where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, ..., $\lambda_{\delta-1} \nless 1$; (ii.) covariant types which have a factor $(ab)^{\lambda}(bc)^{n-\lambda}(ca)^{r}$, where $\lambda \nless \frac{1}{2}n$, $r \nless 2n-3\lambda$; (3) products of covariants of lower degree. This theorem at once gives a system of types in terms of which all irreducible types of grade not exceeding $\frac{1}{2}n$ may be expressed.

8. It appears practically certain that the theorem for perpetuant types is exact, *i.e.*, that the covariant types for binary forms of infinite order $(a_1 a_2)^{\lambda_1}(a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_\delta}$, where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, ..., $\lambda_\delta \nless 1$, and the order of the letters is fixed beforehand, are both independent and irreducible (this system being equivalent to that used by Grace). If this is the case, covariants of this form must be independent and irreducible for quantics of finite order. It does not follow, however, that they cannot be expressed in terms of covariants of higher grade, or else in terms of covariants belonging to the second class. In fact, as it is easy to verify, if $\lambda_1+\lambda_r \geqslant n$, such a covariant type of a system of binary n-ics can be expressed in terms of members of the second class, and of covariants $(a_1 a_2)^{\lambda'_1}(a_2 a_3)^{\lambda'_2} \dots (a_{\delta-1} a_\delta)^{\lambda'_{\delta-1}}$, where one or more of the differences $\lambda'_1-\lambda_1$, $\lambda'_2-\lambda_2$, ..., $\lambda'_{r-1}-\lambda_{r-1}$ is positive, and the rest are zero.

NOTE ON THE FOREGOING PAPER.

By J. H. GRACE.

Consider any number of forms of order not exceeding n.

Following Jordan, we add to the system every covariant of the second degree whose order does not exceed n.

Thus, if a_x^p and b_x^q be two forms, we add the covariant $(ab)^{\lambda} a_x^{p-\lambda} b_x^{q-\lambda}$ to the system whenever $p+q-2\lambda \leqslant n$, and therefore it is always added to the original system when $\lambda \geqslant \frac{1}{2}p$ or $\frac{1}{2}q$.

Hence by the elementary theory of transvectants, if $\lambda \geqslant \frac{1}{2}n$, a covariant involving the factor $(ab)^{\lambda}$ can be expressed as an aggregate of covariants in each of which the letters a and b are replaced by a single symbol also belonging to a form of order not greater than n.

It follows at once that in seeking for the covariant of highest order such a covariant can be neglected because the same order would occur for a lower degree.

The covariants of Class II. in the preceding paper can therefore be neglected because (see § 1) either λ or $n_k-\lambda$ must be at least equal to $\frac{1}{2}n_k$.

Hence a covariant of the highest order will appear in the first class.

Now the highest order for degree ι is $n\iota-2^{\iota}+2$, corresponding to the form $(a_1 a_2)^{2^{\iota-2}}(a_2 a_3)^{2^{\iota-3}} \dots (a_{\iota-1} a_\iota)$.

We therefore have the following rule for finding the maximum order:—

Choose the greatest of the integers n, $2n-2$, $3n-6$, $4n-14$, $5n-30$, $6n-62$, ...*

Since the form from which this order arises is the fundamental perpetuant of degree ι, and this certainly irreducible (although not rigorously proved to be so), it follows that the order given above is a true maximum. It is actually attained by the covariants of a single quantic of order n, and it is not surpassed by the covariants of any number of forms whose orders do not exceed n. The value of ι which makes $n\iota-2^{\iota}+2$ a maximum is the integer next greater than $\log_2 n$.

9. It will be seen that, if $2^{j+1} > n > 2^j$, the maximum order of a covariant of a quantic or quantics of order n is $(j+1)n-2^{j+1}+2$. Thus for values of n from 1 to 100 we have the values of the maximum order as follows:—

n	Max. order.	n	Max. order.	n	Max. order.	n	Max. order.	n	Max. order.
1	1	21	75	41	184	61	304	81	441
2	2	22	80	42	190	62	310	82	448
3	4	23	85	43	196	63	316	83	455
4	6	24	90	44	202	64	322	84	462
5	9	25	95	45	208	65	329	85	469
6	12	26	100	46	214	66	336	86	476
7	15	27	105	47	220	67	343	87	483
8	18	28	110	48	226	68	350	88	490
9	22	29	115	49	232	69	357	89	497
10	26	30	120	50	238	70	364	90	504
11	30	31	125	51	244	71	371	91	511
12	34	32	130	52	250	72	378	92	518
13	38	33	136	53	256	73	385	93	525
14	42	34	142	54	262	74	392	94	532
15	46	35	148	55	268	75	399	95	539
16	50	36	154	56	274	76	406	96	546
17	55	37	160	57	280	77	413	97	553
18	60	38	166	58	286	78	420	98	560
19	65	39	172	59	292	79	427	99	567
20	70	40	178	60	298	80	434	100	574

* See a note to a preceding paper of my own, *supra*, p. 151.