A PSEUDO-SPHERE WHOSE EQUATION IS EXPRESSIBLE IN TERMS OF ELLIPTIC FUNCTIONS

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[Received March 4th, 1915.—Read March 11th, 1915.]

As the general form of the pseudo-sphere has not yet been obtained there is some interest in the determination of simple particular cases, such as that of Serret's surface, in which the two radii of curvature are equal, the common value being $\iota\mu$. A pseudo-sphere whose equation in its most general form contains eight arbitrary constants, of which six are due merely to change of position and orientation in space, may be obtained very simply, as follows.

If we attempt to find the developable surfaces which satisfy the equation

$$s = f(z, p, q), \tag{1}$$

we shall have to determine the functions ϕ and ψ in

$$z = \lambda x + \phi(\lambda) y + \psi(\lambda),$$

$$0 = x + \phi' y + \psi',$$

so that (1) is satisfied. On eliminating x this is easily seen to lead to the relation

$$\phi' = (\phi'' y + \psi'') f\{(\phi - \lambda\phi') y + \psi - \lambda\psi', \phi, \psi\},$$

which, for general forms of f, can only be satisfied by taking $\phi = \lambda\phi'$, and therefore $\phi'' = 0$.

Thus, putting $\phi = b\lambda/a$, we obtain a particular solution of (1) in the form

$$z = \psi(ax + by),$$

where

$$ab\psi'' = f(\psi, a\psi', b\psi').$$

In the particular case of the equation $s = \mu^{-2}\sin z$, which, in conformity with the notation of Forsyth's *Differential Geometry*, § 54, we shall take in the form

$$\frac{\partial^2 \omega}{\partial p \, \partial q} = \frac{1}{\mu^2} \sin \omega,$$

the solution of this type may be expressed (i) in the form

$$\left.\begin{aligned} \mu k(-ab)^{\frac{1}{2}}\phi &= ap+bq+c, \\ \sin\tfrac{1}{2}\omega &= \operatorname{sn}\phi \quad (\text{mod } k), \end{aligned}\right\}$$

the arbitrary constants being a, b, c and k; (ii) in the rather more convenient Weierstrassian form

$$\left.\begin{aligned} u &= ap+bq+c, \\ \wp(u) &= e_2\cos^2\tfrac{1}{2}\omega+e_3\sin^2\tfrac{1}{2}\omega, \end{aligned}\right\} \tag{2}$$

with the relation $$\mu^2 ab\,(e_2-e_3) = 1. \tag{3}$$

For the sake of definiteness we may assume that the roots of $\wp'$ are real and that $e_1 > e_2 > e_3$, so that

$$\wp' = (e_2-e_3)(e_1-\wp)^{\frac{1}{2}}\sin z.$$

We proceed to the determination of the Cartesian form of the equation of the pseudo-sphere defined by equations (2) and (3).

With the notation of Forsyth, *loc. cit.*, § 37, the equations for X and x are

$$\left.\begin{aligned} 2\mu\,(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X_1-(e_1+2\wp)\,x_1+(e_2-e_3)\,x_2 &= 0, \\ x_{11}+2a\mu\,(e_1-\wp)^{\frac{1}{2}}X_1 &= 0, \\ \mu\,(e_2-e_3)x_{12} &= 2\,(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X; \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} 2\mu\,(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X_2+(e_2-e_3)\,x_1-(e_1+2\wp)\,x_2 &= 0, \\ \mu\,(e_2-e_3)\,x_{12} &= 2(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X, \\ x_{22}+2b\mu\,(e_1-\wp)^{\frac{1}{2}}X_2 &= 0. \end{aligned}\right\} \tag{5}$$

In equations (4) we regard q as constant, the dependent variables being X, x_1 and x_2. Putting

$$X = (e_1-\wp)^{\frac{1}{2}}f'(\wp),$$

so that
$$x_1 = \mu a\,\{A-2(e_1-\wp)\,f'-f\},$$
$$x_2 = \mu bf,$$

where A is an arbitrary function of q, we find

$$\wp'^2f''+2(e_1^2+e_2e_3+2e_1\wp-\wp^2)f'+\{e_1+2\wp-(e_2-e_3)\,b/a\}\,f = A\,(e_1+2\wp),$$

in which accents denote differentiation with regard to $\wp$.

The particular integral of this equation is easily seen to lead to a solution of (4) and (5) in the form

$$\left.\begin{aligned} \mu X &= 2C(e_1-\wp)^{\frac{1}{2}}, \\ x_1 &= aC\{e_1+2\wp+(e_2-e_3)\,b/a\}, \\ x_2 &= bC\{e_1+2\wp+(e_2-e_3)\,a/b\}, \end{aligned}\right\} \tag{6}$$

where C is an arbitrary constant.

To complete the determination of the pseudo-sphere we require the complementary function of the equation for f, which may be taken in the form

$$\frac{d^2f}{du^2}+\frac{4}{\wp'}(e_1-\wp)(e_1+2\wp)\frac{df}{du}+\{e_1+2\wp+(e_2-e_3)\,b/a\}\,f=0. \tag{7}$$

It is easy to verify that both integrals of (7) are uniform functions of u over the whole plane, and therefore that we may assume a solution in the form

$$f(u)=\frac{\sigma(u+\alpha)\,\sigma(u+\beta)}{\sigma^2(u)}\,e^{[\lambda-\zeta(\alpha)-\zeta(\beta)]\,u}.$$

Substituting this value we readily find, as, for instance, on pp. 469–70 of Vol. IV of Forsyth's *Theory of Differential Equations*, the relation

$$\begin{aligned} 8\wp &+\frac{4\lambda}{\wp'}(e_1-\wp)(e_1+2\wp)+\left[\lambda-\nu+\frac{2}{\wp'}(e_1-\wp)(e_1+2\wp)\right]\frac{\wp_1'-\wp'}{\wp_1-\wp} \\ &+\left[\lambda+\nu+\frac{2}{\wp'}(e_1-\wp)(e_1+2\wp)\right]\frac{\wp_2'-\wp'}{\wp_2-\wp}+3(\wp_1+\wp_2) \\ &+\lambda^2+e_1+(e_2-e_3)\,b/a\equiv 0, \end{aligned} \tag{8}$$

where the subscripts 1 and 2 denote the arguments α and β respectively, and

$$\nu=\tfrac{1}{2}(\wp_1'+\wp_2')/(\wp_1-\wp_2).$$

The apparent singularities of (8) are $u=0$, ω, ω'; these lead, after reduction, to the equations

$$\begin{gathered} \beta=\alpha+\omega, \\ -\frac{1}{\mu^2b^2}=-(e_2-e_3)\frac{a}{b}=\wp_1+\wp_2+e_1=\frac{1}{4}\frac{\wp_1'^2}{(\wp_1-e_1)^2}, \\ \lambda=-2\frac{(\wp_1-\wp_2)^2}{\wp_1'+\wp_2'}=\frac{\wp_1'}{\wp_1-e_1}-\frac{\wp_1''}{\wp_1'}. \end{gathered} \tag{9}$$

Thus, putting $$\kappa=\zeta_1+\zeta_2-\lambda,$$

we have $$f(u)=H\phi+K\psi,$$

where
$$\left.\begin{aligned}\phi &= \frac{\sigma(u+a)\,\sigma(u+a+\omega)}{\sigma(a)\,\sigma(a+\omega)\,\sigma^2(u)}\,e^{-\kappa u},\\ \psi &= \frac{\sigma(u-a)\,\sigma(u-a-\omega)}{\sigma(a)\,\sigma(a+\omega)\,\sigma^2(u)}\,e^{\kappa u},\end{aligned}\right\} \tag{10}$$

and H and K are functions of q, which by direct substitution in equations (5) are readily found to be

$$H = A e^{-b\lambda q}, \quad K = B e^{b\lambda q},$$

where A and B are arbitrary constants.

Hence the general solution of (4) and (5) is given by

$$\begin{aligned} f_1 &= A_1 e^{-b\lambda q}\phi + B_1 e^{b\lambda q}\psi + C_1(2\wp-\wp_1-\wp_2),\\ x_1 &= -a\left\{\frac{2}{\wp'}(e_1-\wp)\frac{\partial f_1}{\partial u}+f_1+C_1\lambda^2\right\},\\ x_2 &= bf_1,\end{aligned}$$

where use has been made of equations (6); while y and z are given by precisely similar equations with different constants. Let subscripts 2 give the value of y and subscripts 3 the value of z.

The equations connecting the nine constants $A_1, \ldots, C_3$ are most easily determined from the relation

$$x_2^2+y_2^2+z_2^2 = 1,$$

which leads, after rather a long reduction, to

$$\begin{aligned} A_1, A_2, A_3 &= \frac{\mu k}{\lambda}\operatorname{cosec}\frac{\beta-\gamma}{2}(\cos\beta, \sin\beta, \iota),\\ B_1, B_2, B_3 &= \frac{\mu}{\lambda k}\operatorname{cosec}\frac{\beta-\gamma}{2}(\cos\gamma, \sin\gamma, \iota),\\ C_1, C_2, C_3 &= \frac{\iota\mu}{\lambda}\operatorname{cosec}\frac{\beta-\gamma}{2}\left(\cos\frac{\beta+\gamma}{2}, \sin\frac{\beta+\gamma}{2}, \iota\cos\frac{\beta-\gamma}{2}\right),\end{aligned}$$

where k, β and γ are arbitrary.

This is, by Bonnet's theorem, the complete tale of relations satisfied by $A_1, \ldots, C_3$.

From the value of ϕ in (10) we easily deduce

$$\frac{2(e_1-\wp)}{\wp'}\frac{d\phi}{du}+\phi = \frac{e_1-\wp_1}{\wp'_1}\,\frac{\wp'(u+a)}{\wp(u+a)-e_1}\,\phi,$$

and the integral of the right-hand side is readily found to be

$$-\frac{\sigma(u+2a+\omega)}{\sigma(2a+\omega)\,\sigma(u)}\,e^{-\kappa u}.$$

Hence, making use of (9), and putting

$$v = bq,$$

$$\left.\begin{aligned} P &= \frac{\sigma(u+2a+\omega)}{\sigma(2a+\omega)\,\sigma(u)}\,e^{-\kappa u-\lambda v}, \\ Q &= \frac{\sigma(u-2a-\omega)}{\sigma(2a+\omega)\,\sigma(u)}\,e^{\kappa u+\lambda v}, \\ -R &= 2\zeta(u)+\left\{\frac{(e_2-e_3)^2}{\wp_1+\wp_2+e_1}-e_1\right\}(u-v)+(\wp_1+\wp_2)\,v, \end{aligned}\right\} \tag{11}$$

we find the most general form of the pseudo-sphere given by (2), namely,

$$\frac{\lambda}{\mu}(x-l)\sin\frac{\beta-\gamma}{2} = kP\cos\beta+\frac{1}{k}\,Q\cos\gamma+\iota R\cos\frac{\beta+\gamma}{2},$$

$$\frac{\lambda}{\mu}(y-m)\sin\frac{\beta-\gamma}{2} = kP\sin\beta+\frac{1}{k}\,Q\sin\gamma+\iota R\sin\frac{\beta+\gamma}{2},$$

$$\frac{\lambda}{\mu}(z-n)\sin\frac{\beta-\gamma}{2} = \iota kP \qquad +\frac{\iota}{k}\,Q \qquad -R\cos\frac{\beta-\gamma}{2},$$

where a, β, γ, k, l, m, n and the periods ω and ω' are arbitrary constants.

The radii of curvature are

$$\mu\left\{\frac{e_2-\wp(u)}{\wp(u)-e_3}\right\}^{\frac{1}{2}}, \qquad -\mu\left\{\frac{\wp(u)-e_3}{e_2-\wp(u)}\right\}^{\frac{1}{2}}.$$

By change of origin and rotation of the axes the pseudo-sphere just obtained can be reduced to the form

$$\frac{\lambda}{2\mu}(x-\iota y) = -P,$$

$$\frac{\lambda}{2\mu}(x+\iota y) = \quad Q,$$

$$\frac{\lambda}{\mu}\,z \quad = \quad \iota R.$$

The explicit form of the equation can be obtained by actual elimination of the parameters u and v from equations (11).

Using cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we have
$$\lambda^2 r^2/4\mu^2 = -PQ = \wp(u) - \wp(2a+\omega)$$
$$= \wp(u) - \wp(\beta), \text{ say,} \qquad (12)$$

giving u as a function of r; and

$$e^{2\iota\theta} = -\frac{Q}{P} = \frac{\sigma(\beta-u)}{\sigma(\beta+u)} e^{2\kappa u + 2\lambda v},$$

giving v as a function of r and θ.

After some straightforward reductions the explicit equation of the pseudo-sphere is found to be

$$\frac{\iota z}{c} = 2\frac{\wp(\beta)-e_1}{\wp'(\beta)} \{\zeta(u) - [\wp(\beta)+e_1]\, u\} - \zeta(\beta)\, u + \tfrac{1}{2}\log\frac{\sigma(\beta+u)}{\sigma(\beta-u)} + \iota\theta, \quad (13)$$

where
$$c = \mu \left\{ \frac{\wp(\beta)-e_2}{e_1-e_2}\, \frac{\wp(\beta)-e_3}{e_1-e_3} \right\}^{\frac{1}{2}},$$

β is an arbitrary constant, and u is given by (12), *i.e.*,

$$r^2 = \mu^2 \frac{\wp(\beta)-e_1}{(e_1-e_2)(e_1-e_3)} \{\wp(u) - \wp(\beta)\}. \qquad (14)$$

Dr. Bromwich points out that, as equation (13) is of the form

$$z = f(r) + c\theta,$$

the pseudo-sphere might have been obtained with considerably less analysis—once the form is known—by means of the equation

$$1 + \frac{c^2}{r^2} + \left(\frac{df}{dr}\right)^2 = \left(\frac{r^2}{\mu^2} + A\right)^{-1}, \qquad (15)$$

in which c and A are arbitrary constants.

Substituting the actual value of z given by (13), we readily verify that equation (15) is satisfied if

$$A = \frac{\{\wp(\beta)-e_1\}^2}{(e_1-e_2)(e_1-e_3)}.$$

Thus equation (13) must be regarded as containing only two independent arbitrary constants.

Owing to the occurrence of imaginary quantities in equation (13) it is perhaps of interest to note that the surface may be real. A particular case, found by putting $e_1 = e_2$, is that of the pseudo-sphere

$$z = \sqrt{(a^2-r^2)} - a\log[\{a+\sqrt{(a^2-r^2)}\}/r] + c\theta,$$

in which $$\mu^2 = a^2+c^2,$$

the constants a and c being otherwise arbitrary.

Added May 19th, 1915.

It is easy to verify that a surface of constant *mean* curvature $2/a$, of the same form as the surface in (13), is given by the equation

$$\frac{z}{a} = \zeta(u)+\left\{\frac{1-(e_2-e_3)^2}{4}+e_1\right\}u-\{\wp(a)-e_1\}^{\frac{1}{2}}\left\{\tfrac{1}{2}\log\frac{\sigma(u+a)}{\sigma(u-a)}-\zeta(a)u+\iota\theta\right\},$$

where
$$\wp(a) = \tfrac{1}{4}(1-e_1)^2-e_2e_3,$$
$$r^2 = a^2\{\wp(a)-\wp(u)\}.$$

Note by Dr. T. J. I'A. Bromwich.

When the equation to a surface is written in terms of cylindrical coordinates z, r, θ, in the form

$$z = F(r, \theta),$$

it is known that the measure of curvature (Gauss's) is equal to

$$\{r^2F_{11}(F_{22}+rF_1)-(rF_{12}-F_2)^2\}/(r^2+r^2F_1^2+F_2^2)^2,$$

where the suffixes 1 and 2 indicate partial differentiation with respect to r and θ, respectively.

If we take the surfaces of the type found by Dr. Wilton

$$z = f(r)+c\theta,$$

where c is a constant, the condition for a pseudo-sphere becomes

$$-\frac{1}{\mu^2} = \frac{r^3f_1f_{11}-c^2}{(r^2+r^2f_1^2+c^2)^2}.$$

Then, writing $q = 1+f_1^2+c^2/r^2$, we have

$$\frac{dq}{dr} = 2\left(f_1f_{11}-\frac{c^2}{r^3}\right),$$

so that the condition for the pseudo-sphere becomes

$$-\frac{1}{\mu^2} = \frac{1}{2rq^2}\frac{dq}{dr},$$

leading to
$$\frac{r^2}{\mu^2}+A = \frac{1}{q},$$

where A is a constant of integration. This is the equation quoted in (15) of Dr. Wilton's paper; it leads at once to the formula

$$f(r) = \int \frac{dr}{r} \left\{ \frac{\mu^2 r^2 - (r^2+c^2)(r^2+A\mu^2)}{r^2+A\mu^2} \right\}^{\frac{1}{2}};$$

and thus $f(r)$ can be evaluated, by using Dr. Wilton's substitution (14), in the form given in equation (13).

The special case $A = 0$, gives

$$f(r) = \int \frac{dr}{r} (\mu^2 - c^2 - r^2)^{\frac{1}{2}},$$

reducing to the elementary integral

$$f(r) = \sqrt{(a^2 - r^2)} - a \log [\{a + \sqrt{(a^2-r^2)}\}/r],$$

where $$a^2 = \mu^2 - c^2.$$