36. There are two more questions which we must discuss in order to complete this series of investigations. In the first place, when two singular curves intersect within the field of integration, the formula

$$(1) \quad \int_a^A P \int_b^B = \int_b^B P \int_a^A$$

generally ceases to be true. If, for instance,

$$f(x, y) = \frac{\psi(x, y)}{\lambda(x, y)\,\mu(x, y)},$$

where ψ is a function with continuous derivates, and $\lambda = 0$, $\mu = 0$ are two curves which satisfy the conditions of 7, except that they intersect simply at the one point (α, β), the difference between the two sides of (1) will be

$$\frac{2\pi^2\psi(\alpha, \beta)}{\dfrac{\partial(\lambda, \mu)}{\partial(\alpha, \beta)}}.$$

We shall have to discuss this case, and some other similar cases in which the corresponding "correction" or "residue" can be found.

In the second place, we must attempt to extend the theorems of the latter part of this paper to the case in which not only are the limits infinite, but the singular curves infinite in number.

Types of Perpetuants. *By* J. H. Grace.

Received and read June 12th, 1902.

1. I propose to apply the symbolical method of Aronhold directly to the discovery of the irreducible system of covariants of an indefinite number of binary forms of infinite order.

Suppose the forms are a_x^n, b_x^n, c_x^n, ...,

when n is indefinitely large; then the problem is to evolve a system of forms of the type

$$(ab)^\lambda (bc)^\mu (cd)^\nu \ldots a_x^p b_x^q \ldots$$

in terms of which all others can be expressed.

Since a covariant such as that written above is completely determined by its determinantal factors, we need specify these alone, and remark that the reduction of forms is simplified because the presence of the indefinitely large number of factors a_n renders possible the free application of the fundamental identities

$$(bc)\, a_x + (ca)\, b_x + (ab)\, c_x = 0;$$

or, as they stand for our purpose,

$$(bc) + (ca) + (ab) = 0.$$

2. Every covariant may be expressed in terms of those of the type

$$(ab)^\lambda\, (ac)^\mu\, (ad)^\nu\, \ldots,^*$$

because factors of the type (bc) in which a does not occur may be eliminated by the use of the identity

$$(bc)\, a_x = (ba)\, c_x - (ca)\, b_x.$$

The original results of Cayley on the enumeration of these covariants follow at once, as is remarked in a similar way by Stroh.

3. Consider next the covariants of degree 3, namely,

$$(ab)^\lambda\, (bc)^\mu\, (ca)^\nu.$$

In virtue of the above we need only consider the type

$$(ab)^\lambda\, (ac)^\mu,$$

and of these $(ab)(ac)$ is reducible, because

$$2\,(ab)(ac) = (ab)^2 + (ac)^2 - (bc)^2.$$

Thus perpetuants of degree 3 of three different quantics are completely given by

$$(ab)^\lambda\, (ac)^\mu,$$

in which $\mu \geqslant 1$, $\lambda \geqslant 2$.

We shall next show that the perpetuants of degree 4 of four different quantics are given by

$$(ab)^\lambda\, (ac)^\mu\, (ad)^\nu,$$

when $\lambda \geqslant 4$, $\mu \geqslant 2$, $\nu \geqslant 1$; but to establish this we need to recall certain results from the great memoirs of Jordan (*Liouville's Journal*, 1876 and 1879).

* Provided, of course, that the symbol a occurs in the covariant; we shall henceforth consider those containing a and, in fact, a definite set of symbols.

4. If ξ, η, ζ be three quantities such that

$$\xi+\eta+\zeta = 0,$$

then all homogeneous products

$$\xi^p \eta^q \zeta^r,$$

where $$p+q+r = n,$$

can be expressed as linear combinations of

$$\xi^\rho \eta^\sigma, \quad \eta^\rho \zeta^\sigma, \quad \zeta^\rho \xi^\sigma,$$

where ρ, σ take all values subject to the condition that $\sigma \not> \frac{n}{3}$. I do not stop to reproduce the proof of this fundamental lemma of Jordan: it will suffice to refer to his papers cited above, and to remark that a different proof can be given depending on the theory of apolar forms.

5. Consider now the covariants of degree 4. As already observed, we need only take into account

$$(ab)^\lambda (ac)^\mu (ad)^\nu.$$

If now either λ or μ be less than 2—say, $\mu < 2$—then, by transforming the expression

$$(ac)^\mu (ad)^\nu,$$

we can express it in terms of such as contain $(ac)^2$ at least and reducible forms. Hence

$$(ab)^\lambda (ac)^\mu (ad)^\nu$$

can be expressed in terms of covariants containing more factors in a, b, c and reducible forms. Again,

$$(ab)^\lambda (ac)^\mu$$

can be expressed linearly in terms of similar forms, each containing (ab) or (ac) to a power equal to or less than $\frac{\lambda+\mu}{3}$. Supposing this done, we see that, if

$$\lambda+\mu < 6,$$

the form can be expressed in terms of others for which the sum of the exponents of (ab) and (ac) is greater and reducible forms; thus we may assume that $\lambda+\mu \geqslant 6$.

Let us now take $\lambda+\mu = N > 6$ and consider the various products

$$(ab)^\lambda (ac)^\mu.$$

They may be divided into four classes :

(i.) $(ab)^n$, $(ab)^{n-1}(ac)$;

(ii.) $(ac)^n$, $(ac)^{n-1}(ab)$;

(iii.) $(ab)^2(ac)^{n-2}$, $(ab)^3(ac)^{n-3}$;

(iv.) all the remaining products for which $\lambda+\mu=n$.

Now, by means of the identities expressing $(bc)^n$ and $(bc)^{n-1}(ab)$ in terms of (ab) and (ac), we can express

$$(ab)^2(ac)^{n-2},\quad (ab)^3(ac)^{n-3},$$

as linear combinations of

$$\left.\begin{matrix}(ab)^n, & (ab)^{n-1}(ac)\\ (ac)^n, & (ac)^{n-1}(ab)\\ (bc)^n, & (bc)^{n-1}(ab)\end{matrix}\right\}\qquad\text{(A)}$$

and products contained in (iv.), in all of which the exponent of (ab) is 4 at least.

But, if any of the products (A) occur, the covariant is either reducible, or the number of factors involving a, b, c can be increased, and hence we need only consider the others, *i.e.*, every covariant of degree 4 can be expressed in terms of such as contain (ab) to the power 4 at least.

This establishes the result for forms of degree 4, for it manifestly follows that all covariants can be expressed in terms of covariants

$$(ab)^\lambda(ac)^\mu(ad)^\nu,$$

in which $\lambda \geqslant 4$, $\mu \geqslant 2$, $\nu \geqslant 1$.

6. The extension of this to forms of degree n belonging to n different quantics is immediately evident, both as to enunciation and proof. In fact, if we have n quantics $a, b, c, d, \ldots$, the perpetuants of degree n are of the type

$$(ab)^\lambda(ac)^\mu(ad)^\nu\ldots,$$

where $\lambda \geqslant 2^{n-2},\quad \mu \geqslant 2^{n-3},\quad \nu \geqslant 2^{n-4},\ \ldots.$

To put the proof of this by induction into a neater form, let us assume the result for $n+1$ letters and prove it for $(n+2)$ letters.

Suppose the form is $(ab)^\lambda(ac)^\mu R$,

when R is a type belonging to n letters, and for brevity write

$$2^{n-1}=N,$$

then, just as before, $(ab)^{\lambda}(ac)^{\mu}$ can be replaced by a number of products, each containing either (ab) or (ac) to a power not greater than $\frac{\lambda+\mu}{3}$.

But $(ac)^{\mu} R$ is a covariant for $(n+1)$ letters, and can therefore be expressed in terms of reducible forms, and such as contain (ac) to the power 2^{n-1} at least; accordingly, unless $\frac{\lambda+\mu}{3} \geqslant N$, the number of factors containing a, b, c can be increased.

If, then, $$\lambda+\mu = M \geqslant 3N,$$

we have to consider the following expressions, viz.,

$$(ab)^{M},\ (ab)^{M-1}(ac),\ \ldots,\ (ab)^{M-N+1}(ac)^{N-1}, \tag{A}$$

$$(ac)^{M},\ (ac)^{M-1}(ab),\ \ldots,\ (ac)^{M-N+1}(ab)^{N-1}, \tag{B}$$

$$(ab)^{2N}(ac)^{M-2N},\ \ldots,\ (ab)^{M-N}(ac)^{N}, \tag{C}$$

$$(ab)^{N}(ac)^{M-N},\ \ldots,\ (ab)^{2N-1}(ac)^{M-2N+1}. \tag{D}$$

By means of the identical equations expressing

$$(bc)^{M},\ (bc)^{M-1}(ab),\ \ldots,\ (bc)^{M-N+1}(ac)^{N-1}$$

in terms of (ab) and (ac), we can on solution express the N products

$$(ab)^{N}(ac)^{M-N},\ \ldots,\ (ab)^{2N-1}(ac)^{M-2N+1}$$

linearly in terms of the sets A, B, C and the subsidiary set

$$(bc)^{M},\ (bc)^{M-1}(ab),\ \ldots,\ (bc)^{M-N+1}(ac)^{N-1}.^{*}$$

Now, in A, B and the last system of products each term contains either (ab) or (ac) to a power equal to or less than $N = 2^{n-1}$, and accordingly, if one of these products occurs, the number of factors containing a, b, c can be increased. Eventually, then, we can express all the covariants in terms of reducible forms, and such as contain (ab) to the power $2N$ at least. The general result follows at once, and corroborates the result of Stroh relating to the perpetuant of lowest weight and given degree, and also the recent results of MacMahon in the *Camb. Phil. Trans.*

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* The possibility of this depends on the equations in question being linearly independent. With the notation of § 4 we can express all products of ξ, η, ζ of degree n linearly in terms of three sets, viz.,

$$\xi^{n},\ \xi^{n-1}\eta,\ \ldots,\ \xi^{n-g_1+1}\eta^{g_1-1};\ \ \eta^{n},\ \eta^{n-1}\zeta,\ \ldots,\ \eta^{n-g_2+1}\zeta^{g_2-1};\ \ \zeta^{n},\ \zeta^{n-1}\xi,\ \ldots,\ \zeta^{n-g_3+1}\xi^{g_3-1},$$

where g_1, g_2, g_3 are any three integers, such that $g_1+g_2+g_3 = n+1$. The result required in the text is a particular case of this which was first explicitly stated by Stroh, *Math. Ann.*, Vol. XXXIII. [*November* 30*th*, 1902].

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- Jatta, M.—" Ricerche sulla genesi della fibrina nelle membrane difteriche."
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THIRTY-NINTH SESSION, 1902–1903

(since the Formation of the Society, January 16th, 1865).

November 13th, 1902.

THE NINTH ANNUAL GENERAL MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Dr. E. W. HOBSON, F.R.S., President, in the Chair.

Twenty-five members and a visitor present.

Mrs. Alicia Stott was elected a member.

The Treasurer read his report. The reception of the Treasurer's report was moved by Mr. Kempe, seconded by Prof. Hudson, and carried *nem. con.*

The President nominated Mr. J. H. Grace to act as Auditor.

The Secretary reported that the number of members of the Society at the beginning of the previous session was 257. Of these 3 had resigned, and the names of 2 had been omitted from the list. The number of new members elected during the session was 8, bringing the number at the beginning of the present session to 260.

The President stated to the meeting the grounds on which the De Morgan Medal had been awarded by the Council to Prof. A. G. Greenhill, and presented the medal. Prof. Greenhill expressed his thanks to the Council and to the President.

The President spoke on the retirement of Mr. Tucker from the office of Honorary Secretary, and moved the following resolution:—"That the thanks of the London Mathematical Society be offered to Mr. Robert Tucker for the eminent services which he has rendered