On the Expansion of some Infinite Products. By Prof. L. J. ROGERS. Received June 5th, 1893. Read June 8th, 1893.

1. It is a well-known theorem that, if $q < 1$, then

$$1/(1-\lambda)(1-\lambda q)(1-\lambda q^2)\ldots = 1+\frac{\lambda}{1-q}+\frac{\lambda^2}{(1-q)(1-q^2)}+\ldots \quad \ldots(1).$$

It will be found convenient to use the symbol (λ) for the infinite product $(1-\lambda)(1-\lambda q)(1-\lambda q^2)\ldots$, and to write the above equation in the form

$$1/(\lambda) = 1+\Sigma\frac{\lambda^r}{(1-q^r)!},$$

where r is to receive all positive integral values, and where $(1-q^r)!$ denotes the product $(1-q)(1-q^2)\ldots(1-q^r)$.

The following abbreviations will also be used in the following pages:—

$H_r(\lambda_1, \lambda_2, \lambda_3, \ldots)$ will denote the coefficient of x^r in

$$1/(\lambda_1 x)(\lambda_2 x)(\lambda_3 x)\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

while $h_r(\lambda_1, \lambda_2, \ldots)$ will be used for $H_r(\lambda_1, \lambda_2, \ldots)(1-q^r)!$. Moreover $H_r(\mu_1, \mu_2, \ldots/\lambda_1, \lambda_2, \ldots)$ will be written for the coefficient of x^r in

$$(\mu_1 x)(\mu_2 x)\ldots \div (\lambda_1 x)(\lambda_2 x)\ldots,$$

while $h_r(\mu_1, \mu_2\ldots/\lambda_1, \lambda_2, \ldots)$ will $= (1-q^r)!\, H_r(\mu_1, \mu_2, \ldots/\lambda_1, \lambda_2, \ldots)$.

Thus $\quad 1+\Sigma x^r H_r(\lambda, \mu) = 1/(\lambda x)(\mu x)$

$$= \left\{1+\frac{\lambda x}{1-q}+\frac{\lambda^2 x^2}{(1-q)(1-q^2)}+\ldots\right\}\left\{1+\frac{\mu x}{1-q}+\frac{\mu^2 x^2}{(1-q)(1-q^2)}+\ldots\right\},$$

whence

$$h_r(\lambda, \mu) = \lambda^n+\frac{1-q^n}{1-q}\lambda^{n-1}\mu+\frac{(1-q^n)(1-q^{n-1})}{(1-q)(1-q^2)}\lambda^{n-2}\mu^2+\ldots\mu^n \quad \ldots(3),$$

a series having a certain resemblance to the ordinary binomial expansion.

The symbols of operation δ, η with respect to any quantity λ will be defined by the equations

$$\delta f(\lambda) \equiv \frac{f(\lambda)-f(\lambda q)}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\eta f(\lambda) \equiv f(\lambda q).$$

Thus $$\delta\lambda^r = \lambda^{r-1}(1-q^r),$$

$$\delta^2\lambda^r = \lambda^{r-2}(1-q^r)(1-q^{r-1}), \text{ \&c.}$$

The symbols δ_m, η_m will be used in the above sense with reference to the quantity λ_m, so that

$$\eta_1 f(\lambda_1) = f(\lambda_1 q), \text{ \&c.}$$

It is easy to see then that the symbol $\frac{1}{(\lambda\delta_1)}$, *i.e.*

$$1+\frac{\lambda\delta_1}{1-q}+\frac{\lambda^2\delta_1}{(1-q)(1-q^2)}+\ldots,$$

operating on λ_1^r, gives

$$\lambda_1^r+\frac{1-q^r}{1-q}\lambda\lambda_1^{r-1}+\frac{(1-q^r)(1-q^{r-1})}{(1-q)(1-q^2)}\lambda^2\lambda_1^{r-2}+\ldots = h_r(\lambda, \lambda_1), \text{ by (3).}$$

Hence $$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)} = 1+\Sigma\lambda_2^r H_r(\lambda, \lambda_1) = \frac{1}{(\lambda\lambda_2)(\lambda_1\lambda_2)} \quad \ldots\ldots\ldots(5).$$

We may now establish certain lemmata which will be useful hereafter for reference.

Lemma I.—If K_0, K_1 ... are independent of λ_1, then

$$\frac{1}{(\lambda\delta_1)}(K_0+K_1\lambda_1+K_2\lambda_1^2+\ldots) = K_0+K_1h_1(\lambda, \lambda_1)+K_2h_2(\lambda, \lambda_1)+\ldots.$$

This follows obviously from what has gone before.

Lemma II.—The operation $(\lambda\delta_1)$ will be equivalent to

$$\left(\frac{\lambda}{\lambda_1}\right)\frac{1}{\left(\frac{\lambda}{\lambda_1}\eta_1\right)},$$

i.e. $$\left(\frac{\lambda}{\lambda_1}\right)\left\{1-\frac{\lambda}{\lambda_1}\cdot\frac{1}{1-q}\eta_1+\frac{\lambda^2}{\lambda_1^2}\cdot\frac{1}{(1-q)(1-q^2)}\eta_1^2+\ldots\right\}.$$

For $$\delta_1 = \frac{1-\eta_1}{\lambda_1},$$

$$\delta_1^2 = \frac{1-\eta_1}{\lambda_1^2}-\frac{1-\eta_1}{\lambda_1^2 q}\eta_1 = \frac{(1-\eta_1)(q-\eta_1)}{\lambda_1^2 q},$$

$$\delta_1^3 = \frac{(1-\eta_1)(q-\eta_1)(q^2-\eta_1)}{\lambda_1^3 q^3},$$

and $$\delta_1^r = \frac{(1-\eta_1)(q-\eta_1)\dots(q^{r-1}-\eta_1)}{\lambda_1^r q^{\frac{1}{2}r(r-1)}}.$$

Now it is well known that

$$(x) = 1 - \frac{x}{1-q} + \frac{x^2 q}{(1-q)(1-q^2)} - \frac{x^3 q^3}{(1-q)(1-q^2)(1-q^3)} + \dots;$$

therefore

$$(\lambda\delta_1) = 1 + \frac{\lambda}{\lambda_1} \cdot \frac{\eta_1 - 1}{1-q} + \frac{\lambda^2}{\lambda_1^2} \cdot \frac{(\eta_1-1)(\eta_1-q)}{(1-q)(1-q^2)} + \dots.$$

But it is a known theorem that

$$\frac{(ax)}{(x)} = 1 + \frac{1-a}{1-q}x + \frac{(1-a)(1-aq)}{(1-q)(1-q^2)}x^2 + \dots \qquad \dots\dots\dots(6);$$

therefore $$(\lambda\delta_1) = \left(\frac{\lambda}{\lambda_1}\right)\frac{1}{\left(\frac{\lambda}{\lambda_1}\eta_1\right)},$$

if we remember that η_1 is not to operate on the coefficients involving $\frac{\lambda}{\lambda_1}$, *i.e.* $\left(\frac{\lambda}{\lambda_1}\right)$, and the powers of $\left(\frac{\lambda}{\lambda_1}\right)$ always precede the operations η_1, η_1^2, ... in the expansion.

Lemma III. $\frac{1}{(\lambda\delta_1)}(\lambda_1-\lambda)\,\psi(\lambda_1)$ will $= \lambda_1 \frac{1}{(\lambda q\delta_1)}\,\psi(\lambda_1)$.

For $$(\lambda\delta_1)\,\lambda_1 f(\lambda_1)$$

$$= \left(\frac{\lambda}{\lambda_1}\right)\left\{1 + \frac{\lambda}{\lambda_1} \cdot \frac{\eta_1}{1-q} + \frac{\lambda^2}{\lambda_1^2} \cdot \frac{\eta_1^2}{(1-q)(1-q^2)} + \dots\right\}\lambda_1 f(\lambda_1)$$

(by Lemma II.)

$$= \lambda_1\left(\frac{\lambda}{\lambda_1}\right)\left\{1 + \frac{\lambda}{\lambda_1} \cdot \frac{q\eta_1}{1-q} + \frac{\lambda^2}{\lambda_1^2} \cdot \frac{q^2\eta_1^2}{(1-q)(1-q^2)} + \dots\right\}f(\lambda_1)$$

$$= \lambda_1\left(\frac{\lambda}{\lambda_1}\right)\frac{1}{\left(\frac{\lambda q}{\lambda_1}\eta_1\right)}f(\lambda_1)$$

$$= (\lambda_1-\lambda)(\lambda q\delta_1)\,f(\lambda_1) \qquad \text{(by Lemma II.)}.$$

Let
$$\psi(\lambda_1) = (\lambda q\delta_1) f(\lambda_1),$$
so that
$$f(\lambda_1) = \frac{1}{(\lambda q\delta_1)} \psi(\lambda_1),$$
then
$$(\lambda_1-\lambda)\,\psi(\lambda_1) = (\lambda\delta_1)\,\lambda_1 f(\lambda_1),$$
i.e.,
$$\frac{1}{(\lambda\delta_1)}(\lambda_1-\lambda)\,\psi(\lambda_1) = \lambda_1 \frac{1}{(\lambda q\delta_1)}\psi(\lambda_1).$$

Similarly $\frac{1}{(\lambda\delta_1)}(\lambda_1-\lambda)(\lambda_1-\lambda q)\,\psi(\lambda_1) = \lambda_1 \frac{1}{(\lambda q\delta_1)}(\lambda_1-\lambda q)\,\psi(\lambda_1)$
$$= \lambda_1^2 \frac{1}{(\lambda q^2\delta_1)}\psi(\lambda_1), \text{ \&c.}$$

Lemma IV. $\frac{1}{(\lambda\delta_1)}\cdot\frac{\psi(\lambda_1)}{(\lambda_1\mu)} = \frac{1}{(\lambda_1\mu)}\cdot\frac{1}{(\lambda\mu\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\psi(\lambda_1),$

where μ is independent of λ_1.

Since
$$\frac{(\lambda\mu)}{(\lambda_1\mu)} = 1+\frac{\lambda_1-\lambda}{1-q}\mu+\frac{(\lambda_1-\lambda)(\lambda_1-\lambda q)}{(1-q)(1-q^2)}\mu^2+\dots;$$
therefore
$$\frac{1}{(\lambda\delta_1)}\cdot\frac{\psi(\lambda_1)}{(\lambda_1\mu)}$$
$$= \frac{1}{(\lambda\mu)}\cdot\frac{1}{(\lambda\delta_1)}\left\{1+\frac{\lambda_1-\lambda}{1-q}\mu+\dots\right\}\psi(\lambda_1)$$
$$= \frac{1}{(\lambda\mu)}\left\{\frac{1}{(\lambda\delta_1)}+\frac{\lambda_1\mu}{1-q}\cdot\frac{1}{(\lambda q\delta_1)}+\frac{\lambda_1^2\mu^2}{(1-q)(1-q^2)}\cdot\frac{1}{(\lambda q^2\delta_1)}+\dots\right\}\psi$$
(by Lemma III.)
$$= \frac{1}{(\lambda\mu)}\left\{1+\frac{\lambda_1\mu(1-\lambda\delta_1)}{1-q}+\frac{\lambda_1^2\mu^2(1-\lambda\delta_1)(1-\lambda q\delta_1)}{(1-q)(1-q^2)}+\dots\right\}\frac{1}{(\lambda\delta_1)}\psi$$
$$= \frac{1}{(\lambda\mu)}\cdot\frac{1}{(\lambda_1\mu)}(\lambda\mu\lambda_1\delta_1)\frac{1}{(\lambda\delta_1)}\psi$$
$$= \frac{1}{(\lambda_1\mu)}\cdot\frac{1}{(\lambda\mu\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\psi(\lambda_1), \quad \text{(by Lemma II.)}.$$

2. Let us now find the value of
$$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)},$$

which, when expanded as a series, gives

$$\frac{1}{(\lambda\delta_1)}\{1+\Sigma\lambda_1^r H_r(\lambda_2, \lambda_3)\}$$

$$= 1+\Sigma H_r(\lambda_2, \lambda_3)\, h_r(\lambda, \lambda_1), \quad \text{(by Lemma I.)}.$$

Now, by Lemma IV.,

$$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)}$$

$$= \frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_3)}$$

$$= \frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda\lambda_3)(\lambda_1\lambda_3)} \qquad [\text{by § 1, (5)}]$$

$$= \frac{1}{(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)}\left\{1+\frac{1-\lambda_1\lambda_3}{1-q}\lambda\lambda_2+\frac{(1-\lambda_1\lambda_3)(1-\lambda_1\lambda_3 q)}{(1-q)(1-q^2)}\lambda^2\lambda_2^2+\dots\right\}$$

$$= \frac{(\lambda\lambda_1\lambda_2\lambda_3)}{(\lambda\lambda_2)(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)} \qquad [\text{by § 1, (6)}].$$

Hence

$$\frac{(\lambda\lambda_1\lambda_2\lambda_3)}{(\lambda\lambda_2)(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)} = 1+H_1(\lambda_2, \lambda_3)h_1(\lambda, \lambda_1)+H_2(\lambda_2, \lambda_3)h_2(\lambda, \lambda_1)+\dots \qquad \dots\dots(1).$$

If in this identity we write

$$\lambda = xe^{\theta i}, \quad \lambda_1 = xe^{-\theta i}, \quad \lambda_2 = e^{\phi i}, \quad \lambda_3 = e^{-\phi i},$$

we easily obtain 2 (1), on p. 176 of this volume.

3. Let us consider now the result of evaluating

$$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_1\lambda_4)},$$

which, by Lemma I.,

$$= 1+\Sigma H_r(\lambda_2, \lambda_3, \lambda_4)\, h_r(\lambda, \lambda_1), \quad \text{see § 1 (2)}.$$

By Lemma IV., we get

$$\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda_1\lambda_3)}\cdot\frac{1}{(\lambda\lambda_3\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_4)}$$

$$=\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{(\lambda\lambda_1\lambda_3\lambda_4)}{(\lambda_1\lambda_3)(\lambda_1\lambda_4)(\lambda\lambda_3)},\text{ as in the preceding section,}$$

$$=\frac{1}{(\lambda\lambda_3)(\lambda_1\lambda_2)}\left\{1+\frac{\lambda\lambda_2}{1-q}\eta_1+\dots\right\}\frac{(\lambda\lambda_1\lambda_3\lambda_4)}{(\lambda_1\lambda_3)(\lambda_1\lambda_4)}$$

$$=\frac{(\lambda\lambda_1\lambda_3\lambda_4)}{(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_1\lambda_4)}\left\{1+\frac{(1-\lambda_1\lambda_3)(1-\lambda_1\lambda_4)}{(1-q)(1-\lambda\lambda_1\lambda_3\lambda_4)}\lambda\lambda_2+\dots\right\}.$$

This last series is usually called a Heinean series, and has been very fully discussed in Heine's *Kugelfunctionen*, under the symbol

$$\phi\left\{\lambda_1\lambda_3,\ \lambda_1\lambda_4,\ \lambda\lambda_1\lambda_3\lambda_4,\ q,\ \lambda\lambda_2\right\},$$

Heine there shows that

$$\phi\left\{a,b,c,q,x\right\}=\frac{(ax)(b)}{(x)(c)}\,\phi\left\{\frac{c}{b},\ x,\ ax,\ q,\ b\right\},$$

and, by transforming the latter series by continual application of the same formula, he obtains a large series of equivalent expressions.

They may all be easily proved by considering the symmetry existing among λ_2, λ_3, λ_4 in the above expression, and by the symmetry in λ and λ_1, but he does not establish the connection between

$$\phi\left\{\lambda_1\lambda_3,\ \lambda_1\lambda_4,\ \lambda\lambda_1\lambda_3\lambda_4,\ q,\ \lambda\lambda_2\right\}$$

and $$1+\Sigma H_r(\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1).$$

By writing $\lambda=e^{\theta i}$, $\lambda_1=e^{-\theta i}$, we may write the above relation in the form

$$\phi\left\{\lambda_3e^{-\theta i},\ \lambda_4e^{-\theta i},\ \lambda_3\lambda_4,\ q,\ \lambda_2e^{\theta i}\right\}(\lambda_2e^{\theta i})(\lambda_3\lambda_4)$$

$$=\frac{1}{P(\lambda_2)\,P(\lambda_3)\,P(\lambda_4)}\left\{1+A_1(\theta)H_1(\lambda_2,\lambda_3,\lambda_4)+A_2(\theta)H_2(\lambda_2\,\lambda_3,\lambda_4)+\dots\right\},$$

a result first obtained on p. 175 of this volume, where a definition of $A_r(\theta)$ is also given.

By putting $\lambda_1=0$, we, of course, return to the formulae in the last section.

4. If in the formula § 2 (1), we put $\lambda_2 = \mu$, $\lambda_3 = \nu$, $\lambda = e^{\theta i}$, $\lambda_1 = e^{-\theta i}$, we get

$$\frac{(\mu\nu)}{P(\mu)\,P(\nu)} = 1 + \Pi_1(\mu, \nu)\,A_1(\theta) + A_2(\mu, \nu)\,A_2(\theta) + \dots,$$

where
$$1 + \Sigma k^r \Pi_r(\mu, \nu) \equiv \frac{1}{(k\mu)(k\nu)}.$$

This leads to an equation connecting the product of any two of the A's with a linear function of the A's.

For we have, since

$$\frac{1}{P(\mu)} = 1 + \frac{\mu}{1-q} A_1(\theta) + \dots,$$

$$\left\{1 + \Sigma \frac{\mu^r}{(1-q^r)!} A_r(\theta) + \dots\right\} \left\{1 + \Sigma \frac{\nu^r}{(1-q^r)!} A_r(\theta)\right\}$$
$$= \left\{1 + \Sigma \frac{\mu^r \nu^r}{(1-q^r)!}\right\} \left\{1 + \Sigma \Pi_r(\mu, \nu)\,A_r(\theta)\right\}.$$

Equating coefficients of $\mu^r\nu^s$, where, say, $s > r$, we see that

$$\begin{aligned}\frac{A_r(\theta)\,A_s(\theta)}{(1-q^r)!\,(1-q^s)!} = \; & A_{r+s}(\theta)\,\{\text{coeff. of } \mu^r\nu^s \text{ in } \Pi_{r+s}(\mu, \nu)\} \\ & + \frac{1}{1-q} A_{r+s-2}(\theta)\,\{\text{coeff. of } \mu^{r-1}\nu^{s-1} \text{ in } \Pi_{r+s-2}(\mu, \nu)\} \\ & + \dots \\ & + \frac{1}{(1-q^r)!} A_{s-r}(\theta)\,\{\text{coeff. of } \nu^{s-r} \text{ in } \Pi_{s-r}(\mu, \nu)\}.\end{aligned}$$

From this we see that no absolute term occurs unless $r = s$, in which case the absolute term in $A_r(\theta)^2$ is $(1-q^r)!$ ....(1).

These results lead to a very interesting formula expressing the quotient of any series $K_0 + K_1 A_1(\theta) + K_2 A_2(\theta) + \dots$ by the product $P(\lambda)$.

Let this quotient, *i.e.* the product of the K-series with

$$1 + \Sigma \frac{\lambda^r A_r(\theta)}{(1-q^r)!},$$

be denoted by
$$L_0 + L_1 A_1(\theta) + \dots.$$

In multiplying these series, we get a series of terms containing products of the form $A_r(\theta)\,A_s(\theta)$.

By (1), however, we easily see that the absolute term

$$L_0 = K_0 + K_1\lambda + K_2\lambda^2 + \dots \quad \dots\dots\dots\dots\dots\dots\dots(2),$$

which may be called the generating function of the K series, and it is obvious that the generating function in λ of the KA series for $1/P(\lambda_1)\,P(\lambda_2)\,\dots\,P(\lambda_r)$ is symmetrical in $\lambda, \lambda_1, \lambda_2, \dots \lambda_r$, since it is the absolute term in the expansion of $1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,\dots\,P(\lambda_r)$.

The following coefficient may be expressed in a very simple symbolical form involving the operation δ.

Now, supposing the K's not to contain λ, we see that

$$\delta\,\{L_0 + L_1A_1(\theta) + \dots\} = \{K_0 + K_1A_1(\theta) + \dots\}\,\frac{2\cos\theta - \lambda}{P(\lambda)}$$

$$= (2\cos\theta - \lambda)\,\{L_0 + L_1A_1(\theta) + \dots\}.*$$

But $$A_{r+1}(\theta) + (1-q^r)\,A_{r-1}(\theta) = 2\cos\theta\,.\,A_r(\theta).$$

Equating coefficients of A_r, we get

$$(1-q^{r+1})\,L_{r+1} = \lambda L_r - L_{r-1} + \delta L_r \quad \dots\dots\dots\dots\dots\dots(3).$$

Thus $$(1-q)\,L_1 = (\lambda+\delta)\,L_0,$$

$$\begin{aligned}(1-q)(1-q^2)\,L_2 &= \lambda L_1 - (1-q)\,L_0 + (1-q)\,\delta L_1 \\ &= (L_0 - q\eta L_0) + \delta^2 L_0 \\ &= (\lambda^2+\lambda\delta)\,L_0 - (1-q)\,L_0 + L_0 - q\eta L_0 + \delta^2 L_0 \\ &= (\lambda^2+\lambda\delta)\,L_0 + qL_0 - q\,(L_0 - \lambda\delta L_0) + \delta^2 L_0 \\ &= \lambda^2 L_0 + (1+q)\,\lambda\delta L_0 + \delta^2 L_0.\end{aligned}$$

That is, L_1, L_2 are easily seen to be the $H_1(\lambda, \delta)\,L_0$ and $H_2(\lambda, \delta)\,L_0$, where H has the meaning assigned to it at the beginning of this paper. It must be noticed, however, that λ and δ are not inter-

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* For $$1/P(\lambda) = 1 + \Sigma\,\frac{\lambda^r A_r(\theta)}{(1-q^r)\,!};$$

therefore $$1/P(\lambda q) = (1 - 2\lambda\cos\theta + \lambda^2)/P(\lambda) = 1 + \Sigma\,\frac{\lambda^r q^r A_r(\theta)}{(1-q^r)\,!}.$$

Equating coefficient of λ^{r+1}, we get the relation required.

changeable, since δ operates on λ, but that λ must precede δ in all the terms.

We may now establish the general term L_r by induction. Assume

$$L_r = H_r(\lambda, \delta)\, L_0, \text{ for all values of } r \text{ up to } r \quad \ldots\ldots(4).$$

Then $\quad (1-q^{r+1})\, L_{r+1} = (\lambda+\delta)\, H_r(\lambda, \delta)\, L_0 - H_{r-1}(\lambda, \delta)\, L_0.$

The coefficient of $\lambda^m\delta^n$ in $H_r(\lambda, \delta)$ is

$$\frac{1}{(1-q^m)!\,(1-q^n)!},$$

where $\quad m+n = r.$

Now $\quad \delta\lambda^m\delta^n f(\lambda) = \lambda^{m-1}\delta^n f(\lambda) - \lambda^{m-1} q^m \eta \delta^n f(\lambda)$

$$= \lambda^{m-1}\delta^n f(\lambda) - q^m\left\{\lambda^{m-1}\delta^n f(\lambda) - \lambda^m\delta^{n+1} f(\lambda)\right\}$$

$$= (1-q^m)\,\lambda^{m-1}\delta^n f(\lambda) + q^m\lambda^m\delta^{n+1} f(\lambda).$$

The first of these terms obviously cancels with the terms containing $\lambda^{m-1}\delta^n$ in $H_{r-1}(\lambda, \delta)$, so that we get

$$(1-q^{r+1})\, L_{r+1} = \Sigma\left\{\frac{q^m}{(1-q^m)!\,(1-q^n)!} + \frac{1}{(1-q^{m-1})!\,(1-q^{n+1})!}\right\}\lambda^m\delta^{n+1} L_0$$

$$= (1-q^{r+1})\,\Sigma\,\frac{\lambda^m\delta^{n+1} L_0}{(1-q^m)!\,(1-q^{n+1})!}$$

$$= (1-q^{r+1})\, H_{r+1}(\lambda, \delta)\, L_0.$$

But we have established the truth of (4) for $r = 1$, $r = 2$. It is therefore true universally.

We may conveniently state the result thus:—

$$\left\{K_0 + K_1 A_1(\theta) + K_2 A_2(\theta) + \ldots\right\} \div (1 - 2\lambda\cos\theta + \lambda^2)(1 - 2\lambda q\cos\theta + \lambda^2 q^2)\ldots$$

$$= \frac{(\lambda\delta)}{P(\lambda)\,P(\delta)}\left\{K_0 + K_1\lambda + K_2\lambda^2 + \ldots\right\}$$

$$= \left\{1 + H_1(\lambda, \delta)\, A_1(\theta) + H_2(\lambda, \delta)\, A_2(\theta) + \ldots\right\}(K_0 + K_1\lambda + \ldots)\ldots(5).$$

The generating function of

$$\left\{K_0 + K_1 A_1(\theta) + \ldots\right\} / P(\lambda),$$

expressed in powers of k, is found by writing k^r for $A_r(\theta)$ in the right-hand side of (5).

This is $\{1+kH_1(\lambda,\delta)+k^2H_2(\lambda,\delta)+\ldots\}(K_0+K_1\lambda+\ldots)$

$$=\frac{1}{(k\lambda)(k\delta)}(K_0+K_1\lambda+\ldots) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

If, then, the generating function of a series in $A(\theta)$ be known, we have a convenient symbol for the generating function of the series divided by $P(\lambda)$.

5. Now we have seen that $\dfrac{(\lambda_2\lambda_3)}{P(\lambda_2)\,P(\lambda_3)}$ has a generating function (say in ascending powers of λ_1),

$$1+H_1(\lambda_2,\lambda_3)\,\lambda_1+H_2(\lambda_2,\lambda_3)\,\lambda_1^2+\ldots=\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)}.$$

Hence the generating function in λ of $1/P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)$

$$=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_2\lambda_3)},$$

which, by § 2, reduces to

$$(\lambda\lambda_1\lambda_2\lambda_3)/(\lambda\lambda_1)(\lambda\lambda_2)(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_2\lambda_3),$$

a symmetric function in λ, λ_1, λ_2, λ_3, as, by § 4 (2), we should expect to have, since the generating function in λ is the term independent of the A's in $1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)$.

This result may be stated thus :—

$$\frac{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_2\lambda_3)}{P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)}=1+\Sigma H_r(\lambda_1\lambda_2\lambda_3/\lambda_1,\lambda_2,\lambda_3)\,A_r(\theta).$$

6. Similarly, we may find simple expressions in series for the generating functions in λ of $1/P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)\,P(\lambda_4)$ and of

$$1/P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)\,P(\lambda_4)\,P(\lambda_5).$$

Calling these functions $\phi_4(\lambda)$ and $\phi_5(\lambda)$ respectively, we get, by § 4 (6),

$$\phi_4(\lambda)(\lambda_2\lambda_3)(\lambda_2\lambda_4)(\lambda_3\lambda_4)=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_1\lambda_4)}$$

$$=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\delta_1)}\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,\lambda_1^r\}\quad[\text{see § 1 (2)}]$$

$$=\frac{1}{(\lambda\lambda_1)}\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\},\ \text{by Lemma I.}$$

We may, however, also write these expressions in the form

$$\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)}{(\lambda_1\lambda_3)(\lambda_1\lambda_4)} \quad \text{(by Lemma IV.)}$$

$$=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\right\}.$$

Now, the effect of operating with $\frac{1}{(\lambda\lambda_2\eta_1)}$ on $h_r(\lambda,\lambda_1)$ is easily obtained by consideration of the coefficient of x^r in

$$\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(x\lambda)(x\lambda_1)}=\frac{(x\lambda\lambda_1\lambda_2)}{(x\lambda)(x\lambda_1)(\lambda\lambda_2)},$$

which is

$$\frac{1}{(\lambda\lambda_2)}\,h_r(\lambda\lambda_1\lambda_2/\lambda,\lambda_1).$$

Hence

$$\phi_4(\lambda)=\frac{1}{(\lambda\lambda_1)(\lambda_1\lambda_2)(\lambda_2\lambda).(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_2)}$$
$$\times\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_3,\lambda_4)\,h_r(\lambda\lambda_1\lambda_2/\lambda,\lambda_1)\right\}.$$

By 4, § 2, we have seen that $\phi_4(\lambda)$ is symmetrical in $\lambda,\lambda_1,\lambda_2,\lambda_3,\lambda_4$, so that, by an interchange of suffixes, we get a rather more convenient form,

$$\frac{1}{(\lambda\lambda_1)(\lambda\lambda_2)(\lambda\lambda_3)(\lambda\lambda_4)(\lambda_1\lambda_2)(\lambda_3\lambda_4)}\left\{1+\Sigma H_r(\lambda\lambda_1\lambda_2/\lambda_1,\lambda_2)\,h_r(\lambda\lambda_3\lambda_4/\lambda_3,\lambda_4)\right\},$$

in which λ is brought more into prominence.

Equating together these unsymmetrical expressions for the symmetrical ϕ_4, we get a series of identities connecting series built up of pairs of expressions such as $H_r(\lambda\lambda_1\lambda_2/\lambda_1,\lambda_2)$.

Again,
$$\phi_5(\lambda)(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_2)$$

$$=\frac{1}{(\lambda_5\lambda_1)}\cdot\frac{1}{(\lambda_5\delta_1)}\cdot\frac{1}{(\lambda\lambda_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\right\}$$

$$=\frac{1}{(\lambda_5\lambda_1)(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\lambda_5\eta_1)}\cdot\frac{1}{(\lambda_5\delta_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\right\}$$
(by Lemma IV.)

$$=\frac{1}{(\lambda_5\lambda_1)(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\lambda_5\eta_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1,\lambda_5)\right\}$$

$$=\frac{1}{(\lambda_5\lambda_1)(\lambda\lambda_1)(\lambda\lambda_5)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda\lambda_1\lambda_5/\lambda,\lambda_1,\lambda_5)\right\};$$

by the same principle as in the previous section,

$$\phi_5(\lambda) = \frac{1}{(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_2).(\lambda\lambda_1)(\lambda_1\lambda_5)(\lambda_5\lambda)}$$

$$\times \{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2, \lambda_3, \lambda_4)\, h_r(\lambda\lambda_1\lambda_5/\lambda, \lambda_1, \lambda_5)\}.$$

Here again we get an only partially symmetric series for the entirely symmetrical function ϕ_5, so that, by interchanging suffixes, we get a set of transformation-formulae connecting such series in general. These transformations bear a kind of analogy to Heine's.

7. Difference equations for generating functions.

It has been shown that the generating function of $1/P(\lambda_1)P(\lambda_2)$ and $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)$ may be conveniently expressed as infinite products. It is not, however, possible so to express the generating functions of $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)$, &c., but by a simple process we may obtain successively functional equations which such generating functions satisfy.

The generating function, $\phi_3(\lambda)$ say, in λ, for $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)$, found above, in § 5, evidently satisfies the functional equation

$$\phi_3(\lambda)(1-\lambda\lambda_1)(1-\lambda\lambda_2)(1-\lambda\lambda_3) = \phi_3(\lambda q)(1-\lambda\lambda_1\lambda_2\lambda_3) \quad \ldots\ldots(1).$$

Now, $\phi_3(\lambda q)$ can be made to depend on $\delta\phi_3(\lambda)$, while the latter may be made to depend on the coefficient of k in the k-generating function for $1/P(\lambda)P(\lambda_1)P(\lambda_2)P(\lambda_3)$, by § 4.

If we call this latter generating function

$$K_0 + \frac{kK_1}{1-q} + \ldots,$$

the relation (1) can be transformed into an equation connecting K_0 and K_1, which will be symmetrical in $\lambda, \lambda_1, \lambda_2, \lambda_3$.

We get from (1),

$$\phi_3(\lambda)\{\lambda_1+\lambda_2+\lambda_3-\lambda(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+\lambda^2\lambda_1\lambda_2\lambda_3-\lambda_1\lambda_2\lambda_3\}$$

$$= (1-\lambda\lambda_1\lambda_2\lambda_3)\,\delta\phi_3(\lambda).$$

Now, by § 4 (3), $\phi_3(\lambda) = K_0$, and $(\lambda+\delta)\phi_3(\lambda) = K_1$;

whence $$(p_1-p_3)K_0 = (1-p_4)K_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where p_1, p_3, p_4 are the coefficients in the equation whose roots are $\lambda, \lambda_1, \lambda_2, \lambda_3$.

From (2) we shall easily derive a connection between the first three coefficients of the generating function for

$$1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)\,P(\lambda_4),$$

which may be reduced, by § 4 (5), to a functional equation in $\phi_4(\lambda)$, $\delta\phi_4\lambda$, $\delta^2\phi_4(\lambda)$.

Now, by § 4 (6), we have seen that, if

$$K_0 + \frac{K_1}{1-q}\lambda + \dots$$

be the λ-generating function of $1/P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)$, then the k-generating function of $1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)$ is

$$\frac{1}{(k\lambda)(k\delta)}\left\{K_0 + \frac{K_1}{1-q}\lambda + \dots\right\} = L_0 + \frac{L_1}{1-q}k + \dots \text{ say,}$$

and that $$L_0 = K_0 + \frac{K_1}{1-q}\lambda + \dots = \frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_0 \qquad\qquad (3).$$

Now

$$\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_1 = \frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}(\lambda_1+\delta_1)\,K_0, \text{ by § 4 (3),}$$

$$= \frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0 + \frac{\lambda}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0 + \frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}\delta_1 K_0$$

(by Lemma III.)

$$= \frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0 + \lambda L_0 - \frac{1}{\lambda}\cdot\frac{1-\lambda\delta_1}{(\lambda\lambda_1)(\lambda\delta_1)}K_0 + \frac{1}{\lambda\,(\lambda\lambda_1)(\lambda\delta_1)}K_0$$

$$= \frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0 - \frac{1}{\lambda\,(\lambda\lambda_1)(\lambda q\delta_1)}K_0 + \frac{1}{\lambda}\lambda L_0 + \frac{1}{\lambda}L_0$$

$$= -\frac{1}{\lambda\,(\lambda q\lambda_1)(\lambda q\delta_1)}K_0 + \lambda L_0 + \frac{1}{\lambda}L_0$$

$$= (\lambda+\delta)\,L_0 = L_1.$$

Similarly, if we assume

$$\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_r = L_r \qquad\qquad (4),$$

we can prove inductively by precisely similar steps; and, remembering that, by § 4 (3),

$$K_{r+1} = (\lambda_1+\delta_1)\,K_r - (1-q^r)\,K_{r-1},$$

since we have replaced K_r by $K_r/(1-q^r)!$, and that a similar equation in the L's holds good, we get

$$\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_{r+1}=L_{r+1}.$$

Hence (4) holds good for all values of r.

$$\text{Again, }\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}\lambda_1 K_r=\frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_r+\lambda L_r\quad\text{(by Lemma III.)}$$

$$=\lambda L_r+\frac{\lambda_1}{1-\lambda\lambda_1}\eta L_r$$

$$=\frac{(\lambda+\lambda_1)L_r-\lambda\lambda_1(\lambda+\delta)L_r}{1-\lambda\lambda_1}$$

$$=\frac{(\lambda+\lambda_1)L_r-\lambda\lambda_1\{L_{r+1}+(1-q^r)L_{r-1}\}}{1-\lambda\lambda_1}\quad\dots\dots(5).$$

We are now in a position to derive an equation connecting L_0, L_1, L_2. Operating with $\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}$ on the equation

$$(p_1-p_3)K_0-(1-p_4)K_1=0,$$

where now p_1, p_3, p_4, K_0, K_1 refer to λ_1, λ_2, λ_3, λ_4 instead of to λ, λ_1, λ_2, λ_3, we see, by the help of (4) and (5) and § 4 (6), that

$$\begin{aligned}(1-\lambda_2\lambda_3-\lambda_2\lambda_4-\lambda_3\lambda_4)&\{(\lambda+\lambda_1)L_0-\lambda\lambda_1L_1\}\\&+L_0(1-\lambda\lambda_1)(\lambda_2+\lambda_3+\lambda_4-\lambda_2\lambda_3\lambda_4)-(1-\lambda\lambda_1)L_1\\&+\lambda_2\lambda_3\lambda_4\{(\lambda+\lambda_1)L_1-\lambda_1\lambda_2L_2-\lambda_1\lambda_2(1-q)L_0\}=0,\end{aligned}$$

which easily reduces to

$$(p_1-p_3+p_5q)L_0-(1-p_4)L_1-p_5L_2=0\quad\dots\dots\dots\dots\dots(6),$$

where now p_1, p_3, p_4, p_5 are coefficients in the equation whose roots are λ, λ_1, λ_2, λ_3, λ_4, so that $p_1=\lambda+\lambda_1+\lambda_2+\lambda_3+\lambda_4$, &c.

Replacing L_0 by $\phi_4(\lambda)$, L_1 by $(\lambda+\delta)\phi_4(\lambda)$, &c., by § 4 (5), we get

$$\begin{aligned}&(1-\lambda\lambda_1)(1-\lambda\lambda_2)(1-\lambda\lambda_3)(1-\lambda\lambda_4)\phi_4(\lambda)\\&\quad-\{1+\lambda_1\lambda_2\lambda_3\lambda_4q^{-1}-\lambda(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)\\&\qquad\qquad+\lambda^2(1+q)\lambda_1\lambda_2\lambda_3\lambda_4\}\phi_4(\lambda q)\\&\quad+\lambda_1\lambda_2\lambda_3\lambda_4q^{-1}\phi_4(\lambda q^2)=0\quad\dots\dots\dots\dots(7),\end{aligned}$$

a functional equation determining the generating function of $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)$, and identical with the expression given in § 6.

Again, if we increase by unity all the suffixes of the λ's involved in (6), and operate on the equation with $\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}$, we shall get a relation connecting M_0, M_1, M_2, M_3, where

$$M_0+\frac{M_1}{1-q}k+\dots$$

is the generating function of

$$1/P(\lambda)P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)P(\lambda_5).$$

This is found to be

$$(p_1-p_3+p_5q)M_0-\{1-p_4+q(1+q)p_6\}M_1-M_2p_5+M_3p_6=0\dots(8),$$

where $$p_1=\lambda+\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5,\ \&c.$$

This can be reduced to a functional equation connecting $\phi_5(\lambda)$, $\phi_5(\lambda q)$, $\phi_5(\lambda q^2)$, and $\phi_5(\lambda q^3)$.

It will be easily seen, moreover, that, since the p's only contain λ_1 to the first degree, we can always derive new equations of the type of (6) and (8) by help of (4) and (5), and will obtain linear relations between a finite number of coefficients of the generating functions, there being $r-1$ coefficients in the case of $1/P(\lambda)P(\lambda_1)\dots P(\lambda_r)$. These will be multiplied by linear functions of the symmetrical functions of $\lambda, \lambda_1, \dots \lambda_r$.

[8. By employing an operative symbol δ defined by the equation

$$\delta f(\lambda)=\frac{f(\lambda/q)-f(\lambda)}{\lambda},$$

so that $$\delta\lambda^r q^{\frac{1}{2}r(r+1)}=\lambda^{r-1}q^{\frac{1}{2}r(r-1)}(1-q^r),$$

and

$$(-\lambda q\delta_1)\frac{\lambda_1^r q^{\frac{1}{2}r(r+1)}}{(1-q^r)!}\equiv\left\{1+\frac{q}{1-q}\lambda\delta_1+\frac{q^3}{(1-q)(1-q^2)}\lambda^2\delta_1^2+\dots\right\}\frac{\lambda_1^r q^{\frac{1}{2}r(r+1)}}{(1-q^r)!}$$

$$=\text{coef. of } x^r \text{ in } (-\lambda qx)(-\lambda_1 qx),$$

we see that $$(-\lambda q\delta_1)(-\lambda_1\lambda_2 q)=(-\lambda\lambda_2 q)(-\lambda_1\lambda_2 q)\ \dots\dots\dots\dots(1).$$

Moreover, corresponding to Lemmata II., III., and IV., in § 1, we get

$$\frac{1}{(\lambda\delta_1)} = \frac{1}{\left(-\frac{\lambda}{\lambda_1}\right)}\left(-\frac{\lambda}{\lambda_1\eta_1}\right) \quad \text{.....................(2)},$$

$$(-\lambda q\delta_1)(\lambda_1-\lambda)\,\psi(\lambda_1) = \lambda_1(-\lambda\delta_1)\,\psi(\lambda_1) \quad \text{..............(3)},$$

and $(-\lambda q\delta_1)(-\lambda_1\mu q)\,\psi(\lambda_1) = (-\lambda_1\mu q)\left(-\frac{\lambda\mu q}{\eta_1}\right)(-\lambda q\delta_1)\,\psi(\lambda_1)$ ... (4).

By the help of these results we deduce that, if $B_r(\theta)$ be defined by the equation

$$P(-\lambda q) \equiv 1 + \frac{B_1(\theta)}{1-q}\lambda + \frac{B_2(\theta)}{(1-q)(1-q^2)}\lambda^2 + \ldots,$$

then $P(-xq,\,\theta+\phi)\,P(-xq,\,\theta-\phi)/(x^2q)$

$$= 1 + \frac{B_1(\theta)\,B_1(\phi)\,q^{-1}}{1-q} + \frac{B_2(\theta)\,B_2(\phi)}{(1-q)(1-q^2)}q^{-3} + \ldots,$$

and that $\frac{P(-\lambda_1 q)\,P(-\lambda_2 q)\,P(-\lambda_3 q)}{(\lambda_2\lambda_3 q)(\lambda_3\lambda_1 q)(\lambda_1\lambda_2 q)}$

$$= 1 + \Sigma H_r(\lambda_1 q,\,\lambda_2 q,\,\lambda_3 q/\lambda_1\lambda_2\lambda_3 q)\,q^{-\frac{1}{2}r(r+1)}B_r(\theta),$$

corresponding to § 5.]

Note on some Properties of Gauche Cubics. By T. R. Lee.

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1. If a system of quadrics pass through seven points taken arbitrarily in space, they have also one other point common (Salmon's *Geometry of Three Dimensions*, p. 91, first edition, pp. 97, 98, third edition).

This eighth point might be called the eighth point homologous to the seven given points, while the system of eight points might be called eight homologous points.