IV. *On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion.*

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SECTION I.

Introduction.

1. THE equations of motion of viscous fluid (obtained by grafting on certain terms to the abstract equations of the Eulerian form so as to adapt these equations to the case of fluids subject to stresses depending in some hypothetical manner on the rates of distortion, which equations NAVIER* seems to have first introduced in 1822, and which were much studied by CAUCHY† and POISSON‡) were finally shown by St. VENANT§ and Sir GABRIEL STOKES,|| in 1845, to involve no other assumption than that the stresses, other than that of pressure uniform in all directions, are linear functions of the rates of distortion, with a co-efficient depending on the physical state of the fluid.

By obtaining a singular solution of these equations as applied to the case of pendulums in steady periodic motion, Sir G. STOKES¶ was able to compare the theoretical results with the numerous experiments that had been recorded, with the result that the theoretical calculations agreed so closely with the experimental determinations as seemingly to prove the truth of the assumption involved. This was also the result of comparing the flow of water through uniform tubes with the flow calculated from a singular solution of the equations so long as the tubes were small and the velocities slow. On the other hand, these results, both theoretical and practical, were directly at variance with common experience as to the resistance

* 'Mém. de l'Académie,' vol. 6, p. 389.
† 'Mém. des Savants Étrangers,' vol. 1, p. 40.
‡ 'Mém. de l'Académie,' vol. 10, p. 345.
§ 'B.A. Report,' 1846.
|| 'Cambridge Phil. Trans.,' 1845.
¶ 'Cambridge Phil. Trans.,' vol. 9, 1857.

encountered by larger bodies moving with higher velocities through water, or by water moving with greater velocities through larger tubes. This discrepancy Sir G. STOKES considered as probably resulting from eddies which rendered the actual motion other than that to which the singular solution referred and not as disproving the assumption.

In 1850, after JOULE's discovery of the Mechanical Equivalent of Heat, STOKES showed, by transforming the equations of motion—with arbitrary stresses—so as to obtain the equations of ("Vis-viva") energy, that this equation contained a definite function, which represented the difference between the work done on the fluid by the stresses and the rate of increase of the energy, per unit of volume, which function, he concluded, must, according to JOULE, represent the Vis-viva converted into heat.

This conclusion was obtained from the equations irrespective of any particular relation between the stresses and the rates of distortion. Sir G. STOKES, however, translated the function into an expression in terms of the rates of distortion, which expression has since been named by Lord RAYLEIGH the *Dissipation-Function.*

2. In 1883 I succeeded in proving, by means of experiments with colour bands—the results of which were communicated to the Society*—that when water is caused by pressure to flow through a uniform smooth pipe, the motion of the water is *direct*, *i.e.*, parallel to the sides of the pipe, or *sinuous*, *i.e.*, crossing and re-crossing the pipe, according as U_m, the mean velocity of the water, as measured by dividing Q, the discharge, by Δ, the area of the section of the pipe, is below or above a certain value given by

$$K\mu/D\rho,$$

where D is the diameter of the pipe, ρ the density of the water, and K a numerical constant, the value of which according to my experiments and, as I was able to show, to all the experiments by POISEUILLE and DARCY, is for pipes of circular section between

$$1900 \text{ and } 2000,$$

or, in other words, steady direct motion in round tubes is stable or unstable according as

$$\rho\frac{DU_m}{\mu} < 1900 \text{ or } > 2000,$$

the number K being thus a criterion of the possible maintenance of sinuous or eddying motion.

3. The experiments also showed that K was equally a criterion of the law of the resistance to be overcome—which changes from a resistance proportional to the

* 'Phil. Trans.,' 1883, Part III., p. 935.

velocity and in exact accordance with the theoretical results obtained from the singular solution of the equation, when direct motion changes to sinuous, *i.e.*, when

$$\rho \frac{DU_m}{\mu} = K.$$

4. In the same paper I pointed out that the existence of this sudden change in the law of motion of fluids between solid surfaces when

$$DU_m = \frac{\mu}{\rho} K$$

proved the dependence of the manner of motion of the fluid on a relation between the product of the dimensions of the pipe multiplied by the velocity of the fluid and the product of the molecular dimensions multiplied by the molecular velocities which determine the value of

$$\mu$$

for the fluid, also that the equations of motion for viscous fluid contained evidence of this relation.

These experimental results completely removed the discrepancy previously noticed, showing that, whatever may be the cause, in those cases in which the experimental results do not accord with those obtained by the singular solution of the equations, the actual motions of the water are different. But in this there is only a partial explanation, for there remains the mechanical or physical significance of the existence of the criterion to be explained.

5. [My object in this paper is to show that the theoretical existence of an inferior limit to the criterion follows from the equations of motion as a consequence :—

(1) Of a more rigorous examination and definition of the geometrical basis on which the analytical method of distinguishing between molar-motions and heat-motions in the kinetic theory of matter is founded ; and

(2) Of the application of the same method of analysis, thus definitely founded, to distinguish between mean-molar-motions and relative-molar-motions where, as in the case of steady-mean-flow along a pipe, the more rigorous definition of the geometrical basis shows the method to be strictly applicable, and in other cases where it is approximately applicable.

The geometrical relation of the motions respectively indicated by the terms mean-molar-, or MEAN-MEAN-MOTION, and relative-molar or RELATIVE-MEAN-MOTION being essentially the same as the relation of the respective motions indicated by the terms molar-, or MEAN-MOTION, and relative-, or HEAT-MOTION, as used in the theory of gases.

I also show that the limit to the criterion obtained by this method of analysis and by integrating the equations of motion in space, appears as a *geometrical limit* to the *possible simultaneous distribution of certain quantities in space*, and in no wise depends on the physical significance of these quantities. Yet the physical significance of these quantities, as defined in the equations, becomes so clearly exposed as to indicate that further study of the equations would elucidate the properties of matter and mechanical principles involved, and so be the means of explaining what has hitherto been obscure in the connection between thermodynamics and the principles of mechanics.

The geometrical basis of the method of analysis used in the kinetic theory of gases has hitherto consisted :—

(1) Of the geometrical *principle* that the motion of any point of a mechanical system may, at any instant, be abstracted into the mean motion of the whole system at that instant, and the motion of the point relative to the mean-motion ; and

(2) Of the *assumption* that the component, in any particular direction, of the velocity of a molecule may be abstracted into a mean-component-velocity (say u) which is the mean-component velocity of all the molecules in the immediate neighbourhood, and a relative velocity (say ξ), which is the difference between u and the component-velocity of the molecule ;* u and ξ being so related that, M being the mass of the molecule, the integrals of $(M\xi)$, and $(Mu\xi)$, &c., over all the molecules in the immediate neighbourhood are zero, and $\Sigma [M (u + \xi)^2] = \Sigma [M (u^2 + \xi^2)]$.†

The geometrical principle (1) has only been used to distinguish between the energy of the mean-motion of the molecule and the energy of its internal motions taken relatively to its mean motion ; and so to eliminate the internal motions from all further geometrical considerations which rest on the assumption (2).

That this assumption (2) is purely geometrical, becomes at once obvious, when it is noticed that the argument relates solely to the distribution in space of certain quantities at a particular instant of time. And it appears that the questions as to whether the assumed distinctions are possible under any distributions, and, if so, under what distribution, are proper subjects for geometrical solution.

On putting aside the apparent obviousness of the assumption (2), and considering definitely what it implies, the necessity for further definition at once appears.

The mean component-velocity (u) of all the molecules in the immediate neighbourhood of a point, say P, can only be the mean component-velocity of all the molecules in some space (S) enclosing P. u is then the mean-component velocity of the mechanical system enclosed in S, and, for this system, is the mean velocity at every point within S, and multiplied by the entire mass within S is the whole component momentum of the system. But according to the assumption (2), u with its derivatives are to be continuous functions of the position of P, which functions may vary from point to point even within S ; so that u is not taken to represent the mean component-velocity of the system within S, but the mean-velocity at the point P. Although there seems to have been no specific statement to that effect, it is presumable that the space S has

* "Dynamical Theory of Gases," 'Phil. Trans.,' 1866, pp. 67.

† 'Phil. Trans.,' 1866, p. 71.

been assumed to be so taken that P is the centre of gravity of the system within S. The relative positions of P and S being so defined, the shape and size of the space S requires to be further defined, so that u, &c., may vary continuously with the position of P, which is a condition that can always be satisfied if the size and shape of S may vary continuously with the position of P.

Having thus defined the relation of P to S and the shape and size of the latter, expressions may be obtained for the conditions of distribution of u, for which $\Sigma(M\xi)$ taken over S will be zero, *i.e.*, for which the condition of mean-momentum shall be satisfied.

Taking S_1, u_1, &c., as relating to a point P_1 and S, u, &c., as relating to P, another point of which the component distances from P_1 are x, y, z, P_1 is the C.G. of S_1, and by however much or little S may overlap S_1, S has its centre of gravity at x, y, z, and is so chosen that u, &c., may be continuous functions of x, y, z. u may, therefore, differ from u_1 even if P is within S_1. Let u be taken for every molecule of the system S_1. Then according to assumption (2), $\Sigma(Mu)$ over S_1 must represent the component of momentum of the system within S_1, that is, in order to satisfy the condition of mean momentum, the mean-value of the variable quantity u over the system S_1 must be equal to u_1 the mean-component velocity of the system S_1, and this is a condition which in consequence the geometrical definition already mentioned can only be satisfied under certain distributions of u. For since u is a continuous function of x, y, z, $M(u - u_1)$ may be expressed as a function of the derivatives of u at P_1 multiplied by corresponding powers and products of x, y, z, and again by M; and by equating the integral of this function over the space S_1 to zero, a definite expression is obtained, in terms of the limits imposed on x, y, z, by the already defined space S_1 for the geometrical condition as to the distribution of u under which the condition of mean momentum can be satisfied.

From this definite expression it appears, as has been obvious all through the argument, that the condition is satisfied if u is constant. It also appears that there are certain other well-defined systems of distribution for which the condition is strictly satisfied, and that for all other distributions of u the condition of mean-momentum can only be approximately satisfied to a degree for which definite expressions appear.

Having obtained the expression for the condition of distribution of u, so as to satisfy the condition of mean momentum, by means of the expression for $M(u - u')$, &c., expressions are obtained for the conditions as to the distribution of ξ, &c., in order that the integrals over the space S_1 of the products $M(u\xi)$, &c. may be zero when $\Sigma[M(u - u_1)] = 0$, and the conditions of mean energy satisfied as well as those of mean-momentum. It then appears that in some particular cases of distribution of u, under which the condition of mean momentum is strictly satisfied, certain conditions as to the distribution of ξ, &c., must be satisfied in order that the energies of mean-and relative-motion may be distinct. These conditions as to the distribution of ξ, &c., are, however, obviously satisfied in the case of heat motion, and do not present themselves otherwise in this paper.

From the definite geometrical basis thus obtained, and the definite expressions which follow for the condition of distribution of u, &c., under which the method of analysis is strictly applicable, it appears that this method may be rendered generally applicable to any system of motion by a slight adaptation of the meaning of the symbols, and that it does not necessitate the elimination of the internal motion of the molecules, as has been the custom in the theory of gases.

Taking u, v, w to represent the motions (continuous or discontinuous) of the matter passing a point, and ρ to represent the density at the point, and putting $\bar{u}$, &c., for the mean-motion (instead of u as above), and u', &c., for the relative-motion (instead of ξ as before), the geometrical conditions as to the distribution of $\bar{u}$, &c., to satisfy the conditions of mean-momentum and mean-energy are, substituting ρ for M, of precisely the same form as before, and as thus expressed, the theorem is applicable to any mechanical system however abstract.

(1) In order to obtain the conditions of distribution of molar-motion, under which the condition of mean-momentum will be satisfied so that the energy of molar-motion may be separated from that of the heat-motion, u, &c., and ρ are taken as referring to the actual motion and density at a point in a molecule, and S_1 is taken of such dimensions as may correspond to the scale, or periods in space, of the molecular distances, then the conditions of distribution of $\bar{u}$, under which the condition of mean-momentum is satisfied, become the conditions as to the distribution of molar-motion, under which it is possible to distinguish between the energies of molar-motions and heat-motions.

(2) And, when the conditions in (1) are satisfied to a sufficient degree of approximation by taking u to represent the molar-motion ($\bar{u}$ in (1)), and the dimensions of the space S to correspond with the period in space or scale of any possible periodic or eddying motion. The conditions as to the distribution of $\bar{u}$, &c. (the components of mean-mean-motion), which satisfy the condition of mean-momentum, show the conditions of mean-molar-motion, under which it is possible to separate the energy of mean-molar-motion from the energy of relative-molar- (or relative-mean-) motion

Having thus placed the analytical method used in the kinetic theory on a definite geometrical basis, and adapted so as to render it applicable to all systems of motion, by applying it to the dynamical theory of viscous fluid, I have been able to show :— Feb. 18, 1895.]

(a) That the adoption of the conclusion arrived at by Sir GABRIEL STOKES, that the dissipation function represents the rate at which heat is produced, adds a definition to the meaning of u, v, w—the components of mean or fluid velocity—which was previously wanting;

(*b*) That as the result of this definition the equations are true, and are only true as applied to fluid in which the mean-motions of the matter, excluding the heat-motions, are steady;

(*c*) That the evidence of the possible existence of such steady mean-motions, while at the same time the conversion of the energy of these mean-motions into heat is going on, proves the existence of some *discriminative cause* by which the *periods* in space and time of the mean-motion are prevented from approximating in magnitude to the corresponding *periods* of the heat-motions, and also proves the existence of some general action by which the energy of mean-motion is continually *transformed* into the energy of heat-motion without passing through any intermediate stage;

(*d*) That as applied to fluid in unsteady mean-motion (excluding the heat-motions), however steady the mean integral flow may be, the equations are approximately true in a degree which increases with the ratios of the magnitudes of the *periods*, in time and space, of the mean-motion to the magnitude of the corresponding periods of the heat-motions;

(*e*) That if the *discriminative cause* and the *action of transformation* are the result of general properties of matter, and not of properties which affect only the ultimate motions, there must exist evidence of similar actions as between the mean-mean-motion, in directions of mean flow, and the periodic mean-motions taken relative to the mean-mean-motion but excluding heat-motions. And that such evidence must be of a general and important kind, such as the unexplained laws of the resistance of fluid motions, the law of the universal dissipation of energy and the second law of thermodynamics;

(*f*) That the *generality* of the effects of the properties on which the *action of transformation* depends is proved by the fact that resistance, other than proportional to the velocity, is caused by the relative (eddying) mean-motion.

(*g*) That the existence of the *discriminative cause* is directly proved by the existence of the *criterion*, the dependence of which on circumstances which limit the magnitudes of the periods of relative mean-motion, as compared with the heat-motion, also proves the *generality* of the effects of the properties on which it depends.

(*h*) That the proof of the generality of the effects of the properties on which the discriminative cause, and the action of transformation depend, shows that—if in the equations of motion the mean-mean-motion is distinguished from the relative-mean-motion in the same way as the mean-motion is distinguished from the heat-motions—(1) the equations must contain expressions for the *transformation* of the energy of mean-mean-motion to energy of relative-mean-motion; and (2) that the equations, when integrated over a complete system, must show that the possibility of relative-mean-motion depends on the ratio of the possible magnitudes of the periods of relative-mean-motion, as compared with the corresponding magnitude of the periods of the heat-motions.

(*i*) That when the equations are transformed so as to distinguish between the mean-mean-motions, of infinite periods, and the relative-mean-motions of finite periods, there result two distinct systems of equations, one system for mean-mean-motion, as affected by relative-mean-motion and heat-motion, the other system for relative-mean-motion as affected by mean-mean-motion and heat-motions.

(*j*) That the equation of energy of mean-mean-motion, as obtained from the first system, shows that the rate of increase of energy is diminished by conversion into heat, and by transformation of energy of mean-mean-motion in consequence of the relative-mean-motion, which transformation is expressed by a function identical in form with that which expresses the conversion into heat; and that the equation of energy of relative-mean-motion, obtained from the second system, shows that this energy is increased only by transformation of energy from mean-mean-motion expressed by the same function, and diminished only by the conversion of energy of relative-mean-motion into heat.

(*k*) That the difference of the two rates (1) transformation of energy of mean-mean-motion into energy of relative-mean-motion as expressed by the transformation function, (2) the conversion of energy of relative-mean-motion into heat, as expressed by the function expressing dissipation of the energy of relative-mean-motion, affords a discriminating equation as to the conditions under which relative-mean-motion can be maintained.

(*l*) That this discriminating equation is independent of the energy of relative-mean-motion, and expresses a relation between variations of mean-mean-motion of the first order, the space periods of relative-mean-motion and μ/ρ such that any circumstances which determine the maximum periods of the relative-mean-motion determine the conditions of mean-mean-motion under which relative mean-motion will be maintained —determine *the criterion.*

(*m*) That as applied to water in steady mean flow between parallel plane surfaces, the boundary conditions and the equation of continuity impose limits to the maximum space periods of relative-mean-motion such that the discriminating equation affords definite proof that when an indefinitely small sinuous or relative disturbance exists it must fade away if

$$\rho DU_m/\mu$$

is less than a certain number, which depends on the shape of the section of the boundaries, and is constant as long as there is geometrical similarity. While for greater values of this function, in so far as the discriminating equation shows, the energy of sinuous motion may increase until it reaches to a definite limit, and rules the resistance.

(*n*) That besides thus affording a mechanical explanation of the existence of the criterion K, the discriminating equation shows the purely geometrical circumstances on which the value of K depends, and although these circumstances must satisfy geometrical conditions required for steady mean-motion other than those imposed by the conservations of mean energy and momentum, the theory admits of the determination of an inferior limit to the value of K under any definite boundary conditions, which, as determined for the particular case, is

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This is below the experimental value for round pipes, and is about half what might be expected to be the experimental value for a flat pipe, which leaves a margin to meet the other kinematical conditions for steady mean-mean-motion.

(*o*) That the discriminating equation also affords a definite expression for the resistance, which proves that, with smooth fixed boundaries, the conditions of dynamical similarity under any geometrical similar circumstances depend only on the value of

$$\frac{\rho}{\mu^2}\frac{dp}{dx}b^3,$$

where b is one of the lateral dimensions of the pipe; and that the expression for this resistance is complex, but shows that above the critical velocity the relative-mean-motion is limited, and that the resistances increase as a power of the velocity higher than the first.

Section II.

The Mean-motion and Heat-motions as distinguished by Periods.—Mean-mean-motion and Relative-mean-motion.—Discriminative Cause and Action of Transformation.—Two Systems of Equations.—A Discriminating Equation.

6. Taking the general equations of motion for incompressible fluid, subject to no external forces to be expressed by

$$\left.\begin{aligned}
\rho\frac{du}{dt} &= -\left\{\frac{d}{dx}(p_{xx}+\rho uu)+\frac{d}{dy}(p_{yx}+\rho uv)+\frac{d}{dz}(p_{zx}+\rho uw)\right\}\\
\rho\frac{dv}{dt} &= -\left\{\frac{d}{dx}(p_{xy}+\rho vu)+\frac{d}{dy}(p_{yy}+\rho vv)+\frac{d}{dz}(p_{zy}+\rho vw)\right\}\\
\rho\frac{dw}{dt} &= -\left\{\frac{d}{dx}(p_{xz}+\rho wu)+\frac{d}{dy}(p_{yz}+\rho wv)+\frac{d}{dz}(p_{zz}+\rho ww)\right\}
\end{aligned}\right\} \quad . \quad (1),$$

with the equation of continuity

$$0 = du/dx + dv/dy + dw/dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where p_{xx}, &c., are arbitrary expressions for the component forces per unit of area, resulting from the stresses, acting on the negative faces of planes perpendicular to the direction indicated by the first suffix, in the direction indicated by the second suffix.

Then multiplying these equations respectively by u, v, w, integrating by parts, adding and putting

$$2\mathrm{E} \text{ for } \rho(u^2 + v^2 + w^2)$$

and transposing, the rate of increase of kinetic energy per unit of volume is given by

$$\left(\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\mathrm{E} = -\left\{\begin{aligned} &\frac{d}{dx}(up_{xx}) + \frac{d}{dy}(up_{yx}) + \frac{d}{dz}(up_{zx}) \\ &+ \frac{d}{dx}(vp_{xy}) + \frac{d}{dy}(vp_{yy}) + \frac{d}{dz}(vp_{zy}) \\ &+ \frac{d}{dx}(wp_{xz}) + \frac{d}{dy}(wp_{yz}) + \frac{d}{dz}(wp_{zz}) \end{aligned}\right\}$$

$$+\left\{\begin{aligned} &p_{xx}\frac{du}{dx} + p_{yx}\frac{du}{dy} + p_{zx}\frac{du}{dz} \\ &+ p_{xy}\frac{dv}{dx} + p_{yy}\frac{dv}{dy} + p_{zy}\frac{dv}{dz} \\ &+ p_{xz}\frac{dw}{dx} + p_{yz}\frac{dw}{dy} + p_{zz}\frac{dw}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad (3).$$

The left member of this equation expresses the *rate of increase* in the kinetic energy of the fluid per unit of volume at a point moving with the fluid.

The first term on the right expresses the *rate at which work* is being done by the surrounding fluid per unit of volume at a point.

The second term on the right therefore, by the law of conservation of energy, expresses the difference between the rate of increase of kinetic energy and the rate at which work is being done by the stresses. This difference has, so far as I am aware, in the absence of other forces, or any changes of potential energy, been equated to the rate at which heat is being converted into energy of motion, Sir GABRIEL STOKES having first indicated this* as resulting from the law of conservation of energy then just established by JOULE.

7. This conclusion, that the second term on the right of (3) expresses the rate at which heat is being converted, as it is usually accepted, may be correct enough, but there is a consequence of adopting this conclusion which enters largely into the method of reasoning in this paper, but which, so far as I know, has not previously received any definite notice.

* 'Cambridge Phil. Trans.,' vol. 9, p. 57.

The Component Velocities in the Equations of Viscous Fluids.

In no case, that I am aware of, has any very strict definition of u, v, w, as they occur in the equations of motion, been attempted. They are usually defined as the velocities of a particle at a point (x, y, z) of the fluid, which may mean that they are the actual component velocities of the point in the matter passing at the instant, or that they are the mean velocities of all the matter in some space enclosing the point, or which passes the point in an interval of time. If the first view is taken, then the right hand member of the equation represents the rate of increase of kinetic energy, per unit of volume, in the matter at the point; and the integral of this expression over any finite space S, moving with the fluid, represents the total rate of increase of kinetic energy, including heat-motion, within that space; hence the difference between the rate at which work is done on the surface of S, and the rate at which kinetic energy is increasing can, by the law of conservation of energy, only represent the rate at which that part of the heat which does not consist in kinetic energy of matter is being produced, whence it follows :—

(*a*) *That the adoption of the conclusion that the second term in equation* (3) *expresses the rate at which heat is being converted, defines u, v, w, as not representing the component velocities of points in the passing matter.*

Further, if it is understood that u, v, w, represent the mean velocities of the matter in some space, enclosing x, y, z, the point considered, or the mean velocities at a point taken over a certain interval of time, so that $\Sigma(\rho u)$, $\Sigma(\rho v)$, $\Sigma(\rho w)$ may express the components of momentum, and $z\Sigma(\rho v) - y\Sigma(\rho w)$, &c., &c., may express the components of moments of momentum, of the matter over which the mean is taken; there still remains the question as to what spaces and what intervals of time?

(*b*) *Hence the conclusion that the second term expresses the rate of conversion of heat, defines the spaces and intervals of time over which the mean component velocities must be taken, so that* E *may include all the energy of mean-motion, and exclude that of heat-motions.*

Equations Approximate only except in Three Particular Cases.

8. According to the reasoning of the last article, if the second term on the right of equation (3) expresses the rate at which heat is being converted into energy of mean-motion, either ρu, ρv, ρw express the mean components of momentum of the matter, taken at any instant over a space S_0 enclosing the point x, y, z, to which u, v, w refer, so that this point is the centre of gravity of the matter within S_0 and such that ρ represents the mean density of the matter within this space; or ρu, ρv, ρw represent the mean components of momentum taken at x, y, z over an interval of time τ, such that ρ is the mean density over the time τ, and if t marks the instant to which u, v, w refer, and t' any other instant, $\Sigma[(t - t')\rho]$, in which ρ is the actual density, taken over the interval τ is zero. The equations, however, require, that so obtained,

ρ, u, v, w, shall be continuous functions of space and time, and it can be shown that this involves certain conditions between the distribution of the mean-motion and the dimensions of S_0 and τ.

Mean- and Relative-Motions of Matter.

Whatever the motions of matter within a fixed space S may be at any instant, if the component velocities at a point are expressed by u, v, w, the mean component velocities taken over S will be expressed by

$$\bar{u} = \frac{\Sigma(\rho u)}{\Sigma \rho}, \text{ \&c., \&c.} \qquad (4).$$

If then $\bar{u}, \bar{v}, \bar{w}$, are taken at each instant as the velocities of x, y, z, the *instantaneous centre of gravity of the matter within* S, the component momentum at the centre of gravity may be put

$$\rho u = \rho \bar{u} + \rho u' \qquad (5),$$

where u' is the motion of the matter, relative to axes moving with the mean velocity, at the centre of gravity of the matter within S. Since a space S of definite size and shape may be taken about any point x, y, z in an indefinitely larger space, so that x, y, z is the centre of gravity of the matter within S, the motion in the larger space may be divided into two distinct systems of motion, of which $\bar{u}$, $\bar{v}$, $\bar{w}$ represent a mean-motion at each point and u', v', w' a motion at the same point relative to the mean-motion at the point.

If, however, $\bar{u}$, $\bar{v}$, $\bar{w}$ are to represent the real mean-motion, it is necessary that $\Sigma(\rho v')$, $\Sigma(\rho v')$, $\Sigma(\rho \omega')$ summed over the space S, taken about any point, shall be severally zero; and in order that this may be so, certain conditions must be fulfilled.

For taking x, y, z for G the centre of gravity of the matter within S and x', y', z' for any other point within S, and putting a, b, c for the dimensions of S in directions x, y, z, measured from the point x, y, z, since $\bar{u}$, $\bar{v}$, $\bar{w}$ are continuous functions of x, y, z, by shifting S so that the centre of gravity of the matter within it is at x', y', z', the value of $\bar{u}$ for this point is given by

$$\bar{u} = \bar{u}_g + (x' - x)\left(\frac{d\bar{u}}{dx}\right)_g + (y' - y)\left(\frac{d\bar{u}}{dy}\right)_g + (z' - z)\left(\frac{d\bar{u}}{dz}\right)_g + \tfrac{1}{2}(x' - x)^2\left(\frac{d^2\bar{u}}{dx^2}\right)_g + \text{\&c.} \quad (6)$$

where all the differential coefficients on the left refer to the point x, y, z; and in the same way for $\bar{v}$ and $\bar{w}$.

Subtracting the value of $\bar{u}$ thus obtained for the point x', y', z' from that of u at the same point the difference is the value of u' at this point, whence summing these differences over the space S about G at x, y, z, since by definition when summed over the space S about G

$$\Sigma\left[\rho\left(u-\bar{u}_g\right)\right]=0 \text{ and } \Sigma\left[\rho\left(x'-x\right)\right]=0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\left.\begin{aligned}\Sigma\left(\rho u'\right) &= -\left\{\tfrac{1}{2}\Sigma\left[\rho\left(x-x'\right)^2\right]\left(\frac{d^2\bar{u}}{dx^2}\right)_g+\tfrac{1}{2}\Sigma\left[\rho\left(y-y'\right)^2\right]\left(\frac{d^2\bar{u}}{dy^2}\right)_g\right.\\ &\qquad\left.+\tfrac{1}{2}\Sigma\left[\rho\left(z-z'\right)^2\right]\left(\frac{d^2\bar{u}}{dz^2}\right)_g+\&c.\right\}\\ \text{That is}\quad & \\ \frac{\Sigma\left(\rho u'\right)}{\Sigma\left(\rho\right)} &\text{ is } < -\left\{\frac{a^2}{2}\left(\frac{d^2\bar{u}}{dx^2}\right)_g+\frac{b^2}{2}\left(\frac{d^2\bar{u}}{dy^2}\right)_g+\frac{c^2}{2}\left(\frac{d^2\bar{u}}{dz^2}\right)_g+\&c.\right\}\end{aligned}\right\} . \quad (8\text{A}).$$

In the same way if Σ (   ) be taken over the interval of time τ including t; and for the instant t

$$\bar{u}=\frac{\Sigma\left(\rho u\right)}{\Sigma\left(\rho\right)}, \text{ and } \rho\bar{u}=\rho u+\rho u' ;$$

then since for any other instant t'

$$\bar{u}=\bar{u}_t+\left(t-t'\right)\left(\frac{d\bar{u}}{dt}\right)_t+\tfrac{1}{2}\left(t-t'\right)^2\left(\frac{d^2\bar{u}}{dt^2}\right)_t+\&c.,$$

where $\Sigma\left[\rho\left(t-t'\right)\right]=0$, and $\Sigma\left[\rho\left(\bar{u}_t-u\right)\right]=0$.

It appears that

$$\left.\begin{aligned}\Sigma\left(\rho u'\right) &= -\Sigma\left[\tfrac{1}{2}\rho\left(t-t'\right)^2\right]\frac{d^2\bar{u}}{dt^2}+\&c.\\ \frac{\Sigma\left(\rho u'\right)}{\Sigma\left(\rho\right)} &\text{ is } < -\tfrac{1}{2}\tau^2\left(\frac{d^2\bar{u}}{dt^2}\right)_t-\&c.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8\text{B}).$$

From equations (8A) and (8B), and similar equations for $\Sigma\left(\rho v'\right)$ and $\Sigma\left(\rho w'\right)$, it appears that if

$$\Sigma\left(\rho u'\right)=\Sigma\left(\rho v'\right)=\Sigma\left(\rho w'\right)=0,$$

where the summation extends both over the space S and the interval τ, all the terms on the right of equations (8A) and (8B) must be respectively and continuously zero, or, what is the same thing, all the differential coefficients of $\bar{u}$, $\bar{v}$, $\bar{w}$ with respect to x, y, z and t of the first order must be respectively constant.

This condition will be satisfied if the mean-motion is steady, or uniformly varying with the time, and is everywhere in the same direction, being subject to no variations in the direction of motion; for suppose the direction of motion to be that of x, then since the periodic motion passes through a complete period within the distance $2a$, $\Sigma(\rho u')$ will be zero within the space

$$2a\, dy\, dz,$$

however small $dy\, dz$ may be, and since the only variations of the mean-motion are in directions y and z, in which b and c may be taken zero, and du/dt is everywhere constant, the conditions are perfectly satisfied.

The conditions are also satisfied if the mean-motion is that of uniform expansion or contraction, or is that of a rigid body.

These three cases, in which it may be noticed that variations of mean-motion are everywhere uniform in the direction of motion, and subject to steady variations in respect of time, are the only cases in which the conditions (8A), (8B), can be perfectly satisfied.

The conditions will, however, be approximately satisfied, when the variations of $\bar{u}$, $\bar{v}$, $\bar{w}$ of the first order are approximately constant over the space S.

In such case the right-hand members of equations (8A), (8B), are neglected, and it appears that the closeness of the approximations will be measured by the relative magnitude of such terms as

$$a\, d^2\bar{u}/dx^2, \text{ \&c., } \tau\, d^2\bar{u}/dt^2 \text{ as compared with } d\bar{u}/dx,\ d\bar{u}/dt, \text{ \&c.}$$

Since frequent reference must be made to these relative values, and, as in periodic motion, the relative values of such terms are measured by the period (in space or time) as compared with a, b, c and τ, which are, in a sense, the periods of u', v', w', I shall use the term period in this sense, taking note of the fact that when the mean-motion is constant in the direction of motion, or varies uniformly in respect of time, it is not periodic, *i.e.*, its periods are infinite.

9. It is thus seen that the closeness of the approximation with which the motion of any system can be expressed as a varying mean-motion together with a relative-motion, which, when integrated over a space of which the dimensions are a, b, c, has no momentum, increases as the magnitude of the periods of $\bar{u}$, $\bar{v}$, $\bar{w}$ in comparison with the periods of u', v', w', and is measured by the ratio of the relative orders of magnitudes to which these periods belong.

Heat-motions in Matter are Approximately Relative to the Mean-motions.

The general experience that heat in no way affects the momentum of matter, shows that the heat-motions are relative to the mean-motions of matter taken over spaces of sensible size. But, as heat is by no means the only state of relative-motion of matter, if the heat-motions are relative to all mean-motions of matter, whatsoever their periods may be, it follows—that there must be some *discriminative cause* which prevents the existence of relative-motions of matter other than heat, except mean-motions with periods in time and space of greatly higher orders of magnitude than the corresponding periods of the heat-motions—otherwise, by equations (8A), (8B), heat-motions could not be to a high degree of approximation relative to all other motions, and we could not have to a high degree of approximation,

$$\left.\begin{array}{l} p_{xx}\dfrac{du}{dx} + p_{yx}\dfrac{du}{dy} + p_{zx}\dfrac{du}{dz} \\ p_{xy}\dfrac{dv}{dx} + p_{yy}\dfrac{dv}{dy} + p_{zy}\dfrac{dv}{dz} \\ p_{xz}\dfrac{dw}{dx} + p_{yz}\dfrac{dw}{dy} + p_{zz}\dfrac{dw}{dz} \end{array}\right\} = -\frac{d}{dt}({}_p\mathrm{H}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

where the expression on the right stands for the rate at which heat is converted into energy of mean-motion.

Transformation of Energy of Relative-mean-motion to Energy of Heat-motion.

10. The recognition of the existence of a *discriminative cause*, which prevents the existence of relative-mean-motions with periods of the same order of magnitude as heat-motions, proves the existence of another general action by which the energy of relative-mean-motion, of which the periods are of another and higher order of magnitude than those of the heat-motions, is *transformed* to energy of heat-motion.

For if relative-mean-motions cannot exist with periods approximating to those of heat, the conversion of energy of mean-motion into energy of heat, proved by JOULE, cannot proceed by the gradual degradation of the periods of mean-motion until these periods coincide with those of heat, but must, in its final stages, at all events, be the result of some action which causes the energy of relative-mean-motion to be transformed into the energy of heat-motions without intermediate existence in states of relative-motion with intermediate and gradually diminishing periods.

That such change of energy of mean-motion to energy of heat may be properly called transformation becomes apparent when it is remembered that neither mean-motion nor relative-motion have any separate existence, but are only abstract quantities, determined by the particular process of abstraction, and so changes in the actual-motion may, by the process of abstraction, cause transformation of the abstract energy of the one abstract-motion, to abstract energy of the other abstract-motion.

All such transformation must depend on the changes in the actual-motions, and so must depend on mechanical principles and the properties of matter, and hence the direct passage of energy of relative-mean-motion to energy of heat-motions is evidence of a general cause of the condition of actual-motion which results in transformation—which may be called *the cause of transformation.*

The Discriminative Cause, and the Cause of Transformation.

11. The only known characteristic of heat-motions, besides that of being relative to the mean-motion, already mentioned, is that the motions of matter which result from heat are an ultimate form of motion which does not alter so long as the mean-motion is uniform over the space, and so long as no change of state occurs in the matter. In respect of this characteristic, heat-motions are, so far as we know, unique, and it would appear that heat-motions are distinguished from the mean-motions by some ultimate properties of matter.

It does not, however, follow that the cause of transformation, or even the discriminative cause, are determined by these properties. Whether this is so or not can only be ascertained by experience. If either or both these causes depend solely on properties of matter which only affect the heat-motions, then no similar effect would result as between the variations of mean-mean-motion and relative-mean-motion, whatever might be the difference in magnitude of their respective periods. Whereas, if these causes depend on properties of matter which affect all modes of motion, distinctions in periods must exist between mean-mean-motion and relative-mean-motion, and transformation of energy take place from one to the other, as between the mean-motion and the heat-motions.

The mean-mean-motion cannot, however, under any circumstances stand to the relative-mean-motion in the same relation as the mean-motion stands to the heat-motions, because the heat-motions cannot be absent, and in addition to any transformation from mean-mean-motion to relative-mean-motion, there are transformations both from mean- and relative-mean-motion to heat-motions, which transformation may have important effects on both the transformation of energy from mean- to relative-mean-motion, and on the discriminative cause of distinction in their periods.

In spite of the confusing effect of the ever present heat-motions, it would, however, seem that evidence as to the character of the properties on which the cause of transformation and the discriminative cause depend should be forthcoming as the result of observing the mean- and relative-mean-motions of matter.

12. To prove by experimental evidence that the effects of these properties of matter are confined to the heat-motions, would be to prove a negative; but if these properties are in any degree common to all modes of matter, then at first sight it must seem in the highest degree improbable that the effects of these causes on the mean- and relative-mean-motions would be obscure, and only to be observed by delicate tests. For properties which can cause distinctions between the mean- and

heat-motions of matter so fundamental and general, that from the time these motions were first recognized the distinction has been accepted as part of the order of nature, and has been so familiar to us that its cause has excited no curiosity, cannot, if they have any effect at all, but cause effects which are general and important on the mean-motions of matter. It would thus seem that evidence of the general effects of such properties should be sought in those laws and phenomena known to us as the result of experience, but of which no rational explanation has hitherto been found; such as the law that the resistance of fluids moving between solid surfaces and of solids moving through fluids, in such a manner that the general-motion is not periodic, is as the square of the velocities, the evidence covered by the law of the universal tendency of all energy to dissipation and the second law of thermodynamics.

13. In considering the first of the instances mentioned, it will be seen that the evidence it affords as to the general effect of the properties, on which depends transformation of energy from mean- to relative-motion, is very direct. For, since my experiments with colour bands have shown that when the resistance of fluids, in steady mean flow, varies with a power of the velocity higher than the first the fluid is always in a state of sinuous motion, it appears that the prevalence of such resistance is evidence of the existence of a general action by which energy of mean-mean-motion with infinite periods is directly transformed to the energy of relative-mean-motion, with finite periods, represented by the eddying motion, which renders the general mean-motion sinuous, by which transformation the state of eddying-motion is maintained, notwithstanding the continual transformation of its energy into heat-motions.

We have thus direct evidence that properties of matter which determine the cause of transformation, produce general and important effects which are not confined to the heat-motions.

In the same way, the experimental demonstration I was able to obtain, that relative-mean-motion in the form of eddies of finite periods, both as shown by colour bands and as shown by the law of resistances, cannot be maintained except under circumstances depending on the conditions which determine the superior limits to the velocity of the mean-mean-motion, of infinite periods, and the periods of the relative-mean-motion, as defined in the criterion

$$\mathrm{DU}_m/\mu = \mathrm{K},$$

is not only a direct experimental proof of the existence of a discriminative cause which prevents the maintenance of periodic mean-motion except with periods greatly in excess of the periods of the heat-motions, but also indicates that the discriminative cause depends on properties of matter which affect the mean-motions as well as the heat-motions.

Expressions for the Rate of Transformation and the Discriminative Cause.

14. It has already been shown (Art. 8) that the equations of motion approximate to a true expression of the relations between the mean-motions and stresses, when the ratio of the periods of mean-motions to the periods of the heat-motions approximates to infinity. Hence it follows that these equations must of necessity include whatever mechanical or kinematical principles are involved in the transformation of energy of mean-mean-motion to energy of relative-mean-motion. It has also been shown that the properties of matter on which depends the transformation of energy of varying mean-motion to relative-motion are common to the relative-mean-motion as well as to the heat-motion. Hence, if the equations of motion are applied to a condition in which the mean-motion consists of two components, the one component being a mean-mean-motion, as obtained by integrating the mean-motion over spaces S_1 taken about the point x, y, z, as centre of gravity, and the other component being a relative-mean-motion, of which the mean components of momentum taken over the space S_1 everywhere vanish, it follows:—

(1) *That the resulting equations of motion must contain an expression for the rate of transformation from energy of mean-mean-motion to energy of relative-mean-motion, as well as the expressions for the transformation of the respective energies of mean- and relative-mean-motion to energy of heat-motion;*

(2) *That, when integrated over a complete system these equations must show that the possibility of the maintenance of the energy of relative-mean-motion depends, whatsoever may be the conditions, on the possible order of magnitudes of the periods of the relative-mean-motion, as compared with the periods of the heat-motions.*

The Equations of Mean- and Relative Mean-Motion.

15. These last conclusions, besides bringing the general results of the previous argument to the test point, suggest the manner of adaptation of the equations of motion, by which the test may be applied.

Put

$$u = \bar{u} + u', \qquad v = \bar{v} + v', \qquad w = \bar{w} + w' \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where

$$\bar{u} = \Sigma(\rho u)/\Sigma(\rho), \text{ \&c., \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

the summation extending over the space S_1 of which the centre of gravity is at the point x, y, z. Then since u, v, w are continuous functions of x, y, z, therefore $\bar{u}, \bar{v}, \bar{w}$, and u', v', w', are continuous functions of x, y, z. And as ρ is assumed constant, the equations of continuity for the two systems of motion are:

$$\frac{d\bar{u}}{dx} + \frac{d\bar{v}}{dx} + \frac{d\bar{w}}{dx} = 0 \quad \text{and} \quad \frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = 0 \quad . \quad . \quad . \quad . \quad (13);$$

also both systems of motions must satisfy the boundary conditions, whatever they may be.

Further putting $\bar{p}_{xx}$, &c., for the mean values of the stresses taken over the space S_1 and

$$p'_{xx} = p_{xx} - \overline{p_{xx}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and defining S_1 to be such that the space variations of $\bar{u}, \bar{v}, \bar{w}$ are approximately constant over this space, we have, putting $\overline{u'u'}$, &c., for the mean values of the squares and products of the components of relative-mean-motion, for the equations of mean-mean-motion,

$$\left.\begin{aligned} \rho\frac{d\bar{u}}{dt} &= -\left\{\frac{d}{dx}(\overline{p_{xx}} + \rho\bar{u}\bar{u} + \rho\overline{u'u'}) + \frac{d}{dy}(\overline{p_{yx}} + \rho\bar{u}\bar{v} + \rho\overline{u'v'})\right. \\ &\qquad\qquad \left. + \frac{d}{dz}(\overline{p_{zx}} + \rho\bar{u}\bar{w} + \rho\overline{u'w'})\right\} \\ \&\text{c.} &= \qquad \&\text{c.} \\ \&\text{c.} &= \qquad \&\text{c.} \end{aligned}\right\} \quad . \quad . \quad (15),$$

which equations are approximately true at every point in the same sense as that in which the equations (1) of mean-motion are true.

Subtracting these equations of mean-mean-motion from the equations of mean-motion, we have

$$\rho\frac{du'}{dt} = -\left\{\begin{aligned} &\frac{d}{dx}\{p'_{xx} + \rho(\bar{u}u' + u'\bar{u}) + \rho(u'u' - \overline{u'u'})\} \\ &+ \frac{d}{dy}\{p'_{yx} + \rho(\bar{u}v' + u'\bar{v}) + \rho(u'v' - \overline{u'v'})\} \\ &+ \frac{d}{dz}\{p'_{zx} + \rho(\bar{u}w' + u'\bar{w}) + \rho(u'w' - \overline{u'w'})\} \end{aligned}\right\} \quad \&\text{c., \&c.} \ (16),$$

which are the equations of momentum of relative-mean-motion at each point.

Again, multiplying the equations of mean-mean-motion by $\bar{u}$, $\bar{v}$, $\bar{w}$ respectively, adding and putting $2\mathrm{E} = \rho(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)$, we obtain

$$\left(\frac{d}{dt} + \bar{u}\frac{d}{dx} + \bar{v}\frac{d}{dy} + \bar{w}\frac{d}{dz}\right)\bar{\mathrm{E}} =$$

$$-\left\{\begin{aligned} &\frac{d}{dx}[\bar{u}(\bar{p}_{xx} + \overline{u'u'})] + \frac{d}{dy}[\bar{u}(\bar{p}_{yx} + \overline{u'v'})] + \frac{d}{dz}[\bar{u}(\bar{p}_{zx} + \overline{u'w'})] \\ &+ \frac{d}{dx}[\bar{v}(\bar{p}_{xy} + \overline{v'u'})] + \frac{d}{dy}[\bar{v}(\bar{p}_{yy} + \overline{v'v'})] + \frac{d}{dz}[\bar{v}(\bar{p}_{zy} + \overline{v'w'})] \\ &+ \frac{d}{dx}[\bar{w}(\bar{p}_{xz} + \overline{w'u'})] + \frac{d}{dy}[\bar{w}(\bar{p}_{yz} + \overline{w'v'})] + \frac{d}{dz}[\bar{w}(\bar{p}_{zz} + \overline{w'w'})] \end{aligned}\right\}$$

$$+\left\{\begin{aligned} &\bar{p}_{xx}\frac{d\bar{u}}{dx} + \bar{p}_{yx}\frac{d\bar{u}}{dy} + \bar{p}_{zx}\frac{d\bar{u}}{dz} \\ &+ \bar{p}_{xy}\frac{d\bar{v}}{dx} + \bar{p}_{yy}\frac{d\bar{v}}{dy} + \bar{p}_{zy}\frac{d\bar{v}}{dy} \\ &+ \bar{p}_{xz}\frac{d\bar{w}}{dx} + \bar{p}_{yz}\frac{d\bar{w}}{dy} + \bar{p}_{zz}\frac{d\bar{w}}{dz} \end{aligned}\right\} + \left\{\begin{aligned} &\overline{u'u'}\frac{d\bar{u}}{dx} + \overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{d\bar{u}}{dz} \\ &+ \overline{v'u'}\frac{d\bar{v}}{dx} + \overline{v'v'}\frac{d\bar{v}}{dy} + \overline{v'w'}\frac{d\bar{v}}{dz} \\ &+ \overline{w'u'}\frac{d\bar{w}}{dx} + \overline{w'v'}\frac{d\bar{w}}{dy} + \overline{w'w'}\frac{d\bar{w}}{dz} \end{aligned}\right\} \quad (17)$$

which is the approximate equation of energy of mean-mean-motion in the same sense as the equation (3) of energy of mean-motion is approximate.

In a similar manner multiplying the equations (16) for the momentum of relative-mean-motion respectively by u', v', w', and adding, the result would be the equation for energy of relative-mean-motion at a point, but this would include terms of which the mean values taken over the space S_1 are zero, and, since all corresponding terms in the energy of heat are excluded, by summation over the space S_0 in the expression for the rate at which mean-motion is transformed into heat, there is no reason to include them for the space S_1; so that, omitting all such terms and putting

$$2\mathrm{E}' = \rho(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

we obtain

$$\left(\frac{d}{dt} + \bar{u}\frac{d}{dx} + \bar{v}\frac{d}{dy} + \bar{w}\frac{d}{dz}\right)\bar{\mathrm{E}}' =$$

$$- \left\{ \begin{array}{l} \frac{d}{dx}[u'(p'_{xx} + \rho w'u')] + \frac{d}{dy}[u'(p'_{yx} + u'v')] + \frac{d}{dz}[u'(p'_{yx} + u'w')] \\ + \frac{d}{dx}[v'(p'_{yx} + v'u')] + \frac{d}{dy}[v'(p'_{yy} + v'v')] + \frac{d}{dz}[v'(p'_{zy} + v'w')] \\ + \frac{d}{dx}[w'(p_{zx} + w'u')] + \frac{d}{dy}[w'(p'_{yx} + w'v')] + \frac{d}{dz}[w'(p_{zz} + w'w')] \end{array} \right\}$$

$$+ \left\{ \begin{array}{l} p'_{xx}\frac{du'}{dx} + p'_{yx}\frac{du'}{dy} + p'_{zx}\frac{du'}{dz} \\ + p'_{xy}\frac{dv'}{dx} + p'_{yy}\frac{dv'}{dy} + p'_{zy}\frac{dv'}{dz} \\ + p'_{xz}\frac{dw'}{dx} + p'_{yz}\frac{dw'}{dy} + p'_{zz}\frac{dw'}{dz} \end{array} \right\} - \left\{ \begin{array}{l} \rho\overline{u'u'}\frac{d\bar{u}}{dx} + \rho\overline{u'v'}\frac{d\bar{u}}{dy} + \rho\overline{u'w'}\frac{d\bar{u}}{dz} \\ + \rho\overline{v'u'}\frac{d\bar{v}}{dx} + \rho\overline{v'v'}\frac{d\bar{v}}{dy} + \rho\overline{v'w'}\frac{d\bar{v}}{dz} \\ + \rho\overline{w'u'}\frac{d\bar{w}}{dx} + \rho\overline{w'v'}\frac{d\bar{w}}{dy} + \rho\overline{w'w'}\frac{d\bar{w}}{dz} \end{array} \right\} \quad (19)$$

where only the mean values, over the space S_1, of the expressions in the right member are taken into account.

This is the equation for the mean rate, over the space S_1, of change in the energy of relative-mean-motion per unit of volume.

It may be noticed that the rate of change in the energy of mean-mean-motion, together with the mean rate of change in the energy of relative-mean-motion, must be the total mean-rate of change in the energy of mean-motion, and that by adding the equations (17) and (19) the result is the same as is obtained from the equation (3) of energy of mean-motion by omitting all terms which have no mean value as summed over the space S_1.

The Expressions from Transformation of Energy from Mean-mean-motion to Relative-mean-motion.

16. When equations (17) and (19) are added together, the only expressions that do not appear in the equation of mean energy of mean-motion are the last terms on the right of each of the equations, which are identical in form and opposite in sign.

These terms which thus represent no change in the total energy of mean-motion can only represent a transformation from energy of mean-mean-motion to energy of relative-mean-motion. And as they are the only expressions which do not form part of the general expression for the rate of change of the mean energy of mean-motion, they represent the total exchange of energy between the mean-mean-motion and the relative-mean-motion.

It is also seen that the action, of which these terms express the effect, is purely kinematical, depending simply on the instantaneous characters of the mean- and relative-mean-motion, whatever may be the properties of the matter involved, or the mechanical actions which have taken part in determining these characters. The terms, therefore, express the entire result of transformation from energy of mean-mean-motion to energy of relative-mean-motion, and of nothing but the transformation. Their existence thus completely verifies the first of the general conclusions in Art. 14.

The term last but one in the right member of the equation (17) for energy of mean-mean-motion expresses the rate of transformation of energy of heat-motions to that of energy of mean-mean-motion, and is entirely independent of the relative-mean-motion.

In the same way, the term last but one on the right of the equation (19) for energy of relative-mean-motion expresses the rate of transformation from energy of heat-motions to energy of relative-mean-motion, and is quite independent of the mean-mean-motion.

17. In both equations (17) and (19) the first terms on the right express the rates at which the respective energy of mean- and relative-mean-motion are increasing on account of work done by the stresses on the mean- and relative-motion respectively, and by the additions of momentum caused by convections of relative-mean-motion by relative-mean-motion to the mean- and relative-mean-motions respectively.

It may also be noticed that while the first term on the right in the equation (19) of energy of relative-mean-motion is independent of mean-mean-motion, the corresponding term in equation (17) for mean-mean-motion is not independent of relative-mean-motion.

A Discriminating Equation.

18. In integrating the equations over a space moving with the mean-mean-motion of the fluid the first terms on the right may be expressed as surface integrals, which integrals respectively express the rates at which work is being done on, and energy is being received across, the surface by the mean-mean-motion, and by the relative-mean-motion.

If the space over which the integration extends includes the whole system, or such part that the total energy conveyed across the surface by the relative-mean-motion is zero, then the rate of change in the total energy of relative-mean-motion within the space is the difference of the integral, over the space, of the rate of increase of this energy by transformation from energy of mean-mean-motion, less the integral rate at which energy of relative-mean-motion is being converted into heat, or integrating equation (19),

$$\iiint\left(\frac{d}{dt}+\bar{u}\frac{d}{dx}+\bar{v}\frac{d}{dy}+\bar{w}\frac{d}{dz}\right)\mathrm{E}'\,dx\,dy\,dz =$$

$$-\iiint\left\{\begin{array}{l} \rho\overline{u'u'}\dfrac{d\bar{u}}{dx}+\rho\overline{u'v'}\dfrac{d\bar{u}}{dy}+\rho\overline{u'w'}\dfrac{d\bar{u}}{dz} \\ +\rho\overline{v'u'}\dfrac{d\bar{v}}{dx}+\rho\overline{v'v'}\dfrac{d\bar{v}}{dy}+\rho\overline{v'w'}\dfrac{d\bar{v}}{dz} \\ +\rho\overline{w'u'}\dfrac{d\bar{w}}{dx}+\rho\overline{w'v'}\dfrac{d\bar{w}}{dy}+\rho\overline{w'w'}\dfrac{d\bar{w}}{dz} \end{array}\right\}dx\,dy\,dz$$

$$+\iiint\left\{\begin{array}{l} p'_{xx}\dfrac{du'}{dx}+p'_{yx}\dfrac{du'}{dy}+p'_{zx}\dfrac{du'}{dz} \\ p'_{xy}\dfrac{dv'}{dx}+p'_{yy}\dfrac{dv'}{dy}+p'_{zy}\dfrac{dv'}{dz} \\ p'_{xz}\dfrac{dw'}{dx}+p'_{yz}\dfrac{dw'}{dy}+p'_{zz}\dfrac{dw'}{dz} \end{array}\right\}dx\,dy\,dz \quad . \quad . \quad . \quad . \quad . \quad (20).$$

This equation expresses the fundamental relations :—

(1) *That the only integral effect of the mean-mean-motion on the relative-mean-motion is the integral of the rate of transformation from energy of mean-mean-motion to energy of relative-mean-motion.*

(2) *That, unless relative energy is altered by actions across the surface within which the integration extends, the integral energy of relative-mean-motion will be increasing or diminishing according as the integral rate of transformation from mean-mean-motion to relative-mean-motion is greater or less than the rate of conversion of the energy of relative-mean-motion into heat.*

19. For p'_{xx}, &c., are substituted their values as determined according to the theory of viscosity, the approximate truth of which has been verified, as already explained.

Putting

$$\left.\begin{array}{l} p'_{xx}=p+\tfrac{2}{3}\mu\left(\dfrac{du'}{dx}+\dfrac{dv'}{dy}+\dfrac{dw'}{dz}\right)-2\mu\dfrac{du}{dx}\text{, \&c., \&c.} \\ p'_{yx}=-\mu\left(\dfrac{du'}{dy}+\dfrac{dv'}{dx}\right)\text{ \&c., \&c.} \end{array}\right\} \quad . \quad . \quad . \quad (21),$$

we have, substituting in the last term of equation (20), as the expression for the rate of conversion of energy of relative-mean-motion into heat,

$$-\iiint\frac{d}{dt}(\rho\mathrm{H})\,dx\,dy\,dz=\iiint\Bigg[p\left(\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}\right)$$

$$-\mu\left\{-\tfrac{2}{3}\left(\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}\right)^2+2\left[\left(\frac{du'}{dx}\right)^2+\left(\frac{dv'}{dy}\right)^2+\left(\frac{dw'}{dz}\right)^2\right]\right.$$

$$\left.+\left(\frac{dw'}{dy}+\frac{dv'}{dz}\right)^2+\left(\frac{du'}{dz}+\frac{dw'}{dx}\right)^2+\left(\frac{dv'}{dx}+\frac{du'}{dy}\right)^2\right\}\Bigg]dx\,dy\,dz \quad . \quad . \quad (22)$$

in which μ is a function of temperature only ; or since ρ is here considered as constant,

$$-\iiint \frac{d}{dt}\left(p\mathrm{H}'\right) = -\mu \iiint \left\{ 2\left[\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right] + \left(\frac{dw'}{dy} + \frac{dv'}{dx}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2 \right\} dx\, dy\, dz \quad . \quad . \quad . \quad . \quad (23),$$

whence substituting for the last term in equation (20) we have, if the energy of relative-mean-motion is maintained, neither increasing or diminishing,

$$-\rho \iiint \left\{ \begin{array}{l} \overline{u'u'}\,\dfrac{\overline{du}}{dx} + \overline{u'v'}\,\dfrac{\overline{du}}{dy} + \overline{u'w'}\,\dfrac{\overline{du}}{dz} \\ \quad + \overline{v'u'}\,\dfrac{\overline{dv}}{dx} + \overline{v'v'}\,\dfrac{\overline{dv}}{dy} + \overline{v'w'}\,\dfrac{\overline{dv}}{dz} \\ \quad + \overline{w'u'}\,\dfrac{\overline{dw}}{dx} + \overline{w'v'}\,\dfrac{\overline{dw}}{dy} + \overline{w'w'}\,\dfrac{\overline{dw}}{dz} \end{array} \right\} dx\, dy\, dz$$

$$-\mu \iiint \left\{ \begin{array}{l} 2\left[\left(\dfrac{du'}{dx}\right)^2 + \left(\dfrac{dv'}{dy}\right)^2 + \left(\dfrac{dw'}{dz}\right)^2\right] \\ \quad + \left(\dfrac{dw'}{dy} + \dfrac{dv'}{dz}\right)^2 + \left(\dfrac{du'}{dz} + \dfrac{dw'}{dx}\right)^2 \\ \quad + \left(\dfrac{dv'}{dx} + \dfrac{du'}{dy}\right)^2 \end{array} \right\} dx\, dy\, dz = 0 \quad . \quad (24),$$

which is a discriminating equation as to the conditions under which relative-mean-motion can be sustained.

20. Since this equation is homogeneous in respect to the component velocities of the relative-mean-motion, it at once appears that it is independent of the energy of relative-mean-motion divided by the ρ. So that if μ/ρ is constant, the condition it expresses depends only on the relation between variations of the mean-mean-motion and the directional, or angular, distribution of the relative-mean-motion, and on the squares and products of the space periods of the relative-mean-motion.

And since the second term expressing the rate of conversion of heat into energy of relative-mean-motion is always negative, it is seen at once that, whatsoever may be the distribution and angular distribution of the relative-mean-motion and the variations of the mean-mean-motion, this equation must give an inferior limit for the rates of variation of the components of mean-mean-motion, in terms of the limits to the periods of relative-mean-motion, and μ/ρ, within which the maintenance of relative-mean-motion is impossible. And that, so long as the limits to the periods of relative-mean-motion are not infinite, this inferior limit to the rates of variation of the mean-mean-motion will be greater than zero.

Thus the second conclusion of Art. 14, and the whole of the previous argument is verified, and the properties of matter which prevent the maintenance of mean-motion with periods of the same order of magnitude as those of the heat-motion are shown to be amongst those properties of matter which are included in the equations of motion of which the truth has been verified by experience.

The Cause of Transformation.

21. The transformation function, which appears in the equations of mean-energy of mean- and relative-mean-motion, does not indicate the cause of transformation, but only expresses a kinematical principle as to the effect of the variations of mean-mean-motion, and the distribution of relative-mean-motion. In order to determine the properties of matter and the mechanical principles on which the effect of the variations of the mean-mean-motion on the distribution and angular distribution of relative-mean-motion depends, it is necessary to go back to the equations (16) of relative-momentum at a point; and even then the cause is only to be found by considering the effects of the actions which these equations express in detail. The determination of this cause, though it in no way affects the proofs of the existence of the criterion as deduced from the equations, may be the means of explaining what has been hitherto obscure in the connection between thermodynamics and the principles of mechanics. That such may be the case, is suggested by the recognition of the separate equations of mean- and relative-mean-motion of matter.

The Equation of Energy of Relative-mean-motion and the Equation of Thermodynamics.

22. On consideration, it will at once be seen that there is more than an accidental correspondence between the equations of energy of mean- and relative-mean-motion respectively and the respective equations of energy of mean-motion and of heat in thermodynamics.

If instead of including only the effects of the heat-motion on the mean-momentum as expressed by p_{xx}, &c., the effects of relative-mean-motion are also included by putting p_{xx} for $\overline{p_{xx}} + \overline{\rho u'u'}$, &c., and p_{yz} for $\overline{p_{yz}} + \overline{\rho w'v'}$, &c., in equations (15) and (17), the equations (15) of mean-mean-motion become identical in form with the equations (1) of mean-motion, and the equation (17) of energy of mean-mean-motion becomes identical in form with the equation (3) of energy of mean-motion.

These equations, obtained from (15) and (17) being equally true with equations (1) and (3), the mean-mean-motion in the former being taken over the space S_1 instead of S_0 as in the latter, then, instead of equation (9), we should have for the value of the last term—

$$p_{xx}\frac{d\bar{u}}{dx} + \&c., = -\frac{d(\rho\text{H})}{dt} + \overline{u'u'}\frac{d\bar{u}}{dx} + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in which the right member expresses the rate at which heat is converted into energy of mean-mean-motion, together with the rate at which energy of relative-mean-motion is transformed into energy of mean-mean-motion ; while equation (19) shows whence the transformed energy is derived.

The similarity of the parts taken by the transformation of mean-mean-motion into relative-mean-motion, and the conversion of mean-motion into heat, indicates that these parts are identical in form ; or that the conversion of mean-motion into heat is the result of transformation, and is expressible by a transformation function similar in form to that for relative-mean-motion, but in which the components of relative motion are the components of the heat-motions and the density is the actual density at each point. Whence it would appear that the general equations, of which equations (19) and (16) are respectively the adaptations to the special condition of uniform density, must, by indicating the properties of matter involved, afford mechanical explanations of the law of universal dissipation of energy and of the second law of thermodynamics.

The proof of the existence of a criterion as obtained from the equations is quite independent of the properties and mechanical principles on which the effect of the variations of mean-mean-motion on the distribution of relative mean-motion depends. And as the study of these properties and principles requires the inclusion of conditions which are not included in the equations of mean-motion of incompressible fluid, it does not come within the purpose of this paper. It is therefore reserved for separate investigation by a more general method.

The Criterion of Steady Mean-motion.

23. As already pointed out, it appears from the discriminating equation that the possibility of the maintenance of a state of relative-mean-motion depends on μ/ρ, the variation of mean-mean-motion and the periods of the relative-mean-motion.

Thus, if the mean-mean-motion is in direction x only, and varies in direction y only, if u', v', w' are periodic in directions x, y, z, a being the largest period in space, so that their integrals over a distance a in direction x are zero, and if the co-efficients of all the periodic factors are α, then putting

$$\pm\, d\bar{u}/dy = \text{C}^2{}_1\,;$$

taking the integrals, over the space a^3 of the 18 squares and products in the last term on the left of the discriminating equation (24) to be

$$- 18\mu C_2 (2\pi/a)^2 \alpha^2 a^3$$

the integral of the first term over the same space cannot be greater than

$$\rho C_2 \alpha^2 C_1{}^2 a^3.$$

Then, by the discriminating equation, if the mean-energy of relative-mean-motion is to be maintained,

$$\rho C_1{}^2 \text{ is greater than } 700\ \mu/a^2,$$

or

$$\frac{\rho a^2}{\mu} \sqrt{\left(\frac{d\bar{u}}{dy}\right)^2} = 700 \quad \ldots \ldots \ldots \quad (26)$$

is a condition under which relative-mean-motion cannot be maintained in a fluid of which the mean-mean-motion is constant in the direction of mean-mean-motion, and subject to a uniform variation at right angles to the direction of mean-mean-motion. It is not the actual limit, to obtain which it would be necessary to determine the actual forms of the periodic function for u', v', w', which would satisfy the equations of motion (15), (16), as well as the equation of continuity (13), and to do this the functions would be of the form

$$\Sigma \left[A_r \cos \left\{ r \left(nt + \frac{2\pi}{a} x \right) \right\} \right],$$

where r has the values 1, 2, 3, &c. It may be shown, however, that the retention of the terms in the periodic series in which r is greater than unity would increase the numerical value of the limit.

24. It thus appears that the existence of the condition (26) within which no relative-mean-motion, completely periodic in the distance a, can be maintained, is a proof of the existence, for the same variation of mean-mean-motion, of an actual limit of which the numerical value is between 700 and infinity.

In viscous fluids, experience shows that the further kinematical conditions imposed by the equations of motion do not prevent such relative-mean-motion. Hence for such fluids equation (26) proves the actual limit, which discriminates between the possibility and impossibility of relative-mean-motion completely periodic in a space a, is greater than 700.

Putting equation (26) in the form

$$\sqrt{(du/dy)^2} = 700\ \mu/\rho a^2,$$

it at once appears that this condition does not furnish a criterion as to the possibility of the maintenance of relative-mean-motion, irrespective of its periods, for a certain condition of variation of mean-mean-motion. For by taking a^2 large enough, such relative-mean-motion would be rendered possible whatever might be the variation of the mean-mean-motion.

The existence of a criterion is thus seen to depend on the existence of certain restrictions to the value of the periods of relative-mean-motion—on the existence of conditions which impose superior limits on the values of a.

Such limits to the maximum values of a may arise from various causes. If $d\bar{u}/dy$ is periodic, the period would impose such a limit, but the only restrictions which it is my purpose to consider in this paper, are those which arise from the solid surfaces between which the fluid flows. These restrictions are of two kinds—restrictions to the motions normal to the surfaces, and restrictions tangential to the surfaces—the former are easily defined, the latter depend for their definition on the evidence to be obtained from experiments such as those of POISEUILLE, and I shall proceed to show that these restrictions impose a limit to the value of a, which is proportional to D, the dimension between the surfaces. In which case, if

$$\sqrt{(d\bar{u}/dy)^2} = \mathrm{U}/\mathrm{D},$$

equation (26) affords a proof of the existence of a criterion

$$\rho \mathrm{DU}/\mu = \mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

of the conditions of mean-mean-motion under which relative or sinuous-motion can continuously exist in the case of a viscous fluid between two continuous surfaces perpendicular to the direction y, one of which is maintained at rest, and the other in uniform tangential-motion in the direction x with velocity U.

SECTION III.

The Criterion of the Conditions under which Relative-mean-motion cannot be maintained in the case of Incompressible Fluid in Uniform Symmetrical Mean-flow between Parallel Solid Surfaces.—Expression for the Resistance.

25. The only conditions under which definite experimental evidence as to the value of the criterion has as yet been obtained are those of steady flow through a straight round tube of uniform bore; and for this reason it would seem desirable to choose for theoretical application the case of a round tube. But inasmuch as the application of the theory is only carried to the point of affording a proof of the existence of an inferior limit to the value of the criterion which shall be greater than a certain quantity determined by the density and viscosity of the fluid and the conditions of flow, and as the necessary expressions for the round tube are much more complex than those for parallel plane surfaces, the conditions here considered are those defined by such surfaces.

Case I. Conditions.

26. The fluid is of constant density ρ and viscosity μ, and is caused to flow, by a uniform variation of pressure $d\bar{p}/dx$, in direction x between parallel surfaces, given by

$$y = -b_0, \quad y = b_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

the surfaces being of indefinite extent in directions z and x.

The Boundary Conditions.

(1.) There can be no motion normal to the solid surfaces, therefore

$$v = 0 \text{ when } y = \pm b_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

(2.) That there shall be no tangential motion at the surface, therefore

$$u = w = 0 \text{ when } y = \pm b_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30);$$

whence by equation (21), putting u for u', $p_{yx} = -\mu du/dy$.

By the equation of continuity $du/dx + dv/dy + dw/dz = 0$, therefore at the boundaries we have the further conditions, that when $y = \pm b_0$,

$$du/dx = dv/dy = dw/dz = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

Singular Solution.

27. If the mean-motion is everywhere in direction x, then, by the equation of continuity, it is constant in this direction, and as shown (Art. 8) the periods of mean-motion are infinite, and the equations (1), (3), and (9) are strictly true. Hence if

$$\bar{v} = \bar{w} = u' = v' = w' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

we have conditions under which a singular solution of the equations, applied to this case, is possible whatsoever may be the value of b_0, dp/dx, ρ and μ.

Substituting for p_{xx}, p_{yz}, &c., in equations (1) from equations (21), and substituting u for u', &c., these become

$$\rho \frac{du}{dt} = -\frac{dp}{dx} + \mu\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

This equation does not admit of solution from a state of rest;* but assuming a condition of steady motion such that du/dt is everywhere zero, and dp/dx constant, the solution of

$$\left.\begin{aligned} &\frac{\mu}{\rho}\left(\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)-\frac{1}{\rho}\frac{dp}{dx}=0, \\ \text{if}\quad & u = du/dz = 0 \text{ when } y = \pm b_0, \\ \text{is}\quad & u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2-b_0^2}{2} \end{aligned}\right\} \quad \ldots \ldots \ldots (34).$$

This is a possible condition of steady motion in which the periods of u according to Art. 8 are infinite; so that the equations for mean-motion as affected by heat-motion, by Art. 8, are exact, whatever may be the values of

$$u,\ b_0,\ \rho,\ \mu, \text{ and } dp/dx.$$

The last of equations (34) is thus seen to be a singular solution of the equations (15) for steady mean-flow, or steady mean-mean-motion, when u', v', w', p', &c., have severally the values zero, and so the equations (16) of relative-mean-motion are identically satisfied.

In order to distinguish the singular values of u, I put

$$\left.\begin{aligned} & u = \text{U}, \qquad \int_{-b}^{b} u\,dy = 2b_0\text{U}_m\,; \\ \text{whence}\quad & \frac{dp}{dx} = -\frac{3\mu}{b_0^2}\text{U}_m,\ \text{U} = \frac{3}{2}\text{U}_m\frac{b_0^2-y^2}{b_0^2} \end{aligned}\right\} \quad \ldots \ldots \ldots (35).$$

According to the equations such a singular solution is always possible where the conditions can be realized, but the manner in which this solution of the equation (1) of mean-motion is obtained affords no indication as to whether or not it is the only solution—as to whether or not the conditions can be realised. This can only be ascertained either by comparing the results as given by such solutions with the results obtained by experiment, or by observing the manner of motion of the fluid, as in my experiments with colour bands.

* In a paper on the "Equations of Motion and the Boundary Conditions of Viscous Fluid," read before Section A at the meeting of the B.A., 1883, I pointed out the significance of this disability to be integrated, as indicating the necessity of the retention of terms of higher orders to complete the equations, and advanced certain confirmatory evidence as deduced from the theory of gases. The paper was not published, as I hoped to be able to obtain evidence of a more definite character, such as that which is now adduced in Articles 7 and 8 of this paper, which shows that the equations are incomplete, except for steady motion, and that to render them integrable from rest the terms of higher orders must be retained, and thus confirms the argument I advanced, and completely explains the anomaly.

The fact that these conditions are realized, under certain circumstances, has afforded the only means of verifying the truth of the assumptions as to the boundary conditions, that there shall be no slipping, and as to μ being independent of the variations of mean-motion.

Verification of the Assumptions in the Equation of Viscous Fluid.

28. As applied to the conditions of POISEUILLE'S experiments and similar experiments made since, the results obtained from the theory are found to agree throughout the entire range so long as u', v', w' are zero, showing that if there were any slipping it must have been less than the thousandth part of the mean flow, although the tangential force at the boundary was 0·2 gr. per square centimetre, or over 6 lbs. per square foot, the mean flow 376 millims. (1·23 feet) per second, and

$$d\bar{u}/dr = 215{,}000,$$

the diameter of this tube being 0·014 millim., the length 1 25 millims., and the head 30 inches of mercury.

Considering that the skin resistance of a steamer going at 25 knots is not 6 lbs. per square foot, it appears that the assumptions as to the boundary conditions and the constancy of μ have been verified under more exigent circumstances, both as regards tangential resistance and rate of variation of tangential stress, than occur in anything but exceptional cases.

Evidence that other Solutions are possible.

29. The fact that steady mean-motion is almost confined to capillary tubes—and that in larger tubes, except when the motion is almost insensibly slow, the mean-motion is sinuous and full of eddies, is abundant evidence of the possibility, under certain conditions, of solutions other than the singular solutions.

In such solutions u', v', w' have values, which are maintained, not as a system of steady periodic motion, but such as has a steady effect on the mean flow through the tube; and equations (1) are only approximately true.

The Application of the Equations of the Mean- and Relative-mean-motion.

30. Since the components of mean-mean-motion in directions y and z are zero, and the mean flow is steady,

$$\bar{v} = 0, \quad \bar{w} = 0, \quad d\bar{u}/dt = 0, \quad d\bar{u}/dx = 0 \quad . \quad . \quad . \quad . \quad . \quad (36),$$

and as the mean values of functions of u', v', w' are constant in the direction of flow,

$$\frac{d\,(\overline{u'u'})}{dx} = 0, \quad \frac{d\,(\overline{v'u'})}{dx} = 0, \quad \frac{d\,(\overline{w'u'})}{dx} = 0, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (37).$$

By equations (21) and (37) the equations (15) of mean-motion become

$$\left.\begin{aligned} \rho\frac{d\bar{u}}{dt} &= -\frac{dp}{dx} + \mu\left(\frac{d^2\bar{u}}{dy^2} + \frac{d^2\bar{u}}{dz^2}\right) - \rho\left\{\frac{d}{dy}(\overline{u'v'}) + \frac{d}{dz}(\overline{u'w'})\right\} \\ \rho\frac{d\bar{v}}{dt} &= -\frac{dp}{dy} \qquad\qquad\qquad\qquad - \rho\left\{\frac{d}{dy}(\overline{v'v'}) + \frac{d}{dz}(\overline{v'w'})\right\} \\ \rho\frac{d\bar{w}}{dt} &= -\frac{dp}{dz} \qquad\qquad\qquad\qquad - \rho\left\{\frac{d}{dy}(\overline{w'v'}) + \frac{d}{dz}(\overline{w'w'})\right\} \end{aligned}\right\} \quad . \quad . \quad (38).$$

The equation of energy of mean-mean-motion (17) becomes

$$\left.\begin{aligned} \frac{d\,(\mathrm{E})}{dt} = -\bar{u}\frac{d\bar{p}}{dx} + \mu\left\{\frac{d}{dy}\left(\bar{u}\frac{d\bar{u}}{dy}\right) + \frac{d}{dz}\left(\bar{u}\frac{d\bar{u}}{dz}\right)\right\} - \rho\left\{\frac{d}{dy}(\bar{u}\,\overline{u'v'}) + \frac{d}{dz}(\bar{u}\,\overline{u'w'})\right\} \\ -\mu\left\{\left(\frac{d\bar{u}}{dy}\right)^2 + \left(\frac{d\bar{u}}{dz}\right)^2\right\} + \rho\left\{\overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{d\bar{u}}{dz}\right\} \end{aligned}\right\} (39).$$

Similarly the equation of mean-energy of relative-mean-motion (19) becomes

$$\begin{aligned} \frac{d\mathrm{E}'}{dt} = &-\frac{d}{dy}\left[u'(p'_{yx} + \overline{u'v'}) + v'(p'_{yy} + \overline{v'v'}) + w'(p'_{yz} + \overline{w'v'})\right] \\ &-\frac{d}{dz}\left[u'(p'_{zx} + \overline{uw'}) + v'(p'_{zy} + \overline{v'w'}) + w'(p'_{zz} + \overline{w'w'})\right] \\ &-\mu\left[2\left\{\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right\} + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dz}\right)^2\right] \\ &-\rho\left\{\overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{d\bar{u}}{dz}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40). \end{aligned}$$

Integrating in directions y and z between the boundaries and taking note of the boundary conditions by which $\bar{u}$, u', v', w' vanish at the boundaries together with the integrals, in direction z, of

$$\frac{d}{dz}\left(\bar{u}\frac{d\bar{u}}{dz}\right), \quad \frac{d}{dz}\left[\bar{u}\,(\overline{u'w'})\right], \quad \frac{d}{dz}\left[u'\,(p_{zx} + \overline{u'w'})\right], \text{ \&c.,}$$

the integral equation of energy of mean-mean-motion becomes

$$\iint\frac{d\mathrm{E}}{dt}\,dy\,dz = -\iint\left[\bar{u}\frac{dp}{dx} + \mu\left\{\left(\frac{d\bar{u}}{dy}\right)^2 + \left(\frac{d\bar{u}}{dz}\right)^2\right\} - \rho\left\{\overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{d\bar{u}}{dz}\right\}\right]dy\,dz \,. \;(41).$$

The integral equation of energy of relative-mean-motion becomes

$$\iint \frac{d\text{E}'}{dt}\,dy\,dz = -\iint\left[\rho\left\{\overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{d\bar{u}}{dz}\right\}\right]dy\,dz - \mu\iint\left[2\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2 + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2\right]dy\,dz \quad . \quad . \quad . \quad (42).$$

If the mean-mean-motion is steady it appears from equation (41) that

$$-\iint \bar{u}\frac{d\bar{p}}{dx}dy\,dz,$$

the work done on the mean-mean-motion $\bar{u}$, per unit of length of the tube, by the constant variation of pressure, is in part transformed into energy of relative-mean-motion at a rate expressed by the transformation function

$$-\iint \rho\left(\overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{dw}{dy}\right)dy\,dz,$$

and in part transformed into heat at the rate

$$\mu\iint\left[\left(\frac{d\bar{u}}{dy}\right)^2 + \left(\frac{d\bar{u}}{dz}\right)^2\right]dy\,dz.$$

While the equation (42) for the integral energy of relative-mean-motion shows that the only energy received by the relative-mean-motion is that transformed from mean-mean-motion, and the only energy lost by relative-mean-motion is that converted into heat by the relative-mean-motion at the rate expressed by the last term.

And hence if the integral of E′ is maintained constant, the rate of transformation from energy of mean-mean-motion must be equal to the rate at which energy of relative-mean-motion is converted into heat, and the discriminating equation becomes

$$\iint \rho\left(\overline{u'v'}\frac{d\bar{u}}{dy} + \overline{u'w'}\frac{d\bar{u}}{dz}\right)dy\,dz = -\mu\iint\left[2\left\{\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right\} + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dy}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2\right]dy\,dz \quad . \quad (43).$$

The Conditions to be Satisfied by $\bar{u}$ and u', v', w'.

31. If the mean-mean-motion is steady $\bar{u}$ must satisfy :—

(1) The boundary conditions

$$\bar{u} = 0 \text{ when } y = \pm b_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44);$$

(2) The equation of continuity

$$d\bar{u}/dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45);$$

(3) The first of the equations of motion (38)

$$\frac{dp}{dx} = \mu\left(\frac{d^2\bar{u}}{dy^2} + \frac{d^2\bar{u}}{dz^2}\right) - \rho\left\{\frac{d}{dy}(\overline{u'v'}) + \frac{d}{dz}(\overline{u'w'})\right\} \quad . \quad . \quad . \quad . \quad (46);$$

or putting

$$\bar{u} = \mathrm{U} + \bar{u} - \mathrm{U} \text{ and } dp/dx = \mu\, d^2\mathrm{U}/dy^2$$

as in the singular solution, equation (46) becomes

$$\mu\left(\frac{d^2(\bar{u} - \mathrm{U})}{dy^2} + \frac{d^2(\bar{u} - \mathrm{U})}{dz^2}\right) = \rho\left\{\frac{d}{dy}(\overline{u'v'}) + \frac{d}{dz}(\overline{u'w'})\right\} \quad . \quad . \quad . \quad (47);$$

(4) The integral of (47) over the section of which the left member is zero, and

the mean value of $\mu\, d\bar{u}/dy = \mu\, d\mathrm{U}/dy$ when $y = \pm b_0$ . . . (48).

From the condition (3) it follows that if $\bar{u}$ is to be symmetrical with respect to the boundary surfaces, the relative-mean-motion must extend throughout the tube, so that

$$\int_{-\infty}^{\infty}\left(\frac{d\overline{u'v'}}{dy} + \frac{d\overline{u'w'}}{dz}\right) dz \text{ is a function of } y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

And as this condition is necessary, in order that the equations (38) of mean-mean-motion and the equations (16) of relative-mean-motion may be satisfied for steady mean-motion, it is assumed as one of the conditions for which the criterion is sought.

The components of relative-mean-motion must satisfy the periodic conditions as expressed in equations (12), which become, putting $2c$ for the limit in direction z,

(1)

$$\left.\begin{aligned} &\int_0^a u'\,dx = \int_0^a v'\,dx = \int_0^a w'\,dx = 0 \\ &\int_{-b_0}^{b_0}\int_{-c}^{c} u'\,dy\,dz = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50).$$

(2) The equation of continuity

$$du'/dx + dv'/dy + dw'/dz = 0.$$

(3) The boundary conditions which with the continuity give

$$u' = v' = w' = du'/dx = dv'/dy = dw'/dz = 0 \text{ when } y = \pm b_0 \quad . \quad . \quad (51).$$

(4) The condition imposed by symmetrical mean-motion

$$\int_{-c}^{c} \left(\frac{d\overline{u'v'}}{dy} + \frac{d\overline{u'w'}}{dz} \right) dz = 2cf(y^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

These conditions (1 to 4) must be satisfied if the effect on $\bar{u}$ is to be symmetrical however arbitrarily u', v', w' may be superimposed on the mean-motion which results from a singular solution.

(5) If the mean-motion is to remain steady u', v', w' must also satisfy the kinematical conditions obtained by eliminating $\bar{p}$ from the equations of mean-mean-motion (38) and those obtained by eliminating p' from the equations of relative-mean-motion (16).

Conditions (1 *to* 4) *determine an inferior Limit to the Criterion.*

32. The determination of the kinematic conditions (5) is, however, practically impossible; but if they are satisfied, u', v', w' must satisfy the more general conditions imposed by the discriminating equation. From which it appears that when u', v', w' are such as satisfy the conditions (1 to 4), however small their values relative to $\bar{u}$ may be, if they be such that the rate of conversion of energy of relative-mean-motion into heat is greater than the rate of transformation of energy of mean-mean-motion into relative-mean-motion, the energy of relative-mean-motion must be diminishing. Whence, when u', v', w' are taken such periodic functions of x, y, z, as under conditions (1 to 4) render the value of the transformation function relative to the value of the conversion function a maximum, if this ratio is less than unity, the maintenance of any relative-mean-motion is impossible. And whatever further restrictions might be imposed by the kinematical conditions, the existence of an inferior limit to the criterion is proved.

Expressions for the Components of possible Relative-mean-motion.

33. To satisfy the first three of the equations (50) the expressions for u', v', w', must be continuous periodic functions of x, with a maximum periodic distance a, such as satisfy the conditions of continuity.

Putting

$$l = 2\pi/a\text{; and } n \text{ for any number from 1 to } \infty,$$

and

$$\left.\begin{aligned} u' &= \Sigma_0^\infty \left\{\left(\frac{d\alpha_n}{dy} + \frac{d\gamma_n}{dz}\right)\cos(nlx) + \left(\frac{d\beta_n}{dy} + \frac{d\delta_n}{dz}\right)\sin(nlx)\right\} \\ v' &= \Sigma_0^\infty \{nl\alpha_n \sin(nlx) - nl\beta_n \cos(nlx)\} \\ w' &= \Sigma_0^\infty \{nl\gamma_n \sin(nlx) - nl\,\delta_n \cos(nlx)\} \end{aligned}\right\} \quad . \quad . \quad . \quad (53),$$

u', v', w' satisfy the equation of continuity. And, if

$$\left.\begin{aligned} &\alpha = \beta = \gamma = \delta = d\alpha/dy = d\beta/dy = d\gamma/dz = d\delta/dz = 0 \text{ when } y = \pm b_0 \\ &\text{and } \alpha\beta,\ \alpha\gamma,\ \alpha\delta \text{ are all functions of } y^2 \text{ only} \end{aligned}\right\} . \quad (54),$$

it would seem that the expressions are the most general possible for the components of relative-mean-motion.

Cylindrical-relative-motion.

34. If the relative-mean-motion, like the mean-mean-motion, is restricted to motion parallel to the plane of xy,

$$\gamma = \delta = w' = 0, \text{ everywhere},$$

and the equations (53) express the most general forms for u', v' in case of such cylindrical disturbance.

Such a restriction is perfectly arbitrary, and having regard to the kinematical restrictions, over and above those contained in the discriminating equation, would entirely change the character of the problem. But as no account of these extra kinematical restrictions is taken in determining the limit to the criterion, and as it appears from trial that the value found for this limit is essentially the same, whether the relative-mean-motion is general or cylindrical, I only give here the considerably simpler analyses for the cylindrical motion.

The Functions of Transformation of Energy and Conversion to Heat for Cylindrical Motion.

35. Putting

$$\frac{d}{dt}\left({}_p\text{H}'\right)$$

for the rate at which energy of relative-mean-motion is converted to heat per unit of volume, expressed in the right-hand member of the discriminating equation (43),

$$\iiint \frac{d}{dt}({}_p\text{H}')\,dx\,dy\,dz$$
$$= \mu \iiint \left[2\left\{\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2\right\} + \left(\frac{du'}{dy}\right)^2 + \left(\frac{dv'}{dx}\right)^2 + 2\frac{du'}{dy}\frac{dv'}{dx}\right] dx\,dy\,dz \quad . \quad . \quad (56).$$

Then substituting for the values of u', v', w' from equations (53), and integrating in direction x over $2\pi/l$, and omitting terms the integral of which, in direction y, vanishes by the boundary conditions,

$$\iint \frac{d}{dt}\left({}_{\rho}\mathrm{H}'\right) dy\,dz = \frac{\mu}{2}\iint \Sigma\left\{(nl)^4(\alpha_n^2+\beta_n^2) + 2(nl)^2\left[\left(\frac{d\alpha_n}{dy}\right)^2 + \left(\frac{d\beta_n}{dy}\right)^2\right] + \left(\frac{d^2\alpha_n}{dy^2}\right)^2 + \left(\frac{d^2\beta_n}{dy^2}\right)^2\right\} dy\,dz \quad . \quad . \quad . \quad . \quad . \quad (57).$$

In a similar manner, substituting for $u'\ v'$, integrating, and omitting terms which vanish on integration, the rate of transformation of energy from mean-mean-motion, as expressed by the left member in the discriminating equation (43), becomes

$$\iint \rho\overline{u'v'}\frac{d\bar{u}}{dy}\,dy\,dz = \tfrac{1}{2}\iint \Sigma\left[nl\left(\alpha_n\frac{d\beta_n}{dy} - \beta_n\frac{d\alpha_n}{dy}\right)\frac{d\bar{u}}{dy}\right] dy\,dz \quad . \quad . \quad . \quad (58).$$

And, since by Art. 31, conditions (3) equation (47),

$$\mu\frac{d^2}{dy^2}(\bar{u} - \mathrm{U}) = \rho\frac{d}{dy}(\overline{u'v'}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

integrating and remembering the boundary conditions,

$$\mu\frac{d}{dy}(\bar{u} - \mathrm{U}) = \rho\overline{u'v'}, \quad \mu(\bar{u} - \mathrm{U}) = \rho\int_{-b_0}^{y}\overline{u'v'}\,dy \quad . \quad . \quad . \quad . \quad . \quad (60).$$

And since at the boundary $\bar{u} - \mathrm{U}$ is zero,

$$\rho\int_{-b_0}^{b_0}(u'v')\,dy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

Whence, putting $\mathrm{U} + \bar{u} - \mathrm{U}$ for $\bar{u}$ in the right member of equation (58), substituting for $\bar{u} - \mathrm{U}$ from (60), integrating by parts, and remembering that

$$\frac{d^2\mathrm{U}}{dy^2} = -3\frac{\mathrm{U}_m}{b_0^2}, \text{ which is constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62),$$

also that

$$u'v' = \tfrac{1}{2}\Sigma\left\{nl\left(\alpha_n\frac{d\beta_n}{dy} - \beta_n\frac{d\alpha_n}{dy}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

we have for the transformation function

$$\int_{-b_0}^{b_0}\left(\rho\overline{u'v'}\frac{du}{dy}\right)dy$$

$$= \Sigma\left[\tfrac{3}{2}\rho\frac{\mathrm{U}_m}{b_0^2}\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}nl\left(\alpha_n\frac{d\beta_n}{dy} - \beta_n\frac{d\alpha_n}{dy}\right)dy + \frac{\rho^2}{4\mu}\int_{-b_0}^{b}(nl)^2\left(\alpha_n\frac{d\beta_n}{dy} - \beta_n\frac{d\alpha_n}{dy}\right)^2 dy\right] \quad . \quad (64).$$

If u', v' are indefinitely small the last term, which is of the fourth degree, may be neglected.

Substituting in the discriminating equation (43) this may be put in the form

$$\frac{2\rho b_0 \mathrm{U}_m}{\mu} = \frac{2b_0^3 \int_{-b_0}^{b_0} \Sigma \left\{ n^4 l^4 (\alpha_n^2 + \beta_n^2) + 2n^2 l^2 \left[\left(\frac{d\alpha_n}{dy}\right)^2 + \left(\frac{d\beta_n}{dy}\right)^2 \right] + \left(\frac{d^2\alpha_n}{dy^2}\right)^2 + \left(\frac{d^2\beta_n}{dy^2}\right)^2 \right\} dy}{3 \int_{-b_0}^{b_0} dy \int_{-b_0}^{y} \Sigma \left\{ nl \left(\beta_n \frac{d\alpha_n}{dy} - \alpha_n \frac{d\beta_n}{dy} \right) \right\} dy} \qquad (65).$$

Limits to the Periods.

36. As functions of y the variations of α_n, β_n are subject to the restrictions imposed by the boundary conditions, and in consequence their periodic distances are subject to superior limits determined by $2b_0$, the distance between the fixed surfaces.

In direction x, however, there is no such direct connection between the value of b_0 and the limits to the periodic distance, as expressed by $2\pi/nl$. Such limits necessarily exist, and are related to the limits of α_n and β_n in consequence of the kinemetical conditions necessary to satisfy the equations of motion for steady mean-mean-motion; these relations, however, cannot be exactly determined without obtaining a general solution of the equations.

But from the form of the discriminating equation (43) it appears that no such exact determination is necessary in order to prove the inferior limit to the criterion.

The boundaries impose the same limits on α_n, β_n whatever may be the value of nl; so that if the values of α_n, β_n be determined so that the value of

$$\frac{2\rho b_0 \mathrm{U}_m}{\mu} \text{ is a minimum}$$

for every value of nl, the value of rl, which renders this minimum a minimum-minimum may then be determined, and so a limit found to which the value of the complete expression approaches, as the series in both numerator and denominator become more convergent for values of nl differing in both directions from rl.

Putting l, α, β for rl, α_r, β_r respectively, and putting for the limiting value to be found for the criterion

$$\mathrm{K}_1 = \frac{2\rho b_0 \mathrm{U}_m}{\mu} \qquad (66)$$

$$\frac{3}{2}\mathrm{K}_1 = b_0^3 \frac{\int_{-b_0}^{b_0} \left\{ l^4(\alpha^2 + \beta^2) + 2l^2 \left[\left(\frac{d\alpha}{dy}\right)^2 + \left(\frac{d\beta}{dy}\right)^2 \right] + \left(\frac{d^2\alpha}{dy^2}\right)^2 + \left(\frac{d^2\beta}{dy}\right)^2 \right\} dy}{l \int_{-b_0}^{b} dy \int_{-b_0}^{y} \left(\beta \frac{d\alpha}{dy} - \alpha \frac{d\beta}{dy} \right) dy} \qquad (67)$$

when α and β are such functions of y that K_1 is a minimum whatever the value of l, and l is so determined as to render K_1 a minimum-minimum.

Having regard to the boundary conditions, &c., and omitting all possible terms which increase the numerator without affecting the denominator, the most general form appears to be

$$\left.\begin{aligned} \alpha &= \Sigma_0^n [a_{2s+1} \sin (2s+1) p], \\ \beta &= \Sigma_0^n [b_{2t} \sin (2tp)], \\ p &= \pi y / 2b_0 \end{aligned}\right\} \quad . \quad . \quad . \quad (68).$$

where

To satisfy the boundary conditions

$s = 2r$, when s is even, $\qquad s = 2r + 1$, when s is odd.

$t = 2r + 1$, when t is odd, $\qquad t = 2(r + 1)$, when t is even.

Since $\alpha = 0$, when $p = \pm \frac{1}{2}\pi$,

$$\Sigma_0^\infty (a_{4r+1} - a_{4r+3}) = 0,$$

and since $d\beta/dy = 0$, when $p = \pm \frac{1}{2}\pi$,

$$\Sigma_0^\infty \{-(4r+2) b_{4r+2} + 4(r+1) b_{4r+4}\} = 0 \quad . \quad (69).$$

From the form of K_1 it is clear that every term in the series for α and β increases the value of K_1 and to an extent depending on the value of r. K_1 will therefore be minimum, when

$$\left.\begin{aligned} \alpha &= a_1 \sin p + a_3 \sin 3p \\ \beta &= b_2 \sin 2p + b_4 \sin 4p \end{aligned}\right\} \quad . \quad . \quad . \quad (70),$$

which satisfy the boundary conditions if

$$\left.\begin{aligned} a_3 &= a_1 \\ b_2 &= 2b_4 \end{aligned}\right\} \quad . \quad . \quad . \quad (71).$$

Therefore we have, as the values of α and β, which render K_1 a minimum for any value of l

$$\alpha/a_1 = \sin p + \sin 3p, \quad \beta/b_2 = \sin 2p + \tfrac{1}{2} \sin 4p.$$

And

$$\left.\begin{aligned} \frac{2b_0}{\pi a_1}\frac{d\alpha}{dy} &= \cos p + 3 \cos 3p, \quad \frac{2b_0}{\pi b_2}\frac{d\beta}{dy} = 2 \cos 2p + 2 \cos 4p \\ \frac{2b_0}{\pi a_1 b_2}\left(\frac{\alpha d\beta}{dy} - \frac{\beta d\alpha}{dy}\right) &= \frac{1}{4}\{-3 \sin p - 3 \sin 3p + \sin 5p + \sin 7p\} \end{aligned}\right\} \quad . \quad (72)$$

and integrating twice

$$l\int_{-b_0}^{b_0} dy \int_{-b_0}^{y}\left(\beta\frac{d\alpha}{dy} - \alpha\frac{d\beta}{dy}\right) dy = -1\cdot325\, l\frac{2b_0}{\pi}a_1 b_2 \quad . \quad . \quad . \quad . \quad (73).$$

Putting

$$\frac{\pi}{2b_0}\mathrm{L} \text{ for } l,$$

the denominator of $\frac{3}{2}\mathrm{K}_1$, equation (67), becomes

$$-1\cdot325\ \mathrm{L}a_1 b_2.$$

In a similar manner the numerator is found to be

$$b_0^4\left(\frac{\pi}{2b_0}\right)^4\{\mathrm{L}^4(2a_1^2 + 1\cdot25b_2^2) + 2\mathrm{L}^2(10a_1^2 + 8b_2^2) + 82a_1^2 + 80b_2^2\},$$

and as the coefficients of a_1 and b_2 are nearly equal in the numerator, no sensible error will be introduced by putting

$$b_2 = -a_1,$$

then

$$\frac{3}{2}\mathrm{K}_1 = \frac{\mathrm{L}^4 + 2\times 5\cdot53\mathrm{L}^2 + 50}{0\cdot408\mathrm{L}}\left(\frac{\pi}{2}\right)^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

which is a minimum if

$$\mathrm{L} = 1\cdot62 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

and

$$\mathrm{K}_1 = 517 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

Hence, for a flat tube of unlimited breadth, the criterion

$$\rho\, 2b_0\mathrm{U}_m/\mu \text{ is greater than } 517 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77).$$

37. This value must be less than that of the criterion for similar circumstances. How much less it is impossible to determine theoretically without effecting a general solution of the equations; and, as far as I am aware, no experiments have been made in a flat tube. Nor can the experimental value 1900, which I obtained for the round tube, be taken as indicative of the value for a flat tube, except that, both theoretically and practically, the critical value of U_m is found to vary inversely as the hydraulic mean depth, which would indicate that, as the hydraulic mean depth in a flat tube is double that for a round tube, the criterion would be half the value, in which case the limit found for K_1 would be about 0·61 K. This is sufficient to show that the absolute theoretical limit found is of the same order of magnitude as the experimental

value; so that the latter verifies the theory, which, in its turn, affords an explanation of the observed facts.

The State of Steady Mean-motion above the Critical Value.

38. In order to arrive at the limit for the criterion it has been necessary to consider the smallest values of u', v', w', and the terms in the discriminating equation of the fourth degree have been neglected. This, however, is only necessary for the limit, and, preserving these higher terms, the discriminating equation affords an expression for the resistance in the case of steady mean-mean-motion.

The complete value of the function of transformation as given in equation (64) is

$$\int_{-b_0}^{b_0}\left(\overline{\rho u'v'}\,\frac{d\overline{u}}{dy}\right)$$
$$= \Sigma\left[\rho\,\frac{3U_m}{2b_0^2}\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}nl\left(\alpha_n\frac{d\beta_n}{dy}-\beta_n\frac{d\alpha_n}{dz}\right)dy+\frac{\rho^2}{4\mu}\int_{-b_0}^{b_0}(nl)^2\left(\alpha_n\frac{d\beta_n}{dy}-\beta\,\frac{d\alpha}{dy}\right)^2 dy\right]. \quad (77a).$$

Whence putting $U+\overline{u}-U$, for $\overline{u}$ in the left member of equation (77), and integrating by parts, remembering the conditions, this member becomes

$$\frac{3U_m}{b_0^2}\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}\overline{\rho u'v'}\,dy+\frac{\rho^2}{\mu}\int_{-b_0}^{b_0}(u'v')^2\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (78),$$

in which the first term corresponds with the first term in the right member of equation (64), which was all that was retained for the criterion, and the second term corresponds with the second term in equation (64), which was neglected.

Since by equation (35)

$$\frac{3U_m}{b_0^2}=-\frac{1}{\mu}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78a),$$

we have, substituting in the discriminating equation (43), either

$$-\tfrac{2}{3}\,\rho\,\frac{b_0^3}{\mu^2}\frac{dp}{dx}=\frac{2b_0^3}{3}\left\{\frac{\left(\int\frac{d/dt\,(pH')\,dy}{\mu}+\frac{\rho^2}{\mu^2}\int_{-b_0}^{b_0}(\overline{u'v'})^2\,dy\right)}{-\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}\overline{u'v'}\,dy}\right\} . \quad . \quad . \quad . \quad (79),$$

or

$$\mu\,\frac{d^2\overline{u}}{dy^2}-\frac{dp}{dx}=0 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80).$$

Therefore, as long as

$$\tfrac{2}{3}\,\rho\,\frac{b_0^3}{\mu^2}\frac{dp}{dx}$$

is of constant value, there is dynamical similarity under geometrically similar circumstances.

The equation (79) shows that,

$$\text{when } -\tfrac{2}{3}\rho\frac{b_0^3}{\mu^2}\frac{dp}{dx} \text{ is greater than K,}$$

$\overline{u'v'}$ must be finite, and such that the last term in the numerator limits the rate of transformation, and thus prevents further increase of $\overline{u'v'}$.

The last term in the numerator of equation (79) is of the order and degree

$$\rho^2 L^4 \alpha^4/\mu^2 \text{ as compared with } L^4 \alpha^2$$

the order and degree of $\frac{1}{\mu}\frac{d}{dt}({}_p H')$ the first term in the numerator.

It is thus easy to see how the limit comes in. It is also seen from equation (79) that, above the critical value, the law of resistance is very complex and difficult of interpretation, except in so far as showing that the resistance varies as a power of the velocity higher than the first.