is a black interval of G_{m_1} or else there is a black interval of G_{m_1} whose end points lie in (P, P_1) and (Q, Q_1) respectively.

Doing this with each black interval $\geqslant \epsilon$ of Γ, we determine an integer m_2 such that the sum of those black intervals of G_{m_2} which are led up to by all the black intervals of Γ which are $\geqslant \epsilon$ differs from the sum of the latter black intervals by less than σ.

As in § 12, *loc. cit.*, however, G_{m_2} may have other black intervals $\geqslant \epsilon$; but these can be disposed of and the proof completed precisely as was done there for the closed G.

4. Thus, if I and I_n be the contents of Γ and G_n respectively, and $R(\epsilon)$ and $R_n(\epsilon)$ be the sums of those black intervals of Γ and G_n respectively which are $< \epsilon$, we have

$$I - I_n - R_n(\epsilon) + R(\epsilon) < \sigma.$$

Now we can choose ϵ so as to make $R(\epsilon)$ as small as we please, so that *it is evidently necessary and sufficient for the equality of I and $\lim I_n$ ($n = \infty$) that we may be able to choose ϵ so that, for all integers n greater than a certain integer, $R_n(\epsilon)$ may be less than any assigned small quantity.*

Summation of a certain Series. *By* A. C. Dixon.

Received October 10th, 1902. Read November 13th, 1902.*

The object of the present note is to find the sum of the infinite series

$$1 + \frac{\alpha\beta\gamma}{\delta\epsilon} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)\,\gamma(\gamma+1)}{2!\,\delta(\delta+1)\,\epsilon(\epsilon+1)} + \dots$$

in the case when $\beta + \delta = \gamma + \epsilon = \alpha + 1.$

The condition for convergency will be supposed satisfied; that is, the real part of $\delta + \epsilon - \alpha - \beta - \gamma$ will be taken to be positive.

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* The original MS. was lost in transit. The paper has been rewritten by the author, January, 1903.

Let $F(\alpha, \beta, \gamma, x)$ and $F(\alpha, \beta, \gamma, \delta, \epsilon, x)$ stand for the two series

$$1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)}x^2+\dots,$$

$$1+\frac{\alpha\beta\gamma}{\delta\epsilon}x+\frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)}{2!\,\delta(\delta+1)\epsilon(\epsilon+1)}x^2+\dots.$$

Let $\int_{(a,b)}$ denote integration round a closed path consisting of four loops, each of which begins and ends at a certain point c on the straight line between a and b, and which pass respectively round b positively, a positively, b negatively, a negatively; the subject of integration will contain powers of such factors as x, $1-x$ which are initially real and positive; its initial value will be fixed by taking the logarithms of these factors to have their real values.

Let $\int_{a(b)}$ denote integration along a loop starting and ending at a, and passing positively round b, the initial value of the subject being fixed as before. Thus we have, when r is positive,

$$\begin{aligned}\int_{(0,1)} x^{rm-1}(1-x^r)^{n-1}\,dx &= -\int_{(1,0)} x^{rm-1}(1-x^r)^{n-1}\,dx\\ &= \frac{1}{r}(1-e^{2rm\iota\pi})(1-e^{2n\iota\pi})\,B(m,n)\\ &= -\frac{4}{r}\,e^{(rm+n)\iota\pi}\sin rm\pi\,\sin n\pi\,B(m,n),\end{aligned}$$

and, if the real part of m is also positive,

$$\begin{aligned}\int_{0(1)} x^{rm-1}(1-x^r)^{n-1}\,dx &= \frac{1}{r}(1-e^{2n\iota\pi})\,B(m,n)\\ &= -\frac{2\iota}{r}e^{n\iota\pi}\sin n\pi\,B(m,n).\end{aligned}$$

Now
$$\begin{aligned}F(\alpha,\beta,\gamma,\delta,\epsilon,t)\int_{(0,1)} z^{\gamma-1}(1-z)^{\epsilon-\gamma-1}\,dz\\ = \int_{(0,1)} F(\alpha,\beta,\delta,tz)\,z^{\gamma-1}(1-z)^{\epsilon-\gamma-1}\,dz\end{aligned}$$

and
$$F(\alpha,\alpha-\delta+1,\delta,x)=(1+x)^{-\alpha}F\left(\frac{\alpha}{2},\frac{\alpha+1}{2},\delta,\frac{4x}{(1+x)^2}\right).$$

Hence

$$F(a, a-\delta+1, a-\epsilon+1, \delta, \epsilon, t)\int_{(0,1)} z^{a-\epsilon}(1-z)^{2\epsilon-a-2}\,dz$$

$$=\int_{(0,1)} F\left\{\frac{a}{2},\ \frac{a+1}{2},\ \delta,\ \frac{4tz}{(1+tz)^2}\right\}(1+tz)^{-a}z^{a-\epsilon}(1-z)^{2\epsilon-a-2}\,dz$$

$$=(1+t)^{2\epsilon-a-2}\,2^{1-a}t^{1-\epsilon}\int_{[(1-t)/(1+t),\,1]} F\left(\frac{a}{2},\ \frac{a+1}{2},\ \delta,\ 1-u^2\right)$$
$$\times(1-u^2)^{a-\epsilon}\left(u-\frac{1-t}{1+t}\right)^{2\epsilon-a-2}du,$$

by the substitution $u=(1-tz)/(1+tz)$.

Now

$$F\left(\frac{a}{2},\ \frac{a+1}{2},\ \delta,\ 1-u^2\right)$$

$$=\frac{\Gamma\delta\,\Gamma(\delta-a-\frac{1}{2})}{\Gamma\left(\delta-\frac{a}{2}\right)\Gamma\left(\delta-\frac{a+1}{2}\right)}F\left(\frac{a}{2},\ \frac{a+1}{2},\ a-\delta+\tfrac{3}{2},\ u^2\right)$$

$$+\frac{\Gamma\delta\,\Gamma(a+\frac{1}{2}-\delta)}{\Gamma\frac{a}{2}\Gamma\frac{a+1}{2}}u^{2\delta-2a-1}F\left(\delta-\frac{a}{2},\ \delta-\frac{a+1}{2},\ \delta-a+\tfrac{1}{2},\ u^2\right).$$

Let this substitution be made in the integral, and consider the two parts separately. In the first the subject of integration has no singularity at 0, and thus the loop containing $\frac{1-t}{1+t}$ may be taken to contain 0 as well, and we may put $t=1$ at once. The expression to be calculated is therefore

$$\int_{(0,1)} F\left(\frac{a}{2},\ \frac{a+1}{2},\ a-\delta+\tfrac{3}{2},\ u^2\right)(1-u^2)^{a-\epsilon}u^{2\epsilon-a-2}\,du$$

or
$$\int_{(0,1)}\int_{(0,1)} x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}a-\delta+\frac{1}{2}}(1-xu^2)^{-\frac{1}{2}(a+1)}(1-u^2)^{a-\epsilon}u^{2\epsilon-a-2}\,dx\,du$$
$$\div\int_{(0,1)} x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}a-\delta+\frac{1}{2}}\,dx.$$

In the double integral change the order of integration and use the substitution $u^2=y^2/(1-x+xy^2)$.

The expression thus becomes

$$\int_{(0,1)} \int_{(0,1)} (1-y^2)^{a-\epsilon} y^{2\epsilon-a-2} (1-x)^{a-\delta-\epsilon+1} x^{\frac{1}{2}a-1}\, dy\, dx$$
$$\div \int_{(0,1)} x^{\frac{1}{2}a-1} (1-x)^{\frac{1}{2}(a+1)-\delta}\, dx,$$

that is, $-2e^{\epsilon\iota\pi} \sin(a-\epsilon)\pi \sin(2\epsilon-a)\pi\, B\left(a-\epsilon+1,\ \epsilon-\frac{a+1}{2}\right)$

$$\times -4e^{(\frac{3}{2}a-\delta-\epsilon)\iota\pi} \sin(a-\delta-\epsilon)\pi \sin\frac{a\pi}{2} B\left(\frac{a}{2},\ a-\delta-\epsilon+2\right)$$

$$\div -4e^{(a-\delta-\frac{1}{2})\iota\pi} \sin\left(\frac{a+3}{2}-\delta\right)\pi \sin\frac{a\pi}{2} B\left(\frac{a}{2},\ \frac{a+3}{2}-\delta\right).$$

The coefficient of this in $F(a, a-\delta+1, a-\epsilon+1, \delta, \epsilon, 1)$ is

$$2^{2\epsilon-2a-1} \frac{\Gamma\delta\, \Gamma(\delta-a-\frac{1}{2})}{\Gamma\left(\delta-\frac{a}{2}\right) \Gamma\left(\delta-\frac{a+1}{2}\right)}$$
$$\div -4e^{a\iota\pi} \sin(a-\epsilon)\pi \sin(2\epsilon-a)\pi\, B(a-\epsilon+1, 2\epsilon-a-1).$$

Hence the value of the term as a whole is, after reduction,

$$-2^{2\epsilon-2a-2} e^{[\frac{1}{2}(a+1)-\epsilon]\iota\pi}\, \Gamma\delta\, \Gamma\epsilon\, \Gamma(\delta-a+\tfrac{1}{2})\, \Gamma(a-\delta+\tfrac{1}{2})\, \Gamma\left(\epsilon-\frac{a+1}{2}\right)$$
$$\div \Gamma(\delta+\epsilon-a-1)\, \Gamma\left(\delta-\frac{a}{2}\right) \Gamma\frac{a+1}{2}\, \Gamma(\tfrac{3}{2}a-\delta-\epsilon+2)\, \Gamma(2\epsilon-a-1).$$

To find the other term we must evaluate

$$\lim_{\kappa=0} \int_{(\kappa,1)} F\left(\delta-\frac{a}{2},\ \delta-\frac{a+1}{2},\ \delta-a+\tfrac{1}{2},\ u^2\right)$$
$$\times u^{2\delta-2a-1} (1-u^2)^{a-\epsilon} (u-\kappa)^{2\epsilon-a-2}\, du,$$

where κ has been written for $(1-t)/(1+t)$.

Take κ to be small and positive and let the loops start from the point $\sqrt{\kappa}$. Then on the loops round 1 we have $|u| \geqslant \sqrt{\kappa}$, and thus $(u-\kappa)/u = 1$ in the limit, so that we may put $\kappa = 0$ at once in the subject for this part of the path. The loop round κ may be taken to be a circle having the points $\sqrt{\kappa}$ and $\frac{1}{2}\kappa$ as the ends of a diameter. On this circle we have $\left|\frac{u-\kappa}{u}\right| < 1$ and $> 1-\sqrt{\kappa}$; the integral is then a quantity of the same order of magnitude as $\int |u^{2\delta+2\epsilon-3a-3}\, du|$ taken round the same circle; that is, it vanishes in the limit if the real part of $2\delta+2\epsilon-3a-2$ is positive. Hence, under this condition, which we have already supposed fulfilled, we may replace the integral by

$$(1-e^{(2\epsilon-a)2\iota\pi})\int_{0(1)} F\left(\delta-\frac{a}{2},\ \delta-\frac{a+1}{2},\ \delta-a+\tfrac{1}{2},\ u^2\right)$$
$$\times(1-u^2)^{a-\epsilon}\,u^{2\delta+2\epsilon-3a-3}\,du,$$

since the subject of integration at the end of the second loop round 1 has its original value multiplied by $e^{(2\epsilon-a)2\iota\pi}$.

The value of this integral may be found by means of the same substitutions as before, and the second term of the expression $F(a,\ a-\delta+1,\ a-\epsilon+1,\ \delta,\ \epsilon,\ 1)$ is the product of the coefficient

$$2^{2\epsilon-3a-1}\frac{\Gamma\delta\,\Gamma(a+\frac{1}{2}-\delta)}{\Gamma\frac{a}{2}\,\Gamma\frac{a+1}{2}}$$
$$\div -4e^{a\iota\pi}\sin(a-\epsilon)\,\pi\sin(2\epsilon-a)\,\pi B\,(a-\epsilon+1,\ 2\epsilon-a-1),$$

by the value of the integral, namely,

$$(1-e^{(2\epsilon-a)2\iota\pi})\times-\iota e^{(a-\epsilon+1)\iota\pi}\sin(a-\epsilon+1)\,\pi B\left(a-\epsilon+1,\ \delta+\epsilon-\frac{3a}{2}-1\right)$$
$$\times -4e^{(\frac{1}{2}a-\epsilon-\frac{1}{2})\iota\pi}\sin\left(\delta-\frac{a+1}{2}\right)\pi\sin(a-\delta-\epsilon+2)\,\pi$$
$$\times B\left(\delta-\frac{a+1}{2},\ a-\delta-\epsilon+2\right)$$
$$\div -4e^{(\delta-a-\frac{3}{2})\iota\pi}\sin\left(\delta-\frac{a+1}{2}\right)\pi\sin\left(1-\frac{a}{2}\right)\pi$$
$$\times B\left(\delta-\frac{a+1}{2},\ 1-\frac{a}{2}\right).$$

After reduction this product becomes

$$-2^{2\epsilon-2a-2}e^{(\frac{3}{2}a-\delta-\epsilon)\iota\pi}\,\Gamma\delta\,\Gamma\epsilon\,\Gamma(\delta-a+\tfrac{1}{2})\,\Gamma(a-\delta+\tfrac{1}{2})\,\Gamma\left(\delta+\epsilon-\frac{3a}{2}-1\right)$$
$$\div\Gamma(\delta+\epsilon-a-1)\,\Gamma\left(\delta-\frac{a}{2}\right)\Gamma\frac{a+1}{2}\,\Gamma\left(\frac{a+3}{2}-\epsilon\right)\Gamma(2\epsilon-a-1),$$

and, adding this to the former result, we have as the sum of the original series

$$2^{2\epsilon-2a-2}\frac{\Gamma\delta\Gamma\epsilon\Gamma\left(\delta+\epsilon-\frac{3a}{2}-1\right)\Gamma\left(\epsilon-\frac{a+1}{2}\right)}{\Gamma(\delta+\epsilon-a-1)\Gamma\left(\delta-\frac{a}{2}\right)\Gamma\frac{a+1}{2}\Gamma(2\epsilon-a-1)}\frac{\pi}{\sin(\delta-a+\frac{1}{2})\pi}$$

$$\times\left[-\frac{e^{[\frac{1}{2}(a+1)-\epsilon]i\pi}}{\Gamma\left(\delta+\epsilon-\frac{3a}{2}-1\right)\Gamma\left(\frac{3a}{2}-\delta-\epsilon+2\right)}\right.$$

$$\left.-\frac{e^{(\frac{3}{2}a-\delta-\epsilon)i\pi}}{\Gamma\left(\epsilon-\frac{a+1}{2}\right)\Gamma\left(\frac{a+3}{2}-\epsilon\right)}\right]$$

The expression in square brackets

$$=-\frac{1}{\pi}e^{[\frac{1}{2}(a+1)-\epsilon]i\pi}\sin\left(\frac{3a}{2}-\delta-\epsilon\right)\pi+\frac{1}{\pi}e^{(\frac{3}{2}a-\delta-\epsilon)i\pi}\sin\left(\frac{a+1}{2}-\epsilon\right)\pi$$

$$=\frac{1}{\pi}\sin(\delta-a+\tfrac{1}{2})\pi.$$

We also have

$$\Gamma(2\epsilon-a-1)=\Gamma\left(\epsilon-\frac{a}{2}\right)\Gamma\left(\epsilon-\frac{a+1}{2}\right)2^{2\epsilon-a-2}\pi^{-\frac{1}{2}}.$$

Hence, finally,

$$F(a,\ a-\delta+1,\ a-\epsilon+1,\ \delta,\ \epsilon,\ 1)$$

$$=2^{-a}\pi^{\frac{1}{2}}\frac{\Gamma\delta\Gamma\epsilon\Gamma(\delta+\epsilon-\frac{3}{2}a-1)}{\Gamma\left(\delta-\frac{a}{2}\right)\Gamma\left(\epsilon-\frac{a}{2}\right)\Gamma\frac{a+1}{2}\Gamma(\delta+\epsilon-a-1)},$$

if the real part of $\delta+\epsilon-\frac{3}{2}a-1$ is positive.

If in this we write $\delta=\epsilon=1$, we arrive at Prof. Morley's result (*Proceedings*, Vol. xxxiv., p. 401). The present investigation was suggested by the reading of Prof. Morley's paper. I learn from Prof. Morley that he had hoped to sum the series of this paper by his method.

Thursday, December 11th, 1902.

Prof. H. LAMB, F.R.S., President, in the Chair.

Eighteen members present.

Mr. J. H. Grace was admitted into the Society.

The Auditor (Mr. J. H. Grace) having made his report, the President moved that the Treasurer's report (read at the November meeting) be adopted, and that the thanks of the Society be given to the Treasurer and the Auditor. The motion was seconded by Mr. Sheppard, and carried *nem. con.*

The following papers were communicated by their authors:—

Dr. H. F. Baker: (1) On the Calculation of the Finite Equations of a Continuous Group. (2) On the Integration of Linear Differential Equations. (3) On some cases of Matrices with Linear Invariant Factors.

Mr. G. H. Hardy: The expression of the Double Zeta and Gamma Functions in terms of Elliptic Functions.

Prof. M. J. M. Hill: The Continuation of the Power Series for arc sin x.

Mr. E. T. Whittaker: The Functions associated with the Parabolic Cylinder in Harmonic Analysis.

Lieut.-Col. Cunningham took the Chair while the President gave an account of his recent researches on "Wave Motion in Two Dimensions."

The following papers were communicated from the Chair:—

Prof. L. E. Dickson: (1) The Abstract Group Simply Isomorphic with the Group of Linear Fractional Transformations in a Galois Field. (2) Generational Relations of an Abstract Simple Group of Order 4080.

Mr. H. M. Macdonald: Some Applications of Fourier's Theorem.

Rev. F. H. Jackson: Series connected with the Enumeration of Partitions.

Mr. W. H. Young: Sets of Intervals, Part II., Overlapping Intervals.

Mr. J. H. Grace: Perpetuants.

The following presents were made to the Library :—

"Educational Times," Vol. LV., No. 500; 1902.

"Indian Engineering," Vol. XXXII., Nos. 17–20; 1902.

"L'Enseignement Mathématique," Année IV., No. 6; Paris, 1902.

"Société des Naturalistes de Varsovie, Comptes Rendus," 1901.

Martin, Emilie Norton.—"On the Imprimitive Substitution Groups of Degree 15," 4to; Baltimore, 1901.

Mittag-Leffler, G.—"Sur la représentation analytique d'une branche uniforme d'une fonction monogène," 4to; 1902.

The following exchanges were received :—

"Bulletin of the American Mathematical Society," Vol. IX., No. 3; New York, 1902.

"Proceedings of the American Philosophical Society," Vol. XLI., No. 170; Philadelphia, 1902.

"Jornal de Sciencias Mathematicas," Vol. XV., No. 1; Coimbra, 1902.

"Beiblätter zu den Annalen der Physik," Bd. XXVI., Heft 11; Leipzig, 1902.

"Periodico di Matematica," Anno XVIII., Fasc. 3; Livorno, 1902.

"Supplemento al Periodico di Matematica," Anno VI., Fasc. 1; Livorno, 1902.

"Proceedings of the Physical Society," Vol. XVIII., Pt. 3; London, 1902.

"Bulletin des Sciences Mathématiques," Tome XXVI., Oct., Nov., 1902; Paris.

"Reale Accademia dei Lincei—Rendiconti," Vol. XI., Sem. 2, Fasc. 9, 10; Roma, 1902.

"Prace Matematyczno-Fizyczne," Tome XIII.; Warsaw, 1902.