On a Class of Plane Curves. *By* J. H. Grace.

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1. The well known chain of theorems established by Clifford in his "Synthetic Proof of Miquel's Theorem" has been lately obtained by M. Paul Serret* in an outwardly very different manner. Whereas the fundamental consideration in Clifford's proof is a curve of class n touching the line at infinity $(n-1)$ times, the corresponding idea in M. Serret's papers is a curve of the n-th degree, having its asymptotes concurrent and parallel to the sides of a regular polygon. In the one the locus of the foci plays the same part as the locus of the point of concurrence of the asymptotes does in the other. In the following paper, by following out the ideas of M. Serret, I have established an infinite series of propositions regarding lines and circles in a plane. After I had obtained the results hereafter explained a paper was published by Morley† in which the same results are obtained by purely analytical and shorter methods.

2. M. Serret considers, as a generalization of the rectangular hyperbola, curves whose asymptotes meet in a point and are parallel to the sides of a regular polygon. I make use of a somewhat similar, but less restricted, class of curves. In fact, the asymptotes are parallel to the sides of a regular polygon without being concurrent. A slight difference occurs according as the degree of the curve is odd or even. For example, when the degree is 3 the polygon is an equilateral triangle; but when the degree is 4 the polygon is a regular octagon, and not a square; for we require four different directions for the asymptotes. Uniformity is secured by saying that the curve of degree n has its asymptotes parallel to the sides of a regular polygon of m sides, and, for brevity, such a curve will be alluded to as an *isogonal* curve.

3. To discuss such curves we use axes of coordinates OI, OJ, where O is an arbitrary origin and I, J are the circular points at infinity. The equation of n lines through the origin parallel to the sides of a regular polygon of m sides will be of the form

$$x^n = ay^n.$$

* *Comptes Rendus*, 1894, 1895.

† *Proc. of the Amer. Math. Soc.*, Vol. I.

And hence the general equation of an isogonal curve of degree n is

$$f \equiv ax^n + by^n + na_1x^{n-1} + n(n-1)b_1x^{n-2}y - \dots$$
$$\dots n(n-1)c_1xy^{n-2} + d_1y^{n-1} + \dots = 0,$$

the only restrictions being that n coefficients in the general equation vanish. By suitably choosing the origin, *i.e.*, so as to satisfy the equations

$$x + a_1 = 0 \quad \text{or} \quad \frac{\partial^{n-1}f}{\partial x^{n-1}} = 0,$$

$$y + a_1 = 0 \quad \text{or} \quad \frac{\partial^{n-1}f}{\partial y^{n-1}} = 0,$$

the terms in x^{n-1} and y^{n-1} can be made to disappear. On the analogy with conics we are tempted to call this new origin the centre of the curve. As a matter of fact, if we bear in mind that the centre of a curve coincides with the mean centre of the points of intersection of its asymptotes, it can be verified by a short calculation that the point in question is actually the centre in the standard sense introduced by Chasles.

4. By using the equation written above some of the well known properties of rectangular hyperbolas can be extended to all isogonal curves. Thus, if S and S' be two such curves of the n-th degree, then all curves of degree n through their common points are isogonal curves, and the centre locus is a circle. But for our purpose the following theorem is of more importance:—

The first polar of any point at infinity with respect to an isogonal curve of the n-th degree is an isogonal curve of the $(n-1)$-th degree, and the locus of the centres of these curves is a circle concentric with the original curve.

In fact, taking for origin the centre of the curve, we have

$$a_1 = d_1 = 0,$$

and the equation of the first polar of a point at infinity is

$$x_1\frac{\partial f}{\partial x} + y_1\frac{\partial f}{\partial y} = 0$$

or $x_1\left\{ax^{n-1} + (n-1)(n-2)b_1x^{n-3}y + \dots + (n-1)c_1z^{n-2} + \dots\right\}$

$$+ y_1\left\{by^{n-1} + (n-1)b_1x^{n-2} + \dots + (n-1)(n-2)c_1xy^{n-3} + \dots\right\} = 0,$$

and the centre of this isogonal curve is given by

$$ax_1x+b_1y_1 = 0,$$
$$by_1y+c_1x_1 = 0;$$

so that when x_1, y_1 vary the centre locus is the circle whose equation is

$$abxy = b_1c_1.$$

This proves the theorem enunciated above.

5. Consider now three lines whose equations are $P_1 = 0$, $P_2 = 0$, $P_3 = 0$.

Included in $$\lambda_1P_1^2+\lambda_2P_2^2+\lambda_3P_3^2 = 0$$

there is a pencil of isogonal curves, for, in general, $(n-1)$ linear conditions ensure a curve of degree n being isogonal. These curves are, of course, the rectangular hyperbolas self-conjugate with respect to the triangle formed by the three lines, and their centre locus is the circumcircle.

But included in $$\mu_1P_1^3+\mu_2P_2^3+\mu_3P_3^3 = 0$$
there is just one isogonal cubic, and, as all its first polars are of the type
$$\gamma_1P_1^2+\gamma_2P_2^2+\gamma_3P_3^2 = 0,$$

those of the points at infinity must be the rectangular hyperbolas above. Hence, by the theorem of § 4, the centre of the single isogonal cubic is the centre of the circumscribing circle of the triangle.

6. Next consider four lines $P_1 = 0$, $P_2 = 0$, $P_3 = 0$, $P_4 = 0$.

Included in $$\lambda_1P_1^4+\lambda_2P_2^4+\lambda_3P_3^4+\lambda_4P_4^4 = 0$$
there is a single isogonal curve, and the first polars of points at infinity are isogonal curves of the form

$$\mu_1P_1^3+\mu_2P_2^3+\mu_3P_3^3+\mu_4P_4^3 = 0.$$

Now, if we take the first polar of the point at infinity on P_4, we see that $\mu_4 = 0$, and the pencil of polars includes the single isogonal cubic which we have seen to be included in

$$\mu_1P_1^3+\mu_2P_2^3+\mu_3P_3^3 = 0.$$

The centres of all the polars lie on a circle, and, by § 5, this circle passes through the centres of the circumcircles of the four triangles formed by the four lines. Further, its centre is the centre of the isogonal curve included in

$$\lambda_1 P_1^4 + \lambda_2 P_2^4 + \lambda_3 P_3^4 + \lambda_4 P_4^4 = 0.$$

We call this the centre circle of the four lines. Its defining property as here given is well known in elementary geometry.

7. Next consider five lines P_1, P_2, P_3, P_4, P_5.

There is a single isogonal curve included in

$$\lambda_1 P_1^5 + \lambda_2 P_2^5 + \lambda_3 P_3^5 + \lambda_4 P_4^5 + \lambda_5 P_5^5 = 0,$$

and among the first polars of points at infinity is the single isogonal quartic defined by any four of the lines. Thus the centre locus is a circle containing the centres of the centre circles obtained from the lines by leaving out each one in turn, and its centre is the centre of the isogonal quintic. We call this the centre circle of the five lines.

8. In just the same way, by considering the isogonal sextic included in

$$\lambda_1 P_1^6 + \lambda_2 P_2^6 + \ldots + \lambda_6 P_6^6 = 0,$$

we see that the centres of the five centre circles obtained by omitting the lines in turn lie on a circle whose centre is the centre of this sextic, and so on for seven, eight, ... lines *ad inf.*; belonging to n lines we have a circle called the centre circle. Then, taking $(n+1)$ lines, we obtain $(n+1)$ of these circles by omitting each line in turn, and the centres of these $(n+1)$ circles are concyclic.

9. It is easy to see that the $(n+1)$ circles also meet in a point. In fact, take the case of six lines: the centre circle of five lines P_1, P_2, P_3, P_4, P_5 is the locus of the centres of isogonal quartics included in

$$\lambda_1 P_1^4 + \lambda_2 P_2^4 + \lambda_3 P_3^4 + \lambda_4 P_4^4 + \lambda_5 P_5^4 = 0, \qquad \text{(A)}$$

and consequently, when the centre circle belonging to $P_1 P_2 P_3 P_4 P_5$ meets that belonging to $P_1 P_2 P_3 P_4 P_6$, we have a centre of a quartic S included in (A), and of a quartic S' included in

$$\lambda_1 P_1^4 + \lambda_2 P_2^4 + \lambda_3 P_3^4 + \lambda_4 P_4^4 + \lambda_6 P_6^4 = 0.$$

But $$\kappa S + \kappa' S' = 0$$

will then be an isogonal quartic having its centre at this point for all values of κ and κ', and, by suitably choosing κ and κ', we can make this quartic belong to the set derived from any five of the lines, unless $\lambda_5 = \lambda_6 = 0$. Hence, if a point lies on two of the centre circles, it lies on each of them, unless it be the centre of an isogonal quartic included in

$$\lambda_1 P_1^4 + \lambda_2 P_2^4 + \lambda_3 P_3^4 + \lambda_4 P_4^4 = 0.$$

In the case above the centre circle belonging to $P_1 P_2 P_3 P_4 P_5$ meets that belonging to $P_1 P_2 P_3 P_4 P_6$ in two points: one is the centre of the unique isogonal quartic included in

$$\lambda_1 P_1^4 + \lambda_2 P_2^4 + \lambda_3 P_3^4 + \lambda_4 P_4^4 = 0,$$

and the other is the common point spoken of above.

10. The greater part of the above reasoning can be put into the language of synthetic geometry without difficulty.

In an isogonal curve the polar line of I always passes through J, and *vice versa*, while the two polars meet in the centre of the curve. For a pencil of isogonal curves corresponding polars form two projective pencils through I and J, and hence the centre locus is a circle.

Applications to Dynamics of some Algebraical Results. By T. J. I'A. Bromwich.

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The problem considered in this paper is that of the small oscillations of a dynamical system about a state of steady motion. In § 1 the Hamiltonian equations are used to determine the principal coordinates of an oscillation in the neighbourhood of a state of steady motion; the suggestion of using the Hamiltonian function instead of the Lagrangian was made to me by Mr. E. T. Whittaker, and, so far as I know, the results obtained in this way are novel. In § 2 an approximate method for dealing with gyrostatic systems is given; this method has been used by Thomson and Tait, but their work can be abbreviated by using the algebraical results quoted.

It is proved below that the problems in §§ 1, 2 both depend on the algebraic reduction to a canonical shape of two bilinear forms, one being symmetric and the other alternate; this reduction has been