In the numerous distillations which were performed for the purification of the hydrate of cresyl, some circumstances were observed which led to a suspicion that the body undergoes a change of composition, either through the distillation itself, or by some influences accompanying it. These circumstances were,—1st. A tarry residue, from a liquid which when introduced into the retort was perfectly colourless. 2nd. The formation of a small quantity of water in the commencement of such a distillation, though none was contained in the substance used. 3rd. The gradual lowering of the boiling-point of the whole liquid by a great number of distillations. These facts, taken in conjunction, naturally suggested that the oxygen of the air contained in the retort might act upon the substance, and thus gradually reduce it to hydrate of phenyl.

In order to test the correctness of this hypothesis, the atmospheric air was expelled from the distilling apparatus by dry hydrogen gas, and the distillation performed in a pure atmosphere of this gas. A great number of distillations performed in this manner were at exactly the same temperature, and all the other anomalies were simultaneously removed. It was however found that the liquid always boiled at a lower temperature in hydrogen than in atmospheric air, the difference being about 2° Cent., and this without any alteration of the pressure on the surface of the boiling liquid. A similar fact was noticed in the distillation of hydrate of phenyl, and also of some other liquids.

XVII. "On the Formation of Powers from Arithmetical Progressions." By C. Wheatstone, Esq., F.R.S. Received June 15, 1854.

The same sum n^a may be formed by the addition of an arithmetical progression of n terms in various ways. Hence we are enabled to construct a great variety of triangular arrangements of arithmetical progressions, the sums of which are the natural series of square, cube and other powers of numbers. Among these there are several which render evident some remarkable relations.

Each of the following triangles is formed of a series of arithmetical progressions, the number of terms increasing successively by unity.

The first term of an arithmetical progression of n terms having a common difference δ, and whose sum is n^a, is equal to

$$n^{(a-1)}+\frac{\delta}{2}(1-n).$$

§ 1. SQUARE NUMBERS.

$$\text{If } S=n^2, \text{ the first term } =n+\frac{\delta}{2}(1-n).$$

A.

Every square n^2 is the sum of an arithmetical progression of n terms, the first term of which is unity and the difference 2.

$$
\begin{array}{r@{\;\ldots\ldots\;}l}
1 & =1^2 \\
1+3 & =2^2 \\
1+3+5 & =3^2 \\
1+3+5+7 & =4^2 \\
1+3+5+7+9 & =5^2 \\
1+3+5+7+9+11 & =6^2 \\
1+3+5+7+9+11+13 & =7^2
\end{array}
$$

Thus, every square number is formed by the addition of a series of odd numbers commencing with unity; a result universally known.

The difference of any two squares is either an odd number, or the sum of consecutive odd numbers.

Each series may be resolved into two others consisting of alternate odd numbers, the respective sums of which are two adjacent triangular numbers, the addition of which it is well known forms a square. *Ex.*:

$$
\begin{array}{r}
1+5+9+13=28 \\
3+7+11=\underline{21} \\
49=7^2
\end{array}
$$

B.

Every square n^2 is the sum of an arithmetical progression of n terms, the first term of which is $\frac{n+1}{2}$, and the common difference 1.

$$
\begin{array}{rl}
1 \;\ldots\ldots\ldots\ldots\ldots\ldots & =1^2 \\
1\frac{1}{2}+2\frac{1}{2} \;\ldots\ldots\ldots\ldots\ldots & =2^2 \\
2+3+4 \;\ldots\ldots\ldots\ldots\ldots & =3^2 \\
2\frac{1}{2}+3\frac{1}{2}+4\frac{1}{2}+5\frac{1}{2} \;\ldots\ldots\ldots\ldots & =4^2 \\
3+4+5+6+7 \;\ldots\ldots\ldots\ldots & =5^2 \\
3\frac{1}{2}+4\frac{1}{2}+5\frac{1}{2}+6\frac{1}{2}+7\frac{1}{2}+8\frac{1}{2} \;\ldots\ldots\ldots & =6^2 \\
4+5+6+7+8+9+10 \;\ldots\ldots & =7^2
\end{array}
$$

This arrangement renders evident that every square of an odd number is the sum of as many consecutive natural numbers as the root has units.

Every square of an odd number is the difference between two triangular numbers the bases of which are respectively $(3n+1)$ and n. For, the sum of any series of natural numbers is the difference of two series of natural numbers commencing with unity; and since, as it is shown above, every square of an odd number is the sum of a series of natural numbers, it is also the difference between two triangular numbers.

It is also evident that series, the sums of which are squares of odd numbers, may be so taken that, when placed in succession, they will form an uninterrupted progression of natural numbers commencing with unity, the sum of which is a triangular number;

$$(1)+(2+3+4)+(5+6+7+8+9+10+11+12+13)\ldots\&c.=$$
$$(1^2+3^2+9^2+27^2\ldots\ldots+(3^n)^2)=$$

a triangular number the base of which is the series

$$(1+3+9+27\ldots\ldots+3^n).$$

§ 2. CUBE NUMBERS.

If $S=n^3$, the first term $=n^2+\frac{\delta}{2}(1-n)$.

C.

Every cube n^3 is the sum of an arithmetical progression of n terms, the first term of which is unity, and the difference $2(n+1)$.

$$
\begin{aligned}
1 \dots\dots\dots\dots &= 1^3 \\
1+7 \dots\dots\dots\dots &= 2^3 \\
1+9+17 \dots\dots\dots\dots &= 3^3 \\
1+11+21+31 \dots\dots\dots\dots &= 4^3 \\
1+13+25+37+49 \dots\dots\dots\dots &= 5^3 \\
1+15+29+43+57+71 \dots\dots\dots\dots &= 6^3 \\
1+17+33+49+65+81+97 \dots\dots\dots\dots &= 7^3
\end{aligned}
$$

D.

Every cube n^3 is the sum of an arithmetical progression of n terms, the first term of which is the root n, and the difference $2n$.

$$
\begin{aligned}
1 \dots\dots\dots\dots &= 1^3 \\
2+6 \dots\dots\dots\dots &= 2^3 \\
3+9+15 \dots\dots\dots\dots &= 3^3 \\
4+12+20+28 \dots\dots\dots\dots &= 4^3 \\
5+15+25+35+45 \dots\dots\dots\dots &= 5^3 \\
6+18+30+42+54+66 \dots\dots\dots\dots &= 6^3 \\
7+21+35+49+63+77+91 \dots\dots\dots\dots &= 7^3
\end{aligned}
$$

The last terms of these series are the alternate triangular numbers. If they be respectively divided by the first terms, the quotients will be the series of odd numbers.

E.

Every cube n^3 is the sum of an arithmetical progression of n terms, the first term of which is (n^2-n+1), and the difference 2.

$$
\begin{aligned}
1 \dots\dots\dots\dots &= 1^3 \\
3+5 \dots\dots\dots\dots &= 2^3 \\
7+9+11 \dots\dots\dots\dots &= 3^3 \\
13+15+17+19 \dots\dots\dots\dots &= 4^3 \\
21+23+25+27+29 \dots\dots\dots\dots &= 5^3 \\
31+33+35+37+39+41 \dots\dots\dots\dots &= 6^3 \\
43+45+47+49+51+53+55 \dots\dots\dots\dots &= 7^3
\end{aligned}
$$

This, it will be observed, is a triangular arrangement of the uneven numbers in their regular order.

Every cube is the sum of as many consecutive odd numbers as there are units in the root*.

The known theorem, that the sum of the cubes of any succession of the natural numbers commencing with unity is equal to the square of the sum of the roots, or, in other words, to the square of the corresponding triangular number, is an immediate consequence of the above.

$$(1^3+2^3+3^3+4^3\ldots\ldots+n^3)=(1+2+3+4\ldots+n)^2=\left(\frac{n^2+n}{2}\right)^2.$$

The sum of any series of odd numbers commencing with unity being equal to the square of the number of terms (A.), the sum of the numbers in any triangle formed as above is necessarily equal to the square of a triangular number. It is also easy to see that each cube is the difference between the squares of two consecutive triangular numbers; and, that the difference between the squares of any two triangular numbers whatever is the sum of consecutive cubes. The following equations have been found by ascertaining what differences of the squares of two triangular numbers are equal to single cubes:—

$$3^3+4^3+5^3=6^3$$
$$11^3+12^3+13^3+14^3=20^3.$$

F.

Every cube n^3 is the sum of an arithmetical progression of n terms, the first term of which is a triangular number $\frac{n^2+n}{2}$, and the difference $=n$.

$$1 \ldots\ldots\ldots\ldots\ldots\ldots =1^3$$
$$3+5 \ldots\ldots\ldots\ldots\ldots =2^3$$
$$6+9+12 \ldots\ldots\ldots\ldots =3^3$$
$$10+14+18+22 \ldots\ldots\ldots =4^3$$
$$15+20+25+30+35 \ldots\ldots =5^3$$
$$21+27+33+39+45+51 \ldots\ldots =6^3$$
$$28+35+42+49+56+63+70 \ldots =7^3$$

* Since the present note was communicated to the Royal Society, I have found that this relation has been already noticed by Count d'Adhémar (Comptes Rendus, tom. xxiii. p. 501). Cauchy observes, " quoiqu'elle puisse, comme on le voit, se deduire des principes déjà connus, toutefois, elle est assez curieuse et très simple."

Each number contained in this triangle is itself the sum of an arithmetical progression of n terms. Thus, taking the fifth row for example :—

$$\begin{array}{r}
1+2+3+4+5= 15\\
2+3+4+5+6= 20\\
3+4+5+6+7= 25\\
4+5+6+7+8= 30\\
5+6+7+8+9= \underline{35}\\
125=5^3
\end{array}$$

The sum of all the numbers contained in a square thus formed is equal to the cube of the number which occupies the upper right-hand and lower left-hand corners. The sum of the numbers in either of the diagonals is the corresponding square, and in the case of the odd numbers the sum of the middle horizontal or vertical line is also the square.

This last-mentioned relation was pointed out by Lichtenberg*, who stated the theorem thus :—If a be a whole number, and A be the sum of all the natural numbers from 1 to a, then :

$$a^3=A+(A+a)+(A+2a)+(A+3a)\ldots\ldots+(A+[a-1]a).$$

G.

Every cube n^3 above 1 is the sum of an arithmetical progression of n terms, the first term of which is $(n-2)^2$, and the difference $=8$.

$$\begin{array}{rl}
0+8 \quad \ldots\ldots\ldots\ldots\ldots & =2^3\\
1+9+17 \quad \ldots\ldots\ldots\ldots & =3^3\\
4+12+20+28 \quad \ldots\ldots\ldots & =4^3\\
9+17+25+33+41 \quad \ldots\ldots\ldots & =5^3\\
16+24+32+40+48+56 \quad \ldots\ldots & =6^3\\
25+33+41+49+57+65+73 \quad \ldots & =7^3
\end{array}$$

Each progression of this triangle, consisting of an uneven number of terms, contains two consecutive odd square numbers.

An *uninterrupted* arithmetical progression commencing with unity and proceeding by the constant addition of 8, arranged in a triangular form, presents some curious results. 1st. The first terms of

* G. C. Lichtenberg's Vermischte Schriften, Band ix. p. 359. Göttingen 1806.

each line are the squares of the odd numbers in their regular sequence. 2nd. The sum of all the numbers in any two adjacent lines is the cube of an odd number.

$$\begin{array}{r l}
1 \;\ldots\ldots\ldots\ldots & \} \ldots\ldots = 3^3 \\
9+17 \;\ldots\ldots\ldots & \} \ldots\ldots = 5^3 \\
25+33+41 \;\ldots\ldots & \} \ldots\ldots = 7^3 \\
49+57+65+73 \;\ldots\ldots & \} \ldots\ldots = 9^3 \\
81+89+97+105+113 \;\ldots & \} \ldots\ldots = 11^3 \\
121+129+137+145+153+161 &
\end{array}$$

It is evident from the preceding arrangement that

$$(2n+1)^2 = 1+8\left(\frac{n^2+n}{2}\right).$$

Thus any triangular number multiplied by 8 with 1 added is equal to the square of an odd number; or, any square of an uneven number minus 1 is divisible by 8, and the quotient is a triangular number.

§ 3.

Of the higher powers I will confine myself to one example.

H.

Every fourth power n^4 is the sum of an arithmetical progression of n terms, the first term of which is n^2, and the difference $2n^2$.

$$\begin{array}{r l}
1 \;\ldots\ldots\ldots\ldots\ldots\ldots & = 1^4 \\
4+12 \;\ldots\ldots\ldots\ldots\ldots & = 2^4 \\
9+27+45 \;\ldots\ldots\ldots\ldots & = 3^4 \\
16+48+80+112 \;\ldots\ldots\ldots & = 4^4 \\
25+75+125+150+225 \;\ldots\ldots & = 5^4 \\
36+108+180+252+324+396 \;\ldots & = 6^4
\end{array}$$

This triangle consists of the progressions in (D.) multiplied respectively by n, or of those in (A.) multiplied by n^2.