $-\pi < \beta < \pi$, and the upper or lower sign is taken throughout according as $\beta \gtrless 0$.*

15. *The connexion of the asymptotic expansion with Adamoff's formula.*

We shall investigate in detail the form assumed by the asymptotic expansion just obtained, in the special case when $z$ is real ($= x$) and $-2\sqrt{n} < x < 2\sqrt{n}$, $n$ being positive. To interpret $\sqrt{(\sinh\gamma)}$ we suppose that $\zeta$ starts by being greater than $+1$, and then describes a semi-circuit about the point $+1$ in the counter-clockwise direction, so that, at a point where $-1 < \zeta < 1$, $\beta$ is positive, $\alpha$ is zero, and $\arg(\sinh\gamma) = +\frac{1}{2}\pi$, $\sinh\gamma$ reducing to $i\sin\beta$; and $x = 2n^{\frac{1}{2}}\cos\beta$, so that

$$\sin\beta = \tfrac{1}{2}\sqrt{\{(4n - x^2)/n\}},$$

while $$\beta = \tfrac{1}{2}\pi - \sin^{-1}(\tfrac{1}{2}x/\sqrt{n}).$$

The upper sign is to be taken in the final asymptotic expansion of § 14, and so we have

$$D_n(x) \sim \frac{\Gamma(n+1)\, e^{\frac{1}{2}n}}{\{1-(x^2/4n)\}^{\frac{1}{4}} \sqrt{(\pi n^{n+1})}} \left[ \sum_{m=0}^{\infty} \frac{(-)^m A_{m,1}(i\beta) \{\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})\} \cos\phi'_m}{(n^2 - \frac{1}{4}nx^2)^m} + \sum_{m=1}^{\infty} \frac{(-)^m i A_{m,2}(i\beta) \{\Gamma(m+\frac{1}{2})/\Gamma\frac{1}{2}\} \sin\phi'_m}{(n^2-\frac{1}{4}nx^2)^m} \right],$$

where $$\phi'_m = \tfrac{1}{4}x\sqrt{(4n-x^2)} + (n - m - \tfrac{1}{2})\sin^{-1}(\tfrac{1}{2}x/\sqrt{n}) - \tfrac{1}{2}n\pi,$$

$$A_{0,1}(i\beta) = 1, \qquad iA_{0,2}(i\beta) = 0,$$

$$A_{1,1}(i\beta) = \tfrac{1}{8} - \tfrac{5}{24}\cot^2\beta, \qquad iA_{1,2}(i\beta) = -\tfrac{11}{24}\cot\beta,$$

... ... ... ... ... ... ... ...

The dominant term in the expansion of $D_n(x)$ is

$$\Gamma(n+1)\, e^{\frac{1}{2}n} \{1-(x^2/4n)\}^{-\frac{1}{4}} (\pi n^{n+1})^{-\frac{1}{2}} \cos\phi'_0,$$

and, when $x$ is small compared with $n^{\frac{1}{3}}$, we have

$$\phi_0' \sim x\sqrt{n} - \tfrac{1}{2}n\pi,$$

with an error $O(x^3 n^{-\frac{1}{2}})$; from this we deduce the approximate formula

$$D_n(x) \sim \Gamma(n+1)\, e^{\frac{1}{2}n} (\pi n^{n+1})^{-\frac{1}{2}} \cos(x\sqrt{n} - \tfrac{1}{2}n\pi),$$

which is in agreement with Adamoff's approximation.

16. *The asymptotic expansion of $D_n(z)$ in the region hitherto unconsidered.*

In §§ 13 and 14, the asymptotic expansion of $D_n(z)$ was obtained, (i) when $\zeta$ is on the right of the curve $\cosh 2\alpha = 2\beta \operatorname{cosec} 2\beta$, and (ii) when $\zeta$ is in the region between this curve and its image in the imaginary axis. We now consider the expansion when $\zeta$ is on the left of the curve

$$\cosh 2\alpha = 2(\beta \pm \pi) \operatorname{cosec} 2(\beta \pm \pi).$$

We take the formula

$$D_n(z) = e^{\mp n\pi i} D_n(-z) + \frac{\sqrt{(2\pi)}}{\Gamma(-n)} e^{\mp\frac{1}{2}(n+1)\pi i} D_{-n-1}(\pm iz);$$

and we have
$$-z = 2n^{\frac{1}{2}} \cosh(\alpha + i\beta \mp i\pi),$$

the upper or lower sign being taken according as $\beta$ is positive or negative.

Writing $-z$ for $z$ in the formulæ of §§ 13–14, we have

$$D_n(-z) \sim \frac{\Gamma(n+1)\, e^{\frac{1}{2}n} e^{\mp n\pi i} e^{-n(\sinh\gamma\cosh\gamma - \gamma)}}{2\sqrt{(\pi n^{n+1} e^{-\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{(-)^m A_m(\gamma)\, \Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m},$$

$$D_{-n-1}(\pm iz) \sim \frac{e^{\pm\frac{1}{2}(n+1)\pi i} e^{\frac{1}{2}n} e^{\pm n\pi i} e^{-n(\gamma - \sinh\gamma\cosh\gamma)}}{\sqrt{(2n^{n+1} e^{\mp 2\pi i} e^{\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\, \Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{\gamma}\sinh\gamma)^m}.$$

This gives the result

$$D_n(z) = \frac{\Gamma(n+1)\, e^{\frac{1}{2}n} e^{\mp 2n\pi i} e^{-n(\sinh\gamma\cosh\gamma - \gamma)}}{2\sqrt{(\pi n^{n+1} e^{-\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{(-)^m A_m(\gamma)\, \Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m}$$

$$+ \frac{\Gamma(n+1)\sin(n\pi)\, e^{\frac{1}{2}n} e^{\pm n\pi i} e^{-n(\gamma - \sinh\gamma\cosh\gamma)}}{\sqrt{(\pi n^{n+1} e^{\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\, \Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{\gamma}\sinh\gamma)^m}.$$

Putting $z = 2n^{\frac{1}{2}} \cosh(\alpha \pm i\pi)$, we get

$$D_n(-2n^{\frac{1}{2}} \cosh \alpha) = \frac{\Gamma(n+1) e^{\frac{1}{2}n} e^{\mp n\pi i} e^{-n(\sinh \alpha \cosh \alpha - \alpha)}}{2\sqrt{(\pi n^{n+1} e^{-\alpha} \sinh \alpha)}} \times \sum_{m=0}^{\infty} \frac{(-)^m A_m(\alpha) \Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\alpha} \sinh \alpha)^m}$$

$$- \frac{\Gamma(n+1) \sin(n\pi) e^{\frac{1}{2}n} e^{-n(\alpha - \sinh \alpha \cosh \alpha)}}{\sqrt{(\pi n^{n+1} e^{\alpha} \sinh \alpha)}} \sum_{m=0}^{\infty} \frac{A_m(-\alpha) \Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{\alpha} \sinh \alpha)^m};$$

This is the form of the result which is useful when the argument of the function is real and negative; and here the positive sign has to be given to each radical. [The reader will remember that, in the former equation, $\arg(e^{\gamma} \sinh \gamma)$ was $\pm 2\pi$].

It will be observed that $\operatorname{cosec}(n\pi) D_n(-x)$ is negative when $x$ is sufficiently large and positive, and $n$ is not an integer; this is in agreement with the result given by the ordinary asymptotic expansion.*

17. *Asymptotic expansions connected with $D_n(z)$ when $z \sim 2\sqrt{n}$.*

We shall now obtain the asymptotic expansion of $D_n(z)$ when $\zeta\ (= \frac{1}{2}n^{-\frac{1}{2}}z)$ is nearly equal to one of the two critical values $\pm 1$.

We write $$\tau = \tfrac{3}{2} + \log s - 2s + \tfrac{1}{2}s^2,$$

and consider $$\int e^{-n\tau + \epsilon(s-1)} s^{-1} ds,$$

where $\epsilon$ is a small number, not restricted to be real, which will be specified precisely later, and the path of integration is one of the following three portions of the contours shewn in Fig. 2: (i) from $s = 1$ to $s = +\infty$, (ii) from $s = 1$ to $s = \infty e^{\pi i}$, (iii) from $s = 1$ to $s = \infty e^{-\pi i}$.

In each case we may write the integral in the form

$$\int_0^{\infty} e^{-n\tau} \exp\{\epsilon(e^v - 1)\} \frac{dv}{d\tau} d\tau,$$

where $v = \log s$, and so

$$\tau = \tfrac{3}{2} + v - 2e^v + \tfrac{1}{2}e^{2v} = \sum_{m=0}^{\infty} c_m v^{m+3},$$

and $$c_m = (2^{m+2} - 2)/(m+3)!.$$

We may revert the series expressing $\tau$ in terms of $v$, and when we

* *Proceedings*, Ser. 2, Vol. 8, p. 397.

substitute the result in $\exp\{\epsilon(e^v-1)\}(dv/d\tau)$, we find from Lagrange's expansion that

$$\exp\{\epsilon(e^v-1)\}(dv/d\tau) = \tfrac{1}{3}\sum_{m=0}^{\infty} \tau^{\frac{1}{3}(m-2)} c_0^{-\frac{1}{3}(m+1)} B_m(\epsilon),$$

where $c_0^{-\frac{1}{3}(m+1)} B_m(\epsilon)$ is the coefficient of $v^m$ in the expansion of

$$v^{m+1}(\tfrac{3}{2}+v-2e^v+\tfrac{1}{2}e^{2v})^{-\frac{1}{3}(m+1)} \exp\{\epsilon(e^v-1)\}$$

in ascending powers of $v$.

The following formulæ give the values of the first few coefficients:

$$B_0(\epsilon) = 1, \quad B_1(\epsilon) = -\tfrac{1}{2}+\epsilon, \quad B_2(\epsilon) = \tfrac{17}{80}-\tfrac{1}{4}\epsilon+\tfrac{1}{2}\epsilon^2,$$

$$B_3(\epsilon) = \tfrac{17}{240}+\tfrac{3}{40}\epsilon+0.\epsilon^2+\tfrac{1}{6}\epsilon^3,$$

$$B_4(\epsilon) = -\tfrac{4693}{13440}+\tfrac{1}{6}\epsilon+0.\epsilon^2+\tfrac{1}{24}\epsilon^3+\tfrac{1}{24}\epsilon^4.$$

It is easy to see, from an inspection of Fig. 2, that we must define $c_0^{-\frac{1}{3}}$ for the respective contours to mean $3^{\frac{1}{3}}$, $3^{\frac{1}{3}}e^{\frac{2}{3}\pi i}$, $3^{\frac{1}{3}}e^{-\frac{2}{3}\pi i}$.

Moreover, the expansion in powers of $\tau$ defines a function which, with its continuations, is analytic except where $v$ fails to be a monogenic function of $\tau$; since $dv/d\tau = 1/(1-2e^v+e^{2v})$, the singularities of the function are the points where $e^v = 1$, *i.e.* where $\tau = 2r\pi i$; the radius of convergence of the expansion is consequently $2\pi$.

Applying the Lemma of § 11, we find that

$$\int_0^{\infty} e^{-n\tau} \exp\{\epsilon(e^v-1)\} \frac{dv}{d\tau}\, d\tau \sim \tfrac{1}{3}\sum_{m=0}^{\infty} c_0^{-\frac{1}{3}(m+1)} \frac{B_m(\epsilon)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{3}(m+1)}},$$

and a few terms of this series will give a good approximation to the integral, provided that $\epsilon = o(n^{\frac{1}{3}})$.

Now $$D_n(\omega+2\sqrt{n}) = \frac{\Gamma(n+1)e^{\frac{1}{2}n}}{2\pi i n^{\frac{1}{2}n}} \int_{-\infty}^{(0+)} e^{-n\phi(s,\omega)} s^{-1}\, ds,$$

where $$\phi(s,\omega) = (1+\tfrac{1}{2}\omega n^{-\frac{1}{2}})^2+\tfrac{1}{2}+\log s-2(1+\tfrac{1}{2}\omega n^{-\frac{1}{2}})s+\tfrac{1}{2}s^2$$
$$= \tau-n^{-1}\epsilon(s-1)+\tfrac{1}{4}\omega^2 n^{-1},$$

provided that $$\omega = \epsilon n^{-\frac{1}{2}}.$$

From this result, we get at once

$$D_n(\omega+2\sqrt{n}) \sim \frac{\Gamma(n+1)e^{\frac{1}{2}n-\frac{1}{4}\omega^2}}{\pi n^{\frac{1}{2}n}} \sum_{m=0}^{\infty} 3^{\frac{1}{3}(m-2)} \sin\{\tfrac{2}{3}\pi(m+1)\} \frac{B_m(n^{\frac{1}{2}}\omega)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{3}(m+1)}},$$

where $\omega$ must be $o(n^{-\frac{1}{6}})$, and $\frac{1}{2}\omega/\sqrt{n} = o(n^{-\frac{2}{3}})$.

Similarly, from the formulæ of § 10, we get

$$D_{-n-1}(\pm i\omega \pm 2i\sqrt{n})$$

$$\sim (2\pi n^n)^{-\frac{1}{2}} e^{\mp\frac{1}{2}(n+1)\pi i} e^{\frac{1}{2}n-\frac{1}{4}\omega^2} \sum_{m=0}^{\infty} 3^{\frac{1}{6}(m-2)} \{1-e^{\pm\frac{2}{3}(m+1)\pi i}\} \frac{B_m(n^{\frac{1}{6}}\omega)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{6}(m+1)}},$$

and hence

$$D_n(-\omega-2\sqrt{n})$$

$$\sim \frac{\Gamma(n+1)e^{\frac{1}{2}n-\frac{1}{4}\omega^2}}{\pi n^{\frac{1}{2}n}}$$

$$\times \sum_{m=0}^{\infty} 3^{\frac{1}{6}(m-2)} \{2\sin\tfrac{1}{3}(m+1)\pi\cos(\tfrac{1}{3}m\pi+\tfrac{1}{3}\pi-n\pi)\} \frac{B_m(n^{\frac{1}{6}}\omega)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{6}(m+1)}}.$$

We have now obtained all the results analogous to those obtained by Debye for Bessel functions.

18. *Series of Kapteyn's type.*

The analogue of Kapteyn's type of series of Bessel functions is

$$\Sigma a_n D_n(2\xi\sqrt{n}),$$

where $n$ takes the values $\nu$, $\nu+1$, $\nu+2$, $\nu+3$, ..., and $\nu$ is positive, but not necessarily an integer.

From the result of § 8, combined with the formula (cf. § 17)

$$D_n(2\sqrt{n}) = \Gamma(n+1)(e/n)^{\frac{1}{2}n} n^{-\frac{1}{6}} \{\lambda_0+\lambda_1 n^{-\frac{1}{3}}+\lambda_2 n^{-1}+O(n^{-\frac{4}{3}})\},$$

where $\lambda_0$, $\lambda_1$, $\lambda_2$ are independent of $n$, it follows, exactly as in the corresponding investigation for Bessel functions (*Proceedings*, Ser. 2, Vol. 16, pp. 171–173), that the convergence of the series $\Sigma a_n\Gamma(n+1)(e/n)^{\frac{1}{2}n} n^{-\frac{1}{6}}$ is the necessary and sufficient condition both for the convergence and for the uniform convergence of the series $\Sigma a_n D_n(2\xi\sqrt{n})$ throughout the range $1 \leqslant \xi \leqslant 1{\cdot}2 \ldots$. It is also a sufficient condition for the uniform convergence of the same series when $\xi \geqslant 1{\cdot}2 \ldots$, by reason of the inequalities of § 3.

19. *The stationary points of $D_n(x)$.*

Since $D_n^2(x)+D_n'^2(x)/(n+\frac{1}{2}-\frac{1}{4}x^2)$ has a positive differential coefficient, namely, $\frac{1}{2}xD_n'^2(x)/(n+\frac{1}{2}-\frac{1}{4}x^2)^2$, when $x$ is positive, it follows as in *Proceedings*, Ser. 2, Vol. 16, p. 169, that the values of $|D_n(x)|$ at its stationary points form an increasing sequence as $x$ takes in turn the critical values between 0 and $2\sqrt{n}$ going from left to right. This seems a little remarkable when it is considered that $D_n(x)$ is the product of the negative exponential $\exp(-\frac{1}{4}x^2)$ and a polynomial (when $n$ is an integer), but it is quite consistent with the extension of Adamoff's formula given in § 15.

It would be possible, but it seems unnecessary, to obtain limits for the greatest value of $|D_n(x)|$ by methods which I have previously applied to Bessel functions; the results appear to be of no special interest.

20. *The completeness of the system of functions formed by Bessel functions and parabolic cylinder functions.*

The results contained in this paper, together with those given by Debye, complete the discussion of integrals of the type

$$\int s^{-n-1} \exp\{-nAs^m - nBs^p\}\, ds,$$

where $m$ and $p$ are integers (other than zero) by means of the method of steepest descents, when the construction of the appropriate contour involves the solution of only a quadratic equation in $s$; the case $m = 0$ is reducible to the Eulerian integral for the Gamma function. The memoir published recently by Brillouin contains the simplest results when an equation of degree higher than the second has to be solved, but owing to the logarithmic term present in $\log s + As^m + Bs^p$, the analysis necessary to discuss the integral when $m$ and $p$ have not the values $+1, -1$; $1, 2$; or $-1, -2$, seems, in general, to be extremely elaborate, and it is unlikely that it leads to results of much interest.

[*Added June 15th*, 1918.—I have found that, since my paper was communicated to the Society, a paper by Perron has appeared in the *Munchener Sitzungsberichte*, 1917, pp. 191–220. This paper, which was read in May 1917, deals with the general problem of applying the method of steepest descents to functions represented by contour integrals.]