III. "On the Orders and Genera of Quadratic Forms containing more than three Indeterminates." By H. J. STEPHEN SMITH, M.A., F.R.S., Savilian Professor of Geometry in the University of Oxford.—Second Notice. Received October 30, 1867.

The principles upon which quadratic forms are distributed into orders and genera have been indicated in a former notice (Proceedings of the Royal Society, vol. xiii. p. 199). Some further results relating to the same subject are contained in the present communication.

I. *The Definition of the Orders and Genera.*

Retaining, with some exceptions to which we shall now direct attention, the notation and nomenclature of the former notice, we represent by f_1 a primitive quadratic form containing n indeterminates, of which the matrix is $\| \overset{n\times n}{A_{i,j}} \|$; by $f_2, f_3, \ldots f_{n-1}$, the fundamental concomitants of f_1, of which the last is the contravariant. The matrices of these concomitants are the matrices derived from the matrix of f_1, so that the first coefficients of $f_2, f_3, \ldots f_{n-1}$, are respectively the determinants $| \overset{2\times 2}{A_{i,j}} |$, $| \cdot \overset{3\times 3}{A_{i,j}} |$, $\ldots | \overset{n-1\times n-1}{A_{i,j}} |$, taken with their proper signs. The discriminant of f_1, i. e. the determinant of the matrix $| \overset{n\times n}{A_{i,j}} |$, which is supposed to be different from zero, and which is to be taken with its proper sign, is represented by ∇_n. The greatest common divisors of the minors of the orders $n-1$, $n-2$, ... 2, 1 in the same matrix are denoted by $\nabla_{n-1}, \nabla_{n-2}, \ldots \nabla_2, \nabla_1$, of which the last is a unit; we shall presently attribute signs to each of these greatest common divisors. The quotients

$$\frac{\nabla_n}{\nabla_{n-1}} \div \frac{\nabla_{n-1}}{\nabla_{n-2}}, \frac{\nabla_{n-1}}{\nabla_{n-2}} \div \frac{\nabla_{n-2}}{\nabla_{n-3}}, \ldots \ldots \frac{\nabla_2}{\nabla_1} \div \frac{\nabla_1}{1},$$

which are always integral, we represent by $I_{n-1}, I_{n-2}, \ldots I_1$; so that

$$I_k = \frac{\nabla_{k+1}}{\nabla_k} \div \frac{\nabla_k}{\nabla_{k-1}}.$$

The numbers $I_1, I_2, \ldots I_{n-1}$ are the first, second, ... last invariants of the form f_1, and remain unchanged when f_1 is transformed by any substitution of which the determinant is unity and the coefficients integral numbers. Forms which have the same invariants have of course the same discriminant; but (if the number of indeterminates is greater than two) forms which have the same discriminant do not necessarily have the same invariants; for example, the quaternary forms

$$x_1^2 + x_2^2 + 2x_3^2 + 6x_4^2, \; x_1^2 + x_2^2 + x_3^2 + 12\,x_4^2$$

have the same discrminant 12, but their invariants I_1, I_2, I_3 are respectively 1, 2, 3, and 1, 1, 12. As forms which have the same discriminant, but different invariants, do not necessarily have any close relation to one another, we shall not employ the discriminant in the classification of quadratic forms; but we shall regard the infinite number of forms, which have the same invariants, as corresponding, in the general theory, to the infinite number of forms which have the same determinant, in the theory of binary quadratic forms.

If the index of inertia of the form f is k, i. e. if f_1 can be transformed by a substitution of which the coefficients are real into a sum of k positive and $n-k$ negative squares, we attribute to the invariant I_k the sign $-$, and to every other invariant the sign $+$. Thus the numbers $\nabla_1, \nabla_2, \ldots \nabla_k$ are all positive; $\nabla_{k+1}, \nabla_{k+2}, \ldots \nabla_n$ are alternately negative and positive, so that the discriminant ∇_n is of the same sign as $(-1)^{n-k}$, as it ought to be. This convention with respect to the signs of the invariants will enable us to comprehend in the same formulæ the theory of the generic characters of forms of any index of inertia. We shall, however, suppose that the index of inertia is at least 1, *i. e.* we shall exclude negative definite forms. The invariants of a positive definite form are all positive; and the index of inertia of any indefinite form, of which the invariants are given, is always indicated by the ordinal index of its negative invariant. We shall represent by D the product $-I_1 \times -I_3 \times -I_5 \times \ldots$, the last factor being $-I_{n-1}$, or $-I_{n-2}$, according as n is even or uneven.

If $\theta^i = \frac{1}{\nabla_i} f^i$, the forms $\theta_1, \theta_2, \theta_3, \ldots \theta_{n-1}$ are the primitive concomitants, and the last the primitive contravariant, of f_1 or θ_1; each one of them is either uneven, *i. e.* properly primitive, or even, *i. e.* improperly primitive. Two forms, which have the same invariants, are said to belong to the same order when the corresponding primitive concomitants of the two forms are alike uneven or alike even. When the invariants are all uneven, and the number of the indeterminates is also uneven, there is but one order, none of the primitive concomitants being in this case even. Again, when the invariants are all uneven, and the number n of the indeterminates is even, there is either one order or two, according as $D \equiv -1$, or $\equiv +1$, mod 4; for in both cases there is an order in which all the primitive concomitants are uneven, and in the latter case, besides this uneven order, there is an even order, in which these forms are alternately even and uneven, the two extreme forms θ_1 and θ_{n-1} being even. In the general case, when the invariants have any values even or uneven, if I_i is even, θ_i cannot be even; again, if I_i is one of a sequence of an even number of uneven invariants, preceded and followed by even invariants, θ_i cannot be even. But if there be a sequence of an uneven number of uneven invariants $I_i, I_{i+1}, \ldots I_{i+2j}$, preceded and followed by even invariants, the sequence of primitive concomitants $\theta_i, \theta_{i+1}, \ldots \theta_{i+2j}$ are all uneven if θ_i is uneven, and are alternately even and uneven if θ_i is even; a *sequence* of forms or invariants may consist of a single form or invariant. We attribute the value 0 to the symbols I_0 and I_n, the value 1 to the symbols θ_0 and θ_n; thus the invariant I_1

is always to be regarded as preceded by an even invariant, and I_{n-1} as followed by an even invariant; similarly the forms θ_1 and θ_{n-1} are to be regarded as respectively preceded and followed by uneven forms. Two even forms cannot be consecutive in the series $\theta_1 \ldots \theta_{n-1}$.

The preceding observations enable us to assign all the orders which may exist for any given invariants; if the series of invariants $I_0, I_1, \ldots I_{n-1}, I_n$ present ω different sequences each consisting of an uneven number of uneven invariants, preceded and followed by even invariants, there are 2^ω assignable orders. These orders, in general, all exist; there are, however, the following exceptions to this statement:—

(1) If, the number of indeterminates being even and equal to 2ν, D is uneven, there is an assignable order in which the concomitants $\theta_1, \theta_2, \ldots \theta_{n-1}$ are alternately even and uneven. But, as has been already said, this order does not exist if $D \equiv -1$, mod 4; and, if the invariants are all squares, it does not exist, even if $D \equiv 1$, mod 4, unless the equation

$$(-1)^{\frac{\nu(\nu-1)}{2}} = (-1)^{\frac{k(k-1)}{2}}$$

(in which k is the index of inertia) is also satisfied.

(2) If, the number of indeterminates being uneven and equal to $2\nu+1$, D is uneven, and $I_{2\nu}$ even, there is again an assignable order in which the concomitants $\theta_1, \ldots \theta_{2\nu}$ are alternately even and uneven. But, when $I_{2\nu}$ is the double of a square, and the other invariants are squares, this order does not exist unless the equation $(-1)^{\frac{1}{8}(n^2-1)} = (-1)^{\frac{1}{2}k(n-k)}$ (in which k is still the index of inertia) is satisfied.

The reciprocal case (that obtained by changing I_s and θ_s into I_{n-s}, and θ_{n-s}, for every value of s from 0 to n) presents a similar exception, which it is not necessary to enunciate separately.

The generic characters of the form θ_1, or, more properly of the system of concomitant forms $\theta_1, \theta_2, \ldots \theta_{n-1}$, so far as they depend on uneven primes dividing the invariants, have been already defined in the former notice, and the definition need not be repeated here. These characters we shall term the *principal* generic characters of the system. When the invariants and primitive concomitants are all uneven, the principal characters are the only generic characters, with the exception of a certain character, which we shall define hereafter, and of which the value is not independent of the principal characters. In other cases, the forms of the concomitant system may acquire generic characters with respect to 4 or 8: these we shall term *supplementary*. What supplementary characters exist in any given case may always be ascertained by applying the following rules. In their enunciation we represent by I'_i the greatest uneven divisor of I_i taken with the same sign as I_i, by μ_i the exponent of the highest power of 2 contained in I_i, increased by 1 if one of the two forms $\theta_{i-1}, \theta_{i+1}$ is even, and by 2 if both those forms are even; we suppose $0 < i < n$.

I. If $\mu_i \overline{\overline{>}} 2$, θ_i has the character $(-1)^{\frac{\theta_i - 1}{2}}$

II. If $\mu_i \overline{\overline{>}} 3$, θ_i, in addition to the character $(-1)^{\frac{\theta_i - 1}{2}}$, has also the character $(-1)^{\frac{\theta_i^2 - 1}{8}}$.

III. If $\mu_i = 1$, and also $\mu_{i-1} \overline{\overline{>}} 2$, $\mu_{i+1} \overline{\overline{>}} 2$, θ_i (which, as well as θ_{i-1} and θ_{i+1}, is necessarily uneven) has the character

$$(-1)^{\frac{\theta_i^2 - 1}{8}} \text{ or } (-1)^{\frac{\theta_i - 1}{2} + \frac{\theta_i^2 - 1}{8}}$$

according as

$$(-1)^{\frac{1}{2}(\theta_{i-1} - 1) + \frac{1}{2}(\theta_{i+1} - 1)} = (-1)^{\frac{1}{2}(I'_i + 1)}, \text{ or } = (-1)^{\frac{1}{2}(I'_i - 1)}.$$

It will be observed that (by I) the forms θ_{i-1} and θ_{i+1} have the characters $(-1)^{\frac{\theta_{i-1} - 1}{2}}$ and $(-1)^{\frac{\theta_{i+1} - 1}{2}}$.

IV. If $\mu_i = 0$, and also $\mu_{i-1} \overline{\overline{>}} 2$, $\mu_{i+1} \overline{\overline{>}} 2$, θ_i, if uneven, has the character $(-1)^{\frac{\theta_i - 1}{2}}$, or no character at all, according as

$$(-1)^{\frac{1}{2}(\theta_{i-1} - 1) + \frac{1}{2}(\theta_{i+1} - 1)} = (-1)^{\frac{1}{2}(I_i - 1)} \text{ or } (-1)^{\frac{1}{2}(I_i + 1)}.$$

No even concomitant has any supplementary character. But if θ_i is an even concomitant, the uneven forms preceding and following it have, by I, the characters

$$(-1)^{\frac{\theta_{i-1} - 1}{2}}, \text{ and } (-1)^{\frac{\theta_{i+1} - 1}{2}}.$$

These characters are not independent but are connected by the equation

$$(-1)^{\frac{1}{2}(\theta_{i-1} - 1) + \frac{1}{2}(\theta_{i+1} - 1)} = (-1)^{\frac{1}{2}(I_i + 1)}.$$

Thus if I_i, I_{i+1}, ... I_{i+2j} is a sequence of an uneven number of uneven invariants preceded and followed by even invariants, and corresponding to a sequence of alternately even and uneven concomitants θ_i, θ_{i+1}, ... θ_{i+2j}, the character, mod 4, of every uneven form of this sequence, and of the next following form θ_{i+2j+1}, is determined by the character of the form θ_{i-1}. We have, in fact, if $s = 1, 2, \ldots j+1$,

$$(-1)^{\frac{1}{2}(\theta_{i+2s-1} - 1)} = (-1)^s \times (-1)^{\frac{1}{2}[I_i \times I_{i+2} \times \cdots I_{i+2s-2} - 1]} \times (-1)^{\frac{1}{2}(\theta_{i-1} - 1)}.$$

Besides these supplementary characters, which, no less than the principal characters, are attributable to individual forms of the concomitant system, there exist, or may exist, other characters, which we shall term simultaneous, attributable to certain sequences of those forms considered conjointly. Such a character is attributable to every sequence of uneven forms, of which none possesses any supplementary character but which are immediately preceded and followed by forms having such characters. The following definition is requisite in order to explain the nature of these simultaneous characters.

If $\left\| \begin{matrix} n-1\times n \\ \alpha_{i', j} \end{matrix} \right\|$ is a matrix of the type $n-1\times n$, of which the determinants are not all zero, and if m_k represent the value acquired by θ_k, when we attribute to the indeterminates of that form the values of the determinants

$$\left\| \begin{matrix} k\times n \\ \alpha_{i, j} \end{matrix} \right\|$$

$$i=1, 2, \ldots k;\ j=1, 2, \ldots n,$$

taken in the same order in which the determinants of any k horizontal rows of the matrix $\left\| \begin{matrix} n\times n \\ A_{i, j} \end{matrix} \right\|$ are taken in forming the matrix of θ_k, the numbers $m_1, m_2, \ldots m_{n-1}$ are said to be simultaneously represented by the forms $\theta_1, \theta_2, \ldots \theta_{n-1}$.

Let $\theta_{i+1}, \ldots \theta_{i+i'}$ be a sequence of i' uneven concomitants, $\mu_{i+1}, \mu_{i+2}, \ldots \mu_{i+i'}$ being 0, or 1, but μ_i and $\mu_{i+i'+1}$ being greater than 1; the uneven numbers simultaneously represented by $\theta_{i+1}, \theta_{i+2}, \ldots \theta_{i+i'}$ are all such as to render the unit

$$(-1)^{\sum\limits_{s=i}^{s=i+i'} \frac{1}{4}(\theta_s-1)(\theta_{s+1}-1)} \times (-1)^{\sum\limits_{s=i+1}^{s=i+i'} \frac{1}{4}(I'_s+1)(\theta_s-1)} \times (-1)^{\sum\limits_{s=i+1}^{s=i+i'} \mu_s \frac{\theta_s^2-1}{8}}$$

(which we shall symbolize by $\psi(i, i')$), equal to $+1$, or else are all such as to render that unit equal to -1. We therefore attribute to the sequence of forms $\theta_{i+1} \ldots \theta_{i+i'}$, the simultaneous character $\psi(i, i')=+1$, or $\psi(i, i')=-1$, according as the former or latter of those equations is satisfied. If $i'=1$, the sequence consists of but one form, so that the character $\psi(i, i')$ ceases to be a simultaneous character; in fact, if $\mu_{i+1}=1$, it coincides with the supplementary character attributable to θ_{i+1} by III.; if $\mu_{i+1}=0$, it either becomes nugatory (*i. e.* identically equal to $+1$, irrespective of the value of m_{i+1}), or it coincides with the supplementary character of θ_{i+1}, according as that form (by IV.) has not or has a supplementary character.

The complex of all the particular characters (principal, supplementary, and simultaneous) constitutes the complete character of the system of concomitants $\theta_1, \theta_2, \ldots \theta_{n-1}$. Not every complete generic character, assignable *à priori*, corresponds to actually existing forms, but only such characters as satisfy a certain condition of possibility. This condition is expressed by the equation

$$\psi(0, n-1) \times \prod_{s=1}^{s=n-1} \left(\frac{\theta_s}{I'_s}\right) = +1, \quad \ldots\ldots \quad \text{(A)}$$

in which, if θ_s is an even form, we understand by the symbol $\left(\frac{\theta_s}{I_s}\right)$ the quadratic character with respect to I_s of the *half* of any number, prime to I_s, which is represented by θ_s. The unit $\psi(0, n-1)$ is formed in the same way as the unit $\psi(i, i')$: we may omit, however, from the exponent of -1 in its expression every term into which an even form enters; if, for example, θ_s is an even form, that exponent contains the terms

$$\left(\frac{\theta_{s-1}-1}{2}+\frac{\theta_{s+1}-1}{2}+\frac{\mathrm{I}_s+1}{2}\right)\times\frac{\theta_s-1}{2}+\mu_s\frac{\theta_s^{\,2}-1}{8},$$

and no other term into which θ_s enters; but $\mu_s=0$, and the coefficient of $\frac{1}{2}(\theta_s-1)$ is even; so that θ_s disappears from the expression of the unit $\psi(0, n-1)$. It will thus be seen that the equation (A) involves only generic characters (principal, supplementary, or simultaneous) of the concomitant system: that equation therefore expresses a relation which the complete character must satisfy.

In using these formulæ, we must attend to the significations which we have assigned to the symbols I_0, I_n, θ_0, and θ_n. Thus

$$(-1)^{\frac{\theta_0-1}{2}}=1=(-1)^{\frac{\theta_0^2-1}{8}},\ \mu_0>3, \text{ etc.}$$

We shall conclude this part of our subject with the two theorems:—

(i) Every genus, of which the character satisfies the condition of possibility, actually exists.

(ii) Two forms, of the same invariants, of the same order, and of the same genus are transformable, each into the other, by rational linear substitutions of which the determinants are units, and in which the denominators of the coefficients are prime to any given number.

The first of these theorems shows that the condition of possibility is sufficient as well as necessary; the second establishes the completeness of the enumeration of ordinal and generic characters.

II. *Determination of the Weight of a given Genus of Definite Forms.*

It has been shown by Gauss, in the digression on ternary forms in the fifth section of the 'Disquisitiones Arithmeticæ,' that the solution of the problems "to obtain all the representations of a given binary form, or of a given number, by a given ternary form," depends on the solution of the problem "to determine whether two given ternary forms are equivalent, and, if they are, to obtain all the transformations of either of them into the other." Similarly the solution of the problem "to obtain all the representations of a given quadratic form of i indeterminates ($i=1, 2, \ldots n-1$) by a given form of n indeterminates" depends on the solution of the problem of equivalence for quadratic forms of n indeterminates. The following proposition is here of primary importance:—

"If the form ϕ_1, of $n-1$ indeterminates and of the invariants $\mathrm{I}_1, \mathrm{I}_2, \ldots \mathrm{I}_{n-3}, \mathrm{MI}_{n-2}$, is capable of primitive representation by the form θ_1, of n indeterminates, and of the invariants $\mathrm{I}_1, \mathrm{I}_2, \ldots \mathrm{I}_{n-3}, \mathrm{I}_{n-2}, \mathrm{I}_{n-1}$, then $-\mathrm{I}_{n-1}\times\phi_{n-2}$ (where ϕ_{n-2} is the primitive contravariant of ϕ_1) is a quadratic residue of M."

The converse is true, subject to certain limitations :—

"If M is prime to I_{n-1}, and not negative except when I_{n-1} is negative, and if $-I_{n-1} \times \phi_{n-2}$ is a quadratic residue of M, ϕ_1 is capable of primitive representation by f_1."

"If, in addition, M is prime to I_{n-2}, there is always either one or two genera of forms of the invariants (I_1, I_2, . . . M I_{n-2}) capable of primitive representation by forms of a given genus of the invariants (I_1, I_2, . . . I_{n-2}, I_{n-1}) ; and if there are two genera capable of such representation, they are of different orders."

These theorems are especially useful in the theory of definite forms, to which, for the remainder of this paper, we shall confine our attention. In the case of such forms we understand by the weight of a form the reciprocal of the number of its positive automorphics, by the weight of a class the weight of any form representing the class ; the weight of a genus, or order, is the sum of the weights of the classes contained in the genus or order ; the weight of a representation of a number by a form is the weight of the representing form ; the weight of a representation of a form by a form is the product of the weights of the representing and represented forms.

Let Γ denote a system of forms, representatives of a given genus of the invariants I_1, I_2, . . . I_{n-1} ; let M be a number divisible by μ different uneven primes, none of which divide any of the invariants, and let M be uneven or unevenly even, according as the contravariants of the forms Γ are uneven or even ; we then have the theorem—

"The sum of the weights of the representations of M by the contravariants of the forms Γ, is 2^{μ} times the weight of the single genus, or the two genera, of invariants I_1, I_2, . . . MI_{n-2}, which admit of representation by the forms Γ."

The method which this theorem may serve to indicate supplies a solution of the problem "to determine the weight of a given genus of definite forms of n indeterminates, and of the invariants I_1, I_2, . . . I_{n-1} . . ." We shall represent the weight of the given genus by the formula

$$W=\zeta_{2\nu+1} \times \Pi \,.\, \chi(\delta) \times B_{2\nu+1} \times \prod_{s=1}^{s=2\nu} I_s^{\frac{1}{2}s(n-s)}$$

when n is uneven and equal to $2\nu+1$, and by the formula

$$W=\zeta_{2\nu} \times \Pi \,.\, \chi(\delta) \times B_{2\nu} \times \prod_{s=1}^{s=2\nu-1} I_s^{\frac{1}{2}s(n-s)} \times \frac{1}{\pi^{\nu}} \sum_{1}^{\infty} \left(\frac{D}{m}\right) \frac{1}{m^{\nu}}$$

when n is even and equal to 2ν ; and we shall consider separately the factors of which these formulæ are composed.

(i) In the infinite series $\frac{1}{\pi^{\nu}} \sum_{1}^{\infty} \left(\frac{D}{m}\right) \frac{1}{m^{\nu}}$ (which enters into the expression of W only when the number of indeterminates is even) D still represents the product $(-1)^{\nu} I_1 \times I_3 \times \ldots \times I_{2\nu-1}$, and the summation extends to all uneven values of m, which are prime to D, from 1 to ∞. The sum of this infinite series can in every case be obtained in a finite form by the methods employed by Dirichlet (in the 21st volume of Crelle's Journal) and by Cauchy (in the 17th volume of the Mémoires de l'Académie des Sciences, p. 679). As the result of the summation does not seem to have been given, we shall present it here in one of many various forms which it may assume. Let D_1 represent the quotient obtained by dividing D by its greatest square divisor; let q be any uneven prime dividing D, but not D_1, and let $V = \frac{1}{\pi^{\nu}} \sum_1^{\infty} \left(\frac{D_1}{m}\right) \frac{1}{m^{\nu}}$, the sign of summation extending to all values of m prime to $2D_1$; we then have the equation

$$\frac{1}{\pi^{\nu}} \sum_1^{\infty} \left(\frac{D}{m}\right) \frac{1}{m^{\nu}} = \Pi \left[1 - \left(\frac{D_1}{q}\right) \frac{1}{q^2}\right] \times V.$$

To obtain the value of V, let Δ represent the positive value of D_1, so that $\Delta = D_1$ when ν is even, and $\Delta = -D_1$ when ν is uneven. Also let

$$F_{2\sigma}(x) = \frac{x^{2\sigma+1}}{2\sigma+1} - \tfrac{1}{2} x^{2\sigma} + \frac{\Pi \,.\, 2\sigma}{\Pi \,.\, 2\sigma - 1 \,.\, \Pi \,.\, 2} \beta_1 x_2^{\sigma-1}$$
$$- \frac{\Pi \,.\, 2\sigma}{\Pi \,.\, 2\sigma - 3 \,.\, \Pi \,.\, 4} \beta_3 x^{2\sigma-3} + \ldots + (-1)^{\sigma-1} \frac{\Pi \,.\, 2\sigma}{\Pi \,.\, 1 \,.\, \Pi \,.\, 2\sigma} \beta_{2\sigma-1} x,$$

$$F_{2\sigma-1}(x) = \frac{x^{2\sigma}}{2\sigma} - \tfrac{1}{2} x^{2\sigma-1} + \frac{\Pi \,.\, 2\sigma - 1}{\Pi \,.\, 2\sigma - 2 \,.\, \Pi \,.\, 2} \beta_1 x^{2\sigma-2}$$
$$- \frac{\Pi \,.\, 2\sigma - 1}{\Pi \,.\, 2\sigma - 4 \,.\, \Pi \,.\, 4} \beta_3 x^{2\sigma-4} + \ldots + (-1)^{\sigma} \frac{\Pi \,.\, 2\sigma - 1}{\Pi \,.\, 2 \,.\, \Pi \,.\, 2\sigma - 2} \beta_{2\sigma-3} x^2,$$

where $\beta_1, \beta_3, \ldots$ are the fractions of Bernoulli, so that $F^k(x)$ is the function which, when x is an integral number, is equivalent to the sum $\sum_{s=1}^{s=x-1} s^k$. Then, if $\epsilon = (-1)^{\frac{1}{2}(\nu+2)}$, or $(-1)^{\frac{1}{2}(\nu+1)}$, according as ν is even or uneven, the value of ϵV is

(1) when $D_1 \equiv 1$, mod 4,

$$\frac{2^{\nu-1}}{\Pi \,.\, \nu - 1} \times \left[1 - \left(\frac{2}{\Delta}\right) \frac{1}{2^{\nu}}\right] \times \frac{1}{\sqrt{\Delta}} \times \sum_{s=1}^{s=\Delta} \left(\frac{s}{\Delta}\right) F_{\nu-1}\left(\frac{s}{\Delta}\right),$$

(2) in every other case,

$$\frac{2^{\nu-1}}{\Pi \,.\, \nu - 1} \times \frac{1}{2\sqrt{\Delta}} \times \sum_{s=1}^{s=4\Delta} \left(\frac{D_1}{s}\right) F_{\nu-1}\left(\frac{s}{4\Delta}\right),$$

the summation $\sum_1^{\Delta}$ extending to every integral value of s inferior to Δ and prime to Δ, the summation $\sum_1^{4\Delta}$ extending to every integral value of s inferior to 4Δ, and prime to 4Δ. The formula (1) is inapplicable when $\Delta = D_1 = 1$;

but in this case ν is even, and the sum of the series $\frac{1}{\pi^\nu}\Sigma\frac{1}{m^\nu}$ is known.

(ii) The factor $\prod_{s=1}^{s=n-1} I_s^{\frac{1}{2}s(n-s)}$ requires no explanation; it is rational when n is uneven, and is a multiple of $\sqrt{\Delta}$ when n is even.

(iii) The factor B_n is determined by the equations

$$B_{2\nu}=\tfrac{1}{2}\beta_1\times\tfrac{1}{2}\beta_3\times\tfrac{1}{2}\beta_5\times\,..\,\tfrac{1}{2}\beta_{2\nu-3},$$

$$B_{2\nu+1}=\tfrac{1}{2}\beta_1\times\tfrac{1}{2}\beta_3\times\tfrac{1}{2}\beta_5\times\,..\,\tfrac{1}{2}\beta_{2\nu-1}\times\frac{1}{\Pi\,.\,\nu},$$

where β_1, β_3, ... are again the fractions of Bernoulli, so that $\beta_1=\frac{1}{6}$, $\beta_3=\frac{1}{30}$, etc.

(iv) The factors (i) and (ii) depend only on the invariants and on the number of the indeterminates, the factor (iii) only on the number of indeterminates. These factors are therefore the same for all genera of the invariants $I_1, I_2, \ldots I_{n-1}$. But the two remaining factors involve, or may involve, certain of the generic characters, and are therefore not always the same for all genera. In the factor $\Pi\,.\,\chi(\delta)$ the sign of multiplication extends to every uneven prime δ, dividing any one or more of the invariants I_1, I_2, ... I_{n-1}: it will suffice therefore to define the function $\chi(\delta)$, which depends on only one of those primes. Let i_1, i_2, ... be the indices of all the invariants which are divisible by δ; let these indices be arranged in order of magnitude, beginning with 0 and ending with n (because I_0 and I_n may be considered as divisible by δ). The positive differences $i_{s+1}-i_s$ we shall term *intervals*. By the *moiety* of any whole number a we understand $\frac{1}{2}\,a$ when a is even, $\frac{1}{2}\,(a-1)$ when a is uneven. Let κ_s be the moiety of the interval $i_{s+1}-i_s$; when that interval is even, let the barred symbol $\bar{\kappa}_s$ represent the product $(-1)^{\kappa_s} I_{1+i_s}\times I_{3+i_s}\times\ldots\times I_{-1+i_{s+1}}$; and let $\Gamma(\bar{\kappa}_s)=1+\left(\frac{\bar{\kappa}_s\times\theta_{i_s}\times\theta_{i_{s+1}}}{\delta}\right)\frac{1}{\delta^{\kappa_s}}$. Lastly, let $\Omega(h)$ represent the product $\prod_{s=1}^{s=h}\left(1-\frac{1}{\delta^{2s}}\right)$; let σ be the moiety of $n-1$, and μ the number of the invariants $I_1, I_2, \ldots I_{n-1}$, which are divisible by δ. Then $\chi(\delta)$ is the integral function of $\frac{1}{\delta}$, defined by the equation

$$\chi(\delta)=\frac{1}{2^\mu}\times\frac{\Omega(\sigma)}{\Pi\,.\,\Omega(\kappa_s)}\times\Pi\,.\,\Gamma(\bar{\kappa}_s)$$

when n is uneven, and by the equation

$$\chi(\delta)=\frac{1}{2_\mu}\times\frac{\Omega(\sigma)}{\Pi\,.\,\Omega(\kappa_s)}\times\Pi\,.\,\Gamma(\bar{\kappa}_s)\times\left[1-\left(\frac{D}{\delta}\right)\frac{1}{\delta^{\frac{1}{2}n}}\right]$$

when n is even. If D is divisible by δ, the symbol $\left(\frac{D}{\delta}\right)$ is zero. In both formulæ the sign of multiplication Π extends to every value of κ_s or $\bar{\kappa}_s$; the value $+1$ is, as before, to be attributed to the symbols θ_0 and θ_n.

(v) Each factor $\chi(\delta)$ of the product $\Pi\,.\,\chi(\delta)$ thus depends on an uneven prime δ dividing the invariants, on the indices of the invariants divisible by δ, on the principal generic characters with respect to δ, and on the quadratic characters with respect to δ of the invariants not divisible by δ. The remaining factor ζ_n may be said to depend on the relation of the concomitants and invariants to the prime 2 and its powers. The determination of this factor presents no theoretical difficulty; but on account of the multiplicity of the cases to be considered, we shall confine ourselves in this place to the two cases in which the invariants are all uneven.

(A) When the invariants are all uneven, and the given genus is of an uneven order, let Σ_n represent the unit $(-1)^h\,\psi(0, n-1)$, where $\psi(0, n-1)$ is the simultaneous character of the given genus, and h is determined by the equation

$$\begin{aligned}4h=(I_1-1)(I_2+1)+(I_2-1)(I_1I_3+1)+(I_1I_3-1)(I_2I_4+1)\\ +(I_2I_4-1)(I_1I_3I_5+1)+\ldots\ldots\\ +(\ldots\ldots I_{n-4}\,I_{n-2}-1)(\ldots\ldots I_{n-3}\,I_{n-1}+1).\end{aligned}$$

The value of ζ_n then is

(1) if $n=4\lambda$,

$$\frac{1}{2^{2\lambda-1}}[2^{2\lambda-1}+(-1)^{\lambda}\Sigma_n],\ \text{or } 1,$$

according as $D\equiv 1$, or $\equiv -1$, mod 4;

(2) if $n=4\lambda+2$,

$$\frac{1}{2^{2\lambda}}[2^{2\lambda}+(-1)^{\lambda}\Sigma_n],\ \text{or } 1,$$

according as $D\equiv 1$, or $\equiv -1$, mod 4;

(3) if $n=4\lambda+1$,

$$\frac{1}{2^{2\lambda}}[2^{2\lambda}+(-1)^{\lambda}\Sigma_n];$$

(4) if $n=4\lambda+3$,

$$\frac{1}{2^{2\lambda+1}}[2^{2\lambda+1}+(-1)^{\lambda+\frac{D-1}{2}}\Sigma_n].$$

(B) When the invariants are all uneven, and the given genus of an even order, so that $n=2\nu$ is even, the value of ζ_n is

$$\frac{1}{2^{n-2}}\times\frac{1}{1-\left(\frac{2}{D}\right)\frac{1}{2^{\nu}}}.$$

It is easy to apply these general formulæ to particular examples; but our imperfect knowledge of quadratic forms containing many indeterminates, renders it practically impossible to test the results by any independent process. The demonstrations are simple in principle, but require attention to a great number of details with respect to which it is very easy to fall into error. As soon as they can be put into a convenient form, they shall be submitted to the Royal Society.

Eisenstein has observed that, when the number of indeterminates does not surpass eight, there is but one class of quadratic forms of the discriminant 1, but that, when the number of indeterminates surpasses eight, there is always more than one such class. This observation is in accordance with our general formulæ, except that they imply the existence of an improperly primitive class of eight indeterminates and of the discriminant 1.

The theorems which have been given by Jacobi, Eisenstein, and recently in great profusion by M. Liouville, relating to the representation of numbers by four squares and other simple quadratic forms, appear to be deducible by a uniform method from the principles indicated in this paper. So also are the theorems relating to the representation of numbers by six and eight squares, which are implicitly contained in the developments given by Jacobi in the 'Fundamenta Nova.' As the series of theorems relating to the representation of numbers by sums of squares ceases, for the reason assigned by Eisenstein, when the number of squares surpasses eight, it is of some importance to complete it. The only cases which have not been fully considered are those of five and seven squares. The principal theorems relating to the case of five squares have indeed been given by Eisenstein (Crelle's Journal, vol. xxxv. p. 368); but he has considered only those numbers which are not divisible by any square. We shall here complete his enunciation of those theorems, and shall add the corresponding theorems for the case of seven squares. We attend only to primitive representations.

Let Δ represent a number not divisible by any square, Ω^2 an uneven square, a any exponent. By $\Phi_5(4^a\Omega^2\Delta)$, $\Phi_7(4^a\Omega^2\Delta)$, we denote the number of representations of $4^a\Omega^2\Delta$ by five and seven squares respectively; by $Q_5(4^a\Omega^2\Delta)$, $Q_7(4^a\Omega^2\Delta)$, we represent the products

$$5\times 2^{3a}\times\Omega^3\times\Pi\left[1-\left(\frac{\Delta}{q}\right)\frac{1}{q^2}\right]\times\frac{1}{\Delta},$$

$$7\times 2^{5a}\times\Omega^5\times\Pi\left[1-\left(\frac{-\Delta}{q}\right)\frac{1}{q^3}\right]\times\frac{1}{\Delta},$$

the sign of multiplication Π extending to every prime dividing Ω, but not dividing Δ; we then have the formulæ

(A) for five squares.

(1) If $\Delta\equiv 1$, mod 4,

$$\Phi_5(4^a\Omega^2\Delta)=Q_5(4^a\Omega^2\Delta)\times\eta\times\sum_1^{\Delta}\left(\frac{s}{\Delta}\right)s(s-\Delta),$$

where, if $\Delta\equiv 1$, mod 8, $\eta=12$; if $\Delta\equiv 5$, mod 8, $\eta=28$ or 20, according as $a=0$, or $a>0$. If, however, $\Delta=1$, we are to replace $\eta\times\Sigma$ by 2.

(2) In every other case,

$$\Phi_5(4^\alpha\Omega^2\Delta)=Q_5(4^\alpha\Omega^2\Delta)\times\eta\times\sum_1^{4\Delta}\left(\frac{\Delta}{s}\right)s(s-4D),$$

where $\eta=1$, or $\frac{1}{2}$, according as $\alpha=0$, or $\alpha>0$.

(B) for seven squares.

(1) If $\Delta\equiv 3$, mod 4,

$$\Phi_7(4^\alpha\Omega^2\Delta)=Q_7(4^\alpha\Omega^2)\Delta\times\eta\times\sum_1^{\Delta}\left(\frac{s}{\Delta}\right)s(s-\Delta)(2s-\Delta),$$

where $\eta=30$, if $\alpha=0$, $\Delta\equiv 3$, mod 8; $\eta=\frac{2}{3}\times 37$, if $\alpha=0$, $\Delta\equiv 7$, mod 8; $\eta=\frac{1}{3}\times 140$, if $\alpha>0$.

(2) In every other case,

$$\Phi^7(4^\alpha\Omega^2\Delta)=Q_7(4^\alpha\Omega^2\Delta\times\eta\times\sum_1^{4\Delta}\left(\frac{-\Delta}{s}\right)s(s-2\Delta)(s-4\Delta),$$

where $\eta=\frac{1}{3}$, or $\frac{5}{12}$, according as $\alpha=0$, or $\alpha>0$.

The sums $\sum_1^{\Delta}$, and $\sum_1^{4\Delta}$ in these formulæ are easily reduced (by distinguishing different linear forms of the number Δ) to others more readily calculated (see the note of Eisenstein, to which we have already referred); but in the present notice we have preferred to retain them in the form in which they first present themselves.

We shall conclude this paper by calling attention to a class of theorems which have a certain resemblance to the important results established by M. Kronecker for binary quadratic forms.

Let $\frac{1}{4}\frac{F_4(M)}{\Pi . 4}$ represent the weight of the quaternary classes of the invariants [1, 1, M]; $\frac{1}{4}\frac{F_6(M)}{\Pi . 6}$ the weight of the senary classes of the invariants [1, 1, 1, 1, M], then

$$F_4(M)+2F_4(M-1^2)+2F_4(M-2^2)+\ldots=\Sigma(-1)^{\frac{d+1}{2}}d^3,$$
$$F_6(2M)+2F_6(2M-1^2)+2F_6(2M-2^2)+\ldots=\Sigma d^3.$$

In the first of these formulæ M is any unevenly even number, or any number $\equiv 3$, mod 4; in the second M is any uneven number: the series in both are to be continued as long as the numbers $M-s^2$, or $2M-s^2$, are positive; d is any uneven divisor of M. The origin of these formulæ (which may serve as examples of many others) is exactly analogous to that which M. Kronecker has pointed out as characteristic of the more elementary of the two classes into which his formulæ are naturally divided. Whether, for forms of four and six indeterminates, similar formulæ exist comparable to the less elementary formulæ of M. Kronecker, and whether, for forms containing more than six indeterminates, such formulæ exist at all, are questions well worthy of the attention of arithmeticians.