Lie, Sophus.—"Untersuchungen über Unendliche Continuirliche Gruppen," Roy. 8vo; Leipzig, 1895.

"Educational Times," March, 1895.

"Annales de la Faculté des Sciences de Toulouse," Tome IX., Fasc. 1; Paris, 1895.

"Philosophical Transactions of the Royal Society," Vol. CLXXXV., Pt. 1.

"Indian Engineering," Vol. XVII., Nos. 4-7.

The Perpetuant Invariants of Binary Quantics. By Major P. A. MACMAHON, R.A., F.R.S. Read March 14th, 1895. Received 16th May, 1895.

It was in Vol. v. of the *American Journal of Mathematics* that Sylvester first proposed the problem of the enumeration of the perpetuants of given degree and weight.* Of a given degree Cayley's rule gives a generating function which enumerates the asyzygetic seminvariants. A knowledge of the perpetuants of lower degrees leads to the generating function for the compound seminvariants of the given degree. Since these forms are not linearly independent, it is necessary to find the generating function of the syzygies which connect them. We have, then, the means for arriving at the generating function of the perpetuants. It is merely necessary to subtract the generating function of the syzygies from that of the compound forms, and then subtract the difference from that of the asyzygetic forms. This procedure was adopted by Sylvester. For the first four degrees no syzygies arise, and the perpetuant generating functions were found to be

$$x^0,\quad \frac{x^2}{1-x^2},\quad \frac{x^3}{(1-x^2)(1-x^3)},\quad \frac{x^7}{(1-x^2)(1-x^3)(1-x^4)},$$

respectively; the enumeration of the perpetuants being given, for a weight w, by the coefficient of x^w in the developments.

---

* "On Sub-Invariants, *i.e.*, Semi-Invariants to Binary Quantics of Unlimited Order," *Amer. Math. Jour.*, Vol. v., p. 79.

Syzygies first present themselves for the degree 5. Sylvester, in the paper quoted, did not succeed in correctly enumerating them. This was accomplished by Hammond,* who established the generating function

$$\frac{x^7}{(1-x^3)(1-x^4)},$$

which immediately led to the true generating function for perpetuants of degree 5, viz.,

$$\frac{x^{15}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.$$

Cayley† continued the investigation on the same lines, but adding the notion, due to the author of the present paper, of the transformation of seminvariants into non-unitary symmetric functions. Considerable light was thus thrown upon the structure of the syzygies in general, and in particular upon those of degree 6. No new generating function was obtained, as the enumeration of the syzygies of degree 6 proved to be impracticable. The simplest perpetuant of degree 6 was first obtained by the author of this paper.‡ It proved to be of weight 31. The research proceeded on the lines laid down by Sylvester, Hammond, and Cayley, and principally by the use of Cayley's exceedingly useful algorithm for the multiplication of symmetric functions, the whole of the syzygies up to the weight 31 inclusive were calculated as far as was necessary for the purpose in hand. The generating function for the syzygies was not obtained. It should be mentioned also that on p. 45 of the paper the perpetuant of weight 31 is correctly identified, but that the non-exemplar perpetuants of this weight are incorrectly enumerated. The number was given as 5, whereas, as will subsequently appear, we now know the number to be 16.

In a second paper§ in the same volume, the author again considered the question, and showed that on a certain hypothesis, the truth of which he was unable to assert, the generating function for perpetuants of degree θ (>2) was

$$\frac{x^{2^{\theta-1}-1}}{(1-x^2)(1-x^3)\ldots(1-x^\theta)}.$$

---

* *Amer. Math. Jour.*, Vol. IV. (1882), pp. 218–228, "On the Solution of the Differential Equation of Sources."

† *Amer. Math. Jour.*, Vol. VII. (1885), pp. 1–25, "A Memoir on Seminvariants."

‡ "On Perpetuants," *Amer. Math. Jour.*, Vol. VII., pp. 26–46.

§ "A Second Paper on Perpetuants," *Amer. Math. Jour.*, Vol. VII.; pp. 259–263.

This prediction was subsequently verified by Stroh,* who, in § 10 of the paper quoted in the foot-note, established the generating function by an ingenious method which differed totally from that adopted by previous investigators in the same field.

Cayley† followed with interesting remarks and developments of Stroh's theory.

Stroh considers the general seminvariants of degree θ and weight w,

$$\Omega_\theta^w = (a_1\beta_1 + a_2\beta_2 + \ldots + a_\theta\beta_\theta)^w,$$

where $\beta_1, \beta_2, \ldots \beta_\theta$ are arbitrary quantities merely subject to the condition

$$\Sigma\beta = 0,$$

and $a_1, a_2, \ldots a_\theta$ are *umbræ*, such that, after expansion,

$$a_1^s = a_2^s = \ldots = a_\theta^s = (1^s) \text{ or } = a_s.$$

Assuming

$$(1+\mu\beta_1)(1+\mu\beta_2)\ldots(1+\mu\beta_\theta) = 1+\mu^2 B_2+\mu^3 B_3+\ldots+\mu^\theta B_\theta,$$

the expanded function Ω_θ^w

can be exhibited as a linear function of products of powers of

$$B_2, B_3, \ldots B_\theta,$$

of weight w. Appearing as a coefficient of each B term of this function, we find a seminvariant of the binary quantic

$$u^n - \binom{n}{1} a_1 u^{n-1} + \binom{n}{2} a_2 u^{n-2} - \ldots = 0,$$

where n may be supposed to be infinite. Stroh shows that the whole of the seminvariants of degree θ and weight w thus present themselves. To exhibit certain of the seminvariants in terms of seminvariants of lower degree by means of products of degree θ, we may, since $\beta_1, \beta_2, \ldots \beta_\theta$ are merely subject to the condition

$$\Sigma\beta = 0,$$

suppose $\beta_1+\beta_2+\ldots+\beta_\phi = \beta_{\phi+1}+\beta_{\phi+2}+\ldots+\beta_\theta = 0,$

where ϕ may be any integer less than θ.

---

* "Ueber die Symbolische Darstellung den Grundsyzyganten einer binären Form sechster Ordnung und eine Erweckerung der Symbolik von Clebsch," *Math. Ann.*, t. XXXVI. (1890), pp. 263–303.

† "On Symmetric Functions and Seminvariants," *Amer. Math. Jour.*, Vol. XV., pp. 1–69.

We have then $$\Omega_\theta^w = (\Omega_\phi + \Omega_{\theta-\phi})^w,$$

and Ω_θ^w is thus shown to be reducible. But Ω_θ^w is no longer the perfectly general seminvariant that it was proved to be before the introduction of the new conditions

$$\beta_1+\beta_2+\ldots+\beta_\phi = \beta_{\phi+1}+\beta_{\phi+2}+\ldots+\beta_\theta = 0.$$

These conditions necessitate the vanishing of a certain function of the quantities $$B_2,\ B_3,\ \ldots\ B_\theta,$$

so that a certain number of B products, and therefore also of seminvariants, have disappeared from

$$\Omega_\theta^w.$$

These are the perpetuants of the degree θ and weight w.

This is very clearly stated by Stroh; and Cayley, with further amplification of statement, actually determines the conditions for the first six degrees.

The above is merely historical.

I am now principally concerned with the two papers of Stroh and Cayley last mentioned, which, from their recent appearance, will be fresh in the memory of mathematicians.

I propose to present Stroh's theory and Cayley's developments from a purely algebraical point of view—that is to say, without the employment of any umbral symbols—and also to actually identify each of the whole series of perpetuants of all degrees and weights.

First, consider a transformation of Stroh's general seminvariant obtained by employing umbræ with a different signification.

In the form
$$\Omega_\theta^w = (a_1\beta_1+a_2\beta_2+\ldots+a_\theta\beta_\theta)^w,$$

let a_s be an umbral symbol, such that after evolution a_s^σ is to be replaced by $\sigma!\ a_\sigma$.

We find $$\Omega_\theta^w = \Sigma \frac{w!}{\pi_1!\ \pi_2!\ \ldots} a_1^{\pi_1} a_2^{\pi_2} \ldots (\pi_1\pi_2\ldots)_\beta,$$

where $(\pi_1\pi_2\ldots)_\beta$ denotes the symmetric function

$$\Sigma\beta_1^{\pi_1}\beta_2^{\pi_2}\ldots.$$

Thence $$\Omega_\theta^w = w!\ \Sigma\, a_{\pi_1} a_{\pi_2} \ldots (\pi_1\pi_2\ldots)_\beta.$$

The symmetric functions on the dexter side are to be expressed in terms of the elementary functions $B_2, B_3, \dots B_\theta$, and the dexter has then to be arranged as a linear function of products

$$B_s B_t \dots .$$

Let
$$(\pi_1\pi_2\dots)_\theta = \Sigma C_{st\dots} B_s B_t \dots ;$$

then
$$\frac{1}{w!}\Omega_\theta^w = \Sigma\{\Sigma C_{st\dots} a_{\pi_1} a_{\pi_2}\dots\} B_s B_t \dots .$$

The whole coefficient of $B_s B_t$ is

$$\Sigma C_{st\dots} a_{\pi_1} a_{\pi_2} \dots .$$

But $C_{st\dots}$ is the coefficient of $B_s B_t \dots$ in the expression of $(\pi_1\pi_2\dots)_\theta$; therefore, by the well known law of reciprocity, it is also the coefficient $B_{\pi_1} B_{\pi_2}\dots$ in the expression of $(st\dots)_\theta$ or of $a_{\pi_1} a_{\pi_2}\dots$ in the expression of $(st\dots)$, where $(st\dots)$ denotes a symmetric function of the quantities of which

$$a_1,\ a_2,\ \dots\ a_{\pi_1}\ \dots\ a_{\pi_2},\ \dots$$

are the elementary symmetric functions. Hence

$$\Sigma C_{st\dots} a_{\pi_1} a_{\pi_2} \dots = (st\dots),$$

and
$$\frac{1}{w!}\Omega_\theta^w = \Sigma (st\dots) B_s B_t \dots .$$

Since $B_1 = 0$, we have on the right a linear function of the non-unitary symmetric functions of weight w and of degree not exceeding θ.

These non-unitariants (Cayley, *loc. cit.*) of the roots of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots = 0$$

are, as is well known, seminvariants of the binary quantic

$$x^n - na_1 x^{n-1} y + n(n-1) a_2 x^{n-2} y^2 - \dots .$$

Thus transformed, Stroh's general seminvariant assumes a simple and elegant form, and suggests the following method of viewing the subject.

I, first of all, retain B_1, so as to consider the reducibility of symmetric functions in general, and subsequently cause B_1 to vanish, so as to restrict the investigation to seminvariants.

Taking an arbitrary quantity μ, let

$$(1+\mu a_1)(1+\mu a_2)(1+\mu a_3) \dots \text{ ad inf.}$$
$$= 1+\mu\,(1)+\mu^2\,(1^2)+\mu^3\,(1^3)+\dots$$
$$= 1+\mu a_1+\mu^2 a_2+\mu^3 a_3+\dots,$$

where $a_1, a_2, a_3, \dots$ are not the umbræ before mentioned, but quantities obeying the *ordinary* laws of algebraical quantity.

Let, also,
$$(1+\mu\beta_1)(1+\mu\beta_2)\dots(1+\mu\beta_\theta)$$
$$= 1+\mu B_1+\mu^2 B_2+\dots+\mu^\theta B_\theta;$$

then
$$(1+\mu a_1\beta_1)(1+\mu a_1\beta_2)\dots(1+\mu a_1\beta_\theta)$$
$$= 1+\mu a_1 B_1+\mu^2 a_1^2 B_2+\dots+\mu^\theta a_1^\theta B_\theta,$$

and
$$\Pi_s\,(1+\mu a_s\beta_1)(1+\mu a_s\beta_2)\dots(1+\mu a_s\beta_\theta)$$
$$= \Pi_s\,(1+\mu a_s B_1+\mu^2 a_s^2 B_2+\dots+\mu^\theta a_s^\theta B_\theta),$$

the products extending to the quantities

$$a_1,\ a_2,\ a_3,\ \dots$$

of unlimited number.

Multiplying out the dexter of this identity and therein representing the coefficients of μ^κ by $Z_{\kappa,\theta}$, we have

$$1+\mu Z_{1,\theta}+\mu^2 Z_{2,\theta}+\mu^3 Z_{3,\theta}+\dots$$
$$= 1+\mu\,(1)\,B_1+\mu^2\left\{(2)\,B_2+(1^2)\,B_1^2\right\}$$
$$+\mu^3\left\{(3)\,B_3+(21)\,B_2B_1+(1^3)\,B_1^3\right\}+\dots,$$

where on the right the coefficient of μ^κ involves linearly all the symmetric functions of $a_1, a_2, a_3, \dots$ of weight κ and degree not exceeding θ.

[Taking Ω_θ with changed umbræ

$$Z_{\kappa,\theta} = \frac{1}{\kappa!}\,\Omega_\theta^\kappa,$$

and the sinister is (when $B_1 = 0$)

$$\exp\,(\mu\Omega_\theta).]$$

Thus
$$Z_{1,\theta} = (1)\,B_1,$$
$$Z_{2,\theta} = (2)\,B_2 + (1^2)\,B_1^2,$$
$$Z_{3,\theta} = (3)\,B_3 + (21)\,B_2B_1 + (1^3)\,B_1^3,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$Z_{\kappa,\theta} = \Sigma\,(p_1^{\pi_1}p_2^{\pi_2}\ldots)\,B_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots,$$

the summation being for all partitions of κ into parts not exceeding θ in magnitude.

Taking $\phi<\theta$, write
$$(1+\mu\beta_1)(1+\mu\beta_2)\ldots(1+\mu\beta_\phi)$$
$$= 1+\mu B_1'+\mu^2B_2'+\ldots+\mu^\phi B_\phi'$$
$$(1+\mu\beta_{\phi+1})(1+\mu\beta_{\phi+2})\ldots(1+\mu\beta_\theta)$$
$$= 1+\mu B_1''+\mu^2B_2''+\ldots+\mu^{\theta-\phi}B_{\theta-\phi}'',$$

and thence
$$\Pi_s\,(1+\mu a_s\beta_1)(1+\mu a_s\beta_2)\ldots(1+\mu a_s\beta_\phi)$$
$$= \Pi_s\,(1+\mu a_sB_1'+\mu^2a_s^2B_2'+\ldots+\mu^\phi a_s^\phi B_\phi')$$
$$\Pi_s\,(1+\mu a_s\beta_{\phi+1})(1+\mu a_s\beta_{\phi+2})\ldots(1+\mu a_s\beta_\theta)$$
$$= \Pi_s\,(1+\mu a_sB_1''+\mu^2a_s^2B_2''+\ldots+\mu^{\theta-\phi}a_s^{\theta-\phi}B_{\theta-\phi}''),$$

whence we derive
$$1+\mu Z_{1,\phi}+\mu^2Z_{2,\phi}+\mu^3Z_{3,\phi}+\ldots$$
$$= 1+\mu\,(1)\,B_1'+\mu^2\left\{(2)\,B_2'+(1^2)\,B_1'^2\right\}+\ldots,$$

the parts of the partitions being limited not to exceed ϕ in magnitude; and
$$1+\mu Z_{1,\theta-\phi}+\mu^2Z_{2,\theta-\phi}+\mu^3Z_{3,\theta-\phi}+\ldots$$
$$= 1+\mu\,(1)\,B_1''+\mu^2\left\{(2)\,B_2''+(1^2)\,B_1''^2\right\}+\ldots,$$

the parts being limited not to exceed $\theta-\phi$ in magnitude.

Moreover,
$$1+\mu B_1+\mu^2B_2+\ldots+\mu^\theta B_\theta$$
$$= (1+\mu B_1'+\mu^2B_2'+\ldots+\mu^\phi B_\phi')(1+\mu B_1''+\mu^2B_2''+\ldots+\mu^{\theta-\phi}B_{\theta-\phi}''),$$

and
$$1+\mu Z_{1,\theta}+\mu^2Z_{2,\theta}+\mu^3Z_{3,\theta}+\ldots$$
$$= (1+\mu Z_{1,\phi}+\mu^2Z_{2,\phi}+\mu^3Z_{3,\phi}+\ldots)(1+\mu Z_{1,\theta-\phi}+\mu^2Z_{2,\theta-\phi}+\mu^3Z_{3,\theta-\phi}+\ldots).$$

Comparing coefficients of μ^κ,

$$Z_{\kappa,\theta} = Z_{\kappa,\phi} + Z_{\kappa-1,\phi} Z_{1,\theta-\phi} + \dots + Z_{1,\phi} Z_{\kappa-1,\theta-\phi} + Z_{\kappa,\theta-\phi},$$

$Z_{\kappa,\theta}$ involves symmetric functions of weight κ and of degree $\leqslant \theta$, while any product on the right

$$Z_{\kappa-s,\phi} Z_{s,\theta-\phi}$$

involves products of two symmetric functions, the one of weight $\kappa-s$ and degree $\leqslant \phi$, and the other of weight s and degree $\leqslant \theta-\phi$.

Moreover, the quantities

$$B_1,\ B_2,\ \dots\ B_\theta$$

are expressible in terms of the quantities

$$B'_1,\ B'_2,\ \dots\ B'_\phi;\quad B''_1,\ B''_2,\ \dots\ B''_{\theta-\phi},$$

by a series of relations of the form

$$B_s = B'_s + B'_{s-1} B''_1 + \dots + B'_1 B''_{s-1} + B''_s,$$

and, by reason of these relations,

$$B_1,\ B_2,\ \dots\ B_\theta$$

are not subject to any condition.

Hence, by comparison of the two sides of the relation

$$Z_{\kappa,\theta} = Z_{\kappa,\phi} + Z_{\kappa-1,\phi} Z_{1,\theta-\phi} + \dots + Z_{\kappa,\theta-\phi},$$

we are able to express certain symmetric functions of weight κ and degree $\leqslant \theta$ as sums of products of pairs of symmetric functions, each pair involving one function of degree $\leqslant \phi$, and one of degree $\leqslant \theta-\phi$.

We have, in fact, a general theorem of reducibility.

Supposing $\theta > 1$ and ϕ any one of the integers 1, 2, 3, ... $\theta-1$, it can be demonstrated that every monomial symmetric function of degree θ is reducible by the aid of symmetric functions whose partitions are subsequent to it in dictionary order, and of products of pairs of functions of degrees $\leqslant \phi$ and $\leqslant \theta-\phi$, respectively.

Consider in $Z_{\kappa,\theta}$ the term

$$(\theta^{\sigma_0},\ \theta-1^{\sigma_1},\ \theta-2^{\sigma_2},\ \dots)\ B_\theta^{\sigma_0} B_{\theta-1}^{\sigma_1} B_{\theta-2}^{\sigma_2} \dots,$$

where the literal part is equal to

$$(B'_\phi B''_{\theta-\phi})^{\sigma_0} (B'_{\phi-1} B''_{\theta-\phi} + B'_\phi B''_{\theta-\phi-1})^{\sigma_1} (B'_{\phi-2} B''_{\theta-\phi} + B'_{\phi-1} B''_{\theta-\phi-1} + B'_\phi B''_{\theta-\phi-2})^{\sigma_2} \dots$$

Of this consider the portion

$$B'^{\sigma_0}_{\phi}B'^{\sigma_1}_{\phi-1}B'^{\sigma_2}_{\phi-2}\dots B''^{\sigma_0+\sigma_1+\sigma_2+\dots}_{\theta-\phi}.$$

The weight of $B'^{\sigma_0}_{\phi}B'^{\sigma_1}_{\phi-1}B'^{\sigma_2}_{\phi-2}\dots$

is $$\phi(\sigma_0+\sigma_1+\sigma_2+\dots)-\sigma_1-2\sigma_2-\dots=\kappa',$$

and that of $B''^{\sigma_0+\sigma_1+\sigma_2+\dots}_{\theta-\phi}$

is $$(\theta-\phi)(\sigma_0+\sigma_1+\sigma_2+\dots)=\kappa'',$$

where $$\kappa'+\kappa''=\kappa.$$

Hence the literal portion considered must arise in the dexter of the identity in the product

$$Z_{\kappa',\phi}Z_{\kappa'',\theta-\phi},$$

as

$$(\phi^{\sigma_0}\phi-1^{\sigma_1}\phi-2^{\sigma_2}\dots)B'^{\sigma_0}_{\phi}B'^{\sigma_1}_{\phi-1}B'^{\sigma_2}_{\phi-2}\dots(\theta-\phi^{\sigma_0+\sigma_1+\sigma_2+\dots})B''^{\sigma_0+\sigma_1+\sigma_2+\dots}_{\theta-\phi}.$$

On the sinister side the *whole* coefficient of

$$B'^{\sigma_0}_{\phi}B'^{\sigma_1}_{\phi-1}B'^{\sigma_2}_{\phi-2}\dots B''^{\sigma_0+\sigma_1+\sigma_2+\dots}_{\theta-\phi}$$

must be the sum of the monomial functions obtained by the multiplication of the two functions

$$(\phi^{\sigma_0}\phi-1^{\sigma_1}\phi-2^{\sigma_2}\dots),\quad(\theta-\phi^{\sigma_0+\sigma_1+\sigma_2+\dots}),$$

and the first of these in dictionary order is

$$(\theta^{\sigma_0}\theta-1^{\sigma_1}\theta-2^{\sigma_2}\dots).$$

Hence this function together with other functions subsequent to it in dictionary order must be equal to the product

$$(\phi^{\sigma_0}\phi-1^{\sigma_1}\phi-2^{\sigma_2}\dots)(\theta-\phi^{\sigma_0+\sigma_1+\sigma_2+\dots}).$$

In other words, the function is reducible, and the actual reduction is given by the identity. For a given value of ϕ symmetric functions are, in general, reducible in more ways than one.

Ex. gr.—Take $\kappa=6$, $\theta=4$, $\phi=2$.

$$Z_{6,4}=Z'_{6,2}+Z'_{5,2}Z''_{1,2}+Z'_{4,2}Z''_{2,2}+Z'_{3,2}Z''_{3,2}+Z'_{2,2}Z''_{4,2}+Z'_{1,2}Z''_{5,2}+Z''_{6,2},$$

single and double accents being introduced to distinguish forms which, from the circumstance that ϕ and $\theta-\phi$ are equal, would be otherwise indistinguishable.

$$Z_{6,4} = (42)\,B_4 B_2 + (41^2)\,B_4 B_1^2 + (3^2)\,B_3^2 + (321)\,B_3 B_2 B_1 + (31^3)\,B_3 B_1^3$$
$$+ (2^3)\,B_2^3 + (2^2 1^2)\,B_2^2 B_1^2 + (21^4)\,B_2 B_1^4 + (1^6)\,B_1^6,$$

$$Z'_{6,2} = (2^3)\,B_2'^3 + (2^2 1^2)\,B_2'^2 B_1'^2 + (21^4)\,B_2' B_1'^4 + (1^6)\,B_1'^6,$$

$$Z'_{5,2} = (2^2 1)\,B_2'^2 B_1' + (21^3)\,B_2' B_1'^3 + (1^5)\,B_1'^5, \qquad Z''_{1,2} = (1)\,B_1'';$$

$$Z'_{4,2} = (2^2)\,B_2'^2 + (21^2)\,B_2' B_1'^2 + (1^4)\,B_1'^4, \qquad Z''_{2,2} = (2)\,B_2'' + (1^2)\,B_1''^2;$$

$$Z'_{3,2} = (21)\,B_2' B_1' + (1^3)\,B_1'^3, \qquad Z''_{3,2} = (21)\,B_2'' B_1'' + (1^3)\,B_1''^3;$$

$$Z'_{2,2} = (2)\,B_2' + (1^2)\,B_1'^2, \qquad Z''_{4,2} = (2^2)\,B_2''^2 + (21^2)\,B_2'' B_1''^2 + (1^4)\,B_1''^4;$$

$$Z_{1,2} = (1)\,B_1', \qquad Z''_{5,2} = (2^2 1)\,B_2''^2 B_1'' + (21^3)\,B_2'' B_1''^3 + (1^5)\,B_1''^5;$$

$$Z''_{6,2} = (2^3)\,B_2''^3 + (2^2 1^2)\,B_2''^2 B_1''^2 + (21^4)\,B_2'' B_1''^4 + (1^6)\,B_1''^6,$$

and the relations

$$B_4 = B_2' B_2'',$$
$$B_3 = B_2' B_1'' + B_1' B_2'',$$
$$B_2 = B_2' + B_1' B_1'' + B_2'',$$
$$B_1 = B_1' + B_1''.$$

Comparison of the coefficients (1) of $B_2'^2 B_2''$, (2) of $B_2' B_1' B_2'' B_1''$ on the sides of the resulting identity yields the reductions

$$(42) + 3\,(2^3) = (2^2)(2),$$

$$(42) + 2\,(41^2) + 2\,(3^2) + 2\,(321) + 6\,(2^3) + 4\,(2^2 1^2) = (21)^2.$$

A similar process with regard to the term $B_2' B_1'^2 B_2''$ yields

$$(41^2) + (321) + 2\,(2^2 1^2) = (21^2)(2).$$

Reductions of forms of lower degrees are also given by the same identity. It is not necessary to give them, because they can be obtained more simply by formation of the identities for the data

$$(\kappa, \theta, \phi) = (6, 3, 2),$$

$$(\kappa, \theta, \phi) = (6, 2, 1).$$

We thus obtain the reductions

$$(3^2) + (321) + (2^2 1^2) = (2^2)(1^2),$$

$$(321) + 3\,(2^3) + 2\,(2^2 1^2) = (2^2 1)(1),$$

$$(321) + 3\,(31^3) + 3\,(2^3) + 4\,(2^2 1^2) + 6\,(21^4) = (21^2)(1^2),$$

$$(321) + 3\,(31^3) + 2\,(2^2 1^2) + 4\,(21^4) = (21)(1^3),$$

$$(31^3)+2\,(2^21^2)+4\,(21^4) = (21^3)(1),$$

$$(31^3)+(21^4) = (2)(1^4),$$

$$(2^3)+2\,(2^21^2)+6\,(21^4)+20\,(1^6) = (1^3)^2,$$

$$(2^21^2)+4\,(21^4)+15\,(1^6) = (1^4)(1^2),$$

$$(21^4)+6\,(1^6) = (1^5)(1).$$

The identity manifestly also involves a theorem for the multiplication of any two symmetric functions whatever.

I pass on to the discussion of the reduction of non-unitariants, viz., those symmetric functions the parts of whose partitions are all greater than unity.

If a non-unitariant be reducible *quâ* non-unitariants, it must obviously be reducible by means of products of pairs of non-unitariants; this fact follows from the circumstance that the product of two non-unitariants is itself a non-unitariant; the forms, in fact, constitute a closed system. It should be observed that this would not be the case with some other systems that might present themselves for consideration.

If we had to discuss the functions which contain no part 2 in their partitions, we have no closed system; for two forms, such as (31) and (41), which are included in the system, give rise to forms, containing a part 2, which are exterior to the system. Suppose that the quantities β above considered are not all independent, but are connected by a relation

$$f(B_1, B_2, B_3, \dots B_s) = 0;$$

then the expression

$$Z_{\kappa,\theta}$$

will not involve the complete system of symmetric functions of the quantities

$$a_1, a_2, a_3, \dots,$$

for certain of the products

$$B_{p_1}^{\pi_1} B_{p_2}^{\pi_2} \dots$$

can be eliminated between the relations

$$f(B_1, B_2, B_3, \dots B_s) = 0,$$

$$Z_{\kappa,\theta} = \Sigma\,(p_1^{\pi_1} p_2^{\pi_2} \dots)\, B_{p_1}^{\pi_1} B_{p_2}^{\pi_2} \dots,$$

and, as a consequence, the number of symmetric functions

$$(p_1^{\pi_1} p_2^{\pi_2} \dots)$$

in the expression of $Z_{\kappa,\theta}$ will suffer a reduction. We would then have a *particular* system of symmetric functions under consideration, whose nature we may take to be exactly defined by the conditional relation

$$f(B_1, B_2, B_3, \dots B_\theta) = 0.$$

For the theory of the reducibility of this system we are led to the identity

$$\begin{aligned} &1+\mu B_1+\mu^2 B_2+\mu^3 B_3+\dots+\mu^\theta B_\theta \\ =\ &(1+\mu B_1'+\mu^2 B_2'+\mu^3 B_3'+\dots+\mu^\phi B_\phi') \\ &\times(1+\mu B_1''+\mu^2 B_2''+\mu^3 B_3''+\dots+\mu^{\theta-\phi} B_{\theta-\phi}''), \end{aligned}$$

ϕ being any integer equal or less than $\frac{1}{2}\theta$, with the three conditions

$$\begin{aligned} &f(B_1, B_2, B_3, \dots B_\theta) = 0, \\ &f(B_1', B_2', B_3', \dots B_\phi', 0, 0, \dots) = 0, \\ &f(B_1'', B_2'', B_3'', \dots B_{\theta-\phi}'', 0, 0, 0, \dots) = 0. \end{aligned}$$

Hence there are $\theta-1$ independent quantities B,

$\phi-1$ „ „ B',

$\theta-\phi-1$ „ „ B'',

and, since $$(\theta-1)-(\phi-1)-(\theta-\phi-1) = 1,$$

the satisfaction of the identity necessitates another relation between

$$B_1, B_2, B_3, \dots B_\theta,$$

say $$\psi_\phi(B_1, B_2, B_3, \dots B_\theta) = 0.$$

Write this for brevity $\psi_\phi = 0$.

This is the condition of reduction for a given value of the integer ϕ. Considering merely this particular mode of reduction, we find that the condition

$$\psi_\phi = 0$$

causes a further diminution in the number of symmetric functions appearing in the expression of

$$Z_{\kappa,\theta}.$$

These disappearing functions are those which cannot be reduced in the particular manner we are considering.

If $\phi_1, \phi_2, \dots \phi_s$ be s particular values of ϕ, the condition

$$\psi_{\phi_1}\psi_{\phi_2}\dots\psi_{\phi_s} = 0$$

leads to the functions that cannot be reduced in any of the modes defined by the integers

$$\phi_1, \phi_2, \dots \phi_s.$$

For complete irreducibility we have the condition

$$\psi_1 \psi_2 \dots \psi_{\leqslant \frac{1}{2}\theta} = 0.$$

Non-unitariants constitute the simplest restricted system that it is possible to devise. They are the solutions of the partial differential equation

$$d_1 u = \left(\frac{d}{da_1} + a_1 \frac{d}{da_2} + a_2 \frac{d}{da_3} + \dots \right) u = 0.$$

It will subsequently appear that other restricted systems corresponding to other differential equations may be usefully considered, but, for the present, non-unitariants are alone under view.

Hence $$f(B_1, B_2, \dots B_\theta) = B_1,$$

and therefore $$B_1 = B_1' = B_1'' = 0,$$

$Z_{s,\theta}$ now only involves non-unitariants

$$Z_{2,\theta} = (2) B_2,$$

$$Z_{3,\theta} = (3) B_3,$$

$$Z_{4,\theta} = (4) B_4 + (2^2) B_2^2,$$

$$Z_{5,\theta} = (5) B_5 + (32) B_3 B_2,$$

&c.

In order that $$1 + \mu^2 Z_{2,\theta} + \mu^3 Z_{3,\theta} + \dots$$

may be broken up into factors, involving non-unitariants only, we must have

$$1 + \mu^2 B_2 + \mu^3 B_3 + \dots + \mu^\theta B_\theta$$

$$= (1 + \mu^2 B_2' + \mu^3 B_3' + \dots + \mu^\phi B_\phi')$$

$$\times (1 + \mu^2 B_2'' + \mu^3 B_3'' + \dots + \mu^{\theta-\phi} B_{\theta-\phi}''),$$

for some value of $\phi \leqslant \frac{1}{2}\theta$.

It is easy to see that

$$\psi_\phi = \Pi\, (\beta_1 + \beta_2 + \dots + \beta_\phi),$$

when on the right we have a symmetrical function.

Hence the complete condition of reduction is

$$\prod_{\phi=1}^{\phi \leqq \frac{1}{2}\theta} \{\Pi(\beta_1+\beta_2+\ldots+\beta_\phi)\} = 0.$$

The weight of this condition in the quantities $B_2, B_3, \ldots$ is as shown by Stroh and Cayley, and, as it is easy to verify,

$$2^{\theta-1}-1 = w_\theta.$$

This condition causes one B product, containing a factor B_θ, of weight w_θ, to disappear from

$$Z_{w_\theta, \theta},$$

and indicates the irreducibility of the corresponding non-unitariant.

The same procedure as was adopted for the unrestricted system shows that every other form not thus shown to be irreducible is in fact reducible.

It is now easy to show that the number of perpetuants of degree θ and weight w is given by the coefficient of x^w in

$$\frac{x^{2^{\theta-1}-1}}{(1-x^2)(1-x^3)\ldots(1-x^\theta)}.$$

Passing to the simplest particular cases, I put $\mu = \frac{1}{x}$, and consider the factorizations of the polynomial

$$x^\theta + B_2 x^{\theta-2} + B_3 x^{\theta-3} + \ldots + B_\theta,$$

which exhibit it as the product of two polynomials each wanting the second term.

Degree 2, $\theta = 2$.

We have $$x^2 + B_2,$$

and a factor, if it exist, must be simply x, which necessitates

$$\beta_1\beta_2 = B_2 = 0.$$

Hence, in the reducing identity above considered, the terms in B_2 vanish, and no symmetric function which appears as a coefficient of any power of B_2 can be exhibited in a reduced form; such functions are comprised in the series

$$(2),\ (2^2),\ (2^3),\ \ldots,$$

which therefore are all irreducible.

Hence $$(2^\lambda)$$

expresses all perpetuants of degree 2, and the generating function is

$$\frac{x^2}{1-x^2}.$$

Degree 3, $\theta = 3$.

We have $$x^3 + B_2 x + B_3,$$

and one factor must be x.

Thence $$\beta_1\beta_2\beta_3 = -B_3 = 0.$$

All terms of the form $$(3^{\kappa+1}2^\lambda)\, B_3^{\kappa+1} B_2^\lambda$$

disappear from the reducing identity.

The series of perpetuants of degree 3 are included in

$$(3^{\kappa+1}2^\lambda),$$

and the generating function is

$$\frac{x^3}{(1-x^2)(1-x^3)}.$$

Degree 4, $\theta = 4$.

We have $$x^4 + B_2 x^2 + B_3 x + B_4.$$

The factors may have the forms

$$x, \quad x^2 + P.$$

For the factor x, we have

$$\beta_1\beta_2\beta_3\beta_4 = B_4 = 0,$$

while, for the factor x^2+P,

$$\Pi(\beta_1+\beta_2) \equiv B_3 = 0,$$

or the whole condition is

$$\beta_1\beta_2\beta_3\beta_4\,\Pi(\beta_1+\beta_2) \equiv B_4 B_3 = 0.$$

Hence all the terms of the form

$$(4^{\kappa+1}3^{\lambda+1}2^\mu)\, B_4^{\kappa+1} B_3^{\lambda+1} B_2^\mu$$

disappear from the reducing identity, and the whole series of perpetuants of degree 4 are comprised in the expression

$$(4^{\kappa+1}3^{\lambda+1}2^\mu).$$

and the generating function is

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Degree 5, $\theta = 5$.

We have $$x^5+B_2x^3+B_3x^2+B_4x+B_5,$$

and the required factors can assume the forms

$$x,\ x^2+P.$$

For the factor x, $$\Pi\beta_1 = -B_5 = 0,$$

the condition for the factor x^2+P is clearly the eliminant of

$$x^5+B_2x^3+B_3x^2+B_4x+B_5$$

and $$x^5+B_2x^3-B_3x^2+B_4x-B_5,$$

or of $$x^4+B_2x^2+B_4$$

and $$B_3x^2+B_5,$$

which is $$\Pi(\beta_1+\beta_2) \equiv B_5^2-B_5B_3B_2+B_4B_3^2;$$

hence the complete condition is

$$\Pi\beta_1\Pi(\beta_1+\beta_2) \equiv B_5^3-B_5^2B_3B_2+B_5B_4B_3^2=0.$$

On the left-hand side of the reducing identity, we have, with others, the three terms

$$(5^3)\,B_5^3+(5^232)\,B_5^2B_3B_2+(543^2)\,B_5B_4B_3^2,$$

and each term separately would be reducible but for the condition

$$B_5^3-B_5^2B_3B_2+B_5B_4B_3^2=0,$$

which indicates that we can eliminate from the reducing identity *either* of the products

$$B_5^3,\quad B_5^2B_3B_2,\quad B_5B_4B_3^2,$$

and thus obtain two instead of three reducible non-unitariants from the three

$$(5^3),\quad (5^232),\quad (543^2).$$

Eliminating B_5^3, we have, in the reducing identity

$$\{(5^3)+(5^232)\}\,B_5^2B_3B_2+\{-(5^3)+(543^2)\}\,B_5B_4B_3^2,$$

indicating the reducibility of the non-unitariants

$$(5^3)+(5^2 32),$$
$$-(5^3)+(543^2).$$

If, instead, we eliminate $B_5^2 B_3 B_2$ and $B_5 B_4 B_3^2$, we obtain respectively

$$\{(5^3)+(5^2 32)\} B_5^3 + \{(5^2 32)+(543^2)\} B_5 B_4 B_3^2,$$

and $$\{(5^3)-(543^2)\} B_5^3 + \{(5^2 32)+(543^2)\} B_5^2 B_3 B_2.$$

If we agree to consider a non-unitariant reducible, if it can be expressed in terms of non-unitariants subsequent to it in dictionary order and of compound forms of the same degree, we may regard

$$(5^3) \text{ and } (5^2 32)$$

as reducible, as being capable of reduction by the aid of the form (543^2), which is subsequent to them in dictionary order. Hence we regard

$$(543^2)$$

as the exemplar perpetuant of degree 5 and weight 15.

The forms (5^3), $(5^2 32)$ may be said to be non-exemplar.

All exemplar perpetuants of degree 5 are comprised in the expression

$$(5^{\kappa+1} 4^{\lambda+1} 3^{\mu+2} 2^{\nu}).$$

For a given weight w we have a number of equations of condition between the products of the quantities B_2, B_3, B_4, B_5 equal to the number of ways of composing the number $w-15$ with the parts 2, 3, 4, 5; these are formed by multiplying the left-hand side of the equation

$$B_5^3 - B_5^2 B_3 B_2 + B_5 B_4 B_3^2 = 0$$

by each product of the quantities B_2, B_3, B_4, B_5 of weight $w-15$. There are also precisely the same number of exemplar perpetuant forms of degree 5 and weight w. From these equations of condition and the equation of reducibility of weight w, we can eliminate all the products which contain the factor

$$B_5 B_4 B_3^2,$$

and thus exhibit the reduction of all non-exemplar non-unitariants by the aid of the exemplar forms.

Thus the generating function for degree 5 is

$$\frac{x^{15}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.$$

The reducibility of forms of degree 5 by means of products of quadric and cubic forms is, as we saw, governed by the relation

$$B_5^2 - B_5 B_3 B_2 + B_4 B_3^2 = 0,$$

which proves that of a weight lower than 10 all forms are so expressible; *ex. gr.*,

$$(53) = (3^2)(2) - (3^2 2)(a^2),$$

where a^2 has been introduced and in a covariant identity would represent the square of the quantic itself.

For the weight 10, however, the relation shows that only the combinations

$$(5^2) - (43^2)\,a,$$

$$(532) + (43^2)\,a$$

are so expressible.

Moreover, the form (43^2) is not expressible by means of products of quadric and cubic forms (say by products 2 . 3), and thus is not a quintic syzygant. It immediately follows that the form (543^2) is a perpetuant, for, had this form been reducible, the operation of decapitation (see *A. M. J.*, Vol. VII.), or in other words, the performance of

$$D_5 = \frac{1}{5!}(\partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \ldots)^5$$

on the two sides of the equation exhibiting the reduction, would have shown (43^2) as a quintic syzygant.

Algebraical results of this nature are not yielded by Stroh's untransformed theory.

Degree 6, $\theta = 6$.

This case has been worked out in detail by Professor Cayley (*loc. cit.*), but I do not hesitate to give it here, as I wish to introduce some new methods of arriving at the equations of condition.

The quantic is

$$x^6 + B_2 x^4 + B_3 x^3 + B_4 x^2 + B_5 x + B_6,$$

the required factor assuming either of the forms

$$x, \quad x^2 + P, \quad x^3 + Px + Q.$$

For the factor x, $\qquad \Pi\beta_1 = B_6 = 0,$

and, for the factor $x^2 + P$, the condition is the vanishing of the eliminant of

$$x^6 + B_2 x^4 + B_4 x^2 + B_6$$

and $$B_3x^2+B_5.$$

This is
$$\Pi(\beta_1+\beta_2)\equiv\begin{pmatrix}1, & B_2, & B_4, & B_6\\ B_3, & B_5 & & \end{pmatrix}$$
$$\equiv B_6B_3^3-B_5^3+B_5^2B_3B_2-B_5B_4B_3^2$$
$$=0.$$

The notation
$$\begin{pmatrix}1, & B_2, & B_4, & B_6\\ & & B_3, & B_5\end{pmatrix}$$
for the eliminant in question will be found convenient in what follows.

The condition introduced by the third type of factor is obtainable in a variety of simple ways, of which a few of the most interesting will be given.

The condition is, of course, equivalent to
$$\Pi(\beta_1+\beta_2+\beta_3)=0.$$

First Method.

We have the identity
$$x^6+B_2x^4+B_3x^3+B_4x^2+B_5x+B_6=(x^3+Px+Q)(x^3+Rx+S),$$
leading to the relations
$$B_2-P-R=B_3-Q-S=B_4-PR=B_5-PS-QR=B_6-QS=0.$$
Multiplying the two zero determinants
$$\begin{vmatrix}P & R & 0\\ Q & S & 0\\ 1 & 1 & 0\end{vmatrix},\qquad\begin{vmatrix}R & P & 0\\ S & Q & 0\\ 1 & 1 & 0\end{vmatrix},$$
we obtain
$$\begin{vmatrix}2PR & PS+QR & P+R\\ PS+QR & 2QS & Q+S\\ P+Q & Q+S & 2\end{vmatrix}=0,$$
or
$$\begin{vmatrix}2B_4 & B_5 & B_2\\ B_5 & 2B_6 & B_3\\ B_2 & B_3 & 2\end{vmatrix}=0,$$
or
$$(B_2^2-4B_4)(B_3^2-4B_6)-(B_2B_3-2B_5)^2=0,$$
the condition required (*cf.* Cayley, *l.c.*).

Second Method.

The quantic $\quad x^6+B_2x^4+B_3x^3+B_4x^2+B_5x+B_6,$

equated to zero, determines the abscissæ of the six points of intersection of the conic

$$y^2+(B_2x+B_3)\,y+B_4x^2+B_5x+B_6=0$$

with the cubic curve $\qquad y-x^3=0.$

If the conic break up into two right lines

$$(y+Px+Q)(y+Rx+S)=0,$$

the quantic considered is replaceable by

$$(x^3+Px+Q)(x^3+Rx+S).$$

Hence the discriminant of the conic with regard to y must be a perfect square.

This discriminant is

$$(B_2^2-4B_4)\,x^2+2\,(B_2B_3-2B_5)\,x+B_3^2-4B_6.$$

Hence $\qquad (B_2^2-4B_4)(B_3^2-4B_6)-(B_2B_3-2B_5)^2=0,$

the same result as before.

This may be written

$$4B_4B_6-B_2^2B_6-B_5^2+B_5B_3B_2-B_4B_3^2=0.$$

The complete condition is thus

$$\begin{aligned}&\Pi\beta_1\,\Pi\,(\beta_1+\beta_2)\,\Pi\,(\beta_1+\beta_2+\beta_3)\\ &\equiv B_6\,(B_6B_3^3-B_5^3+B_5^2B_3B_2-B_5B_4B_2^2)\\ &\qquad\times(4B_6B_4-B_6B_2^2-B_5^2+B_5B_3B_2-B_4B_3^2)\\ &=0,\end{aligned}$$

or, as calculated by Professor Cayley,

$$\begin{aligned}&4B_6^3B_4B_3^3-B_6^3B_3^3B_2^2-4B_6^2B_5^3B_4+B_6^2B_5^3B_2^2+4B_6^2B_5^2B_4B_3B_2\\ &-B_6^2B_5^2B_3^3-B_6^2B_5^3B_3B_2^3-4B_6^2B_5B_4^2B_3^2+B_6^2B_5B_4B_3^3B_2^2\\ &+B_6^2B_5B_3^4B_2-B_6^2B_4B_3^5+B_6B_5^5-2B_6B_5^4B_3B_2+2B_6B_5^3B_4B_3^2\\ &+B_6B_5^3B_3^3B_2^2-2B_6B_5^2B_4B_3^3B_2+B_6B_5B_4^2B_3^4=0.\end{aligned}$$

The last term, in dictionary order, being $B_6 B_5 B_4^2 B_3^4$, we see that the non-unitariant

$$(654^2 3^4)$$

is the exemplar perpetuant of degree 6 and weight 31. Eliminating the remaining terms in succession between the equation of condition and the equation of reduction, we obtain the reduction of

$$6^3 4 3^3 - 4\,(654^2 3^4),$$

$$(6^2 3^3 2^2) + (654^2 3^4),$$

$$(6^2 5^3 4) + 4\,(654^2 3^4),$$

&c.

Altogether of weight 31 there are 16 non-exemplar forms reducible by the aid of $(654^2 3^4)$.

All perpetuants of degree 6 are included in the expression

$$(6^{\kappa+1} 5^{\lambda+1} 4^{\mu+2} 3^{\nu+4} 2^{\pi}),$$

and the enumeration is given by the generating function

$$\frac{x^{31}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)}.$$

Degree θ.

At this point it will be convenient to determine the general expression for all exemplar perpetuants of degree θ.

Expressed in terms of $\beta_1, \beta_2, \ldots \beta_\theta$ the equation of condition is

$$J_\theta = \overset{\theta}{\Pi}\beta_1 \overset{\theta}{\Pi}(\beta_1+\beta_2) \overset{\theta}{\Pi}(\beta_1+\beta_2+\beta_3) \ldots \overset{\theta}{\Pi}(\beta_1+\beta_2+\ldots+\beta_m),$$

where $m \ngtr \tfrac{1}{2}\theta$.

When $\beta_m = 0$,

$$\frac{J_\theta}{B_\theta} = B_{\theta-1}\left\{\overset{\theta-1}{\Pi}(\beta_1+\beta_2)\overset{\theta-1}{\Pi}(\beta_1+\beta_2+\beta_3)\ldots\overset{\theta-1}{\Pi}(\beta_1+\beta_2+\ldots+\beta_n)\right\}^2,$$

where $n \ngtr \tfrac{1}{2}(\theta-1)$;

therefore $J_\theta = \dfrac{B_\theta}{B_{\theta-1}} J_{\theta-1}^2 +$ terms involving higher powers of B_θ.

Let P_θ denote the B product which corresponds to the simplest exemplar perpetuant of degree θ. Then

$$P_\theta = \frac{B_\theta}{B_{\theta-1}} P_{\theta-1}^2,$$

and, assuming $P_{\theta-1} = B_{\theta-1} B_{\theta-2} B^2_{\theta-3} B^4_{\theta-4} B^8_{\theta-5} \dots B_3^{2^{\theta-5}}$,

we find $P_\theta = B_\theta B_{\theta-1} B^2_{\theta-2} B^4_{\theta-3} B^8_{\theta-4} \dots B_3^{2^{\theta-4}}$,

justifying the assumption and establishing that the exemplar perpetuant of degree θ, and of weight $2^{\theta-1}-1$ is, when $\theta > 2$,

$$(\theta,\ \theta-1,\ \theta-2^2,\ \theta-3^4 \dots 3^{2^{\theta-4}}),$$

where, commencing from the left, $\theta-2$ *different* symbols are written down to make up the partition.

Also the general form of exemplar perpetuants is

$$(\theta^{\kappa_1+1},\ \theta-1^{\kappa_2+1},\ \theta-2^{\kappa_3+2},\ \theta-3^{\kappa_4+4}, \dots 3^{\kappa_{\theta-2}+2^{\theta-4}},\ 2^{\kappa_{\theta-1}}).$$

If we know the whole of the non-exemplar perpetuants of degree $\theta-1$, we can derive the whole of those non-exemplar perpetuants of degree θ which involve in their partitions the number θ unrepeated.

For $J_\theta = \dfrac{B_\theta}{B_{\theta-1}} J^2_{\theta-1} +$ terms involving higher powers of B_θ, therefore,

$$\begin{aligned} J_6 &= \frac{B_6}{B_5}(B_5^3 - B_5^2 B_3 B_2 + B_5 B_4 B_3^2)^2 + \dots \\ &= B_6 B_5^5 - 2B_6 B_5^4 B_3 B_2 + 2B_6 B_5^3 B_4 B_3^2 \\ &\quad + B_6 B_5^3 B_3^2 B_2^2 - 2B_6 B_5^2 B_4 B_3^3 B_2 + B_6 B_5 B_4^2 B_3^4 \\ &\quad + \dots, \end{aligned}$$

which a reference to the value of J_6, already calculated, shows to be correct.

A considerable portion of J_θ may be written down from the results already obtained; for

$$J_\theta = \frac{B_\theta}{B_{\theta-1}} J^2_{\theta-1} + \dots$$

leads to $$J_\theta = \frac{B_\theta B_{\theta-1} B^2_{\theta-2} B^4_{\theta-3} \dots B_{\mu+1}^{2^{\theta-\mu-2}}}{B_\mu^{2^{\theta-\mu-1}}} J_\mu^{2^{\theta-\mu}} + \dots,$$

whence $$J_\theta = \frac{B_\theta B_{\theta-1} B^2_{\theta-2} B^4_{\theta-3} \dots B_7^{2^{\theta-8}}}{B_6^{2^{\theta-7}}} J_6^{2^{\theta-6}} + \dots,$$

when we know the complete value of J_6.

The simplest *non-exemplar* perpetuant of degree θ is easily found, for

$$J_\theta = \ldots - 2B_6 B_5^2 B_4 B_3^3 B_2 + B_6 B_5 B_4^2 B_3^4,$$

and the term in J_θ which precedes the B product corresponding to the exemplar form, in dictionary order, is

$$\frac{B_\theta B_{\theta-1} B_{\theta-2}^2 B_{\theta-3}^4 \ldots B_7^{2^{\theta-8}}}{B_6^{2^{\theta-7}}} B_6 B_5^2 B_4 B_3^3 B_2 \left(B_6 B_5 B_4^2 B_3^4\right)^{2^{\theta-6}-1},$$

or $$B_\theta B_{\theta-1} B_{\theta-2}^2 B_{\theta-3}^4 \ldots B_7^{2^{\theta-8}} B_6^{2^{\theta-7}} B_5^{2^{\theta-6}+1} B_4^{2^{\theta-5}-1} B_3^{2^{\theta-4}-1} B_2,$$

for $$\theta > 6.$$

The simplest non-exemplar perpetuant thus has the partition

$$(\theta\, \theta-1\, \theta-2^2\, \theta-3^4 \ldots 7^{2^{\theta-8}}\, 6^{2^{\theta-7}}\, 5^{2^{\theta-6}+1}\, 4^{2^{\theta-5}-1}\, 3^{2^{\theta-4}-1}\, 2),$$

for $$\theta > 6,$$

and this gives for

$\theta = 7$ $\quad (7 6 5^3 4^3 3^7 2),$

$\theta = 8$ $\quad (8 7 6^2 5^5 4^7 3^{15} 2),$

$\theta = 9$ $\quad (9 8 7^2 6^4 5^9 4^{15} 3^{31} 2),$

and so on.

The calculation of the complete value of J_7 is a very laborious matter, as it contains several hundreds of terms. Moreover, special methods of elimination lead to extraneous factors which are very troublesome.

In a similar manner it is possible to find the perpetuant solutions of the partial differential equation

$$\frac{d}{da_\lambda} + a_1 \frac{d}{da_{\lambda+1}} + a_2 \frac{d}{da_{\lambda+2}} + \ldots = 0.$$