IV. *A Sixth Memoir upon Quantics.* *By* ARTHUR CAYLEY, *Esq., F.R.S.*

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I PROPOSE in the present memoir to consider the geometrical theory: I have alluded to this part of the subject in the articles Nos. 3 and 4 of the Introductory Memoir. The present memoir relates to the geometry of one dimension and the geometry of two dimensions, corresponding respectively to the analytical theories of binary and ternary quantics. But the theory of binary quantics is considered for its own sake; the geometry of one dimension is so immediate an interpretation of the theory of binary quantics, that for its own sake there is no necessity to consider it at all; it is considered with a view to the geometry of two dimensions. A chief object of the present memoir is the establishment, upon purely descriptive principles, of the notion of distance. I had intended in this introductory paragraph to give an outline of the theory, but I find that in order to be intelligible it would be necessary for me to repeat a great part of the contents of the memoir in relation to this subject, and I therefore abstain from entering upon it. The paragraphs of the memoir are numbered consecutively with those of my former Memoirs on Quantics.

147. It will be seen that in the present memoir, the geometry of one dimension is treated of as a geometry of points in a line, and the geometry of two dimensions as a geometry of points and lines in a plane. It is, however, to be throughout borne in mind, that, in accordance with the remarks No. 4 of the Introductory Memoir, the terms employed are not (unless this is done expressly or by the context) restricted to their ordinary significations. In using the geometry of one dimension in reference to geometry of two dimensions considered as a geometry of points and lines in a plane, it is necessary to consider,—1°, that the word point may mean *point* and the word line mean *line*; 2°, that the word point may mean *line* and the word line mean *point*. It is, I say, necessary to do this, for in such geometry of two dimensions we have systems of points in a line and of lines through a point, and each of these systems is in fact a system belonging to, and which can by such extended signification of the terms be included in, the geometry of one dimension. And precisely because we can by such extension comprise the correlative theorems under a common enunciation, it is not in the geometry of one dimension necessary to enunciate them separately; it may be and very frequently is necessary and proper in the geometry of two dimensions, where we are concerned with systems of each kind, to enunciate such correlative theorems separately. It may, by way of further illustration, be remarked, that in using the geometry of one dimension in reference to geometry of three dimensions considered as a geometry of points, lines,

and planes in space, it would be necessary to consider,—1°, that the words point and line may mean respectively *point* and *line*; 2°, that the word line may mean *point in a plane**, and the word point mean *line*, viz. the expression points in a line mean *lines through a point and in a plane*; 3rd, that the word line may mean *line* and the word point mean *plane*, viz. the expression points in a line mean *planes through a line*. And so in using the geometry of two dimensions in reference to geometry of three dimensions considered as a geometry of points, lines, and planes in space, it would be necessary to consider,—1°, that the words point, line, and plane may mean respectively *point*, *line*, and *plane*; 2°, that the words point, line, and plane may mean respectively *plane*, *line*, and *point*. But I am not in the present memoir concerned with geometry of three dimensions. The thing to be attended to is, that in virtue of the extension of the signification of the terms, in treating the geometry of one dimension as a geometry of points in a line, and the geometry of two dimensions as a geometry of points and lines in a plane, we do in reality treat these geometries respectively in an absolutely general manner. In particular—and I notice the case because I shall have occasion again to refer to it—we do in the geometry of two dimensions include spherical geometry; the words plane, point, and line, meaning for this purpose, spherical surface, arc (of a great circle) and point (that is, pair of opposite points) of the spherical surface. And in like manner the geometry of one dimension includes the cases of points on an arc, and of arcs through a point.

148. I repeat also a remark which is in effect made in the same No. 4; the coordinates x, y of the geometry of one dimension, and the coordinates x, y, z and ξ, η, ζ of the geometry of two dimensions are only determinate to a common factor *près* (that is, it is the ratios only of the coordinates, and not their absolute magnitudes, which are determinate); hence in saying that the coordinates x, y are equal to a, b, or in writing $x, y=a, b$, we mean only that $x:y=a:b$, and we never as a result obtain $x, y=a, b$, but only $x:y=a:b$. And the like with respect to the coordinates x, y, z and ξ, η, ζ. (In the geometry of two dimensions, $x, y=a, b$, is for this reason considered and spoken of as a single equation.) But when this is once understood, there is no objection to treating the coordinates as if they were completely determinate.

On Geometry of One Dimension, Nos. 149 to 168.

149. In geometry of one dimension we have the line as a space or *locus in quo*, which is considered as made up of points. The several points of the line are determined by the coordinates (x, y), viz. attributing to these any specific values, or writing $x, y=a, b$, we have a particular point of the line. And we may say also that the line is the *locus in quo* of the coordinates (x, y).

* It would be more accurate to say that the word line may mean *point-in-and-with-a plane*, viz. the *locus in quo* of lines through the point and in the plane. Added, June 16, 1859.—A. C.

150. A linear equation,

$$(*\between x, y)^1=0,$$

is obviously equivalent to an equation of the before-mentioned form $x, y=a, b$, and represents therefore a point. An equation such as

$$(*\between x, y)^m=0$$

breaks up into m linear equations, and represents therefore a system of m points, or point-system of the order m. The component points of the system, or the linear factors, or the values thereby given for the coordinates, are termed roots. When $m=1$ we have of course a single point, when $m=2$ we have a quadric or point-pair, when $m=3$ a cubic or point-triplet, and so on. The point-system is the only figure or locus occurring in the geometry of one dimension. The quantic $(*\between x, y)^m$, when it is convenient to do so, may be represented by a single letter U, and we then have $U=0$ for the equation of the point-system.

151. The equation

$$(*\between x, y)^m=0$$

may have two or more of its roots equal to each other, or generally there may exist any systems of equalities between the roots of the equation, or what is the same thing, the system may comprise two or more coincident points, or any systems of coincident points. In particular, when the discriminant vanishes the equation will have a pair of equal roots, or the system will comprise a pair of coincident points; in the case of the quadric $(a, b, c\between x, y)^2=0$, the condition is $ac-b^2=0$, or as it may be written, $a, b=b, c$; in the case of the cubic

$$(a, b, c, d\between x, y^3)=0,$$

the condition is

$$a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2=0.$$

The preceding is the only special case for a quadric: for a cubic we have besides the special case where the three roots are equal, or the cubic reduces itself to three coincident points; the conditions for this are

$$ac-b^2=0, \quad ad-bc=0, \quad bd-c^2=0,$$

equivalent to the two conditions

$$a:b=b:c=c:d.$$

For equations of a higher order the analytical question is considered, and as regards the quartic and the quintic respectively completely solved, in my "Memoir on the Conditions for the Existence of given Systems of Equalities between the Roots of an Equation*."

152. Any covariant of the equation

$$(*\between x, y)^m=0,$$

equated to zero, gives rise to a point-system connected in a definite manner with the original point-system. And as regards the invariants, the evanescence of any invariant implies a certain relation between the points of the system; the identical evanescence of any covariant implies relations between the points of the system, such that the derived

* Philosophical Transactions,' vol. cxlvii. (1857), pp. 727– 31.

point-system obtained by equating the covariant to zero is absolutely indeterminate. The like remarks apply to the covariants or invariants of two or more equations, and the point-systems represented thereby.

153. In particular, for the two point-pairs represented by the quadric equations

$$(a, b, c \between x, y)^2=0,$$
$$(a', b', c' \between x, y)^2=0,$$

if the lineo-linear invariant vanishes, that is, if

$$ac'-2bb'+ca'=0,$$

we have the harmonic relation,—the two point-pairs are said to be harmonically related to each other, or the two points of the one pair are said to be harmonics with respect to the two points of the other pair. The analytical theory is fully developed in the "Fifth Memoir upon Quantics*." The chief results, stated under a geometrical form, are as follows:—

1°. If either of the pairs and one point of the other pair are given, the remaining point of such other pair can be found.

2°. A point-pair can be found harmonically related to any two given point-pairs.

154. The last of the two theorems gives rise to the theory of involution. The two given point-pairs, viewed in relation to the harmonic pair, are said to be an involution of four points; and the points of the harmonic pair are said to be the (double or) sibiconjugate points of the involution. A system of three or more pairs, such that the third and every subsequent pair are each of them harmonically related to the sibiconjugate points of the first and second pairs, is said to be a system in involution. In particular, for three pairs we have what is termed an involution of six points; and it is clear that when two pairs and a point of the third pair are given, the remaining point of the third pair can be determined. And in like manner for a greater number of pairs, when two pairs and a point of each of the other pairs are given, the remaining point of each of the other pairs can be determined. Two points of the same pair are said to be conjugate to each other; or if we consider two pairs as given, then the points of the third or any subsequent pair are said to be conjugate to each other in respect to the given pairs. This explains the expression sibiconjugate points; in fact, the two pairs being given, either sibiconjugate point is, as the name imports, conjugate to itself. In other words, any two pairs and one of the sibiconjugate points considered as a pair of coincident points, form a system in involution, or involution of five points.

155. The three point-pairs, $U=0$, $U'=0$, $U''=0$, will be in involution when the quadrics U, U', U'' are connected by the linear relation or syzygy $\lambda U+\lambda' U'+\lambda'' U''=0$. This property, or the relation

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}=0$$

* Philosophical Transactions, vol. cxlviii. (1858), pp. 429–462.

to which it gives rise, might have been very properly adopted as the definition of the relation of involution, but I have on the whole preferred to deduce the theory of involution from the harmonic relation. The notion, however, of the linear relation or syzygy of three or more point-systems gives rise to a much more general theory of involution, but this is a subject that I do not now enter upon; it may, however, be noticed, that if $U=0$, $U'=0$ be any two point-systems of the same order, then we may find a point-system $U''=0$ of the same order, in involution with the given point-systems (that is, satisfying the condition $\lambda U+\lambda' U'+\lambda'' U''=0$), and such that the point-system $U''=0$ comprises a pair of coincident points; this is obviously an extension of the notion of the sibiconjugate points of an ordinary involution.

156. It was remarked in the Fifth Memoir, that the theories of the anharmonic ratio and of homography belong analytically to the subject of bipartite (lineo-linear) binary quantics; this may be further illustrated geometrically as follows: we may imagine two distinct spaces of one dimension or lines, one of them the *locus in quo* of the coordinates (x, y), and the other the *locus in quo* of the coordinates (x, y), which are absolutely independent of, and are not in anywise related to, the coordinates of the first-mentioned system. There is no difficulty in the conception of this; for we may in a plane or in space of three dimensions imagine any two lines, and study the relations of analogy between the points of the one line *inter se*, and the points of the other line *inter se*, without in anywise adverting to the space of two or three dimensions which happens to be the common *locus in quo* of the two lines. It is proper to remark, that in speaking of the spaces of one dimension, which are the *loci in quibus* of the coordinates (x, y) and (x, y) respectively, as being each of them a line, we imply a restriction which is altogether unnecessary; the words line and point may, in regard to the two figures respectively, be used in different significations; for instance, one of the spaces may be a *line* and the points in it *points*; while the other of the spaces may be a *point* and the points in it *lines*, or it may be a *line* and the points in it *planes*.

157. A lineo-linear equation,

$$(x-ay)(\mathrm{x}-\alpha\mathrm{y})=0,$$

denotes then the two points $(x, y=a, 1)$ and $(\mathrm{x}, \mathrm{y}=\alpha, 1)$ existing irrespectively of each other in distinct spaces, and only by the equation itself brought into an ideal connexion; and any invariantive relation between the coefficients of any such bipartite functions denotes geometrically a relation between a point-system in the space which is the *locus in quo* of the coordinates (x, y), and a point-system in the space which is the *locus in quo* of the coordinates (x, y); for instance, the equation

$$\begin{vmatrix} 1, & a, & \alpha, & a\alpha \\ 1, & b, & \beta, & b\beta \\ 1, & c, & \gamma, & c\gamma \\ 1, & d, & \delta, & d\delta \end{vmatrix}=0$$

is the relation of homography between the four points $(a, 1)$, $(b, 1)$, $(c, 1)$, $(d, 1)$ in the first line, and the four points $(\alpha, 1)$, $(\beta, 1)$, $(\gamma, 1)$, $(\delta, 1)$ in the second line. The analytical theory is discussed in the Fifth Memoir; and, in particular, it is there shown, that writing

$$\begin{aligned} A&=(d-a)(b-c), & \mathfrak{A}&=(\delta-\alpha)(\beta-\gamma), \\ B&=(d-b)(c-a), & \mathfrak{B}&=(\delta-\beta)(\gamma-\alpha), \\ C&=(d-c)(a-b), & \mathfrak{C}&=(\delta-\gamma)(\alpha-\beta), \end{aligned}$$

then the condition may be expressed under any one of the forms

$$A:B:C=\mathfrak{A}:\mathfrak{B}:\mathfrak{C},$$

equations which denote the equality of the anharmonic ratios of the two point-systems.

158. The number of points in each system may be four, or any greater number; the homographic relation is then conveniently expressed under the form

$$\left\| \begin{matrix} 1\ , & 1\ , & 1\ , & 1\ , & 1, & \dots \\ a\ , & b\ , & c\ , & d\ , & e\ , & \\ \alpha\ , & \beta\ , & \gamma\ , & \delta\ , & \varepsilon\ , & \\ a\alpha, & b\beta, & c\gamma, & d\delta, & e\varepsilon, & \end{matrix} \right\| =0.$$

The relation is such that given three points of the one system and the corresponding three points of the other system, then to any fourth point whatever of the first system there can be found a corresponding fourth point of the second system. It is to be observed, however, that two systems of four points homographically related to each other, always correspond together in four different ways, viz. the two systems being (a, b, c, d) and $(\alpha, \beta, \gamma, \delta)$; then if the four points of the first system are (a, b, c, d), the corresponding four points of the second system may be taken in the four several orders, $(\alpha, \beta, \gamma, \delta)$, $(\beta, \alpha, \delta, \gamma)$, $(\gamma, \delta, \alpha, \beta)$, $(\delta, \gamma, \beta, \alpha)$.

159. What precedes is not to be understood as precluding the existence of a relation between the spaces which are the *loci in quibus* of the coordinates (x, y) and (x, y) respectively: not only may these be spaces of the same kind, but they may be one and the same space or line; and the points of the two systems may then be points of the same kind; and further, the coordinates (x, y) and (x, y) may belong to the same system of coordinates, that is, the equations $(x, y=a, 1)$ and $(\mathrm{x}, \mathrm{y}=a, 1)$ may denote one and the same point.

160. If the two point-systems are systems of the same kind, and are in one and the same line, then there are in general two points of the first system which coincide each of them with the corresponding point of the second system; such two points may be said to be the sibiconjugate points of the homography. In particular, the two sibiconjugate points of the homography may coincide together.

161. A system in involution affords an example of two homographic systems in the same line; in fact, taking arbitrarily a point out of each pair, the points so obtained form a system which is homographic with the system formed with the other points of

the several pairs; and in this case the sibiconjugate points of the involution are also the sibiconjugate points of the homography. Thus if A and A′, B and B′, C and C′, D and D′ are pairs of the system in involution, then (A, B, C, D) and (A′, B′, C′, D′) will be homographic point-systems; and, as a particular case, (A, B, C, C′) and (A′, B′, C′, C) will be homographic point-systems. It is proper to notice that if F is a sibiconjugate point of the involution, then (A, B, F, F) and (A′, B′, F, F) are not (what at first sight they appear to be) homographic point-systems.

162. Imagine an involution of points; take on the line which is the *locus in quo* of the point-system a point O, and consider the point-system formed by the harmonics of O in respect to the several pairs of the involution; and in like manner take on the line any other point O′, and consider the point-system formed by the harmonics of O′ in respect to the several pairs of the involution; these two point-systems are homographically related to each other.—See Fifth Memoir, No. 111.

163. Two involutions may be homographically related to each other; in fact, take on the line which is the *locus in quo* of the first involution a point O, and consider the point-system formed by the harmonics of O in relation to the several pairs of the involution; take in like manner on the line which is the *locus in quo* of the second involution a point Q, and consider the point-system formed by the harmonics of Q with respect to the several pairs of the involution; then if the two point-systems are homographically related, the two involutions are said to be themselves homographically related: the last preceding article shows that the nature of the relation does not in anywise depend on the choice of the points O and Q. And it is not necessary that, as regards the two involutions respectively, the words line and point should have the same significations.—See Fifth Memoir, No. 111.

164. Four or more tetrads of points in a line may be homographically related to the same number of tetrads in another line. This is the case when the anharmonic ratios of the tetrads of the first system are homographically related to the anharmonic ratios of the tetrads of the second system. And it is not material which of the three anharmonic ratios of a tetrad of either system is selected, provided that the same selection is made for each of the other tetrads of the same system. The order of the points of a tetrad must be attended to, but there are in all four admissible permutations of the points of a tetrad, viz. if A, B, C, D are the points of a tetrad, then (A, B, C, D), (B, A, D, C), (C, D, A, B), (D, C, B, A) may be considered as one and the same tetrad. Any three tetrads whatever in the second system may correspond to any three tetrads of the first system; and then given a fourth tetrad of the first system, and three out of the four points of the corresponding tetrad of the second system, the remaining point of the tetrad may be determined. The words line and point need not, as regards the two systems of tetrads respectively, be understood in the same significations.—See Fifth Memoir, No. 112.

165. The foregoing theory of the harmonic relation shows that if we have a point-pair

$$(a, b, c \between x, y)^2=0,$$

the equation of any other point-pair whatever can be expressed, and that in two different ways, in the form

$$(a, b, c \between x, y)^2 + (lx+my)^2 = 0\,;$$

the points $(lx+my=0)$ corresponding to the two admissible values of the linear function being in fact the harmonics of the point-pair in respect to the given point-pair $(a, b, c \between x, y)^2=0$, or what is the same thing, the sibiconjugate points of the involution formed by the two point-pairs (see Fifth Memoir, No. 105). The point-pair represented by the equation in question does not in itself stand in any peculiar relation to the given point pair $(a, b, c \between x, y)^2=0$; but when thus represented it is said to be inscribed in the given point-pair, and the point $lx+my=0$ is said to be the axis of inscription. And the harmonic of this point with respect to the given point-pair (that is, the other sibiconjugate point of the involution of the two point-pairs) is said to be the centre of inscription *.

166. We may, if we please, (x', y') and θ being constants, exhibit the equation of the inscribed point-pair in the form

$$(a, b, c \between x, y)^2 (a, b, c \between x', y')^2 \sin^2\theta - (ac-b^2)(xy'-x'y)^2 = 0,$$

where we have for the axis of inscription and centre of inscription respectively, the equations

$$\begin{aligned} xy'-x'y \qquad\qquad &= 0, \\ (a, b, c \between x, y \between x', y') &= 0. \end{aligned}$$

Or in the equivalent form,

$$(a, b, c \between x, y)^2 (a, b, c \between x', y')^2 \cos^2\theta - \{(a, b, c \between x, y \between x', y')\}^2 = 0,$$

where we have for the axis of inscription and the centre of inscription respectively, the equations

$$\begin{aligned} (a, b, c \between x, y \between x', y') &= 0, \\ xy'-x'y \qquad\qquad &= 0. \end{aligned}$$

167. The equivalence of the two forms depends on the identical equation

$$(a, b, c \between x, y)^2 (a, b, c \between x', y')^2 - \{(a, b, c \between x, y \between x', y')\}^2 = (ac-b^2)(xy'-x'y)^2,$$

which is in fact the equation mentioned, Fifth Memoir, No. 95. If, for shortness, we write

$$\begin{aligned} &(a, b, c \between x, y)^2 = 00, \\ &(a, b, c \between x, y \between x', y') = 01 = 10, \\ &\text{\&c.}, \end{aligned}$$

then the equation may be represented in the form

$$\begin{vmatrix} 00, & 01 \\ 10, & 11 \end{vmatrix} = (ac-b^2) \begin{vmatrix} x, y \\ x, y' \end{vmatrix}^2.$$

168. There is a like equation for the three sets (x, y), (x', y'), (x'', y''); the right-hand side here vanishes, for there are not columns enough to form therewith a determinant,

* The words inscribed, inscription, are used not in opposition to, but as identical with, the words circumscribed, circumscription; and in like manner (*post*, Nos. 203 et seq.) as regards conics.

and the equation is

$$\begin{vmatrix} 00, & 01, & 02 \\ 10, & 11, & 12 \\ 20, & 21, & 22 \end{vmatrix}=0,$$

an equation which may also be written in the form

$$\cos^{-1}\frac{01}{\sqrt{00}\sqrt{11}}+\cos^{-1}\frac{12}{\sqrt{11}\sqrt{22}}=\cos^{-1}\frac{02}{\sqrt{00}\sqrt{22}},$$

as it is easy to verify by reducing this equation to an algebraical form. The various formulæ have been given in relation to the establishment of the notion of distance in the geometry of one dimension, but it will be convenient to defer the consideration of this theory so as to discuss it in connexion with geometry of two dimensions.

On Geometry of Two Dimensions, Nos. 169 to 208.

169. In geometry of two dimensions we have the plane as a space or *locus in quo*, which is considered under two distinct aspects, viz. as made up of points, and as made up of lines. The several points of the plane are determined by means of the point-coordinates (x, y, z), viz. attributing to these any specific values, or writing $x, y, z=a, b, c$, we have a particular point of the plane; and in like manner the several lines of the plane are determined by the line-coordinates (ξ, η, ζ), viz. attributing to these any specific values, or writing $\xi, \eta, \zeta=\alpha, \beta, \gamma$, we have a particular line of the plane. And we may say that the plane is the *locus in quo* of the point-coordinates (x, y, z), and of the line-coordinates (ξ, η, ζ). It is not necessary to consider separately the analytical theories of point-coordinates and of line-coordinates; for the theory of the former in relation to points and lines respectively is identical with the theory of the latter in relation to lines and points respectively; but it is necessary to show how either system of coordinates, say the system of point-coordinates, is applicable to both points and lines, or in fact all loci whatever, and to explain the mutual relation of the two systems of coordinates.

170. Considering then point-coordinates, the equations

$$x, y, z=a, b, c,$$

determine, as already mentioned, a point.

A linear equation,

$$(*\between x, y, z)^1=0,$$

determines a line, viz. the line which is the locus of all the points, the coordinates of which satisfy this equation. And in like manner an equation

$$(*\between x, y, z)^m=0$$

determines a curve of the mth order, viz. the curve which is the locus of all the points, the coordinates of which satisfy this equation. In particular, an equation of the second degree,

$$(*\between x, y, z)^2=0,$$

determines a conic.

171. If the quantic breaks up into rational factors, then the equation of the curve is satisfied by equating to zero any one of these factors, or the curve breaks up into curves of a lower order, and the order of the entire curve is equal to the sum of the orders of the component curves. In particular, a curve of any order may break up into a system of lines, the number of lines being of course equal to the order of the curve, and any two or more of these lines may coincide with each other. A curve not thus breaking up into curves of a lower order is said to be a proper curve.

172. Returning to the linear equation and expressing the coefficients, the equation is

$$(\xi, \eta, \zeta \between x, y, z)=0,$$

or, what is the same thing,

$$\xi x+\eta y+\zeta z=0;$$

and we say as a definition, that the coordinates (line-coordinates) of this line are (ξ, η, ζ).

173. But the same equation, considering (x, y, z) as constant coefficients, and (ξ, η, ζ) as line-coordinates, is the equation of a point, viz. the point which is the locus (envelope) of all those points the coordinates of which satisfy the equation in question; and such point is precisely the point, the coordinates (point-coordinates) of which are (x, y, z). In fact, if (ξ, η, ζ) are considered as variable parameters connected by the equation $\xi x+\eta y+\zeta z=0$, then taking (X, Y, Z) as current point-coordinates, the equation $\xi X+\eta Y+\zeta Z=0$ is satisfied by writing (x, y, z) for (X, Y, Z); or the several lines the coordinates whereof are (ξ, η, ζ), all pass through the point (x, y, z).

174. Hence recapitulating, the equation

$$(\xi, \eta, \zeta \between x, y, z)=0,$$

or

$$\xi x+\eta y+\zeta z \quad =0,$$

considering (x, y, z) as current point-coordinates, and (ξ, η, ζ) as constant coefficients, is the equation of a line the coordinates (line-coordinates) of which are (ξ, η, ζ); and the same equation, considering (ξ, η, ζ) as current line-coordinates, and (x, y, z) as constant coefficients, is the equation of a point the coordinates (point-coordinates) of which are (x, y, z).

175. The expression, the point (a, b, c), means the point whose point-coordinates are (a, b, c); and in like manner the expression, the line (α, β, γ), means the line whose line-coordinates are (α, β, γ). The last-mentioned expression may, without any impropriety or risk of ambiguity, be employed when we are dealing with point-coordinates; but it is of course always allowable, and it is frequently better, to substitute for the definition the thing signified, and say the line having for its equation $\alpha x+\beta y+\gamma z=0$, or more briefly, the line $\alpha x+\beta y+\gamma z=0$. It will be proper to do this in the following articles, Nos. 176 to 184, which contain some formulæ in point-coordinates relating to the theory of the point and the line.

176. The condition that the point (a, b, c) may lie in the line

$$\alpha x+\beta y+\gamma z=0,$$

is of course

$$\alpha a+\beta b+\gamma c=0.$$

177. The equation of the line passing through the points (a, b, c), (a', b', c'), is

$$\begin{vmatrix} x, & y, & z \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}=0;$$

and if in this equation (a', b', c') are considered as indeterminate, we have the equation of a line subjected to the single condition of passing through the point (a, b, c). The equation contains apparently two arbitrary parameters, but these in fact reduce themselves to a single one.

178. The coordinates of the point of intersection of the lines

$$\alpha x+\beta y+\gamma z=0,$$

$$\alpha' x+\beta' y+\gamma' z=0,$$

are given by the equations

$$x, y, z=\beta\gamma'-\beta'\gamma,\ \ \gamma\alpha'-\gamma'\alpha,\ \ \alpha\beta'-\alpha'\beta;$$

and if in these equations we consider α', β', γ' as indeterminate, we have the coordinates of a point subjected to the single condition of lying in the line $\alpha x+\beta y+\gamma z=0$; the result, as in the last case, contains in appearance two arbitrary parameters, but these really reduce themselves to a single one.

179. The condition in order that the points (a, b, c), (a', b', c'), (a'', b'', c'') may lie in a line is

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}=0,$$

which may also be expressed by the equations

$$a'', b'', c''=\lambda a+\mu a',\ \ \lambda b+\mu b',\ \ \lambda c+\mu c',$$

where λ, μ are arbitrary multipliers; these equations give therefore the coordinates of an indeterminate point in the line joining the points (a, b, c) and (a', b', c').

180. The condition that the lines

$$\alpha x+\beta y+\gamma z=0,$$

$$\alpha' x+\beta' y+\gamma' z=0,$$

$$\alpha'' x+\beta'' y+\gamma'' z=0$$

may meet in a point is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}=0,$$

a relation which may also be expressed by the equations

$$\alpha'', \beta'', \gamma''=l\alpha+m\alpha', \quad l\beta+m\beta', \quad l\gamma+m\gamma',$$

where l, m are arbitrary multipliers; and substituting these values in the equation $\alpha''x+\beta''y+\gamma''z=0$, we have for the equation of a line subjected to the single condition of passing through the point of intersection of the lines $\alpha x+\beta y+\gamma z=0$, $\alpha'x+\beta'y+\gamma'z=0$, the equation

$$l(\alpha x+\beta y+\gamma z)+m(\alpha'x+\beta'y+\gamma'z)=0,$$

which is, in fact, at once obtained by the consideration that the values of (x, y, z) which satisfy simultaneously the equations $\alpha x+\beta y+\gamma z=0$ and $\alpha'x+\beta'y+\gamma'z=0$, satisfy also the equation in question.

181. The equation of the line passing through the point of intersection of the lines $\alpha x+\beta y+\gamma z=0$ and $\alpha'x+\beta'y+\gamma'z=0$, and also through the point (a, b, c), is obviously

$$\begin{vmatrix} \alpha x+\beta y+\gamma z, & \alpha'x+\beta'y+\gamma'z \\ \alpha a+\beta b+\gamma c, & \alpha'a+\beta'b+\gamma'c \end{vmatrix}=0,$$

which, or the equivalent form

$$\frac{\alpha x+\beta y+\gamma z}{\alpha a+\beta b+\gamma c}=\frac{\alpha'x+\beta'y+\gamma'z}{\alpha'a+\beta'b+\gamma'c},$$

is usually the most convenient one; but it is to be observed that the equation can also be written in the forms

$$\begin{vmatrix} x, & y, & z \\ a, & b, & c \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{vmatrix}=0,$$

and

$$\begin{vmatrix} bz-cy, & cx-az, & ay-bx \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}=0,$$

or in the form

$$(\beta\gamma'-\gamma\beta')(bz-cy)+(\gamma\alpha'-\gamma'\alpha)(cx-az)+(\alpha\beta'-\beta\alpha')(ay-bx)=0,$$

which might also be represented by

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}\begin{vmatrix} x, & y, & z \\ a, & b, & c \end{vmatrix}=0.$$

182. To find the coordinates of the point of intersection of the line joining the points (a, b, c), (a', b', c'), with the line $\alpha x+\beta y+\gamma z=0$, we have

$$x, y, z=\lambda a+\mu a', \quad \lambda b+\mu b', \quad \lambda c+\mu c',$$

where λ, μ are given by

$$\lambda(\alpha a+\beta b+\gamma c)+\mu(\alpha a'+\beta b'+\gamma c')=0.$$

The preceding are elementary formulæ of almost constant occurrence; it may be proper to add to them the formulæ which follow.

183. To find the equation of the line passing through the point of intersection of the lines

$$\alpha_1x+\beta_1y+\gamma_1z=0,\ \ \alpha_2x+\beta_2y+\gamma_2z=0,$$

and the point of intersection of the lines

$$\alpha_3x+\beta_3y+\gamma_3z=0,\ \ \alpha_4x+\beta_4y+\gamma_4z=0.$$

Write for shortness $u_1=\alpha_1x+\beta_1y+\gamma_1z$, &c.; then we have identically

$$\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix}=0,$$

and the two equations

$$\begin{vmatrix} u_1, & u_2, & ., & . \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix}=0, \qquad \begin{vmatrix} ., & ., & u_3, & u_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix}=0$$

are consequently equivalent to each other, and each of them represents the required line. It is easy to deduce the form

$$x\begin{vmatrix} \alpha_1, & \alpha_2, & ., & . \\ ., & ., & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix}+y\begin{vmatrix} \beta_1, & \beta_2, & ., & . \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ ., & ., & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix}+z\begin{vmatrix} \gamma_1, & \gamma_2, & ., & . \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ ., & ., & \gamma_3, & \gamma_4 \end{vmatrix}=0.$$

184. The condition in order that the points of intersection of the lines $u_1=0$, $u_2=0$, of the lines $u_3=0$, $u_4=0$, and of the lines $u_5=0$, $u_6=0$ (where, as before, u_1 denotes $\alpha_1x+\beta_1y+\gamma_1z$, &c.) may lie in the same line, is

$$\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & ., & . \\ \beta_1, & \beta_2, & \beta_3, & \beta_4, & ., & . \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4, & ., & . \\ ., & ., & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6 \\ ., & ., & \beta_3, & \beta_4, & \beta_5, & \beta_6 \\ ., & ., & \gamma_3, & \gamma_4, & \gamma_5, & \gamma_6 \end{vmatrix}=0,$$

which is of course really symmetrical with respect to the six sets. The last formula was given by me, 'Cambridge Mathematical Journal,' t. iv. p. 18 (1849).

185. Instead of the term point of a curve, it will be convenient to use the term 'ineunt' of the curve.

The line through two consecutive ineunts of the curve is the tangent at the ineunt. The point of intersection of two consecutive tangents is the ineunt on the tangent.

The equation of a curve in point-coordinates, or as it may be termed the point-equation of the curve, is the relation which exists between the point-coordinates of any ineunt of the curve.

The equation of a curve in line-coordinates, or line-equation of the curve, is the relation which exists between the line-coordinates of any tangent of the curve.

186. It has been mentioned, that the order of a curve is the degree of its point-equation: in like manner the class of a curve is the degree of its line-equation; and in the same way that a curve, as represented by a point-equation, may break up into curves having the order of the entire curve for the sum of their orders, so a curve as represented by a line-equation may break up into curves having the class of the entire curve for the sum of their classes. And, in particular, a curve may break up into a system of points, the number of points being equal to the class of the curve, and two or more of these points may coincide together.

187. A line is a curve of the order one and class zero; a point is a curve of the order zero and class one. A proper conic is a curve of the order two and class two; but when the conic breaks up into a pair of lines, the class sinks to zero; and when the conic breaks up into a pair of points, the order sinks to zero. It is to be noticed that a point, or system of points, cannot be represented by an equation in point-coordinates, nor a line or system of lines by an equation in line-coordinates. We may say, in general, that a curve is both a point-curve and a line-curve, but a point or system of points is a line-curve only, and a line or system of lines is a point-curve only.

188. The points of intersection (common ineunts) of two curves are the points the coordinates of which satisfy simultaneously the point-equations of the two curves. Hence the number of common ineunts is equal to the product of the orders of the two curves; and, in particular if one of the curves be a line, the number of points of intersection (common ineunts) is equal to the order of the curve. In like manner the common tangents of the two curves are the lines the coordinates of which satisfy simultaneously the line-equations of the two curves. Hence the number of common tangents is equal to the product of the classes of the two curves; and, in particular, if one of the curves be a point, the number of common tangents (tangents to the curve through the point) is equal to the class of the curve. Since the tangent is the line through two consecutive ineunts, it besides meets the curve, assumed to be of the order m in $(m-2)$ points; and in like manner we may from any ineunt of a curve of the class n draw $(n-2)$ tangents to the curve.

189. The point-equation of a line passing through the points (x', y', z') and (x'', y'', z'') is, as already noticed,

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}=0.$$

Suppose that (x, y, z) are the coordinates of a point (ineunt) of the curve $\mathrm{U}=0$, the coordinates of the consecutive ineunt will be $(x+dx,\ y+dy,\ z+dz)$, and the line joining these two points will be the tangent to the curve at the point (x, y, z). Take (X, Y, Z) as current point-coordinates, the equation of the tangent is

$$\begin{vmatrix} X\ , & Y\ , & Z \\ x\ , & y\ , & z \\ x+dx, & y+dy, & z+dz \end{vmatrix}=0,$$

or what is the same thing,

$$X(ydz-zdy)+Y(zdx-xdz)+Z(xdy-ydx)=0.$$

But since U is a homogeneous function of (x, y, z), we have

$$x\partial_x U+y\partial_y U+z\partial_z U=mU=0\,;$$

and since $(x+dx,\ y+dy,\ z+dz)$ is a point of the curve, we have

$$dx\partial_x U+dy\partial_y U+dz\partial_z U=0\,;$$

and from these two equations

$$ydz-zdy : zdx-xdz : xdy-ydx=\partial_x U : \partial_y U : \partial_z U,$$

and the equation of the tangent consequently is

$$X\partial_x U+Y\partial_y U+Z\partial_z U=0.$$

190. Take (ξ, η, ζ) as the line-coordinates of the tangent, then the equation of the tangent is

$$\xi X+\eta Y+\zeta Z=0\,;$$

and comparing the two forms, we have

$$\xi : \eta : \zeta=\partial_x U : \partial_y U : \partial_z U\,;$$

and if from these equations and the equation $U=0$ (the point-equation of the curve) we eliminate (x, y, z), we obtain an equation between (ξ, η, ζ), which is the line-equation of the curve. We may, if we please, present the system under the form

$$\begin{aligned} \partial_x U+\lambda\xi&=0,\\ \partial_y U+\lambda\eta&=0,\\ \partial_z U+\lambda\zeta&=0,\\ U=\quad&0, \end{aligned}$$

or what is more simple, under the form

$$\begin{aligned} \partial_x U+\lambda\xi&=0,\\ \partial_y U+\lambda\eta&=0,\\ \partial_z U+\lambda\zeta&=0,\\ \xi x+\eta y+\zeta z&=0, \end{aligned}$$

and from either system eliminate x, y, z and λ.

191. If the point-equation of a conic be

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

then its line-equation is

$$-\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi & a, & h, & g \\ \eta & h, & b, & f \\ \zeta & g, & f, & c \end{vmatrix}=0,$$

or writing

$$\begin{aligned} \mathfrak{A}&=bc-f^2, \\ \mathfrak{B}&=ca-g^2, \\ \mathfrak{C}&=ab-h^2, \\ \mathfrak{F}&=gh-af, \\ \mathfrak{G}&=hf-bg, \\ \mathfrak{H}&=fg-ch, \end{aligned}$$

and to complete the system,

$$\mathrm{K}=abc-af^2-bg^2-ch^2+2fgh,$$

then the line-equation of the conic is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2=0.$$

192. The quantities $\mathfrak{A}$, &c. satisfy the relations

$$\mathrm{K}^2=\mathfrak{A}\mathfrak{B}\mathfrak{C}-\mathfrak{A}\mathfrak{F}^2-\mathfrak{B}\mathfrak{G}^2-\mathfrak{C}\mathfrak{H}^2+2\mathfrak{F}\mathfrak{G}\mathfrak{H},$$

$$\begin{array}{lll} \mathfrak{A}a+\mathfrak{H}h+\mathfrak{G}g=\mathrm{K}, & \mathfrak{H}a+\mathfrak{B}h+\mathfrak{F}g=0, & \mathfrak{G}a+\mathfrak{F}h+\mathfrak{C}g=0, \\ \mathfrak{A}h+\mathfrak{H}b+\mathfrak{G}f=0, & \mathfrak{H}h+\mathfrak{B}b+\mathfrak{F}f=\mathrm{K}, & \mathfrak{G}h+\mathfrak{F}b+\mathfrak{C}f=0, \\ \mathfrak{A}g+\mathfrak{H}f+\mathfrak{G}c=0, & \mathfrak{H}g+\mathfrak{B}f+\mathfrak{F}c=0, & \mathfrak{G}g+\mathfrak{F}f+\mathfrak{C}c=\mathrm{K}, \end{array}$$

and moreover

$$\begin{array}{ll} \mathrm{K}a=\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2, & \mathrm{K}f=\mathfrak{G}\mathfrak{H}-\mathfrak{A}\mathfrak{F}, \\ \mathrm{K}b=\mathfrak{C}\mathfrak{A}-\mathfrak{G}^2, & \mathrm{K}g=\mathfrak{H}\mathfrak{F}-\mathfrak{B}\mathfrak{G}, \\ \mathrm{K}c=\mathfrak{A}\mathfrak{B}-\mathfrak{H}^2, & \mathrm{K}h=\mathfrak{F}\mathfrak{G}-\mathfrak{C}\mathfrak{H}. \end{array}$$

193. A system of points in a line is said to be a range, and a system of lines through a point is said to be a pencil. The theories of ranges and pencils, considered irrespectively of each other, are in fact a single theory, constituting the geometry of one dimension. It has been seen how in geometry of one dimension a range of points and a pencil of lines, although considered (as they must be considered) as existing in distinct spaces, may nevertheless stand in certain relations to each other. In geometry of two dimensions, the range and pencil may of course coexist in one and the same plane as their common *locus in quo*; and such coexistence occurs in fact very frequently: thus if we

have a line and a point, and if lines are drawn joining the point with the several points of the line, these lines constitute a pencil, and the points of the line constitute a range, and such pencil and range are homographically related.

194. The theory of homography in geometry of two dimensions may be made to depend upon the corresponding theory in geometry of one dimension, or what is the same thing, upon the homography of ranges or pencils. For consider two figures existing in distinct planes or spaces of two dimensions, any four points (not in a line) in the second figure may correspond to any four points (not in a line) in the first figure; and when this is so, we may, by the process about to be explained, given any other point of the first figure, construct the corresponding point of the second figure; and the two figures are then, by definition, homographically related. Suppose that the points A′, B′, C′, D′ of the second figure correspond respectively to the points A, B, C, D of the first figure, and let E be any other point of the first figure; suppose that E′ is the corresponding point of the second figure; the pencils AB, AC, AD, AE and A′B′, A′C′, A′D′, A′E′ should be homographic to each other, that is, E′ must lie on a given line through A′; and in like manner the pencils BA, BC, BD, BE and B′A′, B′C′, B′D′, B′E′ should be homographic to each other, that is, E′ must lie on a given line through B′. And then, as a theorem, CA, CB, CD, CE and C′A′, C′B′, C′D′, C′E′, or DA, DB, DC, DE and D′A′, D′B′, D′C′, D′E′ will be homographic pencils, that is, the construction will be a determinate one whichever two of the four points are selected for the points A and B. The foregoing construction leads to an analytical relation, which I think constitutes a better foundation of the theory. Consider the first plane as the *locus in quo* of the coordinates (x, y, z), and the second plane as the *locus in quo* of the coordinates (X, Y, Z), these two coordinate systems being absolutely independent of each other. Consider any point of the first plane and a corresponding point of the second plane such that its coordinates (X, Y, Z) are given linear functions of the coordinates (x, y, z) of the point in the first plane. Any figure whatever in the first plane gives rise to a corresponding figure in the second plane, and the two figures are said to be homographic to each other. To a point of the first figure there corresponds in the second figure a point, to a line a line, to a range of points or pencil of lines, a homographic range of points or pencil of lines; the line or point which is the *locus in quo* of the range or pencil in the one figure corresponding with the line or point which is the *locus in quo* of the range or pencil in the other figure. And generally, to any curve of any order and class in the first figure, and to its ineunts and tangents, there correspond in the second figure a curve of the same order and class, and the ineunts and tangents of such curve.

195. It is to be remarked, that it is not by any means necessary that the word or the words plane, point, and line, or consequently the words order and class, should have the same significations as regards the two figures respectively. The theory of homography, as above explained, in fact comprises what is commonly termed the theory of homography and also the theory of reciprocity.

196. Let the word plane have the ordinary signification as regards the two figures

respectively; and Suppose, first, that the words point and line, and therefore order and class, have also the ordinary significations as regards the two figures respectively: we have here the ordinary theory of homography, in which, to any range of points or pencil of lines in the first figure, there corresponds a homographic range of points or pencil of lines in the second figure, and to a curve of any order and class in the first figure there corresponds a curve of the same order and class in the second figure.

197. We may, as a specialization giving rise to further developments, assume that the two figures exist in one and the same plane. There is here in general a triangle, each of whose angles or sides, considered as a point or line in the first figure, corresponds to itself, considered as a point or line in the second figure: such triangle may be called the sibiconjugate triangle. Any one point of the plane, considered as belonging to the first figure, may correspond to any other point of the plane, considered as belonging to the second figure, and the second figure can be completely constructed by means of the sibiconjugate triangle and such pair of corresponding points. In certain special cases the sibiconjugate triangle becomes wholly or in part indeterminate; thus if the two figures are identical, each point of the plane, considered as belonging to the first figure, coincides with itself, considered as belonging to the second figure. But I reserve the further discussion of the theory of homography for another occasion.

198. Suppose, secondly, that in the foregoing general theory, as regards the first figure, the words point and line, and therefore order and class, signify *point* and *line*, *order* and *class*; while as regards the second figure, the words point and line signify *line* and *point* respectively, and therefore the words order and class, *class* and *order* respectively. We have in the present case the ordinary theory of reciprocity, viz. using all the words in the same significations as regards the two figures respectively; to a point in the first figure there corresponds in the second figure a line; to a line, a point; to a range of points or pencil of lines, a pencil of lines or range of points; to a curve of any order and class, and its ineunts and tangents, a curve of the same class and order, and the tangents and ineunts of such curve.

199. As a specialization giving rise to further developments, we may assume that the two figures exist in one and the same plane. In this case, the points which, considered indifferently as belonging to the first or the second figure, lie upon the corresponding lines in the second or first figure, generate a conic which may be termed the pole-conic; and the lines which, considered indifferently as belonging to the first or the second figure, pass through the corresponding points in the second or first figure, envelope a conic which may be termed the polar-conic, and these two conics have double contact with another. The further consideration of this subject is reserved for another occasion; but I remark that in the particular case where the two conics coincide, we have the ordinary theory of poles and polars in regard to a conic; a theory, which, in a different point of view, may be considered as arising out of the harmonic relation, and which must here be noticed.

200. Consider a conic and a point; any line through the point meets the conic in two

points (ineunts of the conic), and the harmonic in relation to these two points of the given point has for its locus a line which is the polar of the given point. The polar passes through the points of contact of the conic with the tangents through the given point.

In like manner considering the conic and a line; from any point of the line we may draw two tangents to the conic, and the harmonic of the given line with respect to the two tangents envelopes a point which is the pole of the given line. The pole is the point of intersection of the tangents of the conic at the points of intersection with the given line.

The polars of the several points of a line envelope a point which is the pole of the line; and the poles of the several lines through a point generate a line which is the polar of the point; and this proposition shows how the theory of poles and polars gives rise to a theory of reciprocity.

201. If the point-equation of a conic be

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

the point-equation of the polar with respect to this conic of the point (x', y', z') is

$$(a, b, c, f, g, h \between x, y, z \between x', y', z)=0.$$

But it has been seen that the line-equation of the same conic is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2=0,$$

and the line-equation of the pole with respect to this conic of the line (ξ', η', ζ') (that is, the line whose point-equation is $\xi'x+\eta'y+\zeta'z=0$) is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta \between \xi', \eta', \zeta')=0,$$

in other words, the point-coordinates of the pole are

$$\mathfrak{A}\xi'+\mathfrak{H}\eta'+\mathfrak{G}\zeta', \quad \mathfrak{H}\xi'+\mathfrak{B}\eta'+\mathfrak{F}\zeta', \quad \mathfrak{G}\xi'+\mathfrak{F}\eta'+\mathfrak{C}\zeta'.$$

202. If $U=0$, $V=0$ be the point-equations of any two curves of the same order, then λ, μ being arbitrary coefficients,

$$\lambda U+\mu V=0$$

is the equation of a curve of the same order passing through the points of intersection (common ineunts) of the two curves; such curve is said to be in involution with the given curves. The discussion of the general theory of involution is reserved for another occasion.

203. In particular, if $U=0$ be the equation of a conic, and $P=0$, $Q=0$ the equations of two lines, then

$$U+\lambda PQ=0$$

is the equation of a conic passing through the points of intersection of the conic with the two lines; and if the two lines coincide, then

$$U+\lambda P^2=0$$

is the equation of a conic having double contact with the conic $U=0$ at its points of intersection with the line $P=0$. Such conic is said to be inscribed in the conic $U=0$;

the line $P=0$ is the axis of inscription; this line has the same pole with respect to each of the two conics, and the pole is termed the centre of inscription: the relation of the two conics is completely expressed by saying that the four common ineunts coincide in pairs upon the axis of inscription, and that the four common tangents coincide in pairs through the centre of inscription; it is consequently a similar relation in regard to ineunts and tangents respectively; and it is to be inferred *à priori*, that if $\Upsilon=0$ be the line-equation of the conic $U=0$, and $\Pi=0$ the line-equation of the centre of inscription, then the line-equation of the inscribed conic is $\Upsilon+\mu\Pi^2=0$.

204. To verify this, I remark that if the equation of the axis of inscription be

$$\xi'x+\eta'y+\zeta'z=0,$$

then (*ante*, No. 201) we have for the line-equation of the centre of inscription

$$\Pi=(\mathfrak{A},..\between\xi,\eta,\zeta\between\xi',\eta',\zeta')=0.$$

The line-equation of the inscribed conic is in the first instance obtained in the form

$$(\mathfrak{A},..\between\xi,\eta,\zeta)^2+\lambda(a,..\between\eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^2=0\,;$$

but we have identically,

$$(\mathfrak{A},..\between\xi,\eta,\zeta)^2(\mathfrak{A},..\between\xi',\eta',\zeta')^2-\{(\mathfrak{A},..\between\xi,\eta,\zeta\between\xi',\eta',\zeta')\}^2=K(a,..\between\eta\zeta'-\eta'\zeta,\zeta\xi'-\zeta'\xi,\xi\eta'-\xi'\eta)^2,$$

and the equation thus becomes

$$[K+\lambda(\mathfrak{A},..\between\xi',\eta',\zeta')^2](\mathfrak{A},..\between\xi,\eta,\zeta)^2-\lambda\{(\mathfrak{A},..\between\xi',\eta',\zeta'\between\xi,\eta,\zeta)\}^2=0,$$

which is of the form in question.

205. Take (x',y',z') as the point-coordinates of the centre of inscription, the equation of the axis of inscription is

$$(a,b,c,f,g,h\between x,y,z\between x',y',z')=0.$$

And we may, if we please, exhibit the equation of the inscribed conic in the form

$$(a,..\between x,y,z)^2(a,..\between x',y',z')^2\cos^2\theta-(\{a,..\between x,y,z\between x',y',z')\}^2=0,$$

where θ is a constant. This equation may also be written

$$(a,..\between x,y,z)^2(a,..\between x',y',z')^2\sin^2\theta-(\mathfrak{A},..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2=0,$$

the two forms being equivalent in virtue of the identity,

$$(a,..\between x,y,z)^2(a,..\between x',y',z')^2-\{(a,..\between x',y',z'\between x,y,z)\}^2=(\mathfrak{A},..\between yz'-y'z,zx'-z'x,xy'-x'y)^2.$$

206. The line-coordinates (ξ',η',ζ') of the axis of inscription are

$$ax'+hy'+gz',\ \ hx'+by'+fz',\ \ gx'+fy'+cz',$$

and we thence deduce the relation

$$(\mathfrak{A},..\between\xi',\eta',\zeta')^2=K(a,..\between x',y',z')^2.$$

In order that the form

$$(a,..\between x,y,z)^2(a,..\between x',y',z')^2\cos^2\theta-\{(a,..\between x',y',z'\between x,y,z)\}^2=0$$

may agree with the originally assumed form

$$(a,..\between x,y,z)^2+\lambda(\xi'x+\eta'y+\zeta'z)^2,$$

or what is the same thing,

$$(a, \ldots)(x, y, z)^2 + \lambda\{(a, \ldots)(x, y, z)(x', y', z')\}^2 = 0,$$

we must have

$$\lambda = \frac{-1}{(a, \ldots)(x', y', z')^2 \cos^2\theta},$$

which may also be written

$$\lambda = \frac{-\mathrm{K}}{(\mathfrak{A}, \ldots)(\xi', \eta', \zeta')^2 \cos^2\theta},$$

or what is the same thing,

$$\mathrm{K} + \lambda(\mathfrak{A}, \ldots)(\xi', \eta', \zeta')^2 - \lambda(\mathfrak{A}, \ldots)(\xi', \eta', \zeta')^2 \sin^2\theta = 0;$$

and we thence, by a preceding formula, obtain the line-equation of the inscribed conic, viz.

207. The point-equation being

$$(a, \ldots)(x, y, z)^2 (a, \ldots)(x', y', z')^2 \cos^2\theta - \{(a, \ldots)(x, y, z)(x', y', z')\}^2 = 0,$$

or

$$(a, \ldots)(x, y, z)^2 (a, \ldots)(x', y', z')^2 \sin^2\theta - (\mathfrak{A}, \ldots)(yz' - y'z,\ zx' - z'x,\ xy' - x'y)^2 = 0,$$

equivalent in virtue of

$$(a, \ldots)(x, y, z)^2 (a, \ldots)(x', y', z')^2 - \{(a, \ldots)(x, y, z)(x', y', z')\}^2 = (\mathfrak{A}, \ldots)(yz' - y'z,\ zx' - z'x,\ xy' - x'y)^2.$$

The corresponding forms of the line-equation are

$$(\mathfrak{A}, \ldots)(\xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots)(\xi', \eta', \zeta')^2 \sin^2\theta - \{(\mathfrak{A}, \ldots)(\xi, \eta, \zeta)(\xi', \eta', \zeta')\}^2 = 0,$$

and

$$(\mathfrak{A}, \ldots)(\xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots)(\xi', \eta', \zeta')^2 \cos^2\theta - \mathrm{K}(a, \ldots)(\eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta)^2 = 0,$$

equivalent to each other in virtue of the before mentioned identity,

$$(\mathfrak{A}, \ldots)(\xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots)(\xi', \eta', \zeta')^2 - \{(\mathfrak{A}, \ldots)(\xi, \eta, \zeta)(\xi', \eta', \zeta')\}^2 = \mathrm{K}(a, \ldots)(\eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta)^2.$$

208. Write for shortness

$$(a, \ldots)(x, y, z)^2 = 00,$$
$$(a, \ldots)(x, y, z)(x', y', z) = 01 = 10,$$
&c.,

then we have identically,

$$\begin{vmatrix} 00, & 01, & 02 \\ 10, & 11, & 12 \\ 20, & 21, & 22 \end{vmatrix} = \mathrm{K} \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}^2$$

and if the determinant on the right hand vanishes, that is if (x, y, z), (x', y', z'), (x'', y'', z'') are points in a line, then we have

$$\begin{vmatrix} 00, & 01, & 02 \\ 10, & 11, & 12 \\ 20, & 21, & 22 \end{vmatrix} = 0,$$

an equation, which, as already remarked, is equivalent to

$$\cos^{-1}\frac{01}{\sqrt{00}\sqrt{11}}+\cos^{-1}\frac{12}{\sqrt{11}\sqrt{22}}=\cos^{-1}\frac{02}{\sqrt{00}\sqrt{22}}.$$

The foregoing investigations in relation to the inscribed conic are given for the sake of the application thereof to the theory of distance, and it has been necessary to make use of analytical formulæ of some complexity which are introduced out of their natural place.

On the Theory of Distance, Nos. 209 to 229.

209. I return to the geometry of one dimension. Imagine in the line or *locus in quo* of the range of points, a point-pair, which I term the Absolute. Any point-pair whatever may be considered as inscribed in the Absolute, the centre and axis of inscription being the sibiconjugate points of the involution formed by the points of the given point-pair and the points of the Absolute; the centre and axis of inscription *quà* sibiconjugate points are harmonics with respect to the Absolute. A point-pair considered as thus inscribed in the Absolute is said to be a *point-pair circle*, or simply a *circle*; the centre of inscription and the axis of inscription are termed the centre and the axis. Either of the two sibiconjugate points may be considered as the centre, but the selection when made must be adhered to. It is proper to notice that, given the centre and one point of the circle, the other point of the circle is determined in a unique manner. In fact the axis is the harmonic of the centre in respect to the Absolute, and then the other point is the harmonic of the given point in respect to the centre and axis.

210. As a definition, we say that the two points of a circle are equidistant from the centre. Now imagine two points P, P'; and take the point P'' such that P, P'' are a circle having P' for its centre; take in like manner the point P''' such that P', P''' are a circle having P'' for its centre; and so on: and again in the opposite direction, a point $P^{\backprime}$ such that $P', P^{\backprime}$ are a circle having P for its centre; a point $P^{\backprime\backprime}$ such that $P, P^{\backprime\backprime}$ are a circle having $P^{\backprime}$ for its centre, and so on. We have a series of points $\ldots P^{\backprime\backprime}, P^{\backprime}, P, P', P'', \ldots$ at equal intervals of distance: and if we take the points P, P' indefinitely near to each other, then the entire line will be divided into a series of equal infinitesimal elements; the number of these elements included between any two points measures the distance of the two points. It is clear that, according to the definition, if P, P', P'' be any three points taken in order, then

$$\text{Dist.}(P, P')+\text{Dist.}(P', P'')=\text{Dist.}(P, P''),$$

which agrees with the ordinary notion of distance.

211. To show how the foregoing definition leads to an analytical expression for the distance of two points in terms of their coordinates, take

$$(a, b, c)(x, y)^2=0$$

for the equation of the Absolute. The equation of a circle having the point (x', y') for its centre is

$$(a, b, c)(x, y)^2(a, b, c)(x', y')^2\cos^2\theta-\{(a, b, c)(x, y)(x', y')\}^2=0;$$

and consequently if (x, y), (x'', y'') are the two points of the circle, then

$$\frac{(a, b, c \between x, y \between x', y')}{\sqrt{(a, b, c \between x, y)^2}\sqrt{(a, b, c \between x', y')^2}}=\frac{(a, b, c \between x', y' \between x'', y'')}{\sqrt{(a, b, c \between x', y')^2}\sqrt{(a, b, c \between x'', y'')^2}},$$

an equation which expresses that the points (x'', y'') and (x, y) are equidistant from the point (x', y'). It is clear that the distance of the points (x, y) and (x', y') must be a function of

$$\frac{(a, b, c \between x, y \between x', y')}{\sqrt{(a, b, c \between x, y)^2}\sqrt{a, b, c \between x', y')^2}},$$

and the form of the function is determined from the before-mentioned property, viz. if P, P′, P″ be any three points taken in order, then

$$\text{Dist. }(\text{P, P}')+\text{Dist. }(\text{P}', \text{P}'')=\text{Dist. }(\text{P, P}'').$$

This leads to the conclusion that the distance of the points (x, y), (x', y') is equal to a multiple of the arc having for its cosine the last-mentioned expression (see *ante*, No. 168); and we may in general assume that the distance is equal to the arc in question, viz. that the distance is

$$\cos^{-1}\frac{(a, b, c \between x, y \between x', y')}{\sqrt{(a, b, c \between x, y)^2}\sqrt{(a, b, c \between x', y')^2}},$$

or what is the same thing,

$$\sin^{-1}\frac{(ac-b^2)(xy'-x'y)}{\sqrt{(a, b, c \between x, y)^2}\sqrt{(a, b, c \between x', y')^2}}.$$

It follows that the two forms

$$(a, b, c \between x, y)^2(a, b, c \between x', y')^2\cos^2\theta-\{(a, b, c \between x, y \between x', y')\}^2=0,$$

$$(a, b, c \between x, y)^2(a, b, c \between x', y')^2\sin^2\theta-(ac-b^2)(xy'-x'y)\qquad=0,$$

of the equation of a circle, each of them express that the distances of the two points from the centre are respectively equal to the arc θ; or, if we please, that θ is the radius of the circle.

212. When $\theta=0$, we have

$$xy'-x'y=0,$$

an equation which expresses that (x, y) and (x', y') are one and the same point. When $\theta=\frac{\pi}{2}$, we have

$$(a, b, c \between x, y \between x', y')=0,$$

an equation which expresses that the points (x, y) and (x', y') are harmonics with respect to the Absolute. The distance between any two points harmonics with respect to the Absolute is consequently a quadrant, and such points may be said to be quadrantal to each other. The quadrant is the unit of distance.

213. The foregoing is the general case, but it is necessary to consider the particular case where the Absolute is a pair of coincident points. The harmonic of any point whatever in respect to the Absolute is here a point coincident with the Absolute itself:

the definition of a circle is consequently simplified; viz. any point-pair whatever may be considered as a circle having for its centre the harmonic of the Absolute with respect to the point-pair; we may, as before, divide the line into a series of equal infinitesimal elements, and the number of elements included between any two points measures the distance between the two points. As regards the analytical expression, in the case in question $ac-b^2$ vanishes, or the distance is given as the arc to an evanescent sine. Reducing the arc to its sine and omitting the evanescent factor, we have a finite expression for the distance. Suppose that the equation of the Absolute is

$$(qx-py)^2=0,$$

or what is the same thing, let the Absolute (treated as a single point) be the point (p, q), then we find for the distance of the points (x, y) and (x', y') the expression

$$\frac{xy'-x'y}{(qx-py)(qx'-py')};$$

or introducing an arbitrary multiplier,

$$\frac{(q\alpha-p\beta)(xy'-x'y)}{(qx-py)(qx'-py')},$$

which is equal to

$$\frac{\beta x-\alpha y}{qx-py}-\frac{\beta x'-\alpha y'}{qx'-py'}.$$

It is hardly necessary to remark, that in the present case the notion of the quadrantal relation of two points has altogether disappeared, and that the unit of distance is arbitrary.

214. Passing now to geometry of two dimensions, we have here to consider a certain conic, which I call the Absolute. Any line whatever determines with the Absolute (cuts it in) two points which are the Absolute in regard to such line considered as a space of one dimension, or *locus in quo* of a range of points, and in like manner any point whatever determines with the Absolute (has for tangents of the Absolute through the point) two lines which are the Absolute in regard to such point considered as a space of one dimension, or *locus in quo* of a pencil of lines. The foregoing theory for geometry of one dimension establishes the notion of distance as regards each of these ranges and pencils considered apart by itself; in order to bring the different ranges and pencils in relation to each other, it is necessary to assume that the quadrant which is the unit of distance for these several systems respectively, is one and the same distance for each system (of course, when, as in the analytical theory, we actually represent the quadrant by the ordinary symbol $\frac{\pi}{2}$, the above assumption is tacitly made; but substituting the thing signified for the definition, and looking at the quadrant merely as the distance between two points, or as the case may be, lines, harmonically related to the point-pair, or as the case may be, line-pair, constituting the Absolute, the assumption is at once seen to be an assumption, and it needs to be made explicitly). But the assumption being made, the foregoing theory of distance in geometry of one dimension enables the comparison not only

of the distances of points upon different lines, or of lines through different points, but of the distances of points on a line and of lines through a point. The pole of any line in relation to the Absolute may be termed simply the pole, and in like manner the polar of any line in relation to the Absolute may be termed simply the polar, and we have the theorem that the distance of two points or lines is equal to the distance of their polars or poles, or what is the same thing, that the distance of two poles and the distance of the two corresponding polars are equal. And we may, as a definition, establish the notion of the distance of a point from a line, viz. it is the complement of the distance of the polar of the point from the line, or what is the same thing, the complement of the distance of the point from the pole of the line. The distance of a pole and polar is therefore the complement of zero, that is, it is the quadrant.

215. It has, by means of the preceding assumption as to the quadrant, been possible to establish the notion of distance, without the assistance of the circle, but this figure must now be considered. A conic inscribed in the Absolute is termed a circle; the centre of inscription (or point of intersection of the common tangents) and the axis of inscription (or line of junction of the common ineunts) are the centre and axis of the circle. All the points of a circle are equidistant from the centre; all the tangents are equidistant from the axis, and this distance is the complement of the former distance.

216. These properties of the circle lead immediately to the analytical expressions for the distances of points or lines in terms of the coordinates. In fact, take

$$(a, b, c, f, g, h \between x, y, z)^2=0$$

for the point-equation of the Absolute; its line-equation will be

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2=0.$$

The point-equation of the circle having the point (x', y', z') for its centre, is

$$(a, .. \between x, y, z)^2(a, .. \between x', y', z')^2 \cos^2\theta - \{(a, ... \between x, y, z \between x', y', z')\}^2=0,$$

or

$$(a, .. \between x, y, z)^2(a, ... \between x', y', z')^2 \sin^2\theta - (\mathfrak{A}, .. \between yz'-y'z, zx'-z'x, xy'-x'y)^2=0,$$

from which (by the same reasoning as for the case of geometry of one dimension) it follows that the distance of the points (x, y, z), (x', y', z') is

$$\cos^{-1}\frac{(a, .. \between x, y, z \between x', y', z')}{\sqrt{(a, .. \between x, y, z)^2}\sqrt{(a, .. \between x', y', z')^2}},$$

or what is the same thing,

$$\sin^{-1}\frac{\sqrt{(\mathfrak{A}, .. \between yz'-y'z, zx'-z'x, xy'-x'y)^2}}{\sqrt{(a, .. \between x, y, z)^2}\sqrt{(a, .. \between x', y', z')^2}};$$

and it appears from the cosine formula (see *ante*, No. 208), that if P, P′, P″ be points on the same line, then we have, as we ought to have,

$$\text{Dist.}\,(P, P')+\text{Dist.}\,(P', P'')=\text{Dist.}\,(P, P'').$$

217. In like manner, the line-equation of the same circle, the line-coordinates of the

axis being (ξ', η', ζ'), is

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \sin^2\theta - \{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')\}^2 = 0,$$

or

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \cos^2\theta - \mathrm{K}(a, \ldots \between \eta\zeta' - \eta'\zeta, \zeta\xi' - \zeta'\xi, \xi\eta' - \xi'\eta)^2 = 0,$$

from which it follows that the distance of the lines (ξ, η, ζ) and (ξ', η', ζ') is

$$\cos^{-1} \frac{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')}{\sqrt{\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2} \sqrt{(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2}},$$

or what is the same thing,

$$\sin^{-1} \frac{\sqrt{\mathrm{K}(a, \ldots \between \eta\zeta' - \eta'\zeta, \zeta\xi' - \zeta'\xi, \xi\eta' - \xi'\eta)^2}}{\sqrt{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2} \sqrt{(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2}}.$$

218. And we may from the first formula of either set, deduce for the distance of the point (x, y, z) and the line (ξ', η', ζ'), the expression

$$\sin^{-1} \frac{\sqrt{\mathrm{K}}(\xi' x + \eta' y + \zeta' z)}{\sqrt{(a, \ldots \between x, y, z)^2} \sqrt{(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2}},$$

as may be easily seen by writing $\mathfrak{A}\xi' + \mathfrak{H}\eta' + \mathfrak{G}\zeta', \ldots$ for x', y', z', or $ax + hy + gz, \ldots$ for ξ, η, ζ, and putting $\sin^{-1}$ for $\cos^{-1}$.

219. It may be noticed that there are certain lines, viz. the tangents of the Absolute, in regard to which, considered as a space of one dimension, the Absolute is a pair of coincident points; and in like manner certain points, viz. the ineunts of the Absolute, in regard to which, considered as a space of one dimension, the Absolute is a pair of coincident lines.

220. We may, in particular, suppose that the Absolute, instead of being a proper conic, is a pair of points. The line through the two points may be called the Absolute line; such line is to be considered as a pair of coincident lines. Any point whatever determines with the Absolute, two lines, viz. the lines joining the point with the two points of the Absolute; this line-pair is the Absolute for the point considered as a space of one dimension or *locus in quo* of a pencil of lines, and the theory of the distances of lines through a point is therefore precisely the same as in the general case. But any line whatever determines with the Absolute (meets the Absolute line in) a pair of coincident points, which pair of coincident points is the Absolute in regard to such line considered as a space of one dimension or *locus in quo* of a range of points, and the theory of the distance of points on a line is therefore the theory before explained for this special case. But we cannot, in the same way as before, compare the distances of points upon different lines, since we have not in the present case the quadrant as a unit of distance. The comparison must be made by means of the circle, viz. in the present case any conic passing through the two points of the Absolute is termed a circle, and the point of intersection of the tangents to the circle at the two points of the Absolute (or what is the same thing, the pole of the Absolute line in respect to the circle) is the centre of the circle. The Absolute line itself may, if it is necessary to do so, be considered as the axis of the circle. It

is assumed that the points of the circle are all of them equidistant from the centre, and by this assumption we are enabled to compare distances upon different lines. In fact we may, by a construction precisely similar to that of EUCLID, Book I. Prop. II., from a given point A draw a finite line equal to a given finite line BC, and thence also upon a given line through A, determine the finite line AD equal to the given finite line BC. Since the unit of distance for points on a line is arbitrary, we cannot of course compare the distances of points with the distances of lines. The distance of a point from a line does, however, admit of comparison with the distance of two points; we have only to assume as a definition that the distance of a point from a line is the distance of the point from the point of intersection of the line with the quadrantal line through the point.

221. As regards the analytical theory, suppose that the point-coordinates of the two points of the Absolute are (p, q, r), (p_0, q_0, r_0), then the line-equation of the Absolute is

$$2(p\xi+q\eta+r\zeta)(p_0\xi+q_0\eta+r_0\zeta)=0;$$

so that we have $\mathfrak{A}=2pp_0$, $\mathfrak{B}=2qq_0$, $\mathfrak{C}=2rr_0$, $\mathfrak{F}=qr_0+rq_0$, $\mathfrak{G}=rp_0+pr_0$, $\mathfrak{H}=pq_0+qp_0$, and thence $K=0$; but

$$K(a, b, c, f, g, h\between x, y, z)^2 = \begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}^2$$

where obviously

$$\begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}=0$$

is the equation of the Absolute line.

222. The expression for the distance of the two points (x, y, z), (x', y', z') is given as the arc to an evanescent sine; but reducing the arc to its sine, and omitting the evanescent factor, the resulting expression is

$$\sqrt{2\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ p, & q, & r \end{vmatrix}\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ p_0, & q_0, & r_0 \end{vmatrix}} \div \begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}\begin{vmatrix} x', & y', & z' \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}$$

and the expression for the distance of the two lines (ξ, η, ζ), (ξ', η', ζ') is

$$\cos^{-1}\frac{(p\xi+q\eta+r\zeta)(p_0\xi'+q_0\eta'+r_0\zeta')+(p\xi'+q\eta'+r\zeta')(p_0\xi+q_0\eta+r_0\zeta)}{\sqrt{2(p\xi+q\eta+r\zeta)(p_0\xi+q_0\eta+r_0\zeta)}\sqrt{2(p\xi'+q\eta'+r\zeta')(p_0\xi'+q_0\eta'+r_0\zeta')}},$$

or what is the same thing,

$$\sin^{-1}\frac{(qr_0-rq_0)(\eta\zeta'-\eta'\zeta)+(rp_0-pr_0)(\zeta\eta'-\zeta'\eta)+(pq_0-qp_0)(\xi\eta'-\xi'\eta)}{\sqrt{2(p\xi+q\eta+r\zeta)(p_0\xi+q_0\eta+r_0\zeta)}\sqrt{2(p\xi'+q\eta'+r\zeta')(p_0\xi'+q_0\eta'+r_0\zeta')}};$$

and finally, the expression for the distance of the point x, y, z from the line (ξ', η', ζ'), reducing the arc to its sine and omitting the evanescent factor, is

$$(\xi'x+\eta'y+\zeta'z) \div \left| \begin{array}{ccc} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{array} \right| \sqrt{2(p\xi'+q\eta'+r\zeta')(p_0\xi'+q_0\eta'+r_0\zeta')}.$$

223. If in the above formula we put $(p, q, r)=(1, i, 0)$, $(p_0, q_0, r_0)=(1, -i, 0)$, where as usual $i=\sqrt{-1}$, then the line-equation of the Absolute is $\xi^2+\eta^2=0$, or what is the same thing, the Absolute consists of the two points in which the line $z=0$ intersects the line-pair $x^2+y^2=0$; the last-mentioned line-pair, as passing through the Absolute, is by definition a circle; it is in fact the circle radius zero, or an evanescent circle. If we put also the coordinate z equal to unity, then the preceding assumption as to the coordinates of the points of the Absolute must be understood to mean only $x:y:1=1:i:0$, or $1:-i:0$; that is, we must have x and y infinite, and, as before, $x^2+y^2=0$, or in other words, the Absolute will consist of the points of intersection of the line infinity by the evanescent circle $x^2+y^2=0$. With the values in question,

224. The expression for the distance of the points (x, y) and (x', y') is

$$\sqrt{(x-x')^2+(y-y')^2};$$

that for the distance of the lines (ξ, η, ζ) and (ξ', η', ζ') is

$$\cos^{-1}\frac{\xi\xi'+\eta\eta'}{\sqrt{\xi^2+\eta^2}\sqrt{\xi'^2+\eta'^2}}$$

$$=\sin^{-1}\frac{\xi\eta'-\xi'\eta}{\sqrt{\xi^2+\eta^2}\sqrt{\xi'^2+\eta'^2}},$$

which may also be written

$$=\tan^{-1}\frac{\xi}{\eta}-\tan^{-1}\frac{\xi'}{\eta'};$$

and the expression for the distance of the point (x, y) from the line (ξ', η', ζ') is

$$\frac{\xi'x+\eta'y+\zeta'}{\sqrt{\xi'^2+\eta'^2}},$$

which are obviously the formulæ of ordinary plane geometry, (x, y) being ordinary rectangular coordinates.

225. The general formulæ suffer no *essential* modification, but they are greatly simplified in form by taking for the point-equation of the Absolute,

$$x^2+y^2+z^2=0,$$

or what is the same, for the line-equation

$$\xi^2+\eta^2+\zeta^2=0.$$

In fact, we then have for the expression of the distance of the points (x, y, z), (x', y', z'),

$$\cos^{-1}\frac{xx'+yy'+zz'}{\sqrt{x^2+y^2+z^2}\sqrt{x'^2+y'^2+z'^2}};$$

for that of the lines (ξ, η, ζ), (ξ', η', ζ'),

$$\cos^{-1}\frac{\xi\xi'+\eta\eta'+\zeta\zeta'}{\sqrt{\xi^2+\eta^2+\zeta^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2}};$$

and for that of the point (x, y, z) and the line (ξ', η', ζ'),

$$\sin^{-1}\frac{\xi' x+\eta' y+\zeta' z}{\sqrt{x^2+y^2+z^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2}}.$$

226. Suppose (x, y, z) are ordinary rectangular coordinates in space satisfying the condition

$$x^2+y^2+z^2=1,$$

the point having (x, y, z) for its coordinates will be a point on the surface of the sphere, and (the last-mentioned equation always subsisting) the equation $\xi x+\eta y+\zeta z=0$ will be a great circle of the sphere; and since we are only concerned with the ratios of ξ, η, ζ, we may also assume $\xi^2+\eta^2+\zeta^2=1$. We may of course retain in the formulæ the expressions $x^2+y^2+z^2$ and $\xi^2+\eta^2+\zeta^2$, without substituting for these the values unity, and it is in fact convenient thus to preserve all the formulæ in their original forms. We have thus a system of spherical geometry; and it appears that the Absolute in such system is the (spherical) conic, which is the intersection of the sphere with the concentric cone or evanescent sphere $x^2+y^2+z^2=0$. The circumstance that the Absolute is a proper conic, and not a mere point-pair, is the real ground of the distinction between spherical geometry and ordinary plane geometry, and the cause of the complete duality of the theorems of spherical geometry.

227. I have, in all that has preceded, given the analytical theory of distance along with the geometrical theory, as well for the purpose of illustration, as because it is important to have the analytical expression of a distance in terms of the coordinates; but I consider the geometrical theory as perfectly complete in itself: the general result is as follows, viz. assuming in the plane (or space of geometry of two dimensions) a conic termed the Absolute, we may by means of this conic, by descriptive constructions, divide any line or range of points whatever, and any point or pencil of lines whatever, into an infinite series of infinitesimal elements, which are (as a definition of distance) assumed to be equal; the number of elements between two points of the range or two lines of the pencil, measures the distance between the two points or lines; and by means of the quadrant, as a distance which exists as well with respect to lines as points, we are enabled to compare the distance of two lines with that of two points; and the distance of a point and a line may be represented indifferently as the distance of two points, or as the distance of two lines.

228. In ordinary spherical geometry, the general theory undergoes no modification whatever; the Absolute is an actual conic, the intersection of the sphere with the concentric evanescent sphere.

229. In ordinary plane geometry, the Absolute degenerates into a pair of points, viz. the points of intersection of the line infinity with any evanescent circle, or what is the

same thing, the Absolute is the two circular points at infinity. The general theory is consequently modified, viz. there is not, as regards points, a distance such as the quadrant, and the distance of two lines cannot be in any way compared with the distance of two points; the distance of a point from a line can be only represented as a distance of two points.

230. I remark in conclusion, that, *in my own point of view*, the more systematic course in the present introductory memoir on the geometrical part of the subject of quantics, would have been to ignore altogether the notions of distance and metrical geometry; for the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connexion with another figure, viz. the conic termed the Absolute. The original figure might comprise a conic; for instance, we might consider the properties of the figure formed by two or more conics, and we are then in the region of pure descriptive geometry: we pass out of it into metrical geometry by fixing upon a conic of the figure as a standard of reference and calling it the Absolute. Metrical geometry is thus a part of descriptive geometry, and descriptive geometry is *all* geometry and reciprocally; and if this be admitted, there is no ground for the consideration, in an introductory memoir, of the special subject of metrical geometry; but as the notions of distance and of metrical geometry could not, without explanation, be thus ignored, it was necessary to refer to them in order to show that they are thus included in descriptive geometry.