VOLUME XII NOVEMBER, 1918 Nos. 1 & 2

BIOMETRIKA

ON THE STANDARD DEVIATIONS OF ADJUSTED AND INTERPOLATED VALUES OF AN OBSERVED *POLYNOMIAL FUNCTION* AND ITS CONSTANTS AND THE GUIDANCE THEY GIVE TOWARDS A PROPER CHOICE OF THE DISTRIBUTION OF OBSERVATIONS.

BY KIRSTINE SMITH, Copenhagen.

CONTENTS

INTRODUCTION

In all sorts of experiments which are not simple repetitions but have at least one varying essential circumstance or indefinite variate the experimentalist is confronted with a choice in regard to the values of that variate. If the experiments be quite simple the question may be without great importance; but when their requirements as to time or expenditure come into account the problem arises, how the observations should be chosen in order that a limited number of them may give the maximum amount of knowledge. It clearly depends upon the relationship between the observed quantity, which we shall name the primary variate, and its essential circumstances, the secondary variates, and upon the variation of the errors of the observations.

When we deal with, for example, a linear function which it is possible to observe with the same accuracy for all values of the indefinite variate we should not hesitate to put the observations in two equally big groups as far apart from each other as feasible. But if the standard deviation of the observations be a function of the indefinite variate and increases with the distance from the middle of the range, where is then the point in which the advantage of removing the two groups of observations from each other just counterbalances the disadvantages of increasing the error of observations? The problem becomes very complicated for functions of higher degrees.

We shall in this memoir try to contribute to the solution in the case of polynomial functions by examining the standard deviations of the adjusted and more especially the interpolated values of such functions for different distributions of observations. Those values inside the working range of observations may be considered the sum of knowledge acquired by the experiments. The adjusted values outside the working range may probably in exceptional cases be of interest, but as only by some other type of experiment we can make sure that the form of function holds outside the range they are in ordinary cases without great value. We shall therefore aim at finding the distribution of observations which within the selected range gives the most satisfactory standard deviations of the adjusted values of the function.

To consider the standard deviations satisfactory we must of course demand that they shall be as small as possible, and since a greater accuracy in one part may be expected to be accompanied by a smaller accuracy in another part we want them in addition to be as near constant as possible. In other words the curve of standard deviation with the lowest possible maximum value within the working range of observations is what we shall attempt to find. It appears that the distribution of observations which fulfils this demand consists of specially placed groups in number just sufficient to determine the constants of the function. We shall accordingly pay attention also to the desirability usually present of ascertaining the form of function by means of the observations. As might be expected we find that the standard deviations obtained from a uniform continuous distribution of observations increase towards the ends of the range. By choosing a uniform continuous distribution with additional clusters at the ends of the range we shall try to find a compromise between the two desiderata of a low maximum of standard deviation and of a uniform distribution.

The indefinite variate is supposed to have a vanishing error of observation compared with that of the principal variate. This error may be constant or varying with the indefinite variate, but in either case it is supposed to follow the typical law so closely that the method of least squares may satisfactorily be applied to the observations. After having found first the most advantageous distributions for observations of functions up to the sixth degree with constant standard deviations we examine the case for observations of functions of the first and of the second degree which have standard deviations of the form $\sigma(1+\alpha x)$ and $\sigma(1+\alpha x^2)$. If it is profitable to use the whole of the working range the latter distributions are practically found from the former by multiplying their frequencies by the squared standard deviations of the observations at the corresponding place. But in cases where extrapolation is of advantage, and the whole range therefore not to be used, the law of the frequencies has to be examined anew.

In Section VIII we find for the same two cases of varying error of observation the distributions which make each single constant of a function of the first and of the second degree a minimum.

I. *Adjustment of a polynomial function of one variable; general distribution of observations.*

(1) Let $y_1, y_2 \ldots\ldots y_p \ldots\ldots y_N$ be N observations of a function of nth degree taken at the points $x_1, x_2 \ldots\ldots x_p \ldots\ldots x_N$,

$$y = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let us assume that from earlier experience we know the standard deviation of an observation of y to be $\sigma\sqrt{f(x)}$. The method of least squares will then give us the following system of normal equations in which the sums are to be extended over all the observations:

$$\left.\begin{aligned}
S\left\{\frac{y_p}{f(x_p)}\right\} &= S\left\{\frac{1}{f(x_p)}\right\} a_0 + S\left\{\frac{x_p}{f(x_p)}\right\} a_1 + S\left\{\frac{x_p^2}{f(x_p)}\right\} a_2 + \ldots\ldots + S\left\{\frac{x_p^n}{f(x_p)}\right\} a_n \\
S\left\{\frac{y_p x_p}{f(x_p)}\right\} &= S\left\{\frac{x_p}{f(x_p)}\right\} a_0 + S\left\{\frac{x_p^2}{f(x_p)}\right\} a_1 + S\left\{\frac{x_p^3}{f(x_p)}\right\} a_2 + \ldots\ldots + S\left\{\frac{x_p^{n+1}}{f(x_p)}\right\} a_n \\
S\left\{\frac{y_p x_p^2}{f(x_p)}\right\} &= S\left\{\frac{x_p^2}{f(x_p)}\right\} a_0 + S\left\{\frac{x_p^3}{f(x_p)}\right\} a_1 + S\left\{\frac{x_p^4}{f(x_p)}\right\} a_2 + \ldots\ldots + S\left\{\frac{x_p^{n+2}}{f(x_p)}\right\} a_n \\
&\vdots \\
S\left\{\frac{y_p x_p^n}{f(x_p)}\right\} &= S\left\{\frac{x_p^n}{f(x_p)}\right\} a_0 + S\left\{\frac{x_p^{n+1}}{f(x_p)}\right\} a_1 + S\left\{\frac{x_p^{n+2}}{f(x_p)}\right\} a_2 + \ldots\ldots + S\left\{\frac{x_p^{2n}}{f(x_p)}\right\} a_n
\end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

If $f(x)$ is 1 the sums are the moment coefficients of the places of observations multiplied by N, and in the general case we shall for brevity put

$$S\left\{\frac{x_p^r}{f(x_p)}\right\} = N \cdot m_r.$$

By elimination of the a's between (1) and (2) we find

$$\begin{vmatrix}
N \cdot y & 1 & x & x^2 & \ldots\ldots & x^n \\
S\left\{\frac{y_p}{f(x_p)}\right\} & m_0 & m_1 & m_2 & \ldots\ldots & m_n \\
S\left\{\frac{y_p x_p}{f(x_p)}\right\} & m_1 & m_2 & m_3 & \ldots\ldots & m_{n+1} \\
S\left\{\frac{y_p x_p^2}{f(x_p)}\right\} & m_2 & m_3 & m_4 & \ldots\ldots & m_{n+2} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
S\left\{\frac{y_p x_p^n}{f(x_p)}\right\} & m_n & m_{n+1} & m_{n+2} & \ldots\ldots & m_{2n}
\end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

which determines the adjusted y corresponding to the variable x.

(2) To find the standard deviation σ_{y_r} of an adjusted y_r it will be easiest to start from the equations (2). If the first be multiplied by α_0, the second by α_1 and so on before summing, and if we choose $\alpha_0, \alpha_1 \ldots\ldots \alpha_n$ so that

$$\left.\begin{array}{l} \alpha_0 m_0 + \alpha_1 m_1 + \alpha_2 m_2 + \ldots\ldots + \alpha_n m_n = 1 \\ \alpha_0 m_1 + \alpha_1 m_2 + \alpha_2 m_3 + \ldots\ldots + \alpha_n m_{n+1} = x_r \\ \alpha_0 m_2 + \alpha_1 m_3 + \alpha_2 m_4 + \ldots\ldots + \alpha_n m_{n+2} = x_r^2 \\ \vdots \\ \alpha_0 m_n + \alpha_1 m_{n+1} + \alpha_2 m_{n+2} + \ldots\ldots + \alpha_n m_{2n} = x_r^n \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(4),$$

we find that $$y_r = \frac{1}{N} S \left\{ \frac{y_p}{f(x_p)} [\alpha_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \ldots\ldots + \alpha_n x_p^n] \right\},$$

and therefore $$\sigma_{y_r}^2 = \frac{\sigma^2}{N^2} S \left\{ \frac{1}{f(x_p)} [\alpha_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \ldots\ldots + \alpha_n x_p^n]^2 \right\}.$$

By multiplying out the square this may be written

$$\begin{aligned} \sigma_{y_r}^2 = \frac{\sigma^2}{N} \{ & \alpha_0 [\alpha_0 m_0 + \alpha_1 m_1 + \alpha_2 m_2 + \ldots\ldots + \alpha_n m_n] \\ & + \alpha_1 [\alpha_0 m_1 + \alpha_1 m_2 + \alpha_2 m_3 + \ldots\ldots + \alpha_n m_{n+1}] \\ & + \alpha_2 [\alpha_0 m_2 + \alpha_1 m_3 + \alpha_2 m_4 + \ldots\ldots + \alpha_n m_{n+2}] \\ & \vdots \\ & + \alpha_n [\alpha_0 m_n + \alpha_1 m_{n+1} + \alpha_2 m_{n+2} + \ldots\ldots + \alpha_n m_{2n}] \}, \end{aligned}$$

or applying (4) $$\sigma_{y_r}^2 = \frac{\sigma^2}{N} (\alpha_0 + \alpha_1 x_r + \alpha_2 x_r^2 + \ldots\ldots + \alpha_n x_r^n) \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Hence $\sigma_{y_r}^2$ is found by elimination of the α's between (4) and (5), which results in

$$\begin{vmatrix} \sigma_{y_r}^2 \cdot \dfrac{N}{\sigma^2} & 1 & x_r & x_r^2 & \ldots\ldots & x_r^n \\ 1 & m_0 & m_1 & m_2 & \ldots\ldots & m_n \\ x_r & m_1 & m_2 & m_3 & \ldots\ldots & m_{n+1} \\ x_r^2 & m_2 & m_3 & m_4 & \ldots\ldots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_r^n & m_n & m_{n+1} & m_{n+2} & \ldots\ldots & m_{2n} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This determinant is of fundamental importance for all the following work and it will be useful at once to examine it more closely.

(3) First however it may be pointed out that the standard deviation of any other linear function $\phi = b_0 a_0 + b_1 a_1 + b_2 a_2 + \ldots\ldots b_n a_n$ of the constants of the function y may be determined in quite the same way by

$$\begin{vmatrix} \sigma_{\phi}^2 \cdot \dfrac{N}{\sigma^2} & b_0 & b_1 & b_2 & \ldots\ldots & b_n \\ b_0 & m_0 & m_1 & m_2 & \ldots\ldots & m_n \\ b_1 & m_1 & m_2 & m_3 & \ldots\ldots & m_{n+1} \\ b_2 & m_2 & m_3 & m_4 & \ldots\ldots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b_n & m_n & m_{n+1} & m_{n+2} & \ldots\ldots & m_{2n} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

In particular $\sigma^2_{a_p}$ is found from

$$\begin{vmatrix} \sigma^2_{a_p}\cdot\frac{N}{\sigma^2} & 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ 0 & m_0 & m_1 & m_2 & \dots & m_p & \dots & m_n \\ 0 & m_1 & m_2 & m_3 & \dots & m_{p+1} & \dots & m_{n+1} \\ 0 & m_2 & m_3 & m_4 & \dots & m_{p+2} & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & m_p & m_{p+1} & m_{p+2} & \dots & m_{2p} & \dots & m_{n+p} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & m_n & m_{n+1} & m_{n+2} & \dots & m_{p+n} & \dots & m_{2n} \end{vmatrix} = 0 \quad \dots\dots\dots\dots(8).$$

(4) Let us call a determinant, identical with that of (6) except that it has 0 instead of the element $\sigma^2_{y_r}\cdot\frac{N}{\sigma^2}$, Δ, let $\Delta_{r,s}$ be its minor not containing the rth row and sth column, again let $\Delta_{r,s,p,q}$ be the minor of this not containing the pth row and the qth column of Δ. We then find from (8)

$$\sigma^2_{a_p} = \frac{\sigma^2}{N}\frac{\Delta_{p+2,\,p+2,\,1,\,1}}{\Delta_{1,1}} \dots\dots\dots\dots(9).$$

With this notation we obtain from (6)

$$\sigma^2_{y_r} = \frac{\sigma^2}{N}\left(-\frac{\Delta}{\Delta_{1,1}}\right) \dots\dots\dots\dots(10).$$

In the following we shall drop the index r and indicate by ${}_n\sigma_y$ the standard deviation of a y adjusted by means of a function of the nth degree.

If we were dealing with a function of $(n-1)$st degree and retained the observations distributed as before we should find

$${}_{n-1}\sigma^2_y = \frac{\sigma^2}{N}\left(-\frac{\Delta_{n+2,\,n+2}}{\Delta_{n+2,\,n+2,\,1,\,1}}\right),$$

and therefore

$${}_n\sigma^2_y - {}_{n-1}\sigma^2_y = \frac{\sigma^2}{N}\cdot\frac{\Delta_{1,1}\cdot\Delta_{n+2,\,n+2} - \Delta\cdot\Delta_{n+2,\,n+2,\,1,\,1}}{\Delta_{1,1}\cdot\Delta_{n+2,\,n+2,\,1,\,1}},$$

but Δ is orthosymmetrical and therefore the numerator of this fraction equals $\Delta^2_{n+2,\,1}$, and

$${}_n\sigma^2_y - {}_{n-1}\sigma^2_y = \frac{\sigma^2}{N}\cdot\frac{\Delta^2_{n+2,\,1}}{\Delta_{1,1}\cdot\Delta_{n+2,\,n+2,\,1,\,1}}.$$

It was shown before that

$$\sigma^2_{a_n} = \frac{\sigma^2}{N}\cdot\frac{\Delta_{n+2,\,n+2,\,1,\,1}}{\Delta_{1,1}},$$

hence $\Delta_{1,1}$ and $\Delta_{n+2,n+2,1,1}$ have the same sign, and ${}_n\sigma^2_y - {}_{n-1}\sigma^2_y$ is therefore a square of a function of x. In the same way we can express ${}_{n-1}\sigma^2_y - {}_{n-2}\sigma^2_y$ and thus further down all the differences till ${}_0\sigma^2_y = \frac{\sigma^2}{N}\cdot\frac{1}{m_0}$ by which means ${}_n\sigma^2_y$ is developed in a sum of squares and takes the shape

$$ {}_n\sigma_y^2 = \frac{\sigma^2}{N}\left\{\frac{1}{m_0} + \frac{\begin{vmatrix} 1 & m_0 \\ x & m_1 \end{vmatrix}^2}{m_0 \cdot \begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix}} + \frac{\begin{vmatrix} 1 & m_0 & m_1 \\ x & m_1 & m_2 \\ x^2 & m_2 & m_3 \end{vmatrix}^2}{\begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix} \cdot \begin{vmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix}} + \ldots\ldots \right. $$

$$ \left. + \frac{\begin{vmatrix} 1 & m_0 & m_1 & \ldots\ldots & m_{n-1} \\ x & m_1 & m_2 & \ldots\ldots & m_n \\ x^2 & m_2 & m_3 & \ldots\ldots & m_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^n & m_n & m_{n+1} & \ldots\ldots & m_{2n-1} \end{vmatrix}^2}{\begin{vmatrix} m_0 & m_1 & \ldots\ldots & m_{n-1} \\ m_1 & m_2 & \ldots\ldots & m_n \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \ldots\ldots & m_{2n-2} \end{vmatrix} \cdot \begin{vmatrix} m_0 & m_1 & \ldots\ldots & m_n \\ m_1 & m_2 & \ldots\ldots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_n & m_{n+1} & \ldots\ldots & m_{2n} \end{vmatrix}} \right\} \quad \ldots\ldots\ldots\ldots(11). $$

It will be seen that *the squared standard deviation of an adjusted y is a function of the 2nth degree of x.* The coefficient of x^{2n} is the square of $\frac{\Delta_{n+2,n+2,1,1}}{\Delta_{1,1}}$ which, as was just seen, is the factor with which $\frac{\sigma^2}{N}$ should be multiplied in order to give $\sigma^2_{a_n}$, it is therefore positive and *can never vanish.*

(5) If all the m's with odd indices are zero it is seen from (6) that σ_y^2 is a function of x^2. This is, at least in theory, a natural thing to aim at, since our general purpose is to find a curve for σ_y^2 giving as nearly as possible a constant value for σ_y^2 throughout the range.

Rearranging the order of rows and columns in (6) we get, when all $m_{2q+1} = 0$ and $n = 2p$,

$$ \begin{vmatrix} {}_{2p}\sigma_y^2 \cdot \frac{N}{\sigma^2} & 1 & x^2 & x^4 & \ldots\ldots & x^{2p} & x & x^3 & x^5 & \ldots\ldots & x^{2p-1} \\ 1 & m_0 & m_2 & m_4 & \ldots\ldots & m_{2p} & 0 & 0 & 0 & \ldots\ldots & 0 \\ x^2 & m_2 & m_4 & m_6 & \ldots\ldots & m_{2p+2} & 0 & 0 & 0 & \ldots\ldots & 0 \\ x^4 & m_4 & m_6 & m_8 & \ldots\ldots & m_{2p+4} & 0 & 0 & 0 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & m_{2p} & m_{2p+2} & m_{2p+4} & \ldots\ldots & m_{4p} & 0 & 0 & 0 & \ldots\ldots & 0 \\ x & 0 & 0 & 0 & \ldots\ldots & 0 & m_2 & m_4 & m_6 & \ldots\ldots & m_{2p} \\ x^3 & 0 & 0 & 0 & \ldots\ldots & 0 & m_4 & m_6 & m_8 & \ldots\ldots & m_{2p+2} \\ x^5 & 0 & 0 & 0 & \ldots\ldots & 0 & m_6 & m_8 & m_{10} & \ldots\ldots & m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{2p-1} & 0 & 0 & 0 & \ldots\ldots & 0 & m_{2p} & m_{2p+2} & m_{2p+4} & \ldots\ldots & m_{4p-2} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(12), $$

from which we find

$$
{}_{2p}\sigma_y^2 = -\frac{\sigma^2}{N}\left\{\frac{\begin{vmatrix} 0 & 1 & x^2 & x^4 & \cdots\cdots & x^{2p} \\ 1 & m_0 & m_2 & m_4 & \cdots\cdots & m_{2p} \\ x^2 & m_2 & m_4 & m_6 & \cdots\cdots & m_{2p+2} \\ x^4 & m_4 & m_6 & m_8 & \cdots\cdots & m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & m_{2p} & m_{2p+2} & m_{2p+4} & \cdots\cdots & m_{4p} \end{vmatrix}}{\begin{vmatrix} m_0 & m_2 & m_4 & \cdots\cdots & m_{2p} \\ m_2 & m_4 & m_6 & \cdots\cdots & m_{2p+2} \\ m_4 & m_6 & m_8 & \cdots\cdots & m_{2p+4} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{2p} & m_{2p+2} & m_{2p+4} & \cdots\cdots & m_{4p} \end{vmatrix}} + x^2\frac{\begin{vmatrix} 0 & 1 & x^2 & x^4 & \cdots\cdots & x^{2p-2} \\ 1 & m_2 & m_4 & m_6 & \cdots\cdots & m_{2p} \\ x^2 & m_4 & m_6 & m_8 & \cdots\cdots & m_{2p+2} \\ x^4 & m_6 & m_8 & m_{10} & \cdots\cdots & m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & m_{2p} & m_{2p+2} & m_{2p+4} & \cdots\cdots & m_{4p-2} \end{vmatrix}}{\begin{vmatrix} m_2 & m_4 & m_6 & \cdots\cdots & m_{2p} \\ m_4 & m_6 & m_8 & \cdots\cdots & m_{2p+2} \\ m_6 & m_8 & m_{10} & \cdots\cdots & m_{2p+4} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{2p} & m_{2p+2} & m_{2p+4} & \cdots\cdots & m_{4p-2} \end{vmatrix}}\right\} \quad \ldots\ldots\ldots(13).
$$

For a function of the degree $2p-1$ we get the same determinant as in (12) except that it does not contain the row and column in which x^{2p} is found. Hence we find

$$
{}_{2p-1}\sigma_y^2 = -\frac{\sigma^2}{N}\left\{\frac{\begin{vmatrix} 0 & 1 & x^2 & x^4 & \cdots\cdots & x^{2p-2} \\ 1 & m_0 & m_2 & m_4 & \cdots\cdots & m_{2p-2} \\ x^2 & m_2 & m_4 & m_6 & \cdots\cdots & m_{2p} \\ x^4 & m_4 & m_6 & m_8 & \cdots\cdots & m_{2p+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & m_{2p-2} & m_{2p} & m_{2p+2} & \cdots\cdots & m_{4p-4} \end{vmatrix}}{\begin{vmatrix} m_0 & m_2 & m_4 & \cdots\cdots & m_{2p-2} \\ m_2 & m_4 & m_6 & \cdots\cdots & m_{2p} \\ m_4 & m_6 & m_8 & \cdots\cdots & m_{2p+2} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{2p-2} & m_{2p} & m_{2p+2} & \cdots\cdots & m_{4p-4} \end{vmatrix}}\right.
$$

$$+\,x^2\,\frac{\begin{vmatrix} 0 & 1 & x^2 & x^4 & \cdots & x^{2p-2} \\ 1 & m_2 & m_4 & m_6 & \cdots & m_{2p} \\ x^2 & m_4 & m_6 & m_8 & \cdots & m_{2p+2} \\ x^4 & m_6 & m_8 & m_{10} & \cdots & m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & m_{2p} & m_{2p+2} & m_{2p+4} & \cdots & m_{4p-2} \end{vmatrix}}{\begin{vmatrix} m_2 & m_4 & m_6 & \cdots & m_{2p} \\ m_4 & m_6 & m_8 & \cdots & m_{2p+2} \\ m_6 & m_8 & m_{10} & \cdots & m_{2p+4} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{2p} & m_{2p+2} & m_{2p+4} & \cdots & m_{4p-2} \end{vmatrix}}\Bigg\} \quad \cdots\cdots\cdots(14).$$

(6) The last two determinant ratios of (13) and (14) are identical, and when the numerator of the first fraction of (13) is indicated by δ we therefore find

$${}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 = \frac{\sigma^2}{N}\left(\frac{\delta_{p+2,\,p+2}}{\delta_{1,\,1,\,p+2,\,p+2}} - \frac{\delta}{\delta_{1,\,1}}\right),$$

or as δ is orthosymmetrical and therefore

$$\delta_{1,\,1}\,.\,\delta_{p+2,\,p+2} - \delta\,.\,\delta_{1,\,1,\,p+2,\,p+2} = \delta^2_{p+2,\,1},$$

$${}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 = \frac{\sigma^2}{N}\cdot\frac{\delta^2_{p+2,\,1}}{\delta_{1,\,1,}\delta_{p+2,\,p+2,\,1,\,1}}.$$

Comparing ${}_{2p-2}\sigma_y^2$ and ${}_{2p-1}\sigma_y^2$ we see that they have the first determinant ratio in common and that when γ stands for the numerator of the other fraction of ${}_{2p-1}\sigma_y^2$ we have

$${}_{2p-1}\sigma_y^2 - {}_{2p-2}\sigma_y^2 = \frac{\sigma^2}{N}\,x^2\left(\frac{\gamma_{p+1,\,p+1}}{\gamma_{p+1,\,p+1,\,1,\,1}} - \frac{\gamma}{\gamma_{1,\,1}}\right),$$

or again, since γ is orthosymmetrical,

$${}_{2p-1}\sigma_y^2 - {}_{2p-2}\sigma_y^2 = \frac{\sigma^2}{N}\,x^2\,\frac{\gamma^2_{p+1,\,1}}{\gamma_{1,\,1}\cdot\gamma_{p+1,\,p+1,\,1,\,1}}.$$

The general formula (11) hence for any $m_{2q+1} = 0$ takes the shape

$${}_{2p}\sigma_y^2 = \frac{\sigma^2}{N}\left\{\frac{1}{m_0} + x^2\frac{1}{m_2} + \frac{\begin{vmatrix} 1 & m_0 \\ x^2 & m_2 \end{vmatrix}^2}{m_0\cdot\begin{vmatrix} m_0 & m_2 \\ m_2 & m_4 \end{vmatrix}} + x^2\frac{\begin{vmatrix} 1 & m_2 \\ x^2 & m_4 \end{vmatrix}^2}{m_2\cdot\begin{vmatrix} m_2 & m_4 \\ m_4 & m_6 \end{vmatrix}} + \cdots\cdots\right.$$

$$+\,x^2\,\frac{\begin{vmatrix} 1 & m_2 & m_4 & \cdots & m_{2p-2} \\ x^2 & m_4 & m_6 & \cdots & m_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & m_{2p} & m_{2p+2} & \cdots & m_{4p-4} \end{vmatrix}^2}{\begin{vmatrix} m_2 & m_4 & \cdots & m_{2p-2} \\ m_4 & m_6 & \cdots & m_{2p} \\ \vdots & \vdots & & \vdots \\ m_{2p-2} & m_{2p} & \cdots & m_{4p-6} \end{vmatrix}\cdot\begin{vmatrix} m_2 & m_4 & \cdots & m_{2p} \\ m_4 & m_6 & \cdots & m_{2p+2} \\ \vdots & \vdots & & \vdots \\ m_{2p} & m_{2p+2} & \cdots & m_{4p-2} \end{vmatrix}}$$

$$+\frac{\begin{vmatrix} 1 & m_0 & m_2 & \cdots\cdots & m_{2p-2} \\ x^2 & m_2 & m_4 & \cdots\cdots & m_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & m_{2p} & m_{2p+2} & \cdots\cdots & m_{4p-2} \end{vmatrix}^2}{\begin{vmatrix} m_0 & m_2 & \cdots\cdots & m_{2p-2} \\ m_2 & m_4 & \cdots\cdots & m_{2p} \\ \vdots & \vdots & & \vdots \\ m_{2p-2} & m_{2p} & \cdots\cdots & m_{4p-4} \end{vmatrix} \cdot \begin{vmatrix} m_0 & m_2 & \cdots\cdots & m_{2p} \\ m_2 & m_4 & \cdots\cdots & m_{2p+2} \\ \vdots & \vdots & & \vdots \\ m_{2p} & m_{2p+2} & \cdots\cdots & m_{4p} \end{vmatrix}}\Bigg\} \quad \ldots\ldots\ldots(15).$$

(7) Before leaving the general case and treating special distributions of observations three auxiliary propositions shall be proved. *We shall first prove that the curve of* ${}_n\sigma_y^2$ *can never be entirely below* $\frac{\sigma^2}{N} \cdot \frac{n+1}{m_0}$. With that purpose ${}_n\sigma_y^2$ will be summed over all the places of observation with the weight $\frac{1}{f(x)}$, i.e. for a continuous distribution of observations, the expression $\frac{\psi(x)}{f(x)}\, {}_n\sigma_y^2\, dx$, where $\psi(x)$ is the number of observations, will be integrated over the range of observations.

Looking first at the numerator of the last term of (11) we find that it can be expanded into

$$(-1)^{n+1}\left\{\begin{vmatrix} 1 & m_0 & m_1 & \cdots\cdots & m_{n-1} \\ x & m_1 & m_2 & \cdots\cdots & m_n \\ x^2 & m_2 & m_3 & \cdots\cdots & m_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^n & m_n & m_{n+1} & \cdots\cdots & m_{2n-1} \end{vmatrix} \times \Delta_{1,\, n+2,\, 2,\, 1} + \begin{vmatrix} x & m_0 & m_1 & \cdots\cdots & m_{n-1} \\ x^2 & m_1 & m_2 & \cdots\cdots & m_n \\ x^3 & m_2 & m_3 & \cdots\cdots & m_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{n+1} & m_n & m_{n+1} & \cdots\cdots & m_{2n-1} \end{vmatrix}\right.$$

$$\left. \times \Delta_{1,\, n+2,\, 3,\, 1} + \cdots\cdots + \begin{vmatrix} x^n & m_0 & m_1 & \cdots\cdots & m_{n-1} \\ x^{n+1} & m_1 & m_2 & \cdots\cdots & m_n \\ x^{n+2} & m_2 & m_3 & \cdots\cdots & m_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2n} & m_n & m_{n+1} & \cdots\cdots & m_{2n-1} \end{vmatrix} \times \Delta_{1,\, n+2,\, n+2,\, 1}\right\}.$$

Now $\int \frac{\psi(x)\, x^r}{f(x)}\, dx$ integrated over all the observations is what we have called $N \, . \, m_r$. When integrating the determinants we therefore find that the first n of them will vanish, two of their columns consisting of proportional elements, whereas the integral of the last determinant is

$$N \begin{vmatrix} m_n & m_0 & m_1 & \cdots\cdots & m_{n-1} \\ m_{n+1} & m_1 & m_2 & \cdots\cdots & m_n \\ m_{n+2} & m_2 & m_3 & \cdots\cdots & m_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{2n} & m_n & m_{n+1} & \cdots\cdots & m_{2n-1} \end{vmatrix} = (-1)^n N \Delta_{1,\,1}.$$

As $\Delta_{1, n+2, n+2, 1} = -\Delta_{n+2, n+2, 1, 1}$, the integral of the last term of (11) equals N. The integration of the other terms, including the first, gives the same result so that

$$\int {}_n\sigma_y^2 \frac{\psi(x)}{f(x)}\, dx = \sigma^2 (n+1),$$

and as

$$\int \frac{\psi(x)}{f(x)}\, dx = N m_0,$$

the mean value of ${}_n\sigma_y^2$ calculated in this special way is

$$\frac{\sigma^2}{N} \cdot \frac{(n+1)}{m_0}.$$

It is therefore clear either that ${}_n\sigma_y^2$ must at all the places of observation be equal to $\frac{\sigma^2}{N} \cdot \frac{n+1}{m_0}$ or ${}_n\sigma_y^2$ must at some of these places be greater. The first case cannot be realised by a distribution of which any part is continuous, as ${}_n\sigma_y^2$ is proved to be of the $2n$th degree in x. If therefore we could find a distribution consisting of groups of observations for which at all the places of observation ${}_n\sigma_y^2$ was equal to $\frac{\sigma^2}{N} \cdot \frac{n+1}{m_0}$, and if further we could choose the places of observation so that ${}_n\sigma_y^2$ at all other places within the range of observations was smaller than that value, we should know that no other distribution of observations with that value for m_0 could provide a curve of standard deviation with a lower maximum.

If the standard deviation of the observations be constant and equal σ, $f(x)$ equals 1, and so does m_0. After what we have just proved the maximum of the ${}_n\sigma_y^2$ curve cannot then be lower than $\frac{\sigma^2}{N}(n+1)$. Now when we choose to distribute our N observations in $(n+1)$ equally big groups the adjusted y at each of these $(n+1)$ places will be the mean of the observations and its squared standard deviation will be $\frac{\sigma^2}{N}(n+1)$. Hence our problem is reduced to find out how to arrange a table of $(n+1)$ values of a function of the nth degree to make the squared standard deviation of any interpolation result inside the range smaller than the squared standard deviation of the values of the table. It will be seen in what follows that this can up to n equal 6—that is so far as the problem here has been investigated—be obtained by one and only one form of grouping.

When the standard deviation of the observations varies over the range, m_0 varies with the different distributions, and we cannot use the same method for finding the best distribution. It even appears that the best distribution has not always its maxima at the places of observation.

(8) A second problem which we want to consider here is *the condition for two adjusted y's being uncorrelated.* In the beginning of this section it has been shown that the adjusted y,

$$y_r = \frac{1}{N} S \left\{ \frac{y_p}{f(x_p)} [\alpha_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \ldots\ldots + \alpha_n x_p^n] \right\},$$

when

$$\left.\begin{array}{l} a_0 m_0 + a_1 m_1 + a_2 m_2 + \ldots\ldots + a_n m_n = 1 \\ a_0 m_1 + a_1 m_2 + a_2 m_3 + \ldots\ldots + a_n m_{n+1} = x_r \\ a_0 m_2 + a_1 m_3 + a_2 m_4 + \ldots\ldots + a_n m_{n+2} = x_r^2 \\ \vdots \\ a_0 m_n + a_1 m_{n+1} + a_2 m_{n+2} + \ldots\ldots + a_n m_{2n} = x_r^n \end{array}\right\} \ldots\ldots\ldots\ldots(16).$$

Let y_s be another adjusted value, then

$$y_s = \frac{1}{N} S \left\{ \frac{y_p}{f(x_p)} [\gamma_0 + \gamma_1 x_p + \gamma_2 x_p^2 + \ldots\ldots + \gamma_n x_p^n] \right\},$$

where

$$\left.\begin{array}{l} \gamma_0 m_0 + \gamma_1 m_1 + \gamma_2 m_2 + \ldots\ldots + \gamma_n m_n = 1 \\ \gamma_0 m_1 + \gamma_1 m_2 + \gamma_2 m_3 + \ldots\ldots + \gamma_n m_{n+1} = x_s \\ \gamma_0 m_2 + \gamma_1 m_3 + \gamma_2 m_4 + \ldots\ldots + \gamma_n m_{n+2} = x_s^2 \\ \vdots \\ \gamma_0 m_n + \gamma_1 m_{n+1} + \gamma_2 m_{n+2} + \ldots\ldots + \gamma_n m_{2n} = x_s^n \end{array}\right\} \ldots\ldots\ldots\ldots(17).$$

Hence the condition that y_r and y_s are uncorrelated is, since the squared standard deviation of the observed y_p equals $\sigma^2 f(x_p)$,

$$S \left\{ \frac{1}{f(x_p)} [a_0 + a_1 x_p + a_2 x_p^2 + \ldots\ldots + a_n x_p^n] \cdot [\gamma_0 + \gamma_1 x_p + \gamma_2 x_p^2 + \ldots\ldots + \gamma_n x_p^n] \right\} = 0,$$

or $$S \left\{ \frac{\gamma_0}{f(x_p)} [a_0 + a_1 x_p + a_2 x_p^2 + \ldots\ldots + a_n x_p^n] \right\} + S \left\{ \frac{\gamma_1}{f(x_p)} [a_0 x_p + a_1 x_p^2 + a_2 x_p^3 + \ldots\ldots + a_n x_p^{n+1}] \right\}$$
$$+ S \left\{ \frac{\gamma_2}{f(x_p)} [a_0 x_p^2 + a_1 x_p^3 + a_2 x_p^4 + \ldots\ldots + a_n x_p^{n+2}] \right\} + \ldots\ldots + S \left\{ \frac{\gamma_n}{f(x_p)} [a_0 x_p^n + a_1 x_p^{n+1} + a_2 x_p^{n+2} + \ldots\ldots + a_n x_p^{2n}] \right\} = 0.$$

Remembering that $S \left\{ \frac{x_p^q}{f(x_p)} \right\} = N m_q$ and applying the relations (16) this reduces to

$$\gamma_0 + \gamma_1 x_r + \gamma_2 x_r^2 + \ldots\ldots + \gamma_n x_r^n = 0,$$

from which the γ's are eliminated by (17).

$$\begin{vmatrix} 0 & 1 & x_r & x_r^2 & \ldots\ldots & x_r^n \\ 1 & m_0 & m_1 & m_2 & \ldots\ldots & m_n \\ x_s & m_1 & m_2 & m_3 & \ldots\ldots & m_{n+1} \\ x_s^2 & m_2 & m_3 & m_4 & \ldots\ldots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_s^n & m_n & m_{n+1} & m_{n+2} & \ldots\ldots & m_{2n} \end{vmatrix} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

is therefore the condition that y_r and y_s are uncorrelated.

(9) Returning to the formula (11) for σ_y^2 written as a sum of squares we shall now prove that *the* $(p+1)$*st term of this put equal to zero determines a set of p abscissae the adjusted y's of which are mutually uncorrelated both for a function of the pth and the* $(p-1)$*st degree.*

The condition for y_1 and y_2 corresponding to the arguments x_1 and x_2 being uncorrelated is for a function of the $(p-1)$st degree

$$\begin{vmatrix} 0 & 1 & x_1 & x_1^2 & \cdots\cdots & x_1^{p-1} \\ 1 & m_0 & m_1 & m_2 & \cdots\cdots & m_{p-1} \\ x_2 & m_1 & m_2 & m_3 & \cdots\cdots & m_p \\ x_2^2 & m_2 & m_3 & m_4 & \cdots\cdots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_2^{p-1} & m_{p-1} & m_p & m_{p+1} & \cdots\cdots & m_{2p-2} \end{vmatrix} = 0,$$

and for the same distribution of observations and for a function of the pth degree the condition is

$$\begin{vmatrix} 0 & 1 & x_1 & x_1^2 & \cdots\cdots & x_1^{p} \\ 1 & m_0 & m_1 & m_2 & \cdots\cdots & m_{p} \\ x_2 & m_1 & m_2 & m_3 & \cdots\cdots & m_{p+1} \\ x_2^2 & m_2 & m_3 & m_4 & \cdots\cdots & m_{p+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_2^{p} & m_{p} & m_{p+1} & m_{p+2} & \cdots\cdots & m_{2p} \end{vmatrix} = 0.$$

Putting

$$\begin{vmatrix} m_0 & m_1 & m_2 & \cdots\cdots & m_p \\ m_1 & m_2 & m_3 & \cdots\cdots & m_{p+1} \\ m_2 & m_3 & m_4 & \cdots\cdots & m_{p+2} \\ \vdots & \vdots & \vdots & & \vdots \\ m_p & m_{p+1} & m_{p+2} & \cdots\cdots & m_{2p} \end{vmatrix} = D,$$

these conditions may be written

$$\sum_0^{p-1} \{x_1^r \,.\, x_2^s D_{p+1,\, p+1,\, r+1,\, s+1}\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

and

$$\sum_0^{p} \{x_1^r \,.\, x_2^s D_{r+1,\, s+1}\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

where the sums include all combinations of powers with r and s lying between 0 and $(p-1)$, and 0 and p respectively.

Now we have for an orthosymmetrical determinant Δ,

$$\Delta_{ss} \,.\, \Delta_{s's''} - \Delta \,.\, \Delta_{sss's''} = \Delta_{ss'} \,.\, \Delta_{ss''}.$$

If therefore (19) is multiplied by D and subtracted from (20) multiplied by $D_{p+1,\, p+1}$ the coefficient of $x_1^r \,.\, x_2^s$ becomes

$$D_{p+1,\, p+1} \,.\, D_{r+1,\, s+1} - D \,.\, D_{p+1,\, p+1,\, r+1,\, s+1} = D_{p+1,\, r+1} \,.\, D_{p+1,\, s+1},$$

as long as both r and s are smaller than p.

When one of them, for example s, equals p the term is

$$x_1^r \cdot x_2^p \cdot D_{p+1,\,p+1} \cdot D_{r+1,\,p+1},$$

which is of the same form and this also holds for $r = s = p$ when the term is

$$x_1^p \cdot x_2^p \cdot D_{p+1,\,p+1}^2.$$

The total result is thus

$$\sum_0^p x_1^r \cdot x_2^s D_{p+1,\,r+1} \cdot D_{p+1,\,s+1} = 0,$$

or in the form of determinants

$$\begin{vmatrix} 1 & m_0 & m_1 & m_2 & \cdots\cdots & m_{p-1} \\ x_1 & m_1 & m_2 & m_3 & \cdots\cdots & m_p \\ x_1^2 & m_2 & m_3 & m_4 & \cdots\cdots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_1^p & m_p & m_{p+1} & m_{p+2} & \cdots\cdots & m_{2p-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & m_0 & m_1 & m_2 & \cdots\cdots & m_{p-1} \\ x_2 & m_1 & m_2 & m_3 & \cdots\cdots & m_p \\ x_2^2 & m_2 & m_3 & m_4 & \cdots\cdots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_2^p & m_p & m_{p+1} & m_{p+2} & \cdots\cdots & m_{2p-1} \end{vmatrix} = 0.$$

Hence x_1 and x_2 must be roots of

$$\begin{vmatrix} 1 & m_0 & m_1 & m_2 & \cdots\cdots & m_{p-1} \\ x & m_1 & m_2 & m_3 & \cdots\cdots & m_p \\ x^2 & m_2 & m_3 & m_4 & \cdots\cdots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^p & m_p & m_{p+1} & m_{p+2} & \cdots\cdots & m_{2p-1} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(21).$$

When x_1 is found from this and substituted in (19) or (20) we get since the coefficient of x_2^p in the latter is zero an equation of the $(p-1)$st degree to determine x_2. It is therefore clear that any pair of roots of (21) determine a pair of uncorrelated y's.

II. *The "best" grouping of observations with constant standard deviation.*

(1) It was shown in the last section under (7) that *the mean of the squared standard deviations of the adjusted y taken over the places of observation and weighted with the number of observations at each place is equal to* $\frac{\sigma^2}{N}(n+1)$ and that therefore the *curve of squared standard deviation can never be entirely below that value.* And further, that since $(n+1)$ equally big groups of observations at the places of observations give the squared standard deviation this minimum, there is *the possibility,* ${}_n\sigma_y^2$ *being of the 2nth degree in x, that by placing the groups at special positions the curve of squared standard deviation could have those values* $\frac{\sigma^2}{N}(n+1)$ *as its maxima* within the range of observations.

Let $x_1, x_2 \ldots x_p \ldots x_{n+1}$ be the places of observations and $\bar{y}_{x_p}$ the mean of the observations at x_p, the interpolation formula of Lagrange is then

$$y = \Sigma \left\{ \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_{n+1})}{(x_p-x_1)(x_p-x_2)\ldots\ldots(x_p-x_{n+1})} \bar{y}_{x_p} \right\},$$

the sum taken over all the places of observation.

From this we find

$$\sigma_y^2 = \frac{\sigma^2}{N}(n+1)\,\Sigma\left\{\frac{(x-x_1)(x-x_2)\,\ldots\ldots\,(x-x_{n+1})}{(x_p-x_1)(x_p-x_2)\,\ldots\ldots\,(x_p-x_{n+1})}\right\}^2 \quad\ldots\ldots\ldots(22),$$

which for $x = x_1,\ x_2 \ldots x_{n+1}$ equals $\frac{\sigma^2}{N}(n+1)$, the n terms of the sum being zero and the $(n+1)$st taking the value 1 as it ought to. If x_p be the greatest of the x's it is hence clear that for $x > x_p$, since

$$\left\{\frac{(x-x_1)(x-x_2)\,\ldots\ldots\,(x-x_{n+1})}{(x_p-x_1)(x_p-x_2)\,\ldots\ldots\,(x_p-x_{n+1})}\right\}^2 > 1,$$

$$\sigma_y^2 > \frac{\sigma^2}{N}(n+1).$$

The same applies to any x smaller than the smallest of the places of observation. Therefore as we want σ_y^2 to be $\lesseqgtr \frac{\sigma^2}{N}(n+1)$ at the ends of the range *we have to place two of our groups of observations there.*

Let us take *the half of the range within which it is possible to make observations as the unit of x so that the range goes from* -1 *to* 1.

(2) Hence for a linear function there is no choice left, the two groups of observations must be at -1 and 1.

According to (22) we have

$${}_1\sigma_y^2 = \frac{\sigma^2}{N}\cdot 2\left\{\frac{(x+1)^2}{4} + \frac{(x-1)^2}{4}\right\}$$

or

$${}_1\sigma_y^2 = \frac{\sigma^2}{N}\cdot 2\{1 - \tfrac{1}{2}(1-x^2)\},$$

which illustrate the well-known fact that *by simple interpolation between two equally good values of a table, we obtain interpolated values with less probable error than those of the table.*

(3) Investigating a function of the second degree we have a third group to place besides the two at -1 and 1, that is if we do not beforehand suppose the distribution to be symmetrical. Let the third group be at α, then the interpolation gives

$$y = \frac{(x-1)(x-\alpha)}{2(1+\alpha)}\,\bar{y}_{-1} + \frac{(x+1)(x-\alpha)}{2(1-\alpha)}\,\bar{y}_1 + \frac{x^2-1}{\alpha^2-1}\,\bar{y}_\alpha,$$

from which

$${}_2\sigma_y^2 = \frac{\sigma^2}{N}\cdot 3\left\{\left[\frac{(x-1)(x-\alpha)}{2(1+\alpha)}\right]^2 + \left[\frac{(x+1)(x-\alpha)}{2(1-\alpha)}\right]^2 + \left[\frac{x^2-1}{\alpha^2-1}\right]^2\right\}.$$

We want this to be a maximum for $x = \alpha$, but $\left(\frac{d\sigma_y^2}{dx}\right)_{x=\alpha}$ can only vanish for $\alpha = 0$, in which case σ_y^2 is reduced to

$${}_2\sigma_y^2 = \frac{\sigma^2}{N}\cdot 3\{1 - \tfrac{3}{2}x^2(1-x^2)\},$$

which shows that we have succeeded in making σ_y^2 a maximum at $x = 0$ and obtained a standard deviation with the maximum value $\frac{\sigma^2}{N} 3$, as we desired.

(4) For a function of the third degree we find from four groups of observations at -1, 1, α and γ that

$$y = \frac{(x-1)(x-\alpha)(x-\gamma)}{-2(1+\alpha)(1+\gamma)} \bar{y}_{-1} + \frac{(x+1)(x-\alpha)(x-\gamma)}{2(1-\alpha)(1-\gamma)} \bar{y}_1 + \frac{(x^2-1)(x-\gamma)}{(\alpha^2-1)(\alpha-\gamma)} \bar{y}_\alpha + \frac{(x^2-1)(x-\alpha)}{(\gamma^2-1)(\gamma-\alpha)} \bar{y}_\gamma$$

and
$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \cdot 4 \left\{ \left[\frac{(x-1)(x-\alpha)(x-\gamma)}{2(1+\alpha)(1+\gamma)}\right]^2 + \left[\frac{(x+1)(x-\alpha)(x-\gamma)}{2(1-\alpha)(1-\gamma)}\right]^2 + \left[\frac{(x^2-1)(x-\gamma)}{(1-\alpha^2)(\alpha-\gamma)}\right]^2 + \left[\frac{(x^2-1)(x-\alpha)}{(1-\gamma^2)(\alpha-\gamma)}\right]^2 \right\}.$$

The condition $\left(\frac{d\sigma_y^2}{dx}\right)_{x=\alpha} = 0$

requires $3\alpha^2 - 2\alpha\gamma - 1 = 0,$

and $\left(\frac{d\sigma_y^2}{dx}\right)_{x=\gamma} = 0$

requires $3\gamma^2 - 2\alpha\gamma - 1 = 0,$

from which is got $\alpha^2 = \gamma^2,$

and, since $\alpha \gtrless \gamma$, $\alpha^2 = \gamma^2 = \frac{1}{5}.$

By introducing this value for α^2 and γ^2 in σ_y^2 we find

$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \cdot 4 \left\{1 - \frac{3 \cdot 5^2}{2^4} (x^2 - \tfrac{1}{5})^2 (1 - x^2)\right\},$$

which has the required maxima at $\pm\sqrt{\frac{1}{5}}$.

(5) For the functions of higher degree we shall at once assume that the distributions sought are symmetrical, since it is pretty clear from the symmetry of y and σ_y^2 with regard to the sought positions that it must be so.

To determine a function of the fourth degree let us put groups of observations at ± 1, $\pm \alpha$ and 0. The expression for σ_y^2 can be written down at once and is such that the terms arising from the groups at $+1$ and -1 can be put together as well as the terms from $+\alpha$ and $-\alpha$, then

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \cdot 5 \left[\left\{\frac{(x^2-1)(x^2-\alpha^2)}{\alpha^2}\right\}^2 + \frac{1}{2}\left\{\frac{x(x^2-\alpha^2)}{1-\alpha^2}\right\}^2 (x^2+1) + \frac{1}{2}\left\{\frac{x(x^2-1)}{\alpha^2(1-\alpha^2)}\right\}^2 (x^2-\alpha^2)\right].$$

$\left(\frac{d\sigma_y^2}{dx}\right)_{x=\alpha} = 0$ provides the condition $\frac{3-7\alpha^2}{\alpha(1-\alpha^2)} = 0$ or

$$\alpha^2 = \tfrac{3}{7},$$

with which value the squared standard deviation becomes

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \cdot 5 \left\{1 - \frac{5 \cdot 7^2}{2^4} x^2 (x^2 - \tfrac{3}{7})^2 (1 - x^2)\right\},$$

which has the required characteristics.

(6) Adjusting by a function of the fifth degree six equally big groups of observations at the arguments ± 1, $\pm \alpha$ and $\pm \gamma$ the squared standard deviation of the adjusted y is

$$_5\sigma_y^2 = \frac{\sigma^2}{N} \cdot 6 \left\{\frac{1}{2}\left[\frac{(x^2-\alpha^2)(x^2-\gamma^2)}{(1-\alpha^2)(1-\gamma^2)}\right]^2 (x^2+1) + \frac{1}{2}\left[\frac{(x^2-\gamma^2)(x^2-1)}{\alpha(1-\alpha^2)(\alpha^2-\gamma^2)}\right]^2 (x^2+\alpha^2) + \frac{1}{2}\left[\frac{(x^2-1)(x^2-\alpha^2)}{\gamma(1-\gamma^2)(\alpha^2-\gamma^2)}\right]^2 (x^2+\gamma^2)\right\}.$$

The condition for maximum at $x = \pm \alpha$ is

$$9\alpha^4 - 5\alpha^2\gamma^2 - 5\alpha^2 + \gamma^2 = 0,$$

which together with the condition for maximum at $x = \pm \gamma$

$$9\gamma^4 - 5\alpha^2\gamma^2 - 5\gamma^2 + \alpha^2 = 0,$$

since α^2 must be $\gtrless \gamma^2$ results in

$$\alpha^2 + \gamma^2 = \tfrac{2}{3} \quad\text{and}\quad \alpha^2\gamma^2 = \tfrac{1}{21}$$

or

$$\left.\begin{matrix}\alpha^2\\ \gamma^2\end{matrix}\right\} = \frac{7 \pm 2\sqrt{7}}{21}.$$

When these values are substituted in the expression above for σ_y^2 this may by somewhat lengthy algebraic operations be brought into the form

$$_5\sigma_y^2 = \frac{\sigma^2}{N} \cdot 6 \left\{1 - \frac{3^3 \cdot 5 \cdot 7^2}{2^7}(x^2-\alpha^2)^2(x^2-\gamma^2)^2(1-x^2)\right\}.$$

(7) For a function of the sixth degree the observations may be supposed to be at ± 1, $\pm \alpha$, $\pm \gamma$ and 0.

The expression for the squared standard deviation of an adjusted y becomes

$$_6\sigma_y^2 = \frac{\sigma^2}{N} \cdot 7 \left\{\left[\frac{(x^2-\alpha^2)(x^2-\gamma^2)(x^2-1)}{\alpha^2\gamma^2}\right]^2 + \frac{1}{2}\left[\frac{x(x^2-\gamma^2)(x^2-1)}{\alpha^2(\alpha^2-\gamma^2)(\alpha^2-1)}\right]^2 (x^2+\alpha^2) + \frac{1}{2}\left[\frac{x(x^2-\alpha^2)(x^2-1)}{\gamma^2(\alpha^2-\gamma^2)(\gamma^2-1)}\right]^2 (x^2+\gamma^2) + \frac{1}{2}\left[\frac{x(x^2-\alpha^2)(x^2-\gamma^2)}{(\alpha^2-1)(\gamma^2-1)}\right]^2 (x^2+1)\right\}.$$

A maximum at $x = \pm \alpha$ requires

$$11\alpha^4 - 7\alpha^2\gamma^2 - 7\alpha^2 + 3\gamma^2 = 0,$$

and a maximum at $x = \pm \gamma$ requires

$$11\gamma^4 - 7\alpha^2\gamma^2 - 7\gamma^2 + 3\alpha^2 = 0,$$

which added and subtracted provide

$$11(\alpha^2+\gamma^2)^2 - 36\alpha^2\gamma^2 - 4(\alpha^2+\gamma^2) = 0,$$

and

$$(\alpha^2-\gamma^2)\{11(\alpha^2+\gamma^2) - 10\} = 0.$$

Since we must have $\alpha^2 \lesssim \gamma^2$,

$$\alpha^2 + \gamma^2 = \tfrac{10}{11} \quad\text{and}\quad \alpha^2\gamma^2 = \tfrac{5}{33},$$

or

$$\left.\begin{matrix}\alpha^2\\ \gamma^2\end{matrix}\right\} = \frac{15 \pm 2\sqrt{15}}{33}.$$

The expression for σ_y^2 may after rather laborious operations be brought into the form

$$_6\sigma_y^2 = \frac{\sigma^2}{N} \cdot 7 \left\{1 - \frac{3^3 \cdot 7 \cdot 11^2}{2^7} x^2(x^2-\alpha^2)^2(x^2-\gamma^2)^2(1-x^2)\right\}.$$

(8) It is thus, as we aimed at, shown for functions up to the sixth degree that *by distributing the observations in* $(n+1)$ *equally big groups and choosing the places of these groups in one special way we can manage to keep the standard deviation of any adjusted* y *within the possible range of observations less than the standard deviation at the places of observation.* There is every reason to believe that the rule holds for any degree of function, but as the general proof would be very complicated and as almost all practical cases will be covered by functions up to the sixth degree, the problem can therefore be left at this stage.

As we have proved, *any other distribution of observations leads to a curve of squared standard deviation that has a higher maximum value within the range.* This special set of $(n+1)$ groups has therefore a very conspicuous advantage over all other distributions of observations. *The application of it is however limited in that it demands that the degree of the function must be known beforehand and thus the observations do not provide any justification for the form of function chosen. If however the function has been fully investigated beforehand and there is no doubt about its form,* $(n+1)$ *equally big groups of observations placed as indicated are the most desirable set of observations possible.* The approximate values of the places of the groups are given in the table below.

TABLE I.

Degree of function	1st	2nd	3rd	4th	5th	6th
Places of observation	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
	—	·0000	·4472	·6547	·7651	·8302
	—	—	—	·0000	·2852	·4689
	—	—	—	—	—	·0000

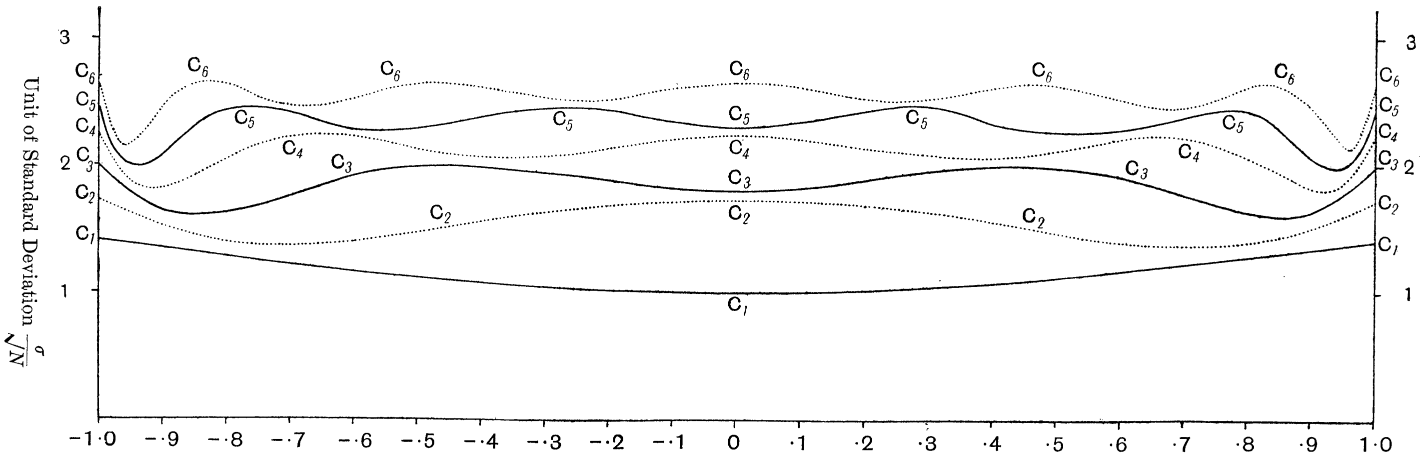
With rougher approximation the intervals between the observations, still expressed by the half range as unit, are as follows:

1st degree of function	2						
2nd " "	1	1					
3rd " "	$\frac{1}{2}$	1	$\frac{1}{2}$				
4th " "	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$			
5th " "	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$		
6th " "	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	

The six curves of standard deviation are represented in Diagram 1. It will be seen that the minima of a curve, if it has more than two, are the lower the greater their distances from the middle of the range, so that the variation of the standard deviation is greatest in the outermost intervals of the range.

III. *Uniform continuous distribution of observations with constant standard deviation. General formulae.*

(1) As was pointed out in the last section the lumping up of observations in groups just necessary to determine the constants of the function in question has some drawbacks and cannot be recommended as a universal rule. In many cases it is through the observations themselves that we first get to know the form of the



Range of observations.

$C_1 C_1$.	Function of	First Degree
$C_2 C_2$.	,,	Second ,,
$C_3 C_3$.	,,	Third ,,
$C_4 C_4$.	,,	Fourth ,,
$C_5 C_5$.	,,	Fifth ,,
$C_6 C_6$.	,,	Sixth ,,

DIAGRAM 1. Curves of Standard Deviations. Equally big clusters at definite points.

function, and thus a full investigation may require more groups of observations than merely a number equal to the assumed number of constants in the formula. Besides, even when we believe we know on theoretical or other grounds beforehand the nature of the function *a priori* we may consider it prudent to distribute the observations so that they supply us with data whereby we may control our hypothesis that the assumed function is the right one.

It is therefore desirable to find other forms of distributions which, at the same time as they make the standard deviation of the adjusted function vary little inside the range of observations, are more uniformly spread over this range.

(2) A uniform continuous distribution at once recommends itself as the simplest assumption. As we suppose the observations to have constant standard deviations the elements of the determinants of (15) are the moment coefficients of the x's at the places of observation.

When the N observations are uniformly spread between $x = -1$ and $x = 1$,

$$\mu_{2r} = \frac{1}{2r+1} \quad \text{and} \quad \mu_{2r+1} = 0,$$

and the expression for ${}_{2p}\sigma_y^2$ is, according to (15),

$$
\begin{aligned}
{}_{2p}\sigma_y^2 = \frac{\sigma^2}{N}\Bigg\{1 + \frac{x^2}{\mu_2} + \frac{\begin{vmatrix} 1 & 1 \\ x^2 & \mu_2 \end{vmatrix}^2}{1 \cdot \begin{vmatrix} 1 & \mu_2 \\ \mu_2 & \mu_4 \end{vmatrix}} + x^2 \frac{\begin{vmatrix} 1 & \mu_2 \\ x^2 & \mu_4 \end{vmatrix}^2}{\mu_2 \cdot \begin{vmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{vmatrix}} + \cdots\cdots \\
+ x^2 \frac{\begin{vmatrix} 1 & \mu_2 & \mu_4 & \cdots\cdots & \mu_{2p-2} \\ x^2 & \mu_4 & \mu_6 & \cdots\cdots & \mu_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & \mu_{2p} & \mu_{2p+2} & \cdots\cdots & \mu_{4p-4} \end{vmatrix}^2}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots\cdots & \mu_{2p-2} \\ \mu_4 & \mu_6 & \cdots\cdots & \mu_{2p} \\ \vdots & \vdots & & \vdots \\ \mu_{2p-2} & \mu_{2p} & \cdots\cdots & \mu_{4p-6} \end{vmatrix} \cdot \begin{vmatrix} \mu_2 & \mu_4 & \cdots\cdots & \mu_{2p} \\ \mu_4 & \mu_6 & \cdots\cdots & \mu_{2p+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2p} & \mu_{2p+2} & \cdots\cdots & \mu_{4p-2} \end{vmatrix}} \\
+ \frac{\begin{vmatrix} 1 & 1 & \mu_2 & \cdots\cdots & \mu_{2p-2} \\ x^2 & \mu_2 & \mu_4 & \cdots\cdots & \mu_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \mu_{2p} & \mu_{2p+2} & \cdots\cdots & \mu_{4p-2} \end{vmatrix}^2}{\begin{vmatrix} 1 & \mu_2 & \cdots\cdots & \mu_{2p-2} \\ \mu_2 & \mu_4 & \cdots\cdots & \mu_{2p} \\ \vdots & \vdots & & \vdots \\ \mu_{2p-2} & \mu_{2p} & \cdots\cdots & \mu_{4p-4} \end{vmatrix} \cdot \begin{vmatrix} 1 & \mu_2 & \cdots\cdots & \mu_{2p} \\ \mu_2 & \mu_4 & \cdots\cdots & \mu_{2p+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2p} & \mu_{2p+2} & \cdots\cdots & \mu_{4p} \end{vmatrix}}\Bigg\} \quad \ldots\ldots\ldots(23).
\end{aligned}
$$

By this formula we may evaluate successively ${}_1\sigma_y^2$, ${}_2\sigma_y^2$ ... ${}_{2p}\sigma_y^2$ when we know the two general terms of which the sum consists.

(3) The determinant of the order p,

$$\overset{q}{{}_p\Delta} = \begin{vmatrix} \frac{1}{2q-1} & \frac{1}{2q+1} & \cdots\cdots & \frac{1}{2q+2p-3} \\ \frac{1}{2q+1} & \frac{1}{2q+3} & \cdots\cdots & \frac{1}{2q+2p-1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2q+2p-3} & \frac{1}{2q+2p-1} & \cdots\cdots & \frac{1}{2q+4p-5} \end{vmatrix},$$

which includes the two types of the denominators in (23), shall first be evaluated.

We find $\overset{q}{{}_1\Delta} = \frac{1}{2q-1}$ and $\overset{q}{{}_2\Delta} = \frac{2^2}{(2q-1)(2q+1)^2(2q+3)}$,

and it shall be proved that if

$$\overset{q}{{}_p\Delta} = \{1^{p-1} \,.\, 2^{p-2} \cdots\cdots (p-2)^2 (p-1)\}^2 \,.\, 2^{p(p-1)} \, \overset{q}{{}_p\Pi} \qquad \text{.........(24)}$$

up to the order p, $\overset{q}{{}_p\Pi}$ being the product of the elements of $\overset{q}{{}_p\Delta}$, the rule holds for determinants of any order.

It is clear that

$$\overset{q}{{}_{p+1}\Delta}_{1,1} = \overset{q+2}{{}_p\Delta}, \quad \overset{q}{{}_{p+1}\Delta}_{p+1,\,p+1} = \overset{q}{{}_p\Delta}, \quad \overset{q}{{}_{p+1}\Delta}_{1,\,p} (-1)^{p+1} = \overset{q+1}{{}_p\Delta}$$

and

$$\overset{q}{{}_{p+1}\Delta}_{p+1,\,p+1,\,1,\,1} = \overset{q+2}{{}_{p-1}\Delta}.$$

If we therefore in the general relation for an orthosymmetrical determinant

$$\Delta = \frac{\Delta_{ss} \,.\, \Delta_{s's'} - \Delta^2_{ss'}}{\Delta_{ss's'}}$$

put $s = 1$ and $s' = p + 1$ and $\Delta = \overset{q}{{}_{p+1}\Delta}$, we find

$$\overset{q}{{}_{p+1}\Delta} = \frac{\overset{q}{{}_p\Delta} \,.\, \overset{q+2}{{}_p\Delta} - \overset{q+1}{{}_p\Delta^2}}{\overset{q+2}{{}_{p-1}\Delta}},$$

and, using (24),

$$\overset{q}{{}_{p+1}\Delta} = \frac{\{1^{p-1} \,.\, 2^{p-2} \cdots\cdots (p-2)^2 (p-1)\}^4}{\{1^{p-2} \,.\, 2^{p-3} \cdots\cdots (p-3)^2 (p-2)\}^2} \,.\, 2^{(p-1)(p+2)} \,.\, \frac{\overset{q}{{}_p\Pi} \,.\, \overset{q+2}{{}_p\Pi} - \overset{q+1}{{}_p\Pi^2}}{\overset{q+2}{{}_{p-1}\Pi}}.$$

Now, according to the definition of Π,

$$\frac{\overset{q}{{}_p\Pi}}{\overset{q}{{}_{p+1}\Pi}} \,.\, \frac{\overset{q+2}{{}_p\Pi}}{\overset{q}{{}_{p+1}\Pi}} = (2q-1)(2q+1)^2(2q+3)^2 \cdots\cdots (2q+4p-3)^2(2q+4p-1) \times (2q+2p-1)^2,$$

$$\frac{\overset{q+1}{{}_p\Pi^2}}{\overset{q}{{}_{p+1}\Pi^2}} = (2q-1)^2(2q+1)^2(2q+3)^2 \cdots\cdots (2q+4p-3)^2(2q+4p-1)^2$$

and

$$\frac{\overset{q+2}{{}_{p-1}\Pi}}{\overset{q}{{}_{p+1}\Pi}} = (2q-1)(2q+1)^2(2q+3)^2 \cdots\cdots (2q+4p-3)^2(2q+4p-1).$$

Hence

$$ {}_{p+1}\overset{q}{\Delta} = \{1^p \,.\, 2^{p-1} \ldots\ldots (p-2)^3 (p-1)^2\}^2 \,.\, 2^{(p-1)(p+2)} \,.\, {}_{p+1}\overset{q}{\Pi} \,.\, [(2q+2p-1)^2 - (2q-1)(2q+4p-1)], $$

$$ {}_{p+1}\overset{q}{\Delta} = \{1^p \,.\, 2^{p-1} \ldots\ldots (p-1)^2 \,.\, p\}^2 \,.\, 2^{p(p+1)} \,.\, {}_{p+1}\overset{q}{\Pi}, $$

which agrees with (24).

(4) Next we have to evaluate the minors of ${}_p\overset{q}{\Delta}$ necessary for calculating the numerators in (23). For this purpose we only need the minors ${}_p\overset{q}{\Delta}_{s,p}$, but to carry through the proof by induction ${}_p\overset{q}{\Delta}_{s,r}$ for any values of s and r is needed.

For ${}_3\overset{q}{\Delta}_{2,3}$ we directly find,

$$ {}_3\overset{q}{\Delta}_{2,3} = \frac{-2^2 \,.\, 2}{(2q-1)(2q+1)(2q+3)(2q+5)} $$

and

$$ {}_3\overset{q}{\Delta}_{2,2} = \frac{2^2 \,.\, 2^2}{(2q-1)(2q+3)^2(2q+7)}, $$

these both agree with the following formula which will be proved by induction,

$$ {}_p\overset{q}{\Delta}_{s,r} = (-1)^{r+s}\beta_{p-1,s-1} \,.\, \beta_{p-1,r-1}\{1^{p-2} \,.\, 2^{p-3} \ldots\ldots (p-3)^2 (p-2)\}^2 \,.\, 2^{(p-1)(p-2)}\, {}_p\overset{q}{\Pi}_{s,r} \quad \ldots\ldots\ldots\ldots\ldots(25), $$

$\beta_{p-1,s-1}$ is the binomial coefficient $\dfrac{|\underline{p-1}}{|\underline{s-1}\; |\underline{p-s}}$ and ${}_p\overset{q}{\Pi}_{s,r}$ the product of all the elements of ${}_p\overset{q}{\Delta}_{s,r}$.

The relation has to be proved first for $r = s$, then for $r = p$ and finally for any combination s and r.

For the first two proofs we use the relation between the minors of an ortho-symmetrical determinant

$$ \frac{\Delta_{ss}}{\Delta_{s's''}} = \frac{\Delta_{ss s's'} \,.\, \Delta_{sss''s''} - \Delta^2_{sss's''}}{\Delta_{s's's''s''} \,.\, \Delta_{sss's''} + \Delta_{s''s''ss'} \,.\, \Delta_{s's'ss''}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(26). $$

This is found from two relations given by Professor Pearson* by dividing one of them by the other.

(5) Let Δ be ${}_{p+1}\overset{q}{\Delta}$, $s' = 1$ and $s'' = p+1$, then

$$ \frac{{}_{p+1}\overset{q}{\Delta}_{ss}}{{}_{p+1}\overset{q}{\Delta}_{1,p+1}} = \frac{{}_{p+1}\overset{q}{\Delta}_{ss11} \,.\, {}_{p+1}\overset{q}{\Delta}_{s,s,p+1,p+1} - {}_{p+1}\overset{q}{\Delta}^2_{s,s,1,p+1}}{{}_{p+1}\overset{q}{\Delta}_{1,1,p+1,p+1} \,.\, {}_{p+1}\overset{q}{\Delta}_{s,s,1,p+1} + {}_{p+1}\overset{q}{\Delta}_{p+1,p+1,s,1} \,.\, {}_{p+1}\overset{q}{\Delta}_{1,1,s,p+1}} \ldots(27). $$

Now

$$ {}_{p+1}\overset{q}{\Delta}_{ss11} = {}_p\overset{q+2}{\Delta}_{s-1,s-1}, $$

$$ {}_{p+1}\overset{q}{\Delta}_{s,s,p+1,p+1} = {}_p\overset{q}{\Delta}_{ss}, $$

$$ {}_{p+1}\overset{q}{\Delta}_{1,1,p+1,p+1} = (-1)^{p+1}\, {}_p\overset{q+1}{\Delta}_{1,p}, $$

$$ {}_{p+1}\overset{q}{\Delta}_{s,s,1,p+1} = (-1)^p\, {}_p\overset{q+1}{\Delta}_{s-1,s}, $$

* *Biometrika*, Vol. XI, pp. 232–3.

$$ {}_{p+1}\overset{q}{\Delta}_{p+1,\,p+1,\,s,\,1} = (-1)^{p+1}\,{}_{p}\overset{q+1}{\Delta}_{p,\,s}, $$
$$ {}_{p+1}\overset{q}{\Delta}_{1,\,1,\,s,\,p+1} = (-1)^{p+1}\,{}_{p}\overset{q+1}{\Delta}_{s-1,\,1}, $$

so that all the determinants on the right side in (27) can be evaluated by (25).

They all have the factor

$$ \{1^{p-2}\,.\,2^{p-3}\,\ldots\ldots\,(p-3)^2\,(p-2)\}^2\,.\,2^{(p-1)\,(p-2)} $$

in common, when that is divided out there remains

$$ \frac{{}_{p+1}\overset{q}{\Delta}_{ss}}{{}_{p+1}\overset{q}{\Delta}_{1,\,p+1}} = \frac{(-1)^p\beta^2_{p-1,\,s-2}\,.\,\beta^2_{p-1,\,s-1}\,({}_{p}\overset{q+2}{\Pi}_{s-1,\,s-1}\,.\,{}_{p}\overset{q}{\Pi}_{ss} - {}_{p}\overset{q+1}{\Pi}{}^2_{s-1,\,s})}{\beta_{p-1,\,s-2}\,.\,\beta_{p-1,\,s-1}\,({}_{p}\overset{q+1}{\Pi}_{s-1,\,1}\,.\,{}_{p}\overset{q+1}{\Pi}_{p,\,s} - {}_{p}\overset{q+1}{\Pi}_{p,\,1}\,.\,{}_{p}\overset{q+1}{\Pi}_{s-1,\,s})} \quad \ldots\ldots\ldots(28). $$

Now indicating by C_r the product of the elements of the rth column or rth row in ${}_{p+1}\overset{q}{\Delta}$ and by e_{rs} the element of the ${}_{p+1}\overset{q}{\Delta}$ common for the rth row and sth column we find

$$ {}_{p+1}\overset{q}{\Pi}_{ss} = {}_{p}\overset{q+2}{\Pi}_{s-1,\,s-1}\,.\,\frac{C_1^2}{e_{11}\,.\,e^2_{s1}}, $$
$$ {}_{p+1}\overset{q}{\Pi}_{ss} = {}_{p}\overset{q}{\Pi}_{ss}\,.\,\frac{C^2_{p+1}}{e_{p+1,\,p+1}\,.\,e^2_{p+1,\,s}}, $$
$$ {}_{p+1}\overset{q}{\Pi}_{ss} = {}_{p}\overset{q+1}{\Pi}_{s-1,\,s}\,.\,\frac{C_1 C_{p+1}}{e_{1,\,p+1}\,.\,e_{s,\,1}\,.\,e_{s,\,p+1}}. $$

Hence the factor of the numerator in (28) is reduced to

$$ {}_{p+1}\overset{q}{\Pi}{}^2_{ss}\,\frac{e^2_{s,\,1}\,.\,e^2_{s,\,p+1}}{C_1^2\,.\,C^2_{p+1}}\{e_{11}\,.\,e_{p+1,\,p+1} - e^2_{1,\,p+1}\}. $$

For the Π's of the denominator we find

$$ {}_{p+1}\overset{q}{\Pi}_{1,\,p+1} = {}_{p}\overset{q+1}{\Pi}_{s-1,\,1}\,.\,\frac{C_1\,.\,C_s}{e_{s1}\,.\,e_{s,\,p+1}\,.\,e_{1,\,1}}, $$
$$ {}_{p+1}\overset{q}{\Pi}_{1,\,p+1} = {}_{p}\overset{q+1}{\Pi}_{p,\,s}\,.\,\frac{C_{p+1}\,.\,C_s}{e_{s,\,p+1}\,.\,e_{p+1,\,p+1}\,.\,e_{1,\,s}}, $$
$$ {}_{p+1}\overset{q}{\Pi}_{1,\,p+1} = {}_{p}\overset{q+1}{\Pi}_{p,\,1}\,.\,\frac{C_{p+1}\,.\,C_1}{e_{1,\,p+1}\,.\,e_{11}\,.\,e_{p+1,\,p+1}}, $$
$$ {}_{p+1}\overset{q}{\Pi}_{1,\,p+1} = {}_{p}\overset{q+1}{\Pi}_{s-1,\,s}\,.\,\frac{C_s^2}{e_{ss}\,.\,e_{s1}\,.\,e_{s,\,p+1}}, $$

the factor containing Π's of the denominator of (28) is therefore equal to

$$ {}_{p+1}\overset{q}{\Pi}{}^2_{1,\,p+1}\,\frac{e_{1s}\,.\,e_{p+1,\,s}\,.\,e_{1,\,1}\,.\,e_{p+1,\,p+1}}{C_1\,.\,C_{p+1}\,.\,C_s^2}\{e_{1s}\,.\,e_{p+1,\,s} - e_{1,\,p+1}\,.\,e_{ss}\}. $$

Introducing these two expressions in (28) and substituting for the one factor $\dfrac{{}_{p+1}\overset{q}{\Pi}_{ss}}{{}_{p+1}\overset{q}{\Pi}_{1,\,p+1}}$ the value $\dfrac{C_1\,.\,C_{p+1}}{C_s^2}\,.\,\dfrac{e_{ss}}{e_{1,\,p+1}}$ we hence find

$$ \frac{{}_{p+1}\overset{q}{\Delta}_{ss}}{{}_{p+1}\overset{q}{\Delta}_{1,\,p+1}} = (-1)^p\beta_{p-1,\,s-2}\,\beta_{p-1,\,s-1}\,\frac{\dfrac{1}{e^2_{1,\,p+1}} - \dfrac{1}{e_{11}\,.\,e_{p+1,\,p+1}}}{\dfrac{1}{e_{ss}\,.\,e_{1,\,p+1}} - \dfrac{1}{e_{1,\,s}\,.\,e_{p+1,\,s}}}\,.\,\frac{{}_{p+1}\overset{q}{\Pi}_{ss}}{{}_{p+1}\overset{q}{\Pi}_{1,\,p+1}}. $$

The fraction containing e's equals

$$\frac{(2q+2p-1)^2-(2q-1)(2q+4p-1)}{(2q+4s-5)(2q+2p-1)-(2q+2s-3)(2q+2s+2p-3)}=\frac{p^2}{(s-1)(p-s+1)},$$

hence
$$\frac{{}_{p+1}\overset{q}{\Delta}_{ss}}{{}_{p+1}\overset{q}{\Delta}_{1,\,p+1}}=(-1)^p\beta^2_{p,\,s-1}\frac{{}_{p+1}\overset{q}{\Pi}_{ss}}{{}_{p+1}\overset{q}{\Pi}_{1,\,p+1}}.$$

As

$${}_{p+1}\overset{q}{\Delta}_{1,\,p+1}=(-1)^p\,{}_p\overset{q+1}{\Delta}=(-1)^p\{1^{p-1}.2^{p-2}\ldots\ldots(p-2)^2(p-1)\}^2.2^{p(p-1)}\,{}_{p+1}\overset{q}{\Pi}_{1,\,p+1},$$

we therefore find

$${}_{p+1}\overset{q}{\Delta}_{ss}=\beta^2_{p,\,s-1}\{1^{p-1}.2^{p-2}\ldots\ldots(p-2)^2(p-1)\}^2.2^{p(p-1)}\,{}_{p+1}\overset{q}{\Pi}_{ss},$$

agreeing with (25).

(6) To evaluate ${}_{p+1}\overset{q}{\Delta}_{s,p+1}$ we shall in (26) put $\Delta={}_{p+1}\overset{q}{\Delta}$, $s=1$, $s'=s$ and $s''=p+1$. Reversing the fractions we then get

$$\frac{{}_{p+1}\overset{q}{\Delta}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Delta}_{11}}=\frac{{}_{p+1}\overset{q}{\Delta}_{s,\,s,\,p+1,\,p+1}\cdot{}_{p+1}\overset{q}{\Delta}_{1,\,1,\,s,\,p+1}+{}_{p+1}\overset{q}{\Delta}_{p+1,\,p+1,\,1,\,s}\cdot{}_{p+1}\overset{q}{\Delta}_{s,\,s\,1,\,p+1}}{{}_{p+1}\overset{q}{\Delta}_{1,\,1,\,s,\,s}\cdot{}_{p+1}\overset{q}{\Delta}_{1,\,1,\,p+1,\,p+1}-{}_{p+1}\overset{q}{\Delta}{}^2_{1,\,1,\,s,\,p+1}}\quad\ldots\ldots\ldots\ldots(29).$$

As
$$\begin{aligned}
{}_{p+1}\overset{q}{\Delta}_{s,\,s,\,p+1,\,p+1}&={}_p\overset{q}{\Delta}_{ss},\\
{}_{p+1}\overset{q}{\Delta}_{1,\,1,\,s,\,p+1}&={}_p\overset{q+2}{\Delta}_{s-1,\,p}=(-1)^{p+1}\,{}_p\overset{q+1}{\Delta}_{1,\,s-1},\\
{}_{p+1}\overset{q}{\Delta}_{p+1,\,p+1,\,1,\,s}&=(-1)^{p+1}\,{}_p\overset{q+1}{\Delta}_{s,\,p},\\
{}_{p+1}\overset{q}{\Delta}_{s,\,s,\,1,\,p+1}&=(-1)^p\,{}_p\overset{q+1}{\Delta}_{s-1,\,s},\\
{}_{p+1}\overset{q}{\Delta}_{1,\,1,\,s,\,s}&={}_p\overset{q+2}{\Delta}_{s-1,\,s-1},\\
{}_{p+1}\overset{q}{\Delta}_{1,\,1,\,p+1,\,p+1}&={}_p\overset{q}{\Delta}_{1,\,1},
\end{aligned}$$

the right side of (29) can be evaluated by (25).

We thus get

$$\frac{{}_{p+1}\overset{q}{\Delta}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Delta}_{1,\,1}}=\frac{(-1)^{s+p-1}\beta_{p-1,\,s-1}\cdot\beta_{p-1,\,s-2}\cdot\beta_{p-1,\,s-1}(-{}_p\overset{q+1}{\Pi}_{s,\,s-1}\cdot{}_p\overset{q+1}{\Pi}_{s,\,p}+{}_p\overset{q}{\Pi}_{ss}\cdot{}_p\overset{q+2}{\Pi}_{s-1,\,p})}{\beta^2_{p-1,\,s-2}({}_p\overset{q+2}{\Pi}_{s-1,\,s-1}\cdot{}_p\overset{q}{\Pi}_{1,\,1}-{}_p\overset{q+1}{\Pi}{}^2_{1,\,s-1})}\quad\ldots\ldots\ldots\ldots(30).$$

We want here to express the Π's of the numerator by ${}_{p+1}\overset{q}{\Pi}_{s,p+1}$ and those of the denominator by ${}_{p+1}\overset{q}{\Pi}_{1,1}$ and we find the following relations

$$\begin{aligned}
{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}&={}_p\overset{q+1}{\Pi}_{s,\,s-1}\cdot\frac{C_1.C_s}{e_{1s}.e_{1,\,p+1}.e_{ss}},\\
{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}&={}_p\overset{q+1}{\Pi}_{sp}\cdot\frac{C_{p+1}.C_1}{e_{1,\,p+1}.e_{p+1,\,p+1}.e_{1,\,s}},\\
{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}&={}_p\overset{q}{\Pi}_{ss}\cdot\frac{C_s.C_{p+1}}{e_{s,\,p+1}.e_{ss}.e_{p+1,\,p+1}},
\end{aligned}$$

$$ {}_{p+1}\overset{q}{\Pi}_{s,\,p+1} = {}_{p}\overset{q+2}{\Pi}_{s-1,\,p} \cdot \frac{C_1 \,.\, C_1}{e_{1,\,1} \cdot e_{1s} \cdot e_{1,\,p+1}}\,, $$

and

$$ {}_{p+1}\overset{q}{\Pi}_{11} = {}_{p}\overset{q+2}{\Pi}_{s-1,\,s-1} \cdot \frac{C_s \,.\, C_s}{e_{ss} \cdot e_{1s}^2}\,, $$

$$ {}_{p+1}\overset{q}{\Pi}_{11} = {}_{p}\overset{q}{\Pi}_{11} \cdot \frac{C_{p+1}^2}{e_{p+1,\,p+1} \cdot e_{1,\,p+1}^2}\,, $$

$$ {}_{p+1}\overset{q}{\Pi}_{11} = {}_{p}\overset{q+1}{\Pi}_{1,\,s-1} \cdot \frac{C_s \,.\, C_{p+1}}{e_{s,\,p+1} \cdot e_{1,\,p+1} \cdot e_{1s}}\,. $$

Substituting the Π's found from these relations into (30) and eliminating the one factor $\dfrac{{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Pi}_{11}}$ by

$$ \frac{{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Pi}_{11}} = \frac{C_1^2}{C_s \,.\, C_{p+1}} \cdot \frac{e_{s,\,p+1}}{e_{1,\,1}} $$

we get

$$ \frac{{}_{p+1}\overset{q}{\Delta}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Delta}_{11}} = \frac{(-1)^{s+p+1}\beta_{p-1,\,s-1}^2}{\beta_{p-1,\,s-2}} \cdot \frac{\dfrac{1}{e_{1,\,s} \,.\, e_{1,\,p+1}} - \dfrac{1}{e_{1,\,1} \cdot e_{s,\,p+1}}}{\dfrac{1}{e_{s,\,p+1}^2} - \dfrac{1}{e_{ss} \,.\, e_{p+1,\,p+1}}} \cdot \frac{{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Pi}_{1,\,1}}\,, $$

or introducing the values of the e's

$$ \frac{{}_{p+1}\overset{q}{\Delta}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Delta}_{1,\,1}} $$

$$ = \frac{(-1)^{s+p+1}\beta_{p-1,\,s-1}^2}{\beta_{p-1,\,s-2}} \cdot \frac{(2q+2s-3)(2q+2p-1)-(2q-1)(2q+2p+2s-3)}{(2q+2p+2s-3)^2-(2q+4s-5)(2q+4p-1)} \cdot \frac{{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Pi}_{1,\,1}} $$

$$ = (-1)^{s+p+1} \,.\, \beta_{p,\,s-1} \frac{{}_{p+1}\overset{q}{\Pi}_{s,\,p+1}}{{}_{p+1}\overset{q}{\Pi}_{1,\,1}}\,. $$

Now

$$ {}_{p+1}\overset{q}{\Delta}_{1,\,1} = {}_{p}\overset{q+2}{\Delta} = \{1^{p-1} \,.\, 2^{p-2} \ldots\ldots (p-2)^2\,(p-1)\}^2 \,.\, 2^{p\,(p-1)}\; {}_{p+1}\overset{q}{\Pi}_{1,\,1}\,, $$

and hence

$$ {}_{p+1}\overset{q}{\Delta}_{s,\,p+1} = (-1)^{s+p+1} \,.\, \beta_{p,\,s-1}\,\{1^{p-1} \,.\, 2^{p-2} \ldots\ldots (p-2)^2\,(p-1)\}^2 \,.\, 2^{p\,(p-1)}\; {}_{p+1}\overset{q}{\Pi}_{s,\,p+1} $$

in agreement with (25).

(7) It now remains to prove that (25) holds for ${}_{p+1}\overset{q}{\Delta}_{s,r}$ when both s and r are different from 1 and $p+1$, and r different from s.

For this shall be used the relation

$$ \Delta \,.\, \Delta_{sss's''} = \Delta_{ss} \,.\, \Delta_{s's''} - \Delta_{ss'} \,.\, \Delta_{ss''} $$

between an orthosymmetrical determinant and its minors.

Putting $\Delta = {}_{p+1}\Delta$, $s = p+1$, $s' = r$ and $s'' = s$ and solving the equation with regard to ${}_{p+1}\Delta_{r,\,s}$ we have

$$ {}_{p+1}\Delta_{r,s} = \frac{1}{{}_{p+1}\Delta_{p+1,\,p+1}} \left({}_{p+1}\Delta \,.\, {}_{p+1}\Delta_{p+1,\,p+1,\,r,\,s} + {}_{p+1}\Delta_{p+1,\,r} \,.\, {}_{p+1}\Delta_{p+1,\,s} \right), $$

where

$$ {}_{p+1}\Delta_{p+1,\,p+1,\,r,\,s} = {}_{p}\Delta_{r,\,s}\,. $$

Evaluating this by (24) and (25) we get

$$ {}_{p+1}\overset{q}{\Delta}_{r,s} = \frac{(-1)^{r+s}}{{}_p\overset{q}{\Pi}} \{1^{p-1} \cdot 2^{p-2} \ldots\ldots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} $$

$$ \times [4p^2 \beta_{p-1, s-1} \cdot \beta_{p-1, r-1} \cdot {}_{p+1}\overset{q}{\Pi} \cdot {}_p\overset{q}{\Pi}_{r,s} + \beta_{p, s-1} \cdot \beta_{p, r-1} \cdot {}_{p+1}\overset{q}{\Pi}_{p+1, r} \cdot {}_{p+1}\overset{q}{\Pi}_{p+1, s}] \ldots (31). $$

But
$$ {}_p\overset{q}{\Pi}_{r,s} = {}_{p+1}\overset{q}{\Pi}_{r,s} \frac{e_{p+1, p+1} \cdot e_{r, p+1} \cdot e_{s, p+1}}{C_{p+1}^2}, $$

$$ {}_{p+1}\overset{q}{\Pi}_{p+1, r} = {}_{p+1}\overset{q}{\Pi}_{r,s} \frac{C_s}{C_{p+1}} \cdot \frac{e_{r, p+1}}{e_{r,s}}, $$

and
$$ {}_{p+1}\overset{q}{\Pi} = {}_p\overset{q}{\Pi} \frac{C_{p+1}^2}{e_{p+1, p+1}}, $$

$$ {}_{p+1}\overset{q}{\Pi}_{p+1, s} = {}_p\overset{q}{\Pi} \frac{C_{p+1}}{C_s} \cdot \frac{e_{s, p+1}}{e_{p+1, p+1}}. $$

Substituting these values in (31) we find

$$ {}_{p+1}\overset{q}{\Delta}_{r,s} = (-1)^{r+s} \{1^{p-1} \cdot 2^{p-2} \ldots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} \cdot \beta_{p-1, s-1} \cdot \beta_{p-1, r-1} \cdot p^2 \cdot {}_{p+1}\overset{q}{\Pi}_{r,s} \times \frac{\left[4 + \dfrac{1}{(p-s+1)(p-r+1)\, e_{r,s} \cdot e_{p+1, p+1}}\right]}{\dfrac{1}{e_{r, p+1}} \cdot \dfrac{1}{e_{s, p+1}}}, $$

and as the last fraction equals

$$ \frac{1}{(p-s+1)(p-r+1)}, $$

$$ {}_{p+1}\overset{q}{\Delta}_{r,s} = (-1)^{r+s} \beta_{p, s-1} \cdot \beta_{p, r-1} \{1^{p-1} \cdot 2^{p-2} \ldots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)}\, {}_{p+1}\overset{q}{\Pi}_{r,s}, $$

with which the proof by induction for (25) is carried through.

(8) We shall now return to (23). It consists of $2p+1$ terms of which the $(2r+1)$st originally was found as $({}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2)$ so that

$$ {}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N} \frac{\begin{vmatrix} 1 & 1 & \frac{1}{3} & \ldots\ldots & \frac{1}{2r-1} \\ x^2 & \frac{1}{3} & \frac{1}{5} & \ldots\ldots & \frac{1}{2r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2r} & \frac{1}{2r+1} & \frac{1}{2r+3} & \ldots\ldots & \frac{1}{4r-1} \end{vmatrix}^2}{\begin{vmatrix} 1 & \frac{1}{3} & \ldots\ldots & \frac{1}{2r-1} \\ \frac{1}{3} & \frac{1}{5} & \ldots\ldots & \frac{1}{2r+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2r-1} & \frac{1}{2r+1} & \ldots\ldots & \frac{1}{4r-3} \end{vmatrix} \cdot \begin{vmatrix} 1 & \frac{1}{3} & \ldots\ldots & \frac{1}{2r+1} \\ \frac{1}{3} & \frac{1}{5} & \ldots\ldots & \frac{1}{2r+3} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2r+1} & \frac{1}{2r+3} & \ldots\ldots & \frac{1}{4r+1} \end{vmatrix}} $$

and

$$ {}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 x^2}{N} \frac{\begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \cdots\cdots & \frac{1}{2r-1} \\ x^2 & \frac{1}{5} & \frac{1}{7} & \cdots\cdots & \frac{1}{2r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2r-2} & \frac{1}{2r+1} & \frac{1}{2r+3} & \cdots\cdots & \frac{1}{4r-3} \end{vmatrix}^2}{\begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \cdots\cdots & \frac{1}{2r-1} \\ \frac{1}{5} & \frac{1}{7} & \cdots\cdots & \frac{1}{2r+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2r-1} & \frac{1}{2r+1} & \cdots\cdots & \frac{1}{4r-5} \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \cdots\cdots & \frac{1}{2r+1} \\ \frac{1}{5} & \frac{1}{7} & \cdots\cdots & \frac{1}{2r+3} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2r+1} & \frac{1}{2r+3} & \cdots\cdots & \frac{1}{4r-1} \end{vmatrix}}. $$

With the notations later adopted we therefore find

$$ {}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N} \frac{\left\{ \overset{s=r}{\underset{s=0}{S}} [x^{2s} \cdot {}_{r+1}\overset{1}{\Delta}_{s+1,\, r+1}] \right\}^2}{{}_r\overset{1}{\Delta} \cdot {}_{r+1}\overset{1}{\Delta}} $$

and

$$ {}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 x^2}{N} \frac{\left\{ \overset{s=r-1}{\underset{s=0}{S}} [x^{2s} \cdot {}_r\overset{2}{\Delta}_{s+1,\, r}] \right\}^2}{{}_{r-1}\overset{2}{\Delta} \cdot {}_r\overset{2}{\Delta}}. $$

Substituting the values for Δ's from (24) and (25) we get

$$ {}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N \cdot 2^{2r} \cdot (\underline{|r})^2} \left\{ \overset{s=r}{\underset{s=0}{S}} \left[(-1)^s \beta_{r,\, s} x^{2s} \frac{{}_{r+1}\overset{1}{\Pi}_{s+1,\, r+1}}{\sqrt{{}_r\overset{1}{\Pi} \cdot {}_{r+1}\overset{1}{\Pi}}} \right] \right\}^2 $$

and

$$ {}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 x^2}{N \cdot 2^{2r-2} \cdot (\underline{|r-1})^2} \left\{ \overset{s=r-1}{\underset{s=0}{S}} \left[(-1)^s \beta_{r-1,\, s} x^{2s} \frac{{}_r\overset{2}{\Pi}_{s+1,\, r}}{\sqrt{{}_{r-1}\overset{2}{\Pi} \cdot {}_r\overset{2}{\Pi}}} \right] \right\}^2, $$

or, as

$$ \frac{{}_{r+1}\overset{1}{\Pi}_{s+1,\, r+1}}{\sqrt{{}_r\overset{1}{\Pi} \cdot {}_{r+1}\overset{1}{\Pi}}} = \frac{e_{s+1,\, r+1}}{C_{s+1} \cdot \sqrt{e_{r+1,\, r+1}}} = \sqrt{4r+1} \cdot (2s+1)(2s+3) \ldots\ldots (2s+2r-1) $$

and

$$ \frac{{}_r\overset{2}{\Pi}_{s+1,\, r}}{\sqrt{{}_{r-1}\overset{2}{\Pi} \cdot {}_r\overset{2}{\Pi}}} = \frac{e_{s+1,\, r}}{C_{s+1} \cdot \sqrt{e_{r,\, r}}} = \sqrt{4r-1} \cdot (2s+3)(2s+5) \ldots\ldots (2s+2r-1)^*, $$

$$ {}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2 (4r+1)}{N (\underline{|r})^2 \cdot 2^{2r}} \left\{ \overset{s=r}{\underset{s=0}{S}} [(-1)^s \beta_{r,\, s} x^{2s} (2s+1)(2s+3) \ldots\ldots (2s+2r-1)] \right\}^2 \quad \ldots\ldots\ldots(32), $$

* The e's and C do not of course have the same value in the two equations as they represent columns and elements in two different determinants.

and

$$ {}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 (4r-1)\, x^2}{N\,(\underline{|r-1})^2 . 2^{2r-2}} \left\{ \overset{s=r-1}{\underset{s=0}{S}} [(-1)^s \beta_{r-1,\,s} x^{2s} (2s+3)(2s+5) \ldots\ldots (2s+2r-1)] \right\}^2 \ldots\ldots\ldots\ldots (33), $$

which enables us to form ${}_n\sigma_y^2$ by successive summations from ${}_0\sigma_y^2 = \frac{\sigma^2}{N}$.

Before investigating the curve for ${}_n\sigma_y^2$ for a special n we shall first look at ${}_n\sigma^2$ for $x = 0$ and $x = \pm 1$.

(9) From (33) we see that when $x = 0$

$$ {}_{2r-1}\sigma_y^2 = {}_{2r-2}\sigma_y^2, $$

${}_{2p}\sigma_y^2$ is for $x = 0$ most easily evaluated from the formula (13).

Remembering that in our case $m_{2r} = \frac{1}{2r+1}$ we find from this

$$ {}_{2p}\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N} \frac{{}_{p+1}\overset{1}{\Delta}_{1,\,1}}{{}_{p+1}\overset{1}{\Delta}}, $$

and hence by (24) and (25)

$$ {}_{2p+1}\overset{x=0}{\sigma_y^2} = {}_{2p}\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N} \left\{ \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \ldots\ldots \frac{2p+1}{2p} \right\}^2 \ldots\ldots\ldots\ldots\ldots\ldots (34). $$

(10) To evaluate ${}_n\sigma_y^2$ for $x = \pm 1$ we use (32) and (33). The sum in (32) may be considered as

$$ S_{2r} = \frac{d}{dx} \frac{1}{x} \frac{d}{dx} \ldots\ldots \frac{1}{x} \frac{d\,\{x^{2r-1} (x^2-1)^r\}}{dx} $$

with a number r of differentiations. If these operations are undertaken directly upon $x^{2r-1}(x^2-1)^r$ the result is

$$ \alpha_r (x^2-1)^r + \alpha_{r-1} (x^2-1)^{r-1} + \ldots\ldots \alpha_1 (x^2-1) + \alpha_0, $$

of which only $\alpha_0 = 2r\,(2r-2) \ldots\ldots 4 . 2 = \underline{|r} . 2^r$

remains for $x = \pm 1$.

Corresponding to this the sum in (33) comes out from

$$ S_{2r-1} = \frac{d}{dx} \frac{1}{x} \frac{d}{dx} \ldots\ldots \frac{d}{dx} \frac{1}{x} \frac{d\,\{x^{2r-1} (x^2-1)^{r-1}\}}{dx} $$

by taking $(r-1)$ differentiations and therefore

S_{2r-1} equals, for $x = \pm 1$, $(2r-2)(2r-4) \ldots\ldots 4 . 2 = \underline{|r-1} . 2^{r-1}$.

Hence
$$ {}_{2r}\overset{x^2=1}{\sigma_y^2} - {}_{2r-1}\overset{x^2=1}{\sigma_y^2} = \frac{\sigma^2}{N}(4r+1) $$

and
$$ {}_{2r-1}\overset{x^2=1}{\sigma_y^2} - {}_{2r-2}\overset{x^2=1}{\sigma_y^2} = \frac{\sigma^2}{N}(4r-1), $$

or since
$$ {}_0\sigma_y^2 = \frac{\sigma^2}{N}, $$

$$ {}_n\overset{x^2=1}{\sigma_y^2} = \frac{\sigma^2}{N} \{1 + 3 + 5 + \ldots\ldots (2n+1)\}, $$

$$ {}_n\overset{x^2=1}{\sigma_y^2} = \frac{\sigma^2}{N} (n+1)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35). $$

(11) In Section I under (7) it was found that

$$\int \frac{\psi(x)}{f(x)} {}_n\sigma_y^2 \, dx = \sigma^2 (n+1)$$

when the integration was taken over the places of observation. For the present distribution $f(x)$ is 1, $\psi(x)$ constant and $\int \psi(x)\, dx = N$, hence the mean of ${}_n\sigma_y^2$ in the range of observations is for a uniform continuous distribution

$$\frac{\sigma^2}{N}(n+1).$$

For the grouped observations in Section II we find by integration of the formulae for functions from the first to the sixth degree that

$$\frac{1}{2}\int_{-1}^{1} {}_n\sigma_y^2 \, dx = \frac{\sigma^2}{N}(n+1)\left(1 - \frac{1}{2n+1}\right).$$

IV. *Uniform continuous distribution of observations with constant standard deviation. Special formulae.*

(1) Let ${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2$ be indicated by S_n, then the formulae (32) and (33) give us

$$\left.\begin{aligned}
S_1 &= \frac{\sigma^2}{N} \cdot 3x^2 \\
S_2 &= \frac{\sigma^2}{N} \cdot \frac{5}{4}(1-3x^2)^2 \\
S_3 &= \frac{\sigma^2}{N} \cdot \frac{7}{4} x^2 (3-5x^2)^2 \\
S_4 &= \frac{\sigma^2}{N} \cdot \frac{9}{64}(3-30x^2+35x^4)^2 \\
S_5 &= \frac{\sigma^2}{N} \cdot \frac{11x^2}{64}(15-70x^2+63x^4)^2 \\
S_6 &= \frac{\sigma^2}{N} \cdot \frac{13}{9\times 256}(15-315x^2+945x^4-693x^6)^2
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(36),$$

from which we form ${}_n\sigma_y^2$ beginning with

$$\begin{aligned}
{}_0\sigma_y^2 &= \frac{\sigma^2}{N} \\
{}_1\sigma_y^2 &= \frac{\sigma^2}{N}(1+3x^2) \\
{}_2\sigma_y^2 &= \frac{\sigma^2}{N}\left(1+3x^2+\tfrac{5}{4}(1-3x^2)^2\right) = \frac{\sigma^2}{N} \cdot \frac{9}{4}(1-2x^2+5x^4),
\end{aligned}$$

and further in the same way

$$\left.\begin{aligned}
{}_3\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{1}{4}(9+45x^2-165x^4+175x^6) \\
{}_4\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{25}{64}(9-36x^2+294x^4-644x^6+441x^8) \\
{}_5\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{9}{64}(25+175x^2-1750x^4+6510x^6-9555x^8+4851x^{10}) \\
{}_6\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{7}{256}(175-1050x^2+17325x^4-93660x^6+225225x^8+ \\
&\qquad - 245322x^{10}+99099x^{12})
\end{aligned}\right\} \ldots(37).$$

(2) Since ${}_n\sigma_y^2 = {}_{n-1}\sigma_y^2 + S_n$ the curve for ${}_n\sigma_y^2$ is entirely above the ${}_{n-1}\sigma_y^2$ curve except where $S_n = 0$.

Solving the equations $S_n = 0$ the following roots are found:

For $S_1 = 0$ $\quad x = 0$

,, $S_2 = 0$ $\quad x = \pm\sqrt{\tfrac{1}{3}} = \pm\,{\cdot}5773$

,, $S_3 = 0$ $\quad x = 0 \quad x = \pm\sqrt{\tfrac{3}{5}} = \pm\,{\cdot}7746$

,, $S_4 = 0$ $\quad x = \pm\sqrt{\dfrac{15 \pm 2\sqrt{30}}{35}} = \pm\begin{cases}{\cdot}8611\\{\cdot}3400\end{cases}$

,, $S_5 = 0$ $\quad x = 0 \quad x = \pm\sqrt{\dfrac{35 \pm 2\sqrt{70}}{63}} = \pm\begin{cases}{\cdot}9030\\{\cdot}5438\end{cases}$

,, $S_6 = 0$ $\quad x = \pm\begin{cases}{\cdot}2386\\{\cdot}6612\\{\cdot}9325\end{cases}$

Since all the roots are rational and all lie between -1 and $+1$, ${}_n\sigma_y^2$ therefore equals ${}_{n-1}\sigma_y^2$ for n values of x all of which are inside the range of the observations.

The adjusted values of the functions at these abscissae appear to be of special interest since they are uncorrelated as was shown in Section I under (9).

(3) Looking at Diagram 2, representing the curves of ${}_n\sigma_y$ up to $n = 6$, it is seen, as was also clear from the formula for $\overset{x=0}{\sigma_y^2}$ and $\overset{x^2=1}{\sigma_y^2}$ given in the last section, that while the standard deviation in the middle of the range increases slowly with the degree of function it increases very rapidly at the ends of the range. At $x = 0$ the curve has a minimum when the degree of function is odd and a maximum when it is even. Besides that the curve has $(2n - 2)$ maxima and minima between -1 and 1. As the curve for ${}_n\sigma_y^2$ is of the $2n$th degree, ${}_n\sigma_y^2$ is therefore increasing for x increasing above 1 or for x decreasing below -1.

The abscissae of the maxima and minima are given in the following table.

Degree of function	Abscissae of maxima	Abscissae of minima
1		0
2	0	$\pm\sqrt{\tfrac{1}{5}} = \pm\,{\cdot}4472$
3	$\pm\sqrt{\tfrac{1}{5}} = \pm\,{\cdot}4472$	$\begin{cases}0\\\pm\sqrt{\tfrac{3}{7}} = \pm\,{\cdot}6547\end{cases}$
4	$\begin{cases}0\\\pm\sqrt{\tfrac{3}{7}} = \pm\,{\cdot}6547\end{cases}$	$\pm\sqrt{\dfrac{7 \pm 2\sqrt{7}}{21}} = \pm\begin{cases}{\cdot}7651\\{\cdot}2852\end{cases}$
5	$\pm\sqrt{\dfrac{7 \pm 2\sqrt{7}}{21}} = \pm\begin{cases}{\cdot}7651\\{\cdot}2852\end{cases}$	$\begin{cases}0\\\pm\sqrt{\dfrac{15 \pm 2\sqrt{15}}{33}} = \pm\begin{cases}{\cdot}8302\\{\cdot}4689\end{cases}\end{cases}$
6	$\begin{cases}0\\\pm\sqrt{\dfrac{15 \pm 2\sqrt{15}}{33}} = \pm\begin{cases}{\cdot}8302\\{\cdot}4689\end{cases}\end{cases}$	$\pm\begin{cases}{\cdot}8718\\{\cdot}5917\\{\cdot}2093\end{cases}$

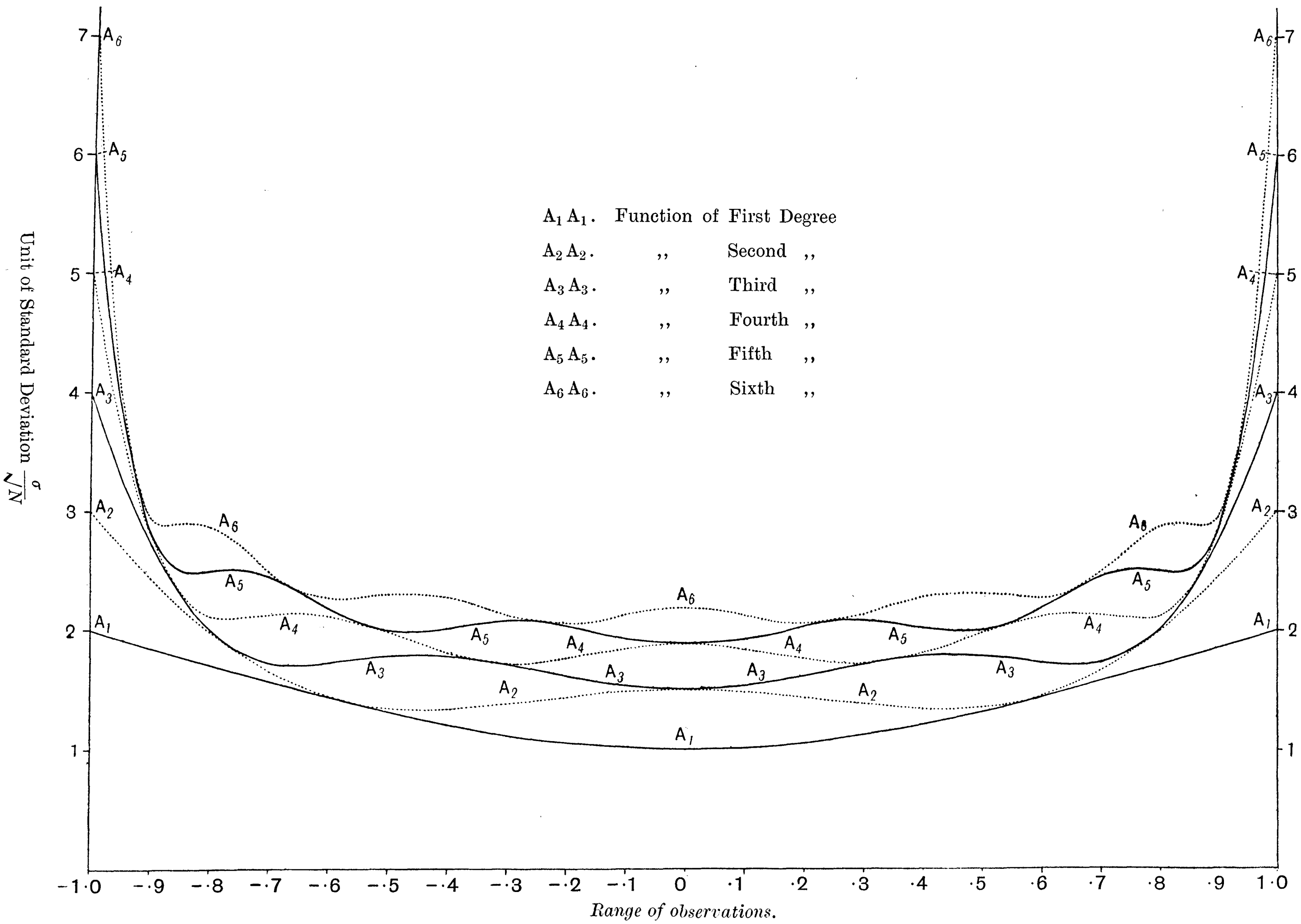


DIAGRAM 2. Curves of Standard Deviations. Uniform Distribution.

Hence the curve for ${}_{2p+1}\sigma_y^2$ has a maximum for the abscissae at which ${}_{2p}\sigma_y^2$ has a minimum. A comparison with the results in Section II shows that the abscissae of the maxima found here are the same as those of the best places of observation for $(n+1)$ equally big groups of observations of a function of the nth degree. These places tally with the places where ${}_n\sigma_y^2$ was a maximum. Thus if we imagine that we had started the investigations with a uniform distribution of observations, and to lower the maxima of the curve of standard deviation had put clusters of observations at those maxima and at the ends of the range we should not get the best curve of standard deviation till all the observations of the continuous distribution had been distributed at the $n-1$ places of maxima and at 1 and -1.

The minima of the standard deviations obtained from a uniform continuous distribution and the $(n+1)$ best groups of observations do not fall at the same abscissae.

(4) The curves are very far from our ideal of a constant standard deviation throughout the range. To obtain the same maximum of standard deviation as $(n+1)$ groups could give us we should have to limit the part of the range used to the following fractions of the range:

for 1st degree	·58
,, 2nd ,,	·73
,, 3rd ,,	·80
,, 4th ,,	·84
,, 5th ,,	·83
,, 6th ,,	·73

It is not likely that the range of values of the function which we investigate would only be of interest inside a range so much smaller than that within which we might actually observe; further it seems likely that observations all of which were taken inside the smaller part of the range would give better information for that special interval. I shall therefore examine in the following sections if a uniform distribution of observations to which is added clusters of observations at the ends of the range will not possibly give a more satisfactory curve of standard deviations.

V. *Uniform continuous distribution of observations with additional observations clustered at the ends of the range; constant standard deviation of observations. General formulae.*

(1) Suppose we have $N \cdot \frac{1}{1+a}$ observations uniformly distributed from -1 to 1 and besides $\frac{N}{2} \cdot \frac{a}{1+a}$ observations at -1 and the same number at 1. We then have

$$\mu_{2r} = \frac{1}{N}\left\{\int_{-1}^{1} \frac{Nx^{2r}}{2(1+a)}\,dx + \frac{Na}{1+a}\right\}$$

or $$\mu_{2r} = \frac{1}{1+\alpha}\left(\frac{1}{2r+1} + \alpha\right)$$

and $$\mu_{2r+1} = 0.$$

According to (13) and (14) we find,

$$
{}_{2p}\sigma_y^2 = \frac{-\sigma^2(1+\alpha)}{N}\left\{\frac{\begin{vmatrix} 0 & 1 & x^2 & \cdots\cdots & x^{2p} \\ 1 & 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ x^2 & \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p+1}+\alpha \end{vmatrix}}{\begin{vmatrix} 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p+1}+\alpha \end{vmatrix}} + x^2\frac{\begin{vmatrix} 0 & 1 & x^2 & \cdots\cdots & x^{2p-2} \\ 1 & \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ x^2 & \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p-1}+\alpha \end{vmatrix}}{\begin{vmatrix} \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p-1}+\alpha \end{vmatrix}}\right\} \quad \ldots (38)
$$

and

$$
{}_{2p-1}\sigma_y^2 = \frac{-\sigma^2(1+\alpha)}{N}\left\{ \frac{\begin{vmatrix} 0 & 1 & x^2 & \cdots\cdots & x^{2p-2} \\ 1 & 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p-1}+\alpha \\ x^2 & \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & \frac{1}{2p-1}+\alpha & \frac{1}{2p+1}+\alpha & \cdots\cdots & \frac{1}{4p-3}+\alpha \end{vmatrix}}{\begin{vmatrix} 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p-1}+\alpha \\ \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p-1}+\alpha & \frac{1}{2p+1}+\alpha & \cdots\cdots & \frac{1}{4p-3}+\alpha \end{vmatrix}} + x^2 \frac{\begin{vmatrix} 0 & 1 & x^2 & \cdots\cdots & x^{2p-2} \\ 1 & \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ x^2 & \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p-1}+\alpha \end{vmatrix}}{\begin{vmatrix} \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p-1}+\alpha \end{vmatrix}} \right\} \quad \text{............(39)},
$$

and, according to I under (6),

$$ {}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 = \frac{\sigma^2}{N}(1+\alpha)\times \frac{\begin{vmatrix} 1 & 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p-1}+\alpha \\ x^2 & \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ x^4 & \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p-1}+\alpha \end{vmatrix}^2}{\begin{vmatrix} 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p-1}+\alpha \\ \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p-1}+\alpha & \frac{1}{2p+1}+\alpha & \cdots\cdots & \frac{1}{4p-3}+\alpha \end{vmatrix} \cdot \begin{vmatrix} 1+\alpha & \frac{1}{3}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p+1}+\alpha \end{vmatrix}} \quad \ldots\ldots\ldots\ldots(40) $$

and

$$ {}_{2p+1}\sigma^2 - {}_{2p}\sigma_y^2 = \frac{\sigma^2(1+\alpha)}{N}x^2\times \frac{\begin{vmatrix} 1 & \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ x^2 & \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ x^4 & \frac{1}{7}+\alpha & \frac{1}{9}+\alpha & \cdots\cdots & \frac{1}{2p+5}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \frac{1}{2p+3}+\alpha & \frac{1}{2p+5}+\alpha & \cdots\cdots & \frac{1}{4p+1}+\alpha \end{vmatrix}^2}{\begin{vmatrix} \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+1}+\alpha \\ \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+\alpha & \frac{1}{2p+3}+\alpha & \cdots\cdots & \frac{1}{4p-1}+\alpha \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{3}+\alpha & \frac{1}{5}+\alpha & \cdots\cdots & \frac{1}{2p+3}+\alpha \\ \frac{1}{5}+\alpha & \frac{1}{7}+\alpha & \cdots\cdots & \frac{1}{2p+5}+\alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+3}+\alpha & \frac{1}{2p+5}+\alpha & \cdots\cdots & \frac{1}{4p+3}+\alpha \end{vmatrix}} \quad \ldots\ldots\ldots\ldots(41). $$

(2) For the reduction of these formulae we have to evaluate the determinant of pth order

$$
{}_p\overset{q}{\delta} = \begin{vmatrix}
\frac{1}{2q-1}+\alpha & \frac{1}{2q+1}+\alpha & \cdots\cdots & \frac{1}{2q+2p-3}+\alpha \\
\frac{1}{2q+1}+\alpha & \frac{1}{2q+3}+\alpha & \cdots\cdots & \frac{1}{2q+2p-1}+\alpha \\
\vdots & \vdots & & \vdots \\
\frac{1}{2q+2p-3}+\alpha & \frac{1}{2q+2p-1}+\alpha & \cdots\cdots & \frac{1}{2q+4p-5}+\alpha
\end{vmatrix}.
$$

By subtracting from the elements of each row the elements of the proceeding and leaving the first row as it is, it is transformed to

$$
{}_p\overset{q}{\delta} = (-1)^{p-1} \times
\begin{vmatrix}
\frac{1}{2q-1}+\alpha & \frac{1}{2q+1}+\alpha & \cdots\cdots & \frac{1}{2q+2p-3}+\alpha \\
\frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots\cdots & \frac{2}{(2q+2p-3)(2q+2p-1)} \\
\vdots & \vdots & & \vdots \\
\frac{2}{(2q+2p-5)(2q+2p-3)} & \frac{2}{(2q+2p-3)(2q+2p-1)} & \cdots\cdots & \frac{2}{(2q+4p-7)(2q+4p-5)}
\end{vmatrix},
$$

which when the columns undergo the same process takes the form

$$
{}_p\overset{q}{\delta} =
\begin{vmatrix}
\frac{1}{2q-1}+\alpha & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-5)(2q+2p-3)} \\
\frac{2}{(2q-1)(2q+1)} & \frac{2.4}{(2q-1)(2q+1)(2q+3)} & \frac{2.4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2.4}{(2q+2p-5)\ldots(2q+2p-1)} \\
\frac{2}{(2q+1)(2q+3)} & \frac{2.4}{(2q+1)(2q+3)(2q+5)} & \frac{2.4}{(2q+3)(2q+5)(2q+7)} & \cdots & \frac{2.4}{(2q+2p-3)\ldots(2q+2p+1)} \\
\vdots & \vdots & \vdots & & \vdots \\
\frac{2}{(2q+2p-5)(2q+2p-3)} & \frac{2.4}{(2q+2p-5)\ldots(2q+2p-1)} & \frac{2.4}{(2q+2p-3)\ldots(2q+2p+1)} & \cdots & \frac{2.4}{(2q+4p-9)\ldots(2q-4p-5)}
\end{vmatrix}.
$$

Let us introduce the notation

$$
{}_p\overset{q}{D} =
\begin{vmatrix}
\frac{1}{(2q-1)(2q+1)(2q+3)} & \frac{1}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{1}{(2q+2p-3)\ldots(2q+2p+1)} \\
\frac{1}{(2q+1)(2q+3)(2q+5)} & \frac{1}{(2q+3)(2q+5)(2q+7)} & \cdots & \frac{1}{(2q+2p-1)\ldots(2q+2p+3)} \\
\vdots & \vdots & & \vdots \\
\frac{1}{(2q+2p-3)\ldots(2q+2p+1)} & \frac{1}{(2q+2p-1)\ldots(2q+2p+3)} & \cdots & \frac{1}{(2q+4p-5)\ldots(2q+4p-1)}
\end{vmatrix}.
$$

Then, since for $\alpha = 0$ ${}_p\overset{q}{\delta}$ equals the determinant ${}_p\overset{q}{\Delta}$, we have

$$
{}_p\overset{q}{\delta} = {}_p\overset{q}{\Delta} + \alpha \,.\, 2^{3(p-1)} \,.\, {}_{p-1}\overset{q}{D} \qquad \ldots\ldots(42),
$$

and the problem is reduced to the evaluation of ${}_p\overset{q}{D}$.

(3) It shall be proved by induction that

$$ {}_{p}\overset{q}{D} = \frac{\{1^p \,.\, 2^{p-1} \ldots (p-1)^2 \,.\, p\}^2 \,.\, 2^{p(p-2)}\,(p+1)}{(2q-1)(2q+1)^2(2q+3)^3 \ldots (2q+2p-5)^{p-1}(2q+2p-3)^p(2q+2p-1)^p(2q+2p+1)^p(2q+2p+3)^{p-1} \ldots (2q+4p-3)^2(2q+4p-1)} \quad \ldots\ldots\ldots\ldots\ldots (43). $$

It contains the $2p+1$ different factors of the elements with indices increasing from 1 at the extreme to p in the middle so that the three factors of which the one diagonal line of the determinant consists occur with the index p.

For $p = 1$ the formula gives

$$ {}_{1}\overset{q}{D} = \frac{1}{(2q-1)(2q+1)(2q+3)} $$

as it ought to.

As the determinant is orthosymmetrical the relation

$$ \Delta = \frac{\Delta_{ss} \,.\, \Delta_{s's'} - \Delta_{ss'}^2}{\Delta_{sss's'}} \text{ holds.} $$

Applied on ${}_{p+1}\overset{q}{D}$ for $s = 1$ and $s' = p+1$ it may be written

$$ {}_{p+1}\overset{q}{D} = \frac{{}_{p}\overset{q}{D} \,.\, {}_{p}\overset{q+2}{D} - {}_{p}\overset{q+1}{D}{}^2}{{}_{p-1}\overset{q+2}{D}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44). $$

Looking first at the numerator of (43) we see that it has the same value for the two terms of the numerator of (44), and divided by the corresponding factor of ${}_{p-1}\overset{q+2}{D}$ it becomes

$$ \left\{\frac{1^{2p} \,.\, 2^{2(p-1)} \ldots\ldots (p-2)^6 (p-1)^4 p^2}{1^{p-1} \,.\, 2^{p-2} \ldots\ldots (p-2)^2 (p-1)}\right\}^2 \frac{(p+1)^2}{p} \,.\, 2^{2p(p-2)-(p-1)(p-3)} $$
$$ = \{1^{p+1} \,.\, 2^p \ldots\ldots (p-2)^4 (p-1)^3 p^2 (p+1)\}^2 \frac{1}{p} \,.\, 2^{p^2-3}. $$

To evaluate the factor in ${}_{p+1}\overset{q}{D}$ arising from the denominator of (43) we shall give a table of the indices with which the different factors occur in the D's and their ratios.

	$2q-1$	$2q+1$	$2q+3$	...	$2q+2p-5$	$2q+2p-3$	$2q+2p-1$	$2q+2p+1$	$2q+2p+3$	$2q+2p+5$	$2q+2p+7$	...	$2q+4p-1$	$2q+4p+1$	$2q+4p+3$
${}_{p}\overset{q}{D}$	1	2	3	...	$p-1$	p	p	p	$p-1$	$p-2$	$p-3$	...	1	—	—
${}_{p}\overset{q+2}{D}$	—	—	1	...	$p-3$	$p-2$	$p-1$	p	p	p	$p-1$	...	3	2	1
${}_{p-1}\overset{q+2}{D}$	—	—	1	...	$p-3$	$p-2$	$p-1$	$p-1$	$p-1$	$p-2$	$p-3$	...	1	—	—
${}_{p}\overset{q+1}{D}{}^2$	—	2	4	...	$2(p-2)$	$2(p-1)$	$2p$	$2p$	$2p$	$2(p-1)$	$2(p-2)$	...	4	2	—
$\frac{{}_{p}\overset{q}{D} \,.\, {}_{p}\overset{q+2}{D}}{{}_{p-1}\overset{q+2}{D}}$	1	2	3	...	$p-1$	p	p	$p+1$	p	p	$p-1$	...	3	2	1
$\frac{{}_{p}\overset{q+1}{D}{}^2}{{}_{p-1}\overset{q+2}{D}}$	—	2	3	...	$p-1$	p	$p+1$	$p+1$	$p+1$	p	$p-1$	...	3	2	—

Hence the factor arising from the denominator of (43) is

$$\frac{(2q+2p-1)(2q+2p+3)-(2q-1)(2q+4p+3)}{(2q-1)(2q+1)^2\ldots(2q+2p-3)^p(2q+2p-1)^{p+1}(2q+2p+1)^{p+1}(2q+2p+3)^{p+1}(2q+2p+5)^p\ldots(2q+4p+1)^2(2q+4p+3)}.$$

The numerator of this equals $4p(p+2)$,

multiplying with the factor previously found we therefore get

$$ {}_{p+1}\overset{q}{D} = \frac{\{1^{p+1}.\,2^p\ldots(p-1)^3p^2(p+1)\}^2.\,2^{(p+1)(p-1)}.\,(p+2)}{(2q-1)(2q+1)^2\ldots(2q+2p-3)^p(2q+2p-1)^{p+1}(2q+2p+1)^{p+1}(2q+2p+3)^{p+1}(2q+2p+5)^p\ldots(2q+4p+1)^2(2q+4p+3)},$$

which is what we wanted to prove.

(4) When the values of Δ and D are introduced in (42) we get

$$ {}_{p}\overset{q}{\delta} = \frac{\{1^{p-1}.\,2^{p-2}\ldots(p-2)^2(p-1)\}^2.\,2^{p(p-1)}}{(2q-1)(2q+1)^2\ldots(2q+2p-5)^{p-1}(2q+2p-3)^p(2q+2p-1)^{p-1}\ldots(2q+4p-7)^2(2q+4p-5)}$$

$$+\,\alpha\,.\,2^{3(p-1)}\times \frac{\{1^{p-1}.\,2^{p-2}\ldots(p-2)^2(p-1)\}^2.\,2^{(p-1)(p-3)}.\,p}{(2q-1)(2q+1)^2\ldots(2q+2p-7)^{p-2}(2q+2p-5)^{p-1}(2q+2p-3)^{p-1}(2q+2p-1)^{p-1}(2q+2p+1)^{p-2}\ldots(2q+4p-7)^2(2q+4p-5)}$$

or

$$ {}_{p}\overset{q}{\delta} = \frac{\{1^{p-1}.\,2^{p-2}\ldots(p-2)^2(p-1)\}^2.\,2^{p(p-1)}[1+\alpha p(2q+2p-3)]}{(2q-1)(2q+1)^2\ldots(2q+2p-5)^{p-1}(2q+2p-3)^p(2q+2p-1)^{p-1}\ldots(2q+4p-7)^2(2q+4p-5)} \quad\ldots\ldots\ldots\ldots(45).$$

The denominators of the formulae (38)—(41) for ${}_n\sigma_y^2$ are now known since they only consist of the factors ${}_p\overset{1}{\delta}$ and ${}_p\overset{2}{\delta}$. To be able to write down the general expression for ${}_n\sigma_y^2$ we should have to evaluate the minors of δ, but their form is so complicated that a direct calculation of the determinants for the degrees of function in question appears to be simpler. With the material in hand we are however able to determine ${}_n\sigma_y^2$ for $x=0$ and $x^2=1$.

(5) From (38) and (39) we see that

$$ {}_{2p}\overset{x=0}{\sigma_y^2} = {}_{2p+1}\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N}\frac{{}_p\overset{3}{\delta}}{{}_{p+1}\overset{1}{\delta}}(1+\alpha),$$ and with the δ's as given by (45)

$$ {}_{2p}\overset{x=0}{\sigma_y^2} = {}_{2p+1}\overset{x=0}{\sigma_y^2} =$$

$$\frac{\sigma^2(1+\alpha)[1+\alpha p(2p+3)]\,1\,.\,3^2\,.\,5^3\,.\,7^4\,.\,9^5\ldots(2p-1)^p(2p+1)^{p+1}(2p+3)^p(2p+5)^{p-1}\ldots(4p-1)^2(4p+1)}{N\{1\,.\,2\,.\,3\ldots p\}^2.\,2^{2p}.\,[1+\alpha(p+1)(2p+1)]\,5\,.\,7^2\,.\,9^3\ldots(2p-1)^{p-2}(2p+1)^{p-1}(2p+3)^p(2p+5)^{p-1}\ldots(4p-1)^2(4p+1)}$$

$$=\frac{\sigma^2(1+\alpha)\,3^2\,.\,5^2\ldots(2p-1)^2(2p+1)^2.\,[1+\alpha p(2p+3)]}{N\{1\,.\,2\,.\,3\ldots p\}^2.\,2^{2p}.\,[1+\alpha(p+1)(2p+1)]}$$

or

$$ {}_{2p}\overset{x=0}{\sigma_y^2} = {}_{2p+1}\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N}\left\{\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2p-1}{2p-2}\cdot\frac{2p+1}{2p}\right\}^2\frac{(1+\alpha)[1+\alpha p(2p+3)]}{[1+\alpha(p+1)(2p+1)]}\ldots\ldots(46$$

(6) To find ${}_n\overset{x^2=1}{\sigma_y^2}$ we have to evaluate the determinant of $(p+1)$st order,

$$\begin{vmatrix} 0 & 1 & 1 & \ldots\ldots & 1 \\ 1 & \frac{1}{2q-1}+\alpha & \frac{1}{2q+1}+\alpha & \ldots\ldots & \frac{1}{2q+2p-3}+\alpha \\ 1 & \frac{1}{2q+1}+\alpha & \frac{1}{2q+3}+\alpha & \ldots\ldots & \frac{1}{2q+2p-1}+\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \frac{1}{2q+2p-3}+\alpha & \frac{1}{2q+2p-1}+\alpha & \ldots\ldots & \frac{1}{2q+4p-5}+\alpha \end{vmatrix}.$$

Treating it as ${}_{p}\overset{q}{\delta}$ was treated under (2) of this section, except that now two rows or columns are left unaltered, it takes the form

$$\begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & \frac{1}{2q-1}+a & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-5)(2q+2p-3)} \\ 0 & \frac{2}{(2q-1)(2q+1)} & \frac{2.4}{(2q-1)(2q+1)(2q+3)} & \frac{2.4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2.4}{(2q+2p-5)\ldots(2q+2p-1)} \\ 0 & \frac{2}{(2q+1)(2q+3)} & \frac{2.4}{(2q+1)(2q+3)(2q+5)} & \frac{2.4}{(2q+3)(2q+5)(2q+7)} & \cdots & \frac{2.4}{(2q+2p-3)\ldots(2q+2p+1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{2}{(2q+2p-5)(2q+2p-3)} & \frac{2.4}{(2q+2p-5)\ldots(2q+2p-1)} & \frac{2.4}{(2q+2p-3)\ldots(2q+2p+1)} & \cdots & \frac{2.4}{(2q+4p-9)\ldots(2q+4p-5)} \end{vmatrix}$$

$$= -2^{3(p-1)}{}_{p-1}\overset{q}{D}.$$

Hence we find from (38),

$$\underset{2p}{\overset{x^2=1}{\sigma_y^2}} = \frac{\sigma^2}{N}(1+a)\left\{\frac{2^{3p}{}_{p}\overset{1}{D}}{{}_{p+1}\overset{1}{\delta}} + \frac{2^{3(p-1)}{}_{p-1}\overset{2}{D}}{{}_{p}\overset{2}{\delta}}\right\}.$$

Now from (43) and (45) we get

$$\frac{2^{3p}{}_{p}\overset{q}{D}}{{}_{p+1}\overset{q}{\delta}} = \frac{(p+1)(2q+2p-1)}{[1+a(p+1)(2q+2p-1)]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

and therefore

$$\underset{2p}{\overset{x^2=1}{\sigma_y^2}} = \frac{\sigma^2}{N}(1+a)\left\{\frac{(p+1)(2p+1)}{1+a(p+1)(2p+1)} + \frac{p(2p+1)}{1+ap(2p+1)}\right\}$$

or

$$\underset{2p}{\overset{x^2=1}{\sigma_y^2}} = \frac{\sigma^2}{N}(1+a)(2p+1)\left\{\frac{p+1}{1+a(p+1)(2p+1)} + \frac{p}{1+ap(2p+1)}\right\} \ldots(48).$$

In the same way we get from (39),

$$\underset{2p-1}{\overset{x^2=1}{\sigma_y^2}} = \frac{\sigma^2}{N}(1+a)\left\{\frac{2^{3(p-1)}{}_{p-1}\overset{1}{D}}{{}_{p}\overset{1}{\delta}} + \frac{2^{3(p-1)}{}_{p-1}\overset{2}{D}}{{}_{p}\overset{2}{\delta}}\right\},$$

which by the relation between ${}_{p}\overset{q}{D}$ and ${}_{p+1}\overset{q}{\delta}$ just found is reduced to

$$\underset{2p-1}{\overset{x^2=1}{\sigma_y^2}} = \frac{\sigma^2}{N}(1+a)\left\{\frac{p(2p-1)}{1+ap(2p-1)} + \frac{p(2p+1)}{1+ap(2p+1)}\right\} \ldots\ldots\ldots(49).$$

Both (48) and (49) are covered by the formula

$$\underset{n}{\overset{x^2=1}{\sigma_y^2}} = \frac{\sigma^2}{N}(1+a)(n+1)\left\{\frac{n}{2+an(n+1)} + \frac{n+2}{2+a(n+1)(n+2)}\right\} \ldots(50).$$

(7) The evaluation of ${}_{n}\sigma_y^2$ for special values of n can be made easier by a transformation of the determinant

$$
{}_{p+1}\overset{q}{d} = \begin{vmatrix}
1 & \frac{1}{2q-1}+a & \frac{1}{2q+1}+a & \cdots\cdots & \frac{1}{2q+2p-3}+a \\
x^2 & \frac{1}{2q+1}+a & \frac{1}{2q+3}+a & \cdots\cdots & \frac{1}{2q+2p-1}+a \\
x^4 & \frac{1}{2q+3}+a & \frac{1}{2q+5}+a & \cdots\cdots & \frac{1}{2q+2p+1}+a \\
\vdots & \vdots & \vdots & & \vdots \\
x^{2p} & \frac{1}{2q+2p-1}+a & \frac{1}{2q+2p+1}+a & \cdots\cdots & \frac{1}{2q+4p-3}+a
\end{vmatrix}.
$$

Leaving the first row unaltered and subtracting from each of the others the proceeding we get a determinant the first column of which is

$$1,\ x^2-1,\ x^2(x^2-1)\ \ldots\ x^{2p-2}(x^2-1),$$

while the other columns are identical with those of the determinant δ previously treated in the same way. When next the two first rows are left as they are and from each of the others is subtracted the proceeding one the result is

$$
{}_{p+1}\overset{q}{d} = (-1)^{2p-1} \times
$$

$$
\begin{vmatrix}
1 & \frac{1}{2q-1}+a & \frac{1}{2q+1}+a & \cdots & \frac{1}{2q+2p-3}+a \\
1-x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-3)(2q+2p-1)} \\
(1-x^2)^2 & \frac{2.4}{(2q-1)(2q+1)(2q+3)} & \frac{2.4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2.4}{(2q+2p-3)\ldots(2q+2p+1)} \\
\vdots & \vdots & \vdots & & \vdots \\
x^{2p-4}(1-x^2)^2 & \frac{2.4}{(2q+2p-5)\ldots(2q+2p-1)} & \frac{2.4}{(2q+2p-3)\ldots(2q+2p+1)} & \cdots & \frac{2.4}{(2q+4p-7)\ldots(2q+4p-3)}
\end{vmatrix}.
$$

Leaving now three rows unaltered, next time four and so on, it is clear that we shall at last after p of these sets of operations get

$$
{}_{p+1}\overset{q}{d} = (-1)^{\frac{p(p+1)}{2}} \times
$$

$$
\begin{vmatrix}
1 & \frac{1}{2q-1}+a & \frac{1}{2q+1}+a & \cdots & \frac{1}{2q+2p-3}+a \\
1-x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-3)(2q+2p-1)} \\
(1-x^2)^2 & \frac{2.4}{(2q-1)(2q+1)(2q+3)} & \frac{2.4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2.4}{(2q+2p-3)\ldots\ldots(2q+2p+1)} \\
\vdots & \vdots & \vdots & & \vdots \\
(1-x^2)^p & \frac{2.4\ldots 2p}{(2q-1)\ldots\ldots(2q+2p-1)} & \frac{2.4\ldots 2p}{(2q+1)\ldots\ldots(2q+2p+1)} & \cdots & \frac{2.4\ldots 2p}{(2q+2p-3)\ldots\ldots(2q+4p-3)}
\end{vmatrix}.
$$

By treating the columns in the same way, leaving first two then three and so on unaltered, we find after the first set of operations

$$
{}_{p+1}\overset{q}{d} = (-1)^{\frac{p(p+1)}{2}+(p-1)} \times
\begin{vmatrix}
1 & \frac{1}{2q-1}+\alpha & \frac{2}{(2q-1)(2q+1)} & \cdots & \frac{2}{(2q+2p-5)(2q+2p-3)} \\
1-x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{2\,.\,4}{(2q-1)(2q+1)(2q+3)} & \cdots & \frac{2\,.\,4}{(2q+2p-5)\dots(2q+2p-1)} \\
(1-x^2)^2 & \frac{2\,.\,4}{(2q-1)(2q+1)(2q+3)} & \frac{2\,.\,4\,.\,6}{(2q-1)(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2\,.\,4\,.\,6}{(2q+2p-5)\dots(2q+2p+1)} \\
\vdots & \vdots & \vdots & & \vdots \\
(1-x^2)^p & \frac{2\,.\,4\dots 2p}{(2q-1)(2q+1)\dots(2q+2p-1)} & \frac{2\,.\,4\dots 2p\,(2p+2)}{(2q-1)(2q+1)\dots(2q+2p+1)} & \cdots & \frac{2\,.\,4\dots 2p\,(2p+2)}{(2q+2p-5)\dots(2q+4p-3)}
\end{vmatrix}
$$

and after $(p-1)$ sets of operations

$$
{}_{p+1}\overset{q}{d} = (-1)^{p^2} \times
\begin{vmatrix}
1 & \frac{1}{2q-1}+\alpha & \frac{2}{(2q-1)(2q+1)} & \cdots & \frac{2\,.\,4\dots(2p-2)}{(2q-1)(2q+1)\dots(2q+2p-3)} \\
1-x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{2\,.\,4}{(2q-1)(2q+1)(2q+3)} & \cdots & \frac{2\,.\,4\dots(2p-2)\,2p}{(2q-1)(2q+1)\dots(2q+2p-1)} \\
(1-x^2)^2 & \frac{2\,.\,4}{(2q-1)(2q+1)(2q+3)} & \frac{2\,.\,4\,.\,6}{(2q-1)(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2\,.\,4\dots 2p\,(2p+2)}{(2q-1)(2q+1)\dots(2q+2p+1)} \\
\vdots & \vdots & \vdots & & \vdots \\
(1-x^2)^p & \frac{2\,.\,4\dots 2p}{(2q-1)(2q+1)\dots(2q+2p-1)} & \frac{2\,.\,4\dots 2p\,(2p+2)}{(2q-1)(2q+1)\dots(2q+2p+1)} & \cdots & \frac{2\,.\,4\dots(4p-4)(4p-2)}{(2q-1)(2q+1)\dots(2q+4p-3)}
\end{vmatrix}
$$

since
$$(-1)^{\frac{p(p+1)}{2}+\frac{(p-1)p}{2}} = (-1)^{p^2}.$$

Here the first element of the last $p-1$ columns is seen to occur as factor for the whole column so that we can put outside the factor

$$
\frac{2^{p-1}\,.\,4^{p-2}\dots(2p-4)^2\,(2p-2)}{(2q-1)^{p-1}(2q+1)^{p-1}(2q+3)^{p-2}(2q+5)^{p-3}\dots(2q+2p-5)^2(2q+2p-3)}
$$
$$
= \frac{1^{p-1}\,.\,2^{p-2}\dots(p-2)^2\,(p-1)\,2^{\frac{p(p-1)}{2}}}{(2q-1)^{p-1}(2q+1)^{p-1}(2q+3)^{p-2}(2q+5)^{p-3}\dots(2q+2p-5)^2(2q+2p-3)},
$$

the resulting expression being

$$
{}_{p+1}\overset{q}{d} = \frac{(-1)^{p^2}\,.\,1^{p-1}\,.\,2^{p-2}\dots(p-2)^2\,(p-1)\,2^{\frac{p(p-1)}{2}}}{(2q-1)^{p-1}(2q+1)^{p-1}(2q+3)^{p-2}\dots(2q+2p-5)^2(2q+2p-3)} \times
$$
$$
\begin{vmatrix}
1 & \frac{1}{2q-1}+\alpha & 1 & \cdots & 1 \\
1-x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{4}{2q+3} & \cdots & \frac{2p}{2q+2p-1} \\
(1-x^2)^2 & \frac{2\,.\,4}{(2q-1)(2q+1)(2q+3)} & \frac{4\,.\,6}{(2q+3)(2q+5)} & \cdots & \frac{2p\,(2p+2)}{(2q+2p-1)(2q+2p+1)} \\
\vdots & \vdots & \vdots & & \vdots \\
(1-x^2)^p & \frac{2\,.\,4\dots 2p}{(2q-1)(2q+1)\dots(2q+2p-1)} & \frac{4\,.\,6\dots(2p+2)}{(2q+3)\dots(2q+2p+1)} & \cdots & \frac{2p\dots(4p-2)}{(2q+2p-1)\dots(2q+4p-3)}
\end{vmatrix}
$$

In our formulae the two cases $q = 1$ or $q = 2$ only occur for which according to this we find

$${}_{p+1}\overset{1}{d} = \frac{(-1)^{p^2} \cdot 1^{p-1} \cdot 2^{p-2} \ldots (p-2)^2 (p-1)\, 2^{\frac{p(p-1)}{2}}}{3^{p-1} \cdot 5^{p-2} \cdot 7^{p-3} \ldots (2p-3)^2 (2p-1)} \times$$

$$\begin{vmatrix} 1 & 1+a & 1 & 1 & \cdots & 1 \\ 1-x^2 & \dfrac{2}{1 \cdot 3} & \dfrac{4}{5} & \dfrac{6}{7} & \cdots & \dfrac{2p}{2p+1} \\ (1-x^2)^2 & \dfrac{2 \cdot 4}{1 \cdot 3 \cdot 5} & \dfrac{4 \cdot 6}{5 \cdot 7} & \dfrac{6 \cdot 8}{7 \cdot 9} & \cdots & \dfrac{2p(2p+2)}{(2p+1)(2p+3)} \\ (1-x^2)^3 & \dfrac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7} & \dfrac{4 \cdot 6 \cdot 8}{5 \cdot 7 \cdot 9} & \dfrac{6 \cdot 8 \cdot 10}{7 \cdot 9 \cdot 11} & \cdots & \dfrac{2p(2p+2)(2p+4)}{(2p+1)(2p+3)(2p+5)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (1-x^2)^p & \dfrac{2 \cdot 4 \ldots 2p}{1 \cdot 3 \ldots (2p+1)} & \dfrac{4 \cdot 6 \ldots (2p+2)}{5 \cdot 7 \ldots (2p+3)} & \dfrac{6 \cdot 8 \ldots (2p+4)}{7 \cdot 9 \ldots (2p+5)} & \cdots & \dfrac{2p(2p+2) \ldots (4p-2)}{(2p+1)(2p+3) \ldots (4p-1)} \end{vmatrix} \quad \ldots\ldots\ldots(51)$$

and

$${}_{p+1}\overset{2}{d} = \frac{(-1)^{p^2} \cdot 1^{p-1} \cdot 2^{p-2} \ldots (p-2)^2 (p-1)\, 2^{\frac{p(p-1)}{2}}}{3^{p-1} \cdot 5^{p-1} \cdot 7^{p-2} \ldots (2p-1)^2 (2p+1)} \times$$

$$\begin{vmatrix} 1 & \dfrac{1}{3}+a & 1 & 1 & \cdots & 1 \\ 1-x^2 & \dfrac{2}{3 \cdot 5} & \dfrac{4}{7} & \dfrac{6}{9} & \cdots & \dfrac{2p}{2p+3} \\ (1-x^2)^2 & \dfrac{2 \cdot 4}{3 \cdot 5 \cdot 7} & \dfrac{4 \cdot 6}{7 \cdot 9} & \dfrac{6 \cdot 8}{9 \cdot 11} & \cdots & \dfrac{2p(2p+2)}{(2p+3)(2p+5)} \\ (1-x^2)^3 & \dfrac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 9} & \dfrac{4 \cdot 6 \cdot 8}{7 \cdot 9 \cdot 11} & \dfrac{6 \cdot 8 \cdot 10}{9 \cdot 11 \cdot 13} & \cdots & \dfrac{2p(2p+2)(2p+4)}{(2p+3)(2p+5)(2p+7)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (1-x^2)^p & \dfrac{2 \cdot 4 \cdot 6 \ldots 2p}{3 \cdot 5 \cdot 7 \ldots (2p+3)} & \dfrac{4 \cdot 6 \ldots (2p+2)}{7 \cdot 9 \ldots (2p+5)} & \dfrac{6 \cdot 8 \ldots (2p+4)}{9 \cdot 11 \ldots (2p+7)} & \cdots & \dfrac{2p(2p+2) \ldots (4p-2)}{(2p+3)(2p+5) \ldots (4p+1)} \end{vmatrix} \quad \ldots\ldots\ldots(52).$$

VI. *Uniform continuous distribution of observations with additional clusters at the ends of the range; constant standard deviation of observations. Special formulae.*

(1) Our first task shall be to work out the formulae for ${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2$ for values of n up to 6, the next to find what values should be given to a in order to make ${}_n\sigma_y^2$ as flat a curve as possible within the range of observations.

With the notations just introduced (40) and (41) take the form

$$S_{2p} = {}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 = \frac{\sigma^2}{N}(1+a)\,\frac{{}_{p+1}\overset{1}{d}{}^2}{{}_p\overset{1}{\delta} \cdot {}_{p+1}\overset{1}{\delta}}$$

and
$$S_{2p+1} = {}_{2p+1}\sigma_y^2 - {}_{2p}\sigma_y^2 = \frac{\sigma^2}{N}(1+\alpha)\,x^2\,\frac{\overset{2}{{}_{p+1}d^2}}{{}_p\overset{2}{\delta}\cdot{}_{p+1}\overset{2}{\delta}}.$$

From these formulae we find, after applying (45), (51) and (52),

$$S_1 = \frac{\sigma^2}{N}\,\frac{3(1+\alpha)\,x^2}{1+3\alpha} \quad\ldots\ldots(53),$$

$$S_2 = \frac{\sigma^2}{N}(1+\alpha)\,\frac{1}{1+1\,.\,1\alpha}\cdot\frac{1\,.\,3^2\,.\,5}{2^2(1+2\,.\,3\alpha)}\begin{vmatrix}1 & 1+\alpha\\ 1-x^2 & \dfrac{2}{3}\end{vmatrix}^2$$
$$= \frac{\sigma^2}{N}\cdot\frac{5}{2^2}\,\frac{[2+3(1+\alpha)(x^2-1)]^2}{1+6\alpha} \quad\ldots\ldots(54),$$

$$S_3 = \frac{\sigma^2}{N}(1+\alpha)\,x^2\,\frac{3}{1+1\,.\,3\alpha}\cdot\frac{3\,.\,5^2\,.\,7}{2^2(1+2\,.\,5\alpha)}\begin{vmatrix}1 & \dfrac{1}{3}+\alpha\\ 1-x^2 & \dfrac{2}{3\,.\,5}\end{vmatrix}^2$$
$$= \frac{\sigma^2}{N}\cdot\frac{7}{2^2}\,\frac{(1+\alpha)\,x^2[2+5(1+3\alpha)(x^2-1)]^2}{(1+3\alpha)(1+10\alpha)} \quad\ldots\ldots(55),$$

$$S_4 = \frac{\sigma^2}{N}(1+\alpha)\,\frac{1\,.\,3^2\,.\,5}{2^2(1+2\,.\,3\alpha)}\cdot\frac{1\,.\,3^2\,.\,5^3\,.\,7^2\,.\,9}{2^2\,.\,2^6(1+3\,.\,5\alpha)}\left(\frac{2}{3}\right)^2\begin{vmatrix}1 & 1+\alpha & 1\\ 1-x^2 & \dfrac{2}{3} & \dfrac{4}{5}\\ (1-x^2)^2 & \dfrac{2\,.\,4}{3\,.\,5} & \dfrac{4\,.\,6}{5\,.\,7}\end{vmatrix}^2$$
$$= \frac{\sigma^2}{N}\cdot\frac{9}{2^6}\,\frac{(1+\alpha)[8+20(2+9\alpha)(x^2-1)+35(1+6\alpha)(x^2-1)^2]^2}{(1+6\alpha)(1+15\alpha)} \quad\ldots\ldots(56),$$

$$S_5 = \frac{\sigma^2}{N}(1+\alpha)\,\frac{3\,.\,5^2\,.\,7}{2^2(1+2\,.\,5\alpha)}\cdot\frac{3\,.\,5^2\,.\,7^3\,.\,9^2\,.\,11}{2^2\,.\,2^6(1+3\,.\,7\alpha)}\left(\frac{2}{3\,.\,5}\right)^2 x^2\begin{vmatrix}1 & \dfrac{1}{3}+\alpha & 1\\ 1-x^2 & \dfrac{2}{3\,.\,5} & \dfrac{4}{7}\\ (1-x^2)^2 & \dfrac{2\,.\,4}{3\,.\,5\,.\,7} & \dfrac{4\,.\,6}{7\,.\,9}\end{vmatrix}^2$$
$$= \frac{\sigma^2}{N}\cdot\frac{11}{2^6}\,\frac{(1+\alpha)\,x^2[8+28(2+15\alpha)(x^2-1)+63(1+10\alpha)(x^2-1)^2]^2}{(1+10\alpha)(1+21\alpha)} \quad\ldots\ldots(57),$$

$$S_6 = \frac{\sigma^2}{N}(1+\alpha)\,\frac{1\,.\,3^2.\,5^3.\,7^2.\,9}{2^2.\,2^6(1+3\,.\,5\alpha)}\cdot\frac{1\,.\,3^2.\,5^3.\,7^4.\,9^3.\,11^2.\,13}{(2^2.\,3)^2.\,2^{12}(1+4\,.\,7\alpha)}\left(\frac{2\,.\,2^3}{3^2.\,5}\right)^2\begin{vmatrix}1 & 1+\alpha & 1 & 1\\ 1-x^2 & \dfrac{2}{3} & \dfrac{4}{5} & \dfrac{6}{7}\\ (1-x^2)^2 & \dfrac{2.4}{3.5} & \dfrac{4.6}{5.7} & \dfrac{6.8}{7.9}\\ (1-x^2)^3 & \dfrac{2.4.6}{3.5.7} & \dfrac{4.6.8}{5.7.9} & \dfrac{6.8.10}{7.9.11}\end{vmatrix}^2$$
$$= \frac{\sigma^2}{N}\cdot\frac{13}{2^8}\,\frac{(1+\alpha)[16+168(1+10\alpha)(x^2-1)+126(3+40\alpha)(x^2-1)^2+231(1+15\alpha)(x^2-1)^3]^2}{(1+15\alpha)(1+28\alpha)}$$
$$\ldots\ldots(58).$$

(2) We shall now look at ${}_n\sigma_y^2$ for special values of n and as a first attempt at finding a flat curve for ${}_n\sigma_y^2$ try to make ${}_n\overset{x=0}{\sigma_y^2} = {}_n\overset{x^2=1}{\sigma_y^2}$.

For a *linear function* we find, since

$${}_1\sigma_y^2 = {}_0\sigma_y^2 + S_1,$$

$${}_1\sigma_y^2 = \frac{\sigma^2}{N}\left(1 + \frac{3(1+\alpha)}{1+3\alpha}x^2\right) \quad \text{..........................(59).}$$

As α is positive it is obvious that we cannot make ${}_1\overset{x=0}{\sigma_y^2} = {}_1\overset{x^2=1}{\sigma_y^2}$ which indeed we knew beforehand. This follows because we have proved that ${}_n\sigma_y^2$ is of $2n$th degree and never lower.

For $x = 0$ we find $${}_1\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N},$$

which holds for any symmetrical distribution of observations with constant standard deviation. α is the ratio between the number of observations at the ends of the range and the number uniformly distributed through the range, it may therefore vary from 0 to ∞. As $\frac{3(1+\alpha)}{1+3\alpha}$ decreases when α increases we get the flattest possible curve when $\alpha = \infty$, that is when the distribution of observations consists of two groups at the ends of the range. Then the curve is, as already shown in Section II,

$${}_1\sigma_y^2 = \frac{\sigma^2}{N}(1 + x^2).$$

To get a check on the degree of the function and at the same time a flatter curve of σ_y^2 than that obtained from a uniform distribution we may choose something between the two extreme cases and take for example $\frac{1}{4}N$ observations at each end of the range and $\frac{1}{2}N$ uniformly distributed through the range.

Then $\alpha = 1$ and, according to (59),

$$\sigma_y^2 = \frac{\sigma^2}{N}(1 + \tfrac{3}{2}x^2),$$

with the maximum $$\overset{x^2=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 1{\cdot}581.$$

(3) For a *function of the second degree* we find, from (46),

$${}_2\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N} \cdot \frac{9}{4}\frac{(1+\alpha)(1+5\alpha)}{1+6\alpha},$$

and from (50), $${}_2\overset{x^2=1}{\sigma_y^2} = \frac{\sigma^2}{N}3(1+\alpha)\left\{\frac{1}{1+3\alpha} + \frac{2}{1+6\alpha}\right\}.$$

We want to make these equal and this requires

$$3(1+5\alpha)(1+3\alpha) = 4\{1 + 6\alpha + 2(1+3\alpha)\}$$

or $$15\alpha^2 - 8\alpha - 3 = 0.$$

This has only one positive root $\alpha = {\cdot}7873500$.

For this value $\frac{a}{2(1+a)}$, which is the ratio between the number of observations at one end of the range and the total number of observations, is ·2202562.

As ${}_2\sigma_y^2 = {}_1\sigma_y^2 + S_2$ we find, from (59) and (54),

$$ {}_2\sigma_y^2 = \frac{\sigma^2}{N}\left(1 + \frac{3(1+a)}{1+3a}x^2 + \frac{5}{4}\frac{[2+3(1+a)(x^2-1)]^2}{1+6a}\right) \quad \ldots\ldots\ldots(60),$$

for $a = \cdot 7873500$ the curve is

$$ {}_2\sigma_y^2 = \frac{\sigma^2}{N}\{3\cdot 46837 - 6\cdot 27862x^2 + 6\cdot 27862x^4\},$$

which has minima at

$$x = \pm\frac{1}{\sqrt{2}}.$$

The extreme values in the range of observations are therefore

$$\sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,1\cdot 8624 \text{ for } x = \begin{cases}0\\ \pm 1\end{cases}$$

and

$$\sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,1\cdot 3779 \text{ for } x = \pm\cdot 70711.$$

(4) For a *function of the third degree* we have, from (46),

$$ {}_3\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N}\cdot\frac{9}{4}\frac{(1+a)(1+5a)}{1+6a},$$

and from (50),

$$ {}_3\overset{x^2=1}{\sigma_y^2} = \frac{\sigma^2}{N}2(1+a)\left\{\frac{3}{1+6a} + \frac{5}{1+10a}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(61).$$

Hence the condition that they are equal is

$$9(1+5a)(1+10a) = 32(2+15a)$$

or

$$90a^2 - 69a - 11 = 0,$$

with one positive root $a = \cdot 9021461$.

From (60) and (55) we find

$$ {}_3\sigma_y^2 = \frac{\sigma^2}{N}\left(1 + \frac{3(1+a)}{1+3a}x^2 + \frac{5}{4}\frac{[2+3(1+a)(x^2-1)]^2}{1+6a} + \frac{7}{4}\frac{(1+a)x^2[2+5(1+3a)(x^2-1)]^2}{(1+3a)(1+10a)}\right) \ldots\ldots\ldots(62),$$

which for $a = \cdot 9021461$ becomes

$$ {}_3\sigma_y^2 = \frac{\sigma^2}{N}\{3\cdot 67775 + 17\cdot 78799x^2 - 48\cdot 56651x^4 + 30\cdot 77852x^6\}.$$

Besides the minimum for $x = 0$ this curve has other minima for $x^2 = \cdot 815820$ and maxima for $x^2 = \cdot 2361366$.

The maxima and minima are as follows:

$$\text{For } x = \begin{cases}\pm 1\\ 0\end{cases} \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,1\cdot 9177,$$

$$\text{,, } x = \pm\cdot 48594 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2\cdot 3612,$$

$$\text{,, } x = \pm\cdot 90323 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,1\cdot 6055.$$

By choosing $\alpha = \cdot 9021461$, that is by taking $\cdot 237139 \times N$ observations at each end of the range, we seem therefore to have overshot our aim since the result is that we have got inside the range a maximum for σ_y greater than the value obtained for $x = \pm 1$.

(5) Our next attempt shall be to make

$$\overset{x^2=1}{{}_3\sigma_y^2} = 2\, \overset{x=0}{{}_3\sigma_y^2}.$$

It requires $$9(1+5\alpha)(1+10\alpha) = 16(2+15\alpha)$$

or $$450\alpha^2 - 105\alpha - 23 = 0.$$

The only positive root is $\alpha = \cdot 3710723$ which gives the curve

$${}_3\sigma_y^2 = \frac{\sigma^2}{N}\{2\cdot 730117 + 12\cdot 89741x^2 - 37\cdot 07612x^4 + 26\cdot 90882x^6\}.$$

The maxima and minima are:

For $x = \cdot 0000$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1\cdot 652$,

,, $x = \pm \cdot 4828$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2\cdot 016$,

,, $x = \pm \cdot 8279$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1\cdot 678$,

,, $x = \pm 1\cdot 0000$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2\cdot 337$.

This distribution of observations makes σ_y for $x = \pm 1$ greater than the maximum at $x = \pm \cdot 4828$. By interpolation between these two cases we shall now try to find an α, lying between those of our two trials, for which σ_y for $x = \pm 1$ equals the maximum value of σ_y which still may be expected at about $x = \cdot 48$.

(6) In our first attempt we found $\overset{x=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 1\cdot 918$ and its difference from the maximum $\frac{\sigma}{\sqrt{N}} \cdot \cdot 444$, in the second attempt $\overset{x=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 2\cdot 337$ and its difference from the maximum $-\frac{\sigma}{\sqrt{N}} \cdot \cdot 321$.

If the relation were linear this difference would be zero for

$$\overset{x=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 2\cdot 161.$$

The α for which $\overset{x=1}{\sigma_y}$ takes this value is found by (61) which leads to

$$8(1+\alpha)(2+15\alpha) = 2\cdot 161^2(1+6\alpha)(1+10\alpha)$$

or $$160\cdot 2\alpha^2 - 61\cdot 28\alpha - 11\cdot 330 = 0,$$

with the positive root $\alpha = \cdot 519$.

For this value (62) becomes

$${}_3\sigma_y^2 = \frac{\sigma^2}{N}\{2\cdot 9866 + 14\cdot 2364x^2 - 40\cdot 0058x^4 + 27\cdot 4521x^6\}.$$

The maxima and minima are:

$$\text{For } x = \cdot 0000 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 1{\cdot}728,$$

$$,, \quad x = \pm \cdot 4843 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2{\cdot}116,$$

$$,, \quad x = \pm \cdot 8585 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 1{\cdot}655,$$

$$,, \quad x = \pm 1{\cdot}0000 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2{\cdot}161,$$

and this distribution which has $\cdot 1708 \times N$ observations at each end of the range may be considered satisfactory.

(7) From (46) and (50) we find, for a *function of the fourth degree*,

$$\overset{x=0}{{}_4\sigma_y^2} = \frac{\sigma^2}{N} \cdot \frac{225}{64} \frac{(1+\alpha)(1+14\alpha)}{1+15\alpha}$$

and

$$\overset{x^2=1}{{}_4\sigma_y^2} = \frac{\sigma^2}{N} 5(1+\alpha) \left\{ \frac{2}{1+10\alpha} + \frac{3}{1+15\alpha} \right\},$$

which are equal when

$$9(1+14\alpha)(1+10\alpha) = 64(1+12\alpha)$$

or

$$1260\alpha^2 - 552\alpha - 55 = 0,$$

that is when

$$\alpha = \cdot 5217564.$$

The formula for ${}_4\sigma_y^2$, found from (62) and (56), is

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \left\{ 1 + \frac{3(1+\alpha)}{1+3\alpha} x^2 + \frac{5}{4} \frac{[2+3(1+\alpha)(x^2-1)]^2}{1+6\alpha} + \frac{7}{4} \frac{(1+\alpha) x^2 [2+5(1+3\alpha)(x^2-1)]^2}{(1+3\alpha)(1+10\alpha)} \right.$$
$$\left. + \frac{9}{64} \frac{(1+\alpha)[8+20(2+9\alpha)(x^2-1)+35(1+6\alpha)(x^2-1)^2]^2}{(1+6\alpha)(1+15\alpha)} \right\} \quad \ldots\ldots(63).$$

For $\alpha = \cdot 5217564$ it becomes

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \{5{\cdot}03367 - 19{\cdot}72772x^2 + 133{\cdot}01711x^4 - 235{\cdot}96817x^6 + 122{\cdot}67868x^8\}.$$

The maxima and minima are as follows:

$$\text{For } x = \begin{cases} 0 \\ \pm 1 \end{cases} \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2{\cdot}244,$$

$$,, \quad x = \pm \cdot 3130 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2{\cdot}041,$$

$$,, \quad x = \pm \cdot 6844 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2{\cdot}575,$$

$$,, \quad x = \pm \cdot 9361 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 1{\cdot}856.$$

We have again as for the function of the third degree brought $\overset{x^2=1}{\sigma_y}$ down below one of the maxima of ${}_4\sigma_y$, although since ${}_4\sigma_y$ has a maximum at $x = 0$ the demand that $\overset{x=0}{\sigma_y} = \overset{x^2=1}{\sigma_y}$ is not so exacting as for ${}_3\sigma_y$ which has a minimum at $x = 0$.

(8) We shall next make ${}_4\overset{x^2=1}{\sigma_y^2} = 1{\cdot}2671861\ {}_4\overset{x=0}{\sigma_y^2}$*.

The condition obtained from (46) and (50) is

$$9 \times 1{\cdot}2671861\,(1 + 10\alpha)\,(1 + 14\alpha) = 64\,(1 + 12\alpha)$$

or

$$\alpha^2 - {\cdot}3095773\alpha - {\cdot}032940969 = 0,$$

with the only positive root $\alpha = {\cdot}3933269$.

Introducing this value of α in (63) we get

$${}_4\sigma_y^2 = \frac{\sigma^2}{N}\{4{\cdot}61918 - 18{\cdot}02388x^2 + 122{\cdot}71833x^4 - 220{\cdot}34099x^6 + 116{\cdot}8807x^8\}.$$

The maxima and minima for this curve are:

$$\text{At } x = 0 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}149,$$

$$,,\ x = \pm\,{\cdot}3116 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,1{\cdot}958,$$

$$,,\ x = \pm\,{\cdot}6839 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}467,$$

$$,,\ x = \pm\,{\cdot}9214 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,1{\cdot}913,$$

$$,,\ x = \pm\,1{\cdot}0000 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}419.$$

We have thus for $\alpha = {\cdot}3933269$, that is by taking ${\cdot}141147 \times N$ observations at each end of the range, succeeded in bringing ${}_4\overset{x^2=1}{\sigma_y}$ down to be approximately equal to the highest of the maxima of the curve, thus fulfilling our purpose.

(9) After our experiences in the cases of the functions of the third and fourth degree we cannot expect for a *function of the fifth degree* by making

$${}_5\overset{x^2=1}{\sigma_y^2} = {}_5\overset{x=0}{\sigma_y^2}$$

to find a curve which has not a greater maximum than that value. We shall therefore start with the attempt

$${}_5\overset{x^2=1}{\sigma_y^2} = 2\ {}_5\overset{x=0}{\sigma_y^2}.$$

The condition found from (46) and (50) is

$$25\,(1 + 14\alpha)\,(1 + 21\alpha) = 64\,(2 + 35\alpha)$$

or

$$7350\alpha^2 - 1365\alpha - 103 = 0,$$

with the only positive root $\alpha = {\cdot}2433100$.

* The ratio $1{\cdot}2671861$ results from consideration of a special ${}_2\sigma_y^2$ curve. It was determined as that curve obtained from three groups of observations for which the standard deviation of σ_y^2's within the range of observations was a minimum. It is not mentioned elsewhere in this memoir as it does not seem to have the interest I at first assumed it to have.

For ${}_5\sigma_y^2$ we find, from (63) and (57),

$$ {}_5\sigma_y^2 = \frac{\sigma^2}{N}\left\{1 + \frac{3(1+\alpha)}{1+3\alpha}x^2 + \frac{5}{4}\frac{[2+3(1+\alpha)(x^2-1)]^2}{1+6\alpha} + \frac{7}{4}\frac{(1+\alpha)x^2[2+5(1+3\alpha)(x^2-1)]^2}{(1+3\alpha)(1+10\alpha)} \right. $$
$$ + \frac{9}{64}\frac{(1+\alpha)[8+20(2+9\alpha)(x^2-1)+35(1+6\alpha)(x^2-1)^2]^2}{(1+6\alpha)(1+15\alpha)} $$
$$ \left. + \frac{11}{64}\frac{(1+\alpha)x^2[8+28(2+15\alpha)(x^2-1)+63(1+10\alpha)(x^2-1)^2]^2}{(1+10\alpha)(1+21\alpha)}\right\} \ldots (64). $$

Introducing $\alpha = \cdot 2433100$ we get

$$ {}_5\sigma_y^2 = \frac{\sigma^2}{N}\{4\cdot 14228 + 28\cdot 47030x^2 - 258\cdot 05238x^4 + 853\cdot 0448x^6 - 1095\cdot 921x^8 + 476\cdot 5990x^{10}\}, $$

from which we find the maxima and minima:

At $x = \quad 0 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2\cdot 035,$

,, $x = \pm\ \cdot 2953 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2\cdot 273,$

,, $x = \pm\ \cdot 5004 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2\cdot 155,$

,, $x = \pm\ \cdot 7853 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2\cdot 762,$

,, $x = \pm\ \cdot 9418 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2\cdot 231,$

,, $x = \pm 1\cdot 0000 \qquad \sigma_y = \frac{\sigma}{\sqrt{N}} \,.\, 2\cdot 878.$

$\overset{x^2=1}{\sigma_y}$ does not differ much from the greatest maximum and we may thus consider the distribution with $\cdot 097848 \times N$ observations at each end of the range for which $\alpha = \cdot 2433100$ as satisfying fairly well our aim.

(10) Considering our previous results we must assume that for a *function of the sixth degree* $\overset{x^2=1}{\sigma_y^2} \Big/ \overset{x=0}{\sigma_y^2}$ ought to be made somewhat smaller than 2 which was the value that gave a satisfying result for a function of the fifth degree.

Let us assume $\overset{x^2=1}{\sigma_y^2} = 1\cdot 75\, \overset{x=0}{\sigma_y^2}$ or, substituting from (46) and (50),

$$ 256(1+24\alpha) = 1\cdot 75 \times 25(1+21\alpha)(1+27\alpha) $$

from which $$ 567\alpha^2 - 92\cdot 43430\alpha - 4\cdot 851429 = 0 $$

and $$ \alpha = \cdot 2048019 $$

are found.

For ${}_6\sigma_y^2$ we get, from (64) and (58),

$$ {}_6\sigma_y^2 = \frac{\sigma^2}{N}\left\{1 + \frac{3(1+\alpha)}{1+3\alpha}x^2 + \frac{5}{4}\frac{[2+3(1+\alpha)(x^2-1)]^2}{1+6\alpha} \right.$$
$$+\frac{7}{4}\frac{(1+\alpha)x^2[2+5(1+3\alpha)(x^2-1)]^2}{(1+3\alpha)(1+10\alpha)}$$
$$+\frac{9}{64}\frac{1+\alpha}{(1+6\alpha)(1+15\alpha)}[8+20(2+9\alpha)(x^2-1)+35(1+6\alpha)(x^2-1)^2]^2$$
$$+\frac{11}{64}\frac{1+\alpha}{(1+10\alpha)(1+21\alpha)}[8+28(2+15\alpha)(x^2-1)+63(1+10\alpha)(x^2-1)^2]^2$$
$$+\frac{13}{256}\frac{1+\alpha}{(1+15\alpha)(1+28\alpha)}[16+168(1+10\alpha)(x^2-1)$$
$$\left. +126(3+40\alpha)(x^2-1)^2+231(1+15\alpha)(x^2-1)^3]^2\right\},$$

which for $\alpha = \cdot 2048019$ becomes

$$ {}_6\sigma_y^2 = \frac{\sigma^2}{N}\{5{\cdot}58984 - 33{\cdot}14234x^2 + 504{\cdot}4523x^4 - 2512{\cdot}673x^6 + 5524{\cdot}186x^8 + $$
$$ - 5452{\cdot}650x^{10} + 1974{\cdot}020x^{12}\}.$$

The maxima and minima are:

At $x = 0$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}364$,

,, $x = \pm\ \cdot 2216$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}216$,

,, $x = \pm\ \cdot 4826$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}515$,

,, $x = \pm\ \cdot 6194$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}427$,

,, $x = \pm\ \cdot 8445$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,3{\cdot}149$,

,, $x = \pm\ \cdot 9615$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,2{\cdot}485$,

,, $x = \pm 1{\cdot}0000$ $\quad \sigma_y = \frac{\sigma}{\sqrt{N}}\,.\,3{\cdot}128$.

It thus appears that this distribution which has $\cdot 08499 \times N$ observations at each end of the range fulfils our demand that $\overset{x^2=1}{\sigma_y}$ shall be approximately equal to the greatest of the maxima.

(11) We bring together our final results in the following table. It gives the distribution of observations, the maximum of σ_y within the range, the value of $\sqrt{n+1}$ or the lowest maximum of $\frac{\sigma_y\sqrt{N}}{\sigma}$ possible, which can only be obtained by distributing the observations of the function of the nth degree into $(n+1)$ groups, and the value of $n+1$ which is the maximum of $\frac{\sigma_y\sqrt{N}}{\sigma}$ for a uniform distribution.

TABLE II.

Degree of function	Ratio of number of observations at each end of the range to the total number	Maximum of $\frac{\sigma_y \sqrt{N}}{\sigma}$	$\sqrt{n+1}$	$n+1$
1	·2500	1·581	1·414	2
2	·2203	1·862	1·732	3
3	·1708	2·161	2·000	4
4	·1411	2·467	2·236	5
5	·0978	2·878	2·449	6
6	·0850	3·149	2·646	7

A comparison between our maximum and $\sqrt{n+1}$ shows the price we have to pay for information about the degree of the function. For lower degrees the maximum only differs quite insignificantly from $\sqrt{n+1}$, but with increasing degree the difference grows relatively greater for the sixth degree, being about one-fifth of $\sqrt{n+1}$.

The curves of standard deviation for the three sets of distributions are given in Diagrams 3—8, while Diagram 9 represents the six curves just reached. It seems likely from the form of the σ_y curves that two clusters of observations placed at the outermost of the maxima besides the two clusters at the ends of the range would produce a σ_y curve with a lower maximum than the one we have succeeded in getting for the functions from the fourth to the sixth degree. But then again the position of these new clusters would depend on the degree of the function and thus make the proceedings more complicated; and what is more at the same time as the maximum of the curve approached $\sqrt{n+1}$ the distribution of observations would incur the disadvantages of the grouping in $(n+1)$ clusters. On the whole the distribution arrived at seems to be satisfactory and certainly marks a great progress from the uniform distribution.

VII. *Observations with varying standard deviation.*

(1) In Section I we have already given the formula for the standard deviation σ_y of an adjusted y when the standard deviation s_y of an observation is $\sigma\sqrt{f(x)}$.

It is

$$\begin{vmatrix} \sigma_y^2 \cdot \dfrac{N}{\sigma^2} & 1 & x & x^2 & \dots\dots & x^n \\ 1 & m_0 & m_1 & m_2 & \dots\dots & m_n \\ x & m_1 & m_2 & m_3 & \dots\dots & m_{n+1} \\ x^2 & m_2 & m_3 & m_4 & \dots\dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^n & m_n & m_{n+1} & m_{n+2} & \dots\dots & m_{2n} \end{vmatrix} = 0,$$

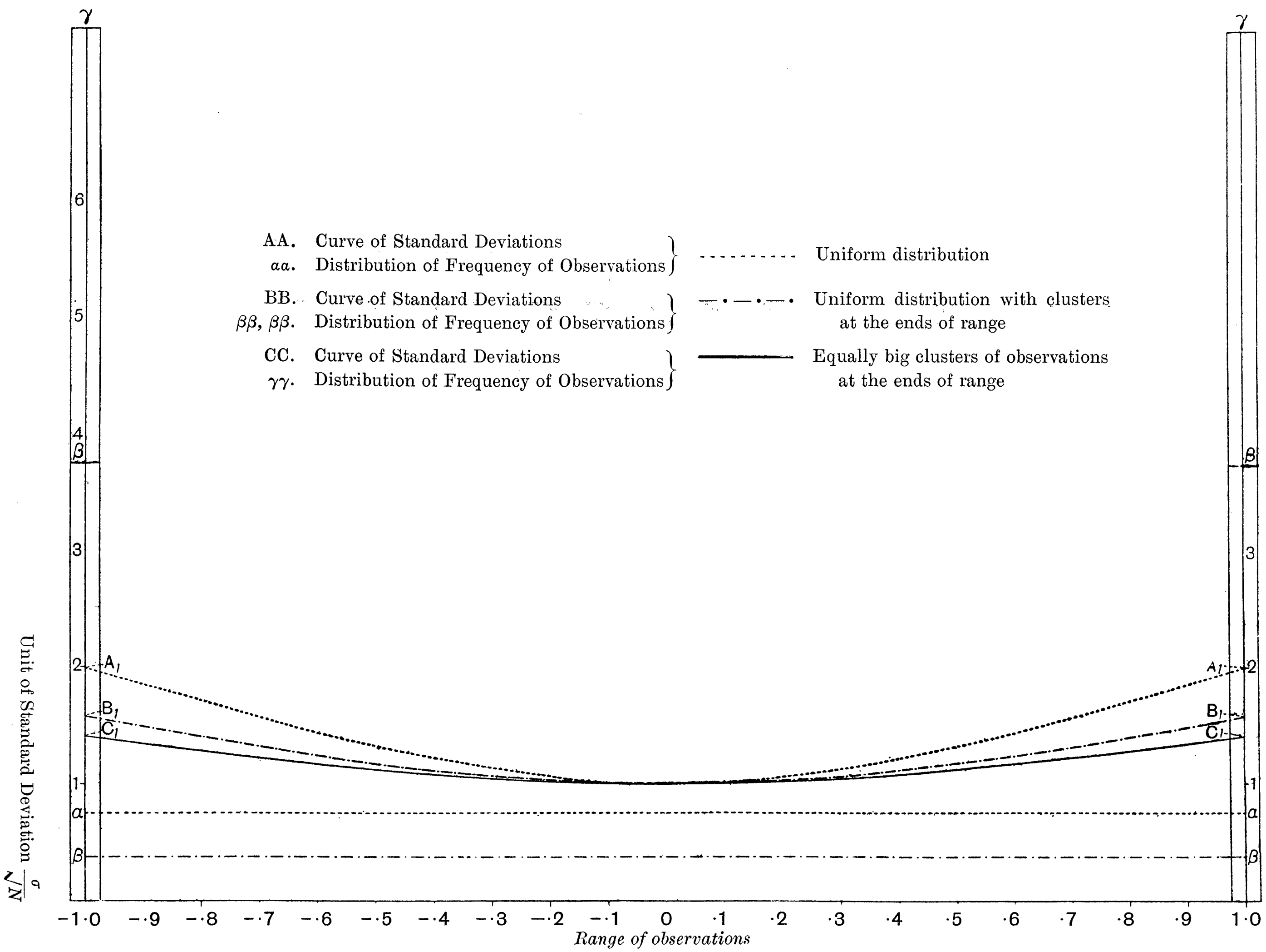
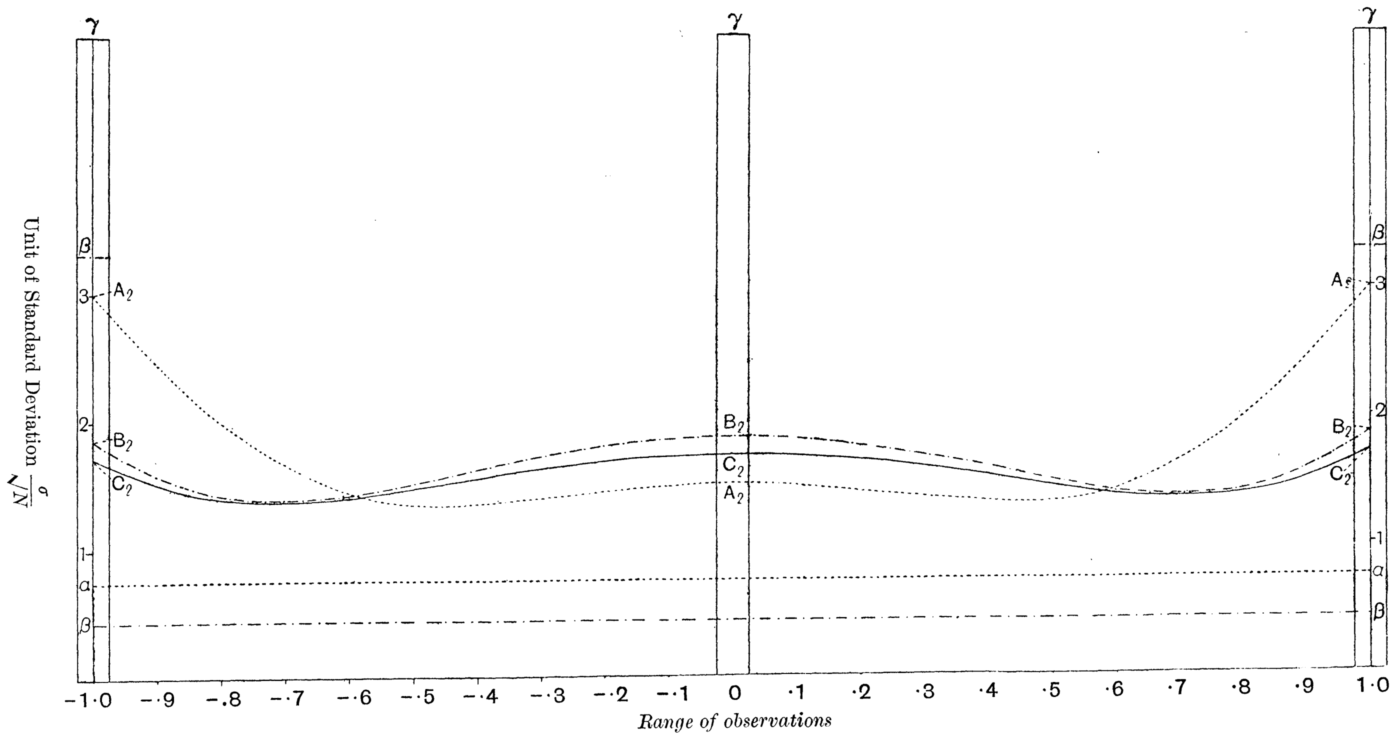


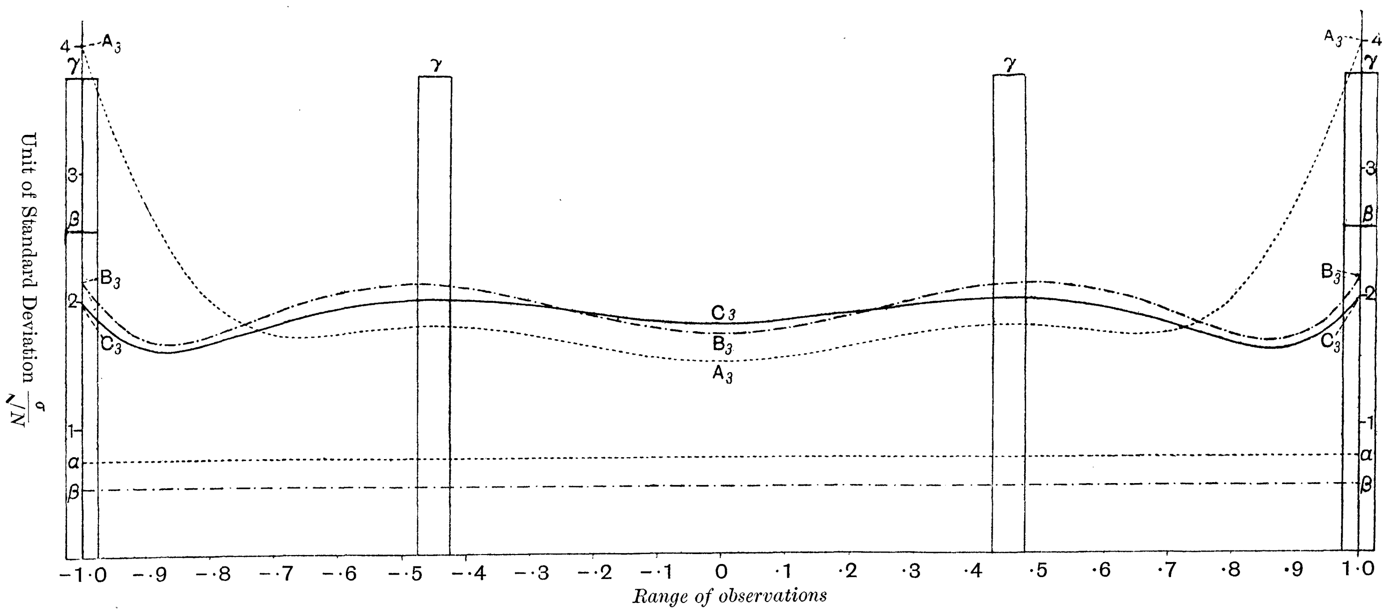
DIAGRAM 3. Linear Function.

4—2



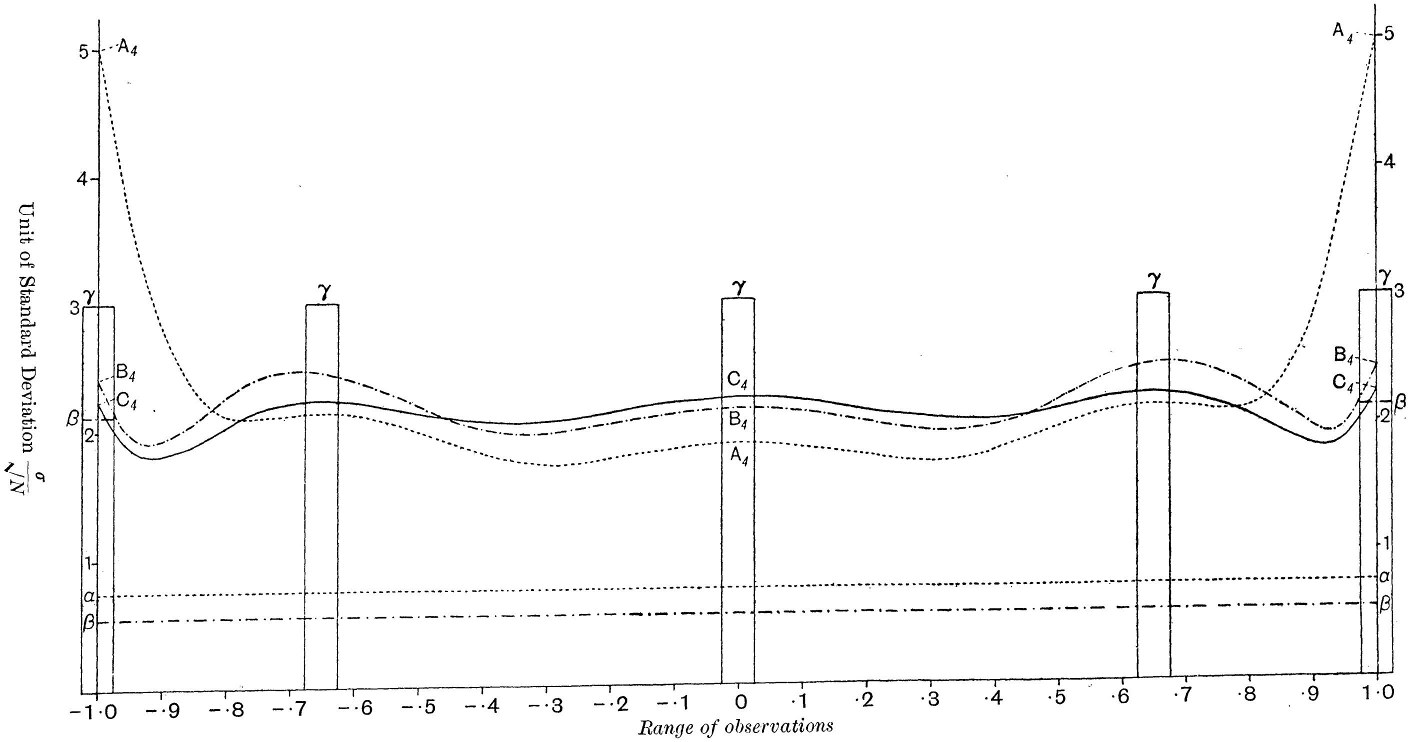
AA.	Curve of Standard Deviations	⎫ - - - - - - - - - -	Uniform distribution
αα.	Distribution of Frequency of Observations	⎭	
BB.	Curve of Standard Deviations	⎫ —·—·—·	Uniform distribution with clusters at the ends of range
ββ, ββ.	Distribution of Frequency of Observations	⎭	
CC.	Curve of Standard Deviations	⎫ ———	Equally big clusters of observations at three points
γγ.	Distribution of Frequency of Observations	⎭	

DIAGRAM 4. Function of Second Degree.



AA.	Curve of Standard Deviations	- - - - - - - - -	Uniform distribution
$\alpha\alpha$.	Distribution of Frequency of Observations		
BB.	Curve of Standard Deviations	—·—·—·	Uniform distribution with clusters at the ends of range
$\beta\beta$, $\beta\beta$.	Distribution of Frequency of Observations		
CC.	Curve of Standard Deviations	———	Equally big clusters of observations at four points
$\gamma\gamma$.	Distribution of Frequency of Observations		

Diagram 5. Function of Third Degree.



AA. Curve of Standard Deviations
αα. Distribution of Frequency of Observations } - - - - - - - - - Uniform distribution

BB. Curve of Standard Deviations
ββ, ββ. Distribution of Frequency of Observations } —·—·—· Uniform distribution with clusters at the ends of range

CC. Curve of Standard Deviations
γγ. Distribution of Frequency of Observations } ——— Equally big clusters of observations at five points

Diagram 6. Function of Fourth Degree.

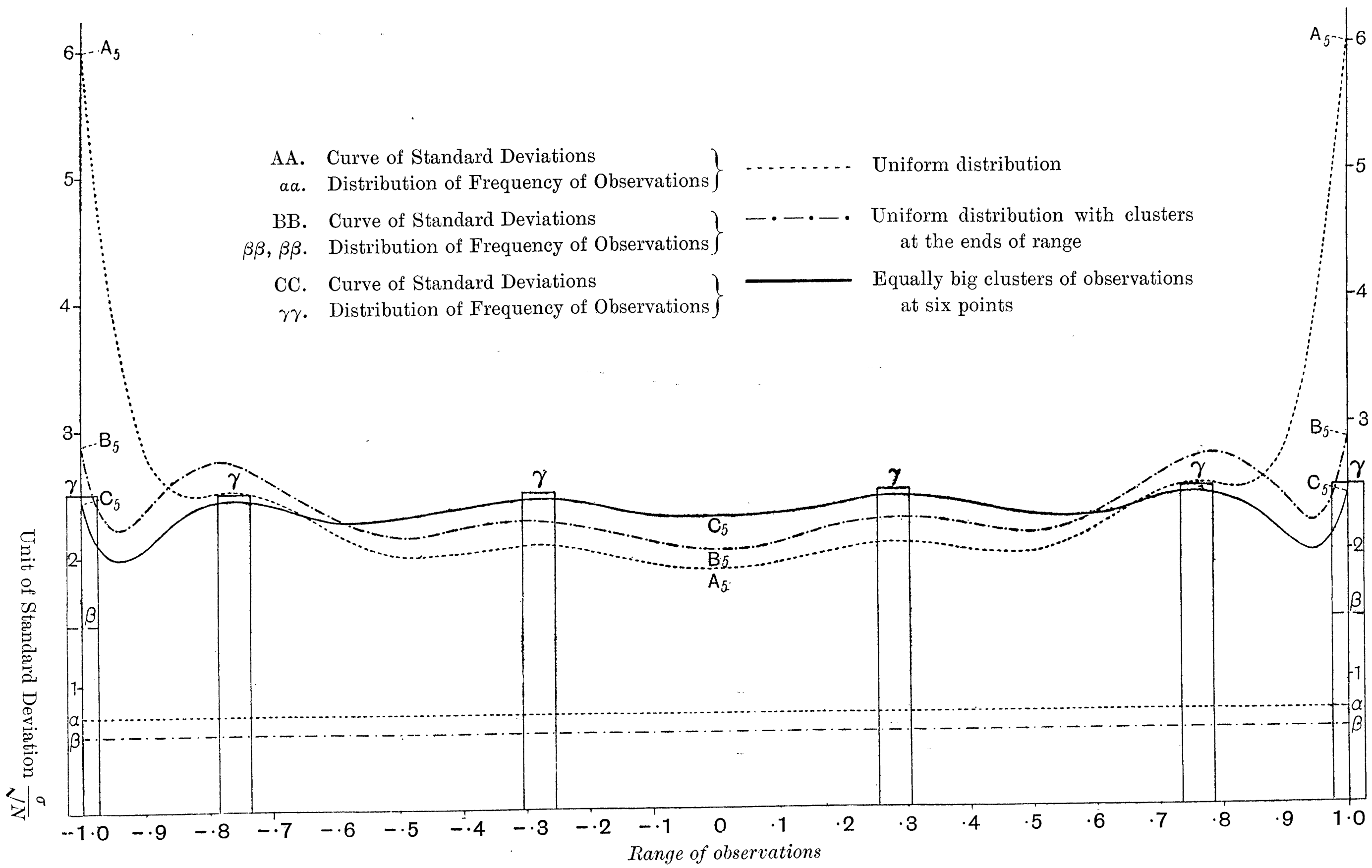


DIAGRAM 7. Function of Fifth Degree.

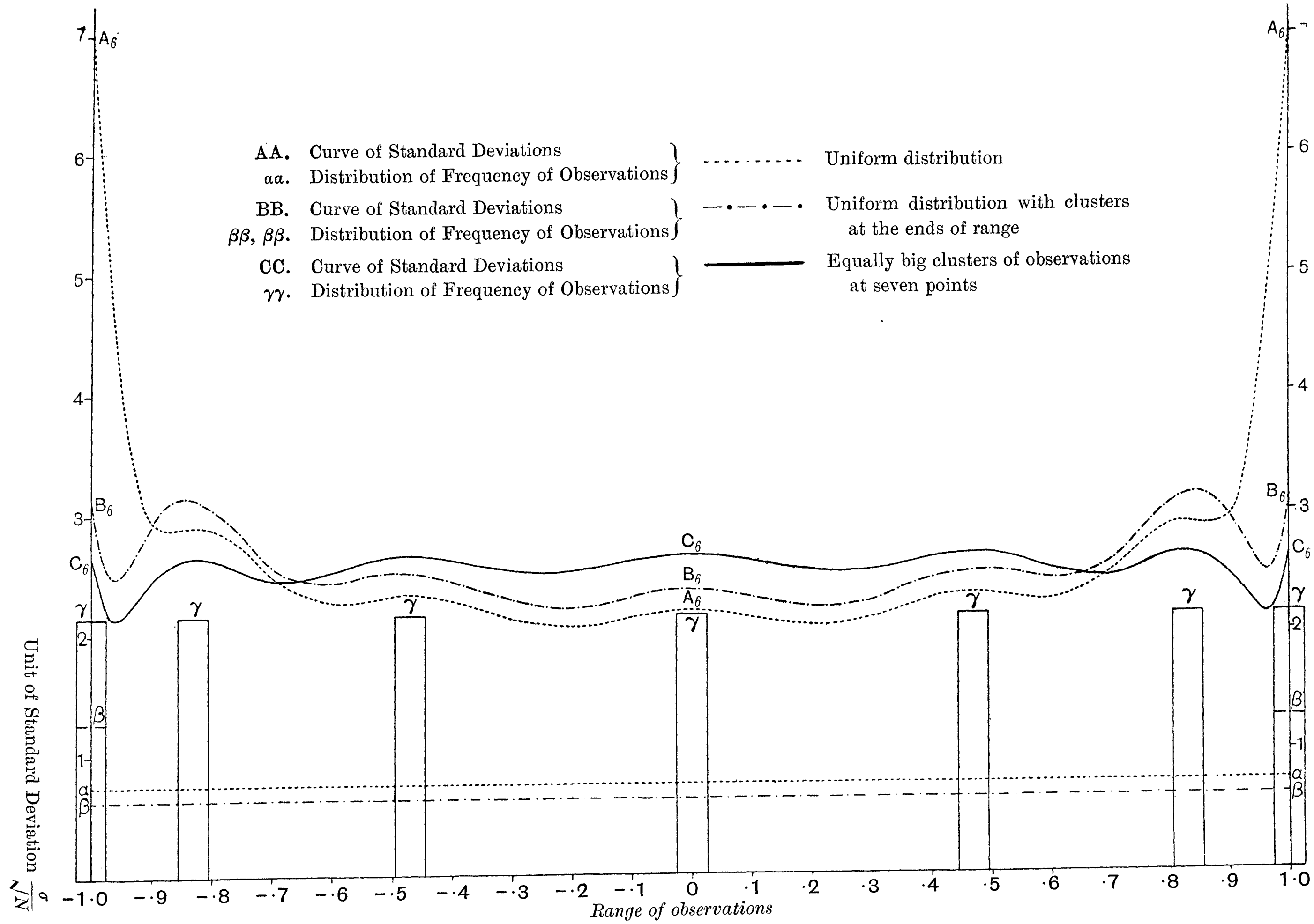
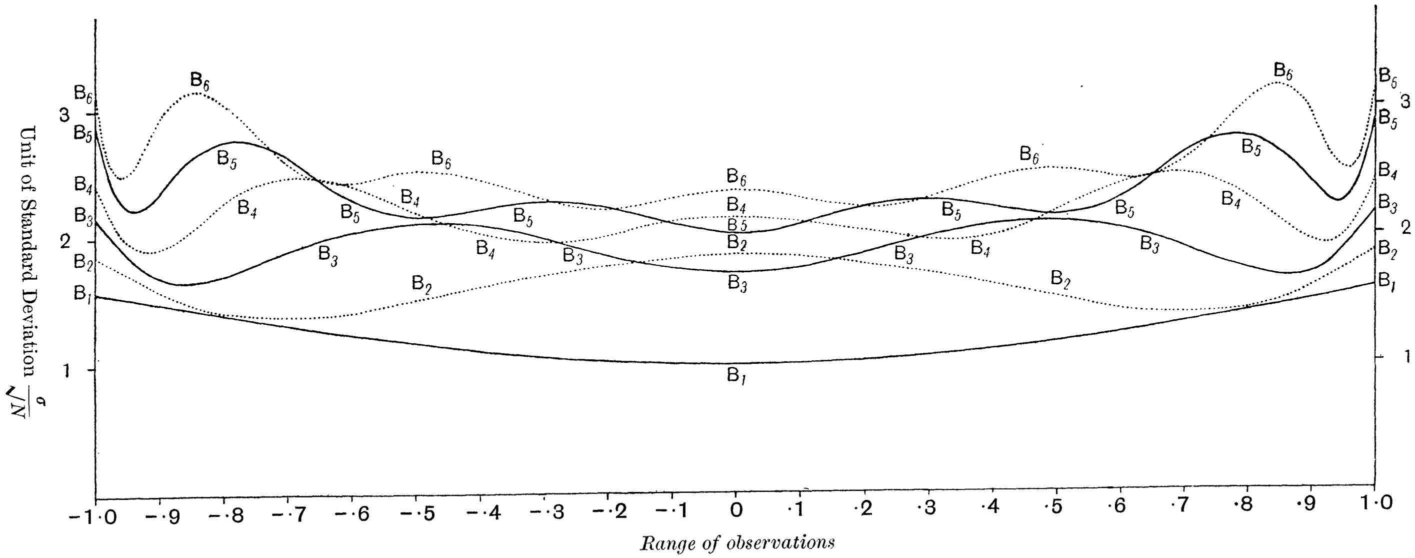


DIAGRAM 8. Function of Sixth Degree.



B_1B_1.	Function of	First	Degree
B_2B_2.	,,	Second	,,
B_3B_3.	,,	Third	,,
B_4B_4.	,,	Fourth	,,
B_5B_5.	,,	Fifth	,,
B_6B_6.	,,	Sixth	,,

DIAGRAM 9. Curves of Standard Deviations. Uniform Distribution with clusters at the ends of range.

where $m_p = \frac{1}{N}\int x^p \frac{\psi(x)}{f(x)}\,dx$, $\psi(x)\,dx$ being the number of observations between x and $x + dx$ and the integration being extended over the range of observations.

It is clear that if we have found a suitable curve of squared standard deviation for adjusted y by taking a distribution $\phi(x)$ of observations with constant standard deviations a corresponding curve can be derived for observations with varying standard deviations by using the distribution

$$\psi(x) = k\phi(x)\,.\,f(x) \qquad \text{.............................(65).}$$

As $\int k\phi(x)\,.\,f(x)\,dx = N$ the constant k must be

$$k = \frac{N}{\int \phi(x)\,.\,f(x)\,dx}.$$

Hence we find $$m_p = \frac{\int x^p\,.\,\phi(x)\,dx}{\int \phi(x)\,.\,f(x)\,dx} = \frac{N\mu_p}{\int \phi(x)\,.\,f(x)\,dx},$$

where μ_p is the pth moment coefficient for the distribution $\phi(x)$, and as

$$\frac{m_p}{\mu_p} = \frac{N}{\int \phi(x)\,.\,f(x)\,dx} = k$$

for any p the determinant may be written

$$\begin{vmatrix} \sigma_y^2\,.\,\frac{N}{\sigma^2}k & 1 & x & x^2 & \cdots\cdots & x^n \\ 1 & 1 & \mu_1 & \mu_2 & \cdots\cdots & \mu_n \\ x & \mu_1 & \mu_2 & \mu_3 & \cdots\cdots & \mu_{n+1} \\ x^2 & \mu_2 & \mu_3 & \mu_4 & \cdots\cdots & \mu_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^n & \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots\cdots & \mu_{2n} \end{vmatrix} = 0 \qquad \text{..................(66).}$$

We thus find the same determinant as the distribution $\phi(x)$ would give for observations with constant error of observation except that the factor k has come in, that is to say the expression for σ_y^2 has been multiplied by

$$\frac{1}{k} = \frac{1}{N}\int \phi(x)\,.\,f(x)\,dx \qquad \text{.............................(67).}$$

The goodness of the distribution therefore will partly depend on the value of $\frac{1}{k}$, and because we have found $\phi(x)$ the best distribution for observations with constant standard deviation it does not follow that

$$\psi(x) = k\phi(x)\,.\,f(x)$$

is the best distribution for observations with the standard deviation $\sigma\sqrt{f(x)}$. But the deriving of $\psi(x)$ from $\phi(x)$ is nevertheless useful as a means of simplifying the investigations and will be applied in the following special inquiries.

We shall consider two forms of $f(x)$ and try to find the best distributions for functions of the first and of the second degree.

(a) $$f(x) = (1 + ax^2)^2, \text{ where } a > -1,$$

for errors of observation increasing or decreasing in both directions from the middle of the range.

(b) $$f(x) = (1 + ax)^2, \text{ where } 1 > a \geqq 0,$$

for error of observations increasing in one direction.

These two forms will roughly cover two distinct and important types of cases, such as occur in practice.

(2) When $f(x) = (1 + ax^2)^2$ we find, according to (67),

$$\frac{1}{k} = 1 + 2a\mu_2 + a^2\mu_4,$$

and as (66) for $n = 1$ gives

$$\sigma_y^2 \cdot \frac{N}{\sigma^2} k = \frac{1}{\mu_2 - \mu_1^2}\{\mu_2 - 2\mu_1 x + x^2\},$$

we have for a *function of the first degree*

$$\sigma_y^2 = \frac{\sigma^2}{N} \frac{1 + 2a\mu_2 + a^2\mu_4}{\mu_2 - \mu_1^2}\{\mu_2 - 2\mu_1 x + x^2\} \quad \text{.................(68).}$$

This curve has a minimum for $x = \mu_1$ and the maximum in the range is, if $\mu_1 > 0$, at $x = -1$, and if $\mu_1 < 0$, at $x = 1$; it equals in both cases

$$\frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\left\{1 + \frac{(1 + [\mu_1])^2}{\mu_2 - \mu_1^2}\right\} \quad \text{....................(69),}$$

$[\mu_1]$ being the numerical value of μ_1.

Now (69) is a minimum for $\mu_1 = 0$; we therefore ought to choose that value for μ_1 and we then get, from (68),

$$\sigma_y^2 = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\left\{1 + \frac{x^2}{\mu_2}\right\} \quad \text{.......................(70),}$$

μ_2 and μ_4 may vary between 0 and 1 independently of each other and are only bound by the conditions that

$$\mu_4 \overline{\gtrless} \mu_2$$

and $$\beta_2 = \frac{\mu_4}{\mu_2^2} \overline{\gtrless} 1.$$

For any set of values which satisfies these conditions we may determine a distribution consisting of $\frac{\gamma}{2}N$ observations at $x = \pm v$ and $(1 - \gamma)N$ at $x = 0$, since from any two such values we could determine

$$v^2 = \frac{\mu_4}{\mu_2} \overline{\gtrless} 1$$

and $$\gamma = \frac{\mu_2^2}{\mu_4} \overline{\gtrless} 1.$$

By introducing v^2 and γ for μ_2 and μ_4 we get two quite independent variables and (70) then takes the form

$$\sigma_y^2 = \frac{\sigma^2}{N}(1 + 2a\gamma v^2 + a^2\gamma v^4)\left(1 + \frac{x^2}{\gamma v^2}\right).$$

We now have to determine γ and v^2 so that the maximum value $\overset{x=1}{\sigma_y^2}$ is as small as possible. We find

$$\left[\frac{d\sigma_y^2}{d\gamma}\right]_{x=1} = \frac{\sigma^2}{N}\left(2\alpha v^2 + \alpha^2 v^4 - \frac{1}{\gamma^2 v^2}\right) \quad \text{.......................(71)}$$

and

$$\left[\frac{d\sigma_y^2}{dv^2}\right]_{x=1} = \frac{\sigma^2}{N}\left(2\alpha\gamma + 2\alpha^2\gamma v^2 + \alpha^2 - \frac{1}{\gamma v^4}\right) \quad \text{.................(72).}$$

Clearly

$$\left[\frac{d\sigma_y^2}{d\gamma}\right]_{x=1} = 0 \text{ leads to } \gamma^2 = \frac{1}{\alpha v^4 (2 + \alpha v^2)} \quad \text{.................(73).}$$

Introducing this value into (72) we obtain

$$\left[\frac{d\sigma_y^2}{dv^2}\right]_{x=1} = \frac{\sigma^2}{N} \cdot \alpha^2 \left(1 + \frac{1}{\sqrt{\alpha}\,(2 + \alpha v^2)}\right),$$

which is > 0.

Hence the minimum for constant v^2 determined by

$$\left[\frac{d\sigma_y^2}{d\gamma}\right]_{x=1} = 0$$

decreases with v^2.

But when v^2 decreases, γ^2, as given by (73), increases and the lowest value of v^2, for which it is real, is that determined by

$$\gamma^2 = \frac{1}{\alpha v^4 (2 + \alpha v^2)} = 1.$$

For v^2 smaller than this (73) gives $\gamma^2 > 1$, and as long as $\gamma^2 \gtreqless 1$ we therefore have

$$\left[\frac{d\sigma_y^2}{d\gamma}\right]_{x=1} \gtreqless 0.$$

Hence the minimum of $\overset{x=1}{\sigma_y^2}$ is to be found for $\gamma^2 = 1$.

For this value (72) may be written as

$$\left[\frac{d\sigma_y^2}{dv^2}\right]_{x=1} = \frac{\sigma^2}{N}\frac{1}{v^4}(1 + \alpha v^2)(2\alpha v^4 + \alpha v^2 - 1) \quad \text{.................(74),}$$

which is zero for

$$v^2 = -\tfrac{1}{4} + \sqrt{\frac{1 + \frac{8}{\alpha}}{16}} \quad \text{.............................(75)}$$

and > 0 for v^2 greater than this value.

When the v^2 found lies between 0 and 1, that is when $\alpha > \frac{1}{3}$, we have thus found the minimum sought. When $\alpha \eqslantless \frac{1}{3}$, then $\left[\frac{d\sigma_y^2}{dv^2}\right]_{x=1}$ as given by (74) is < 0 and the minimum of $\overset{x=1}{\sigma_y^2}$ is found by giving v^2 its maximum value, that is 1.

Returning to the variates μ_2 and μ_4 we see that in all cases

$$\gamma^2 = \frac{\mu_2^2}{\mu_4} = \frac{1}{\beta_2} = 1,$$

from which it follows that no distribution of observations other than those arrived at consisting of two equally big groups can give μ_1, μ_2 and μ_4 the values required.

We accordingly reach the result that: *when observing a function of the first degree for which the standard deviation of the observations is $\sigma(1+\alpha x^2)$, symmetrical about the middle of the range, we get the best function for σ_y^2 by taking two equally big groups of observations, at the ends of the range if $\alpha \leqq \frac{1}{3}$ and at $v = \pm\frac{1}{2}\sqrt{\sqrt{1+\frac{8}{\alpha}}-1}$ if $\alpha > \frac{1}{3}$.*

(3) According to (70) the maximum of σ_y^2 for this distribution is

$$\overset{x=1}{\sigma_y^2} = \frac{\sigma^2}{N}(1+\alpha v^2)^2\left(1+\frac{1}{v^2}\right),$$

v being equal to 1 for $\alpha \leqq \frac{1}{3}$ and v being determined by (75) for $\alpha > \frac{1}{3}$.

We shall next consider the distributions (i) for which $\phi(x)$ is constant from -1 to 1 and (ii) for which $\phi(x)$ consists of $\frac{N}{2}$ observations uniformly distributed from -1 to 1 and $\frac{N}{2}$ into two clusters.

(i) For a uniform distribution from -1 to 1 we have $\mu_2 = \frac{1}{3}$, $\mu_4 = \frac{1}{5}$ and, according to (67),

$$\frac{1}{k} = 1 + \tfrac{2}{3}\alpha + \tfrac{1}{5}\alpha^2,$$

the actual distribution is hence, as $\phi(x) = \frac{N}{2}$,

$$\psi(x) = \frac{N}{2}\,\frac{(1+\alpha x^2)^2}{1+\frac{2}{3}\alpha+\frac{1}{5}\alpha^2},$$

and the maximum σ_y^2 as given by (70) for $x = \pm 1$,

$$\overset{x=\pm 1}{\sigma_y^2} = \frac{\sigma^2}{N}(1+\tfrac{2}{3}\alpha+\tfrac{1}{5}\alpha^2) \,.\, 4.$$

(ii) When $\phi(x) = \frac{N}{4}$ with the additional clusters $\frac{N}{4}$ at $\pm u$ we have

$$\mu_2 = \tfrac{1}{6} + \tfrac{1}{2}u^2 \quad\text{and}\quad \mu_4 = \tfrac{1}{10} + \tfrac{1}{2}u^4.$$

According to (70) the maximum σ_y^2 is then

$$\overset{x=\pm 1}{\sigma_y^2} = \frac{\sigma^2}{N}\left[1+\alpha(\tfrac{1}{3}+u^2)+\alpha^2(\tfrac{1}{10}+\tfrac{1}{2}u^2)\right]\left(1+\frac{6}{3u^2+1}\right).$$

We shall now determine u so as to make this a minimum. We find that

$$\frac{d\overset{x=1}{\sigma_y^2}}{du^2} = \frac{\sigma^2}{N}\cdot\frac{1}{5}\left[5\alpha\{1+\alpha(1+u^2)\} - \frac{2}{(3u^2+1)^2}(45+7\alpha^2)\right] = 0$$

requires

$$45\alpha^2u^6 + 15\alpha(3+5\alpha)u^4 + 5\alpha(6+7\alpha)u^2 - (90-5\alpha+9\alpha^2) = 0 \quad \ldots(76),$$

the root u^2 of which is > 1 for $\alpha < \cdot 5576$.

For $\alpha \leqq \cdot 5576$ we hence get the minimum $\overset{x=1}{\sigma_y^2}$ by taking the clusters at $u = \pm 1$ and for $\alpha > \cdot 5576$ at the places $\pm u$ determined by (76).

Table III contains for a series of values of a the values of v, $(1+av^2)$ and u of the two distributions above and the maximum σ_y for the three distributions.

TABLE III.

a	v	$1+av^2$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from best distribution	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x)=\frac{N}{2}$	u	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x)=\frac{N}{4}$ and clusters of $\frac{N}{4}$ at $\pm u$
0	1·0000	1·000	1·414	2·000	1·0000	1·581
$\frac{1}{6}$	1·0000	1·167	1·650	2·113	1·0000	1·760
$\frac{1}{3}$	1·0000	1·333	1·886	2·231	1·0000	1·944
$\frac{1}{2}$	·8836	1·390	2·100	2·352	1·0000	2·131
$\frac{2}{3}$	·8071	1·434	2·284	2·477	·9289	2·316
$\frac{5}{6}$	·7510	1·470	2·448	2·603	·8502	2·483
1	·7071	1·500	2·598	2·733	·7797	2·637
2	·5559	1·618	3·330	3·540	·5762	3·438
3	·4782	1·686	3·908	4·382	·4925	4·173
4	·4278	1·732	4·404	5·241	·4612	4·899

The difference between the maxima from the two first distributions taken as a proportion of the maximum of the first decreases from 41 per cent. at $a=0$ to the minimum 5 per cent. at $a=1$, and then again increases to 19 per cent. at $a=4$. For small a, that is in practice $a=0$, and again for $a>3$, for which the difference is greater than 12 per cent., the third distribution may therefore be useful as giving a much smaller maximum value than the purely continuous distribution and at the same time offering some justification for the form of the function.

(4) We shall next, still assuming that $f(x)=(1+ax^2)^2$, consider the choice of observations for a *function of the second degree*.

According to (66) and (67) we find

$$\sigma_y^2 = \frac{\sigma^2}{N}\cdot\frac{1}{k}\times \left\{\frac{\mu_2\mu_4-\mu_3^2+2(\mu_2\mu_3-\mu_1\mu_4)x+(\mu_4-3\mu_2^2+2\mu_1\mu_3)x^2+2(\mu_1\mu_2-\mu_3)x^3+(\mu_2-\mu_1^2)x^4}{\mu_2\mu_4-\mu_2^3+2\mu_1\mu_2\mu_3-\mu_1^2\mu_4-\mu_3^2}\right\} \quad \text{...........(77)},$$

and

$$\frac{1}{k}=1+2a\mu_2+a^2\mu_4,$$

where the μ's are the moment coefficients about $x=0$ of the distribution $\phi(x)$ which is connected with the actual distribution $\psi(x)$ by the relation

$$\psi(x)=k\phi(x).f(x).$$

From any distribution $\phi(x)$ which has μ_1 and $\mu_3 \gtrless 0$ we can form a symmetrical $\frac{1}{2}\{\phi(x)+\phi(-x)\}$ which has the same μ_2 and μ_4 as $\phi(x)$. We shall prove that the maximum σ_y^2 obtained from the symmetrical distribution is always lower than that obtained from the skew.

Let the factor in curled brackets in (77) be F_s for a skew distribution $\phi(x)$ and F_0 for the corresponding symmetrical distribution.

We then have

$$F_0 = \frac{\mu_2\mu_4 + (\mu_4 - 3\mu_2^2)\,x^2 + \mu_2 x^4}{\mu_2(\mu_4 - \mu_2^2)}.$$

The condition for a maximum or minimum other than that at $x = 0$ is

$$3\mu_2^2 - \mu_4 > 0,$$

or

$$\beta_2 < 3,$$

and as the denominator is positive we have in that case the maximum at $x = 0$. It is thus clear that the maxima of F_0 between -1 and 1 must be either at $x = 0$ or at $x = \pm 1$.

We shall show that

$$[F_s]_{x=0} > [F_0]_{x=0},$$

and that either $[F_s]_{x=1}$ or $[F_s]_{x=-1} > [F_0]_{x=\pm 1}$.

According to what has been proved in Section I (4) the coefficient of x^4 in (77) is positive, the denominator of (77) is therefore positive and we have

$$[F_s - F_0]_{x=0} = \frac{(\mu_2\mu_3 - \mu_1\mu_4)^2}{(\mu_4 - \mu_2^2)(\mu_2\mu_4 - \mu_2^3 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4 - \mu_3^2)} > 0.$$

We shall next compare F_s and F_0 for $x = \pm 1$.

Putting

$$[F_0]_{x=1} = \frac{N}{D}$$

we have

$$[F_s]_{x=1} = \frac{N - \delta}{D - \epsilon},$$

where

$$\delta = \mu_3^2 - 2\mu_1\mu_3 + \mu_1^2 \pm 2\{\mu_3(1 - \mu_2) - \mu_1(\mu_2 - \mu_4)\}$$

and

$$\epsilon = \mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4.$$

For $\dfrac{\delta}{\epsilon}$ we find

$$\frac{\delta}{\epsilon} = \frac{(\mu_3 - \mu_1)^2 \pm 2\{\mu_3(1 - \mu_2) - \mu_1(\mu_2 - \mu_4)\}}{(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)}.$$

Looking first at the case $\dfrac{\mu_1}{\mu_3} \geqq 0$, we have

$$(\mu_3 - \mu_1)^2 < (\mu_3 - \mu_1\mu_2)^2,$$

and if we choose the value for which the other term of the numerator is < 0,

$$\frac{\delta}{\epsilon} < 1.$$

When $\dfrac{\mu_1}{\mu_3} < 0$ we see, from considering the form

$$\frac{\delta}{\epsilon} = 1 - \frac{\mu_1^2(1 - \mu_4) - 2\mu_1\mu_3(1 - \mu_2) \pm 2\{\mu_3(1 - \mu_2) - \mu_1(\mu_2 - \mu_4)\}}{(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)},$$

that for either $x = 1$ or $x = -1$

$$\frac{\delta}{\epsilon} < 1.$$

As $\epsilon > 0$ we have hence for any μ_1 and μ_3, remembering that $\frac{N}{D}$ being a squared standard deviation multiplied by the number of observations is $\gtreqless 1$,

$$\frac{N-\delta}{D-\epsilon} > \frac{N}{D} > \frac{\delta}{\epsilon},$$

that is, for either $x = 1$ or -1,

$$F_s > F_0.$$

We have thus proved that the maxima of F_0 are below those of F_s.

(5) Our problem is hence reduced to finding the best curve among those represented by

$$\sigma_y^2 = \frac{\sigma^2}{N} \frac{(1 + 2a\mu_2 + a^2\mu_4)}{\mu_2(\mu_4 - \mu_2^2)} \{\mu_2\mu_4 + (\mu_4 - 3\mu_2^2)\, x^2 + \mu_2 x^4\} \quad \text{.........(78).}$$

As was stated in (2) of this section we get all sets of possible values for μ_2 and μ_4 from three groups of observations symmetrical about $x = 0$, and we may therefore limit our search of the best distribution to these.

Let the observations be $\frac{\gamma}{2}N$ at $x = \pm v$, at $(1-\gamma)\,N$ at $x = 0$. The interpolation formula of Lagrange gives, when $\bar{y}_p$ represents the mean of the observations at $x = p$,

$$y = \frac{x^2 - v^2}{-v^2}\bar{y}_0 + \frac{x(x-v)}{2v^2}\bar{y}_{-v} + \frac{x(x+v)}{2v^2}\bar{y}_v,$$

from which we find

$$\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{1}{v^4} \left\{ \frac{(x^2 - v^2)^2}{1-\gamma} + \frac{x^2(x^2+v^2)(1+av^2)^2}{\gamma} \right\} \quad \text{...............(79).}$$

It is obvious that if for a certain distribution we have

$$\overset{x=0}{\sigma_y^2} > \overset{x^2=1}{\sigma_y^2}$$

we can get a better distribution by taking more observations at 0. If on the other hand

$$\overset{x=0}{\sigma_y^2} < \overset{x^2=1}{\sigma_y^2},$$

the curve cannot be the best unless $\overset{x^2=1}{\sigma_y^2}$ is a minimum for the present values of v and γ. From (79) we find

$$\left[\frac{d\sigma_y^2}{d\gamma}\right]_{x^2=1} = \frac{\sigma^2}{N} \cdot \frac{1}{v^4} \left\{ \frac{(1-v^2)^2}{(1-\gamma)^2} - \frac{(1+v^2)(1+av^2)^2}{\gamma^2} \right\} \text{...............(80)}$$

and

$$\left[\frac{d\sigma_y^2}{dv^2}\right]_{x^2=1} = \frac{\sigma^2}{N} \cdot \frac{1}{v^6} \left\{ -\frac{2(1-v^2)}{1-\gamma} - \frac{(2 + v^2 - av^4)(1+av^2)}{\gamma} \right\},$$

from which we obtain the conditions for maximum or minimum

$$v^2 = \frac{1 \pm 2\sqrt{a}}{a}$$

and

$$\frac{\gamma}{1-\gamma} = \frac{2\sqrt{a}(1 \pm \sqrt{a})^2}{a \mp 2\sqrt{a} - 1}.$$

The lower sign requires $3-2\sqrt{2} \gtreqless \alpha \gtreqless \frac{1}{4}$ and the upper sign $\alpha > 3+2\sqrt{2}$ to make $0 \gtreqless v^2 \gtreqless 1$. The case $\alpha < \frac{1}{4}$ has no interest, as we have seen that when $\alpha < \frac{1}{3}$ extrapolation is not even for a linear function advantageous. We have therefore seen that for $\alpha < 3+2\sqrt{2}$* $\overset{x^2=1}{\sigma_y^2}$ has no minimum and we have thus proved that the best distribution requires $\overset{x=0}{\sigma_y^2} = \overset{x^2=1}{\sigma_y^2}$, that is

$$\frac{2v^2-1}{1-\gamma} = \frac{(1+v^2)(1+\alpha v^2)^2}{\gamma},$$

or

$$\frac{1}{1-\gamma} = 1 + \frac{(1+v^2)(1+\alpha v^2)^2}{2v^2-1} \quad \text{......................(81).}$$

The maximum of the curve is

$$\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N} \cdot \frac{1}{1-\gamma}.$$

To find the minimum of this value we differentiate (81) and get

$$\left[\frac{d\sigma_y^2}{dv^2}\right]_{x=0} = \frac{1+\alpha v^2}{(2v^2-1)^2}\{4\alpha v^4 - \alpha v^2 - 2\alpha - 3\},$$

which is zero for

$$v^2 = \frac{1}{8}\left(1 + \sqrt{33 + \frac{48}{\alpha}}\right) \quad \text{.............................(82)}$$

and positive for greater v^2, so that we have found a minimum.

For $\alpha = 3$ we find from (82) $v^2 = 1$, hence for $\alpha \gtreqless 3$ we have to choose $v^2 = 1$, from which, according to (81), follows

$$\frac{1}{1-\gamma} = 1 + 2(1+\alpha)^2 \text{ or } \gamma = \frac{2(1+\alpha)^2}{1+2(1+\alpha)^2}.$$

When $3+2\sqrt{2} > \alpha > 3$, $v^2 = \frac{1}{8}\left(1+\sqrt{33+\frac{48}{\alpha}}\right)$ is < 1, and for the corresponding γ we have

$$\frac{\dfrac{\frac{1}{2}\gamma}{(1+\alpha v^2)^2}}{1-\gamma} = \frac{1+v^2}{2(2v^2-1)} = \frac{5\alpha + 4 + \sqrt{\alpha(33\alpha+48)}}{8(\alpha+2)} \quad \text{...........(83).}$$

Returning to the $\phi(x)$ distribution, which is found from this distribution by dividing the frequencies by $k.(1+\alpha x^2)^2$, we therefore find, when $\frac{\epsilon}{2}N$ is the number of observations at $x = \pm v$ and $(1-\epsilon)N$, that, at $x=0$,

$$\frac{\frac{\epsilon}{2}}{1-\epsilon} = \frac{1+v^2}{2(2v^2-1)}$$

* A further examination shows that for $\alpha > 3+2\sqrt{2}$ $\overset{x^2=1}{\sigma_y^2}$ has a minimum but this is smaller than $\overset{x=0}{\sigma_y^2}$ when $\alpha < 6{\cdot}7$. Up to this value we therefore have $\overset{x^2=1}{\sigma_y^2} = \overset{x=0}{\sigma_y^2}$ for the best curves. For $\alpha > 6{\cdot}7$ the minimum of $\overset{x^2=1}{\sigma_y^2}$ determines the best distribution.

or
$$\epsilon = \frac{1+v^2}{3v^2}.$$

Hence
$$\left.\begin{aligned} \mu_2 &= \tfrac{1}{3}(1+v^2) \\ \mu_4 &= \frac{v^2}{3}(1+v^2) \end{aligned}\right\} \qquad \text{(84)}.$$
and

For $\alpha \lesseqgtr 3$ we have found $v^2 = 1$ which according to (84) involves $\mu_2 = \mu_4$, so that only the distribution above consisting of three groups can realise the requisite conditions.

When $\alpha > 3$ we have $v < 1$ and therefore $\mu_4 < \mu_2$, so that it must be possible to satisfy the equation (84) by a continuous distribution of observations. However v^2 is decreasing so slowly for increasing α that practically the distribution determined by (84) cannot differ much from three groups of observations.

Our results are accordingly that *for a function of the second degree, of which the standard deviation of the observations is* $\sigma(1+\alpha x^2)$, *we get the best function for* σ_y^2 *when* $\alpha \lesseqgtr 3$ *by taking three groups of observations at the middle and the ends of the range, each group proportional to the squared standard deviation at the place, and when* $3 + 2\sqrt{2} > \alpha > 3$ *by taking three groups of observations determined by* (82) *and* (83).

(6) From (78) we find
$$\overset{x=0}{\sigma_y^2} = \frac{\sigma^2}{N}(1 + 2\alpha\mu_2 + \alpha^2\mu_4)\frac{\mu_4}{\mu_4 - \mu_2^2},$$
which, when μ_2 and μ_4 are found in accordance with (82) and (84), determines the maximum σ_y arrived at from our special three groups of observations. Besides the numerical evaluation of this standard deviation, we give in Table IV below the maximum of σ_y obtained from a distribution for which $\phi(x)$ is constant from -1 to 1, that is, since, according to (67),
$$\frac{1}{k} = \left(1 + \tfrac{2}{3}\alpha + \frac{\alpha^2}{5}\right),$$
the distribution
$$\psi(x) = \frac{(1+\alpha x^2)^2}{1 + \tfrac{2}{3}\alpha + \dfrac{\alpha^2}{5}} \cdot \frac{N}{2}.$$

That maximum is determined by
$$\sigma_y^2 = \frac{\sigma^2}{N}\left(1 + \tfrac{2}{3}\alpha + \frac{\alpha^2}{5}\right).9,$$
$\dfrac{\sigma^2}{N}.9$ being the maximum σ_y^2 obtained from a rectangular distribution of observations with the standard deviation σ.

The last column of the same table gives the maximum σ_y arrived at when $\phi(x)$ is the rectangular distribution with clusters at -1 and 1 for which $\overset{x=0}{\sigma_y^2} = \overset{x^2=1}{\sigma_y^2}$. For this distribution consisting of $\cdot 22026\,N$ observations at $+1$ and at -1 and $\cdot 5595\,N = 2cN$ uniformly distributed from -1 to 1, we have found as given in

Table II (p. 50) the maximum $\frac{\sigma}{\sqrt{N}} \cdot 1{\cdot}862$. Hence when μ_2 and μ_4 are the moment coefficient of this $\phi(x)$ the maximum is found from

$$\sigma_y^2 = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4) \cdot 1{\cdot}862.$$

We find $\mu_2 = {\cdot}6270$, $\mu_4 = {\cdot}5524$ and $\frac{1}{k} = 1 + 1{\cdot}2540a + {\cdot}5524a^2$.

The actual distribution is hence

$$\psi(x) = \frac{{\cdot}27975\,(1 + ax^2)^2}{1 + 1{\cdot}2540a + {\cdot}5524a^2} \cdot N,$$

together with the clusters

$$\frac{{\cdot}22026\,(1 + a)^2}{1 + 1{\cdot}2540a + {\cdot}5524a^2} \cdot N,$$

at -1 and 1.

TABLE IV.

a	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for the best distribution	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x) = \frac{N}{2}$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x) = c$ and clusters at ± 1
0	1·732	3·000	1·862
1	3·000	4·099	3·120
2	4·359	5·310	4·453
3	5·745	6·573	5·810
4	7·135	7·861	7·178
5	8·522	9·165	8·551

The difference between the first and second maxima taken as a proportion of the first varies from 79 per cent. at $a = 0$ to 8 per cent. at $a = 5$, while the difference between the first and the third maxima varies from 8 per cent. at $a = 0$ to 0·4 per cent. at $a = 5$. The continuous distribution with clusters is therefore especially useful for smaller a.

For $a = 4$ we find from (82) $v = {\cdot}9816$ and for $a = 5$, $v = {\cdot}9700$, both of these values of v are so close to 1 that if instead of using them we take the observations at 1 and -1 and let the numbers of the three groups of observations be proportional to the squared standard deviations we get the maxima 7·141 and 8·544 which only differ quite insignificantly from the corresponding values of Table IV.

(7) For a *function of the first degree,* of which the standard deviation of the observations is $\sigma(1 + ax)$, where $0 \gtreqless a < 1$, we have, according to (66) and (67),

$$\sigma_y^2 = \frac{\sigma^2}{N}\frac{1 + 2a\mu_1 + a^2\mu_2}{\mu_2 - \mu_1^2}\{\mu_2 - 2\mu_1 x + x^2\} \quad \text{..................(85)}.$$

For $\mu_1 = -c^2$ the maximum of this function is at $x = 1$, and for $\mu_1 = c^2$ at -1. As the maximum of $(\mu_2 - 2\mu_1 x + x^2)$ has the same value in both cases it is clear that the negative μ_1 gives the lower maximum for σ_y^2. We therefore only have to find the conditions for $[\sigma_y^2]_{x=1}$ being a minimum when $\mu_1 < 0$.

We have

$$\overset{x=1}{\sigma_y^2} = \frac{\sigma^2}{N} \frac{[(1+\alpha\mu_1)^2 + \alpha^2(\mu_2 - \mu_1^2)]}{\mu_2 - \mu_1^2} \{\mu_2 - \mu_1^2 + (1-\mu_1)^2\} \quad \text{.........(86)},$$

and differentiating with regard to μ_2,

$$\left[\frac{d\sigma_y^2}{d\mu_2}\right]_{x=1} = \frac{\sigma^2}{N} \cdot \frac{\alpha^2(\mu_2 - \mu_1^2)^2 - (1-\mu_1)^2(1+\alpha\mu_1)^2}{(\mu_2 - \mu_1^2)^2}.$$

As $\alpha < 1$, we have $(1-\mu_1)(1+\alpha\mu_1) > 0$ and

$$\alpha(\mu_2 - \mu_1^2) - (1-\mu_1)(1+\alpha\mu_1) = (\alpha\mu_2 - 1) + \mu_1(1-\alpha) < 0,$$

from which it follows that

$$\left[\frac{d\sigma_y^2}{d\mu_2}\right]_{x=1} < 0$$

for any $\mu_1 \gtreqless 0$.

The greatest value μ_2 can take for our range -1 to $+1$ is 1, the minimum of $\overset{x=1}{\sigma_y^2}$ must therefore be found for $\mu_2 = 1$, for which value (86) passes into

$$[\sigma_y^2]_{x=1} = \frac{\sigma^2}{N} 2 \left\{2\alpha + \frac{(1-\alpha)^2}{1+\mu_1}\right\},$$

which, since $\mu_1 \gtreqless 0$, is a minimum and equals $\frac{\sigma^2}{N} \cdot 2(1+\alpha^2)$ when $\mu_1 = 0$.

The $\phi(x)$ distribution ought accordingly to consist of two equally big groups at the ends of the range and the *actual distribution to be chosen for a function of the first degree, the standard deviation of which is a linear function of the variable, should be two groups at the ends of the working range with numbers proportional to the squared standard deviations at these places.*

(8) For a continuous distribution from -1 to 1 with frequencies proportional to the squared standard deviations we have

$$\mu_1 = 0 \text{ and } \mu_2 = \tfrac{1}{3},$$

and the maximum

$$\overset{x=1}{\sigma_y^2} = \frac{\sigma^2}{N}\left(1 + \frac{\alpha^2}{3}\right)4,$$

the actual distribution is

$$\psi(x) = \frac{(1+\alpha x)^2}{1 + \frac{\alpha^2}{3}} \cdot \frac{N}{2}.$$

Table V contains besides the maxima of σ_y from these two distributions those obtained from a distribution for which $\phi(x)$ is constant with two additional clusters at -1 and 1 each consisting of $\frac{N}{4}$ of the observations.

The actual distribution is, since

$$\mu_2 = \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{2}{3},$$

$$\psi(x) = \frac{(1+\alpha x)^2}{1 + \frac{2}{3}\alpha^2} \cdot \frac{N}{4},$$

with $\frac{(1-\alpha)^2}{1+\frac{2}{3}\alpha^2}\cdot\frac{N}{4}$ observations at -1

and $\frac{(1+\alpha)^2}{1+\frac{2}{3}\alpha^2}\cdot\frac{N}{4}$ at $+1$ in addition.

The maximum of σ_y^2 is

$$\frac{\sigma^2}{N}(1+\tfrac{2}{3}\alpha^2)\tfrac{5}{2}.$$

TABLE V.

α	Maximum of $\sigma_y\frac{\sqrt{N}}{\sigma}$ for best distribution	Maximum of $\sigma_y\frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x)=\frac{N}{2}$	Maximum of $\sigma_y\frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x)=\frac{N}{4}$ and clusters at ±1
·0	1·414	2·000	1·581
·1	1·421	2·003	1·587
·2	1·442	2·013	1·602
·3	1·477	2·030	1·628
·4	1·523	2·053	1·663
·5	1·581	2·082	1·708
·6	1·649	2·117	1·761
·7	1·726	2·157	1·821
·8	1·811	2·203	1·889
·9	1·903	2·254	1·962

(9) For a *function of the second degree* we found in (5) that when the standard deviation of the observations was $s_y=\sigma(1+\alpha x^2)$ and $\alpha \gtreqless 3$ it was advantageous to use the whole working range of observations, much more must this be the case when $s_y=\sigma(1+\alpha x)$ and $0 \gtreqless \alpha < 1$. We shall therefore try to *find the three best groups of observations* taken at -1, v, and 1, supposing v unknown. We do not venture to assert that another form of distribution might not lead to a curve of standard deviation with lower maximum, but the solution of the general problem would involve a more elaborate investigation into the possible variations of μ_1, μ_2, μ_3 and μ_4 for distributions with limited range than seems desirable in this connection. We shall further limit our problem by assuming that the best distribution will be found among those which make $\overset{x=1}{\sigma_y^2}=\overset{x=-1}{\sigma_y^2}$ and both also equal to a maximum situated between $x=-1$ and $x=1$. This would obviously be right if the maximum were found at $x=v$; this in fact is not the case, but still the maximum value is likely to be chiefly determined by the number of observations at $x=v$ and there is therefore every reason to believe that our assumption is justifiable.

Let there be $N\delta$ observations at -1, $N.\gamma$ at 1 and $N(1-\delta-\gamma)$ at v. The interpolation formula of Lagrange then gives

$$y=\frac{(x-v)(x-1)}{(1+v).2}\bar{y}_{-1}+\frac{(x-v)(x+1)}{(1-v).2}\bar{y}_1+\frac{x^2-1}{v^2-1}\bar{y}_v,$$

from which we find

$$\sigma_y^2 = \frac{\sigma^2}{N}\left\{\frac{(x-v)^2(x-1)^2}{4(1+v)^2}\cdot\frac{(1-\alpha)^2}{\delta} + \frac{(x-v)^2(x+1)^2}{4(1-v)^2}\cdot\frac{(1+\alpha)^2}{\gamma} + \frac{(x^2-1)^2}{(v^2-1)^2}\cdot\frac{(1+\alpha v)^2}{1-\delta-\gamma}\right\}.$$

The condition for $\overset{x=1}{\sigma_y^2} = \overset{x=-1}{\sigma_y^2}$

is

$$\frac{(1+\alpha)^2}{\gamma} = \frac{(1-\alpha)^2}{\delta}.$$

Eliminating δ we obtain for $\sigma_y^2 - \overset{x=1}{\sigma_y^2}$ the value

$$\sigma_y^2 - \overset{x=1}{\sigma_y^2} = \frac{\sigma^2}{N}\cdot\frac{(1+\alpha)^2(x^2-1)}{(v^2-1)^2}\left\{\frac{(1+\alpha v)^2(x^2-1)}{(1+\alpha)^2 - 2\gamma(1+\alpha^2)} + \frac{1}{2\gamma}[(1+v^2)x^2 + 2v(1-v^2)x + 2 - 5v^2 + v^4]\right\}$$

or

$$\sigma_y^2 - \overset{x=1}{\sigma_y^2} = \frac{\sigma^2}{N}\cdot\frac{(1+\alpha)^2(x^2-1)}{(v^2-1)^2[(1+\alpha)^2 - 2\gamma(1+\alpha^2)]}\{[(1+\alpha)^2(1+v^2) - 2\gamma(\alpha-v)^2]x^2 + 2v(1-v^2)[(1+\alpha)^2 - 2\gamma(1+\alpha^2)]x + (1+\alpha)^2(2-5v^2+v^4) - 2\gamma[(1+\alpha^2)(2-5v^2+v^4) + (1+\alpha v)^2]\} \quad \text{......(87).}$$

Our assumption that the maximum σ_y^2 shall be equal to $\overset{x=1}{\sigma_y^2}$ requires that the expression in curled brackets shall be a perfect square for which the condition is

$$2\left[\frac{\gamma}{(1+\alpha)^2}\right]^2\{\alpha^2(1+\alpha^2)v^6 + 2\alpha(1+\alpha^2)v^5 + (3-\alpha^2-3\alpha^4)v^4 - 4\alpha(3+2\alpha^2)v^3 + (-2+9\alpha^2+5\alpha^4)v^2 + 2\alpha(3+\alpha^2)v - \alpha^2(3+2\alpha^2)\}$$
$$+ \frac{\gamma}{(1+\alpha)^2}\{-\alpha^2v^6 - 2\alpha v^5 + (-5+2\alpha^2)v^4 + 12\alpha v^3 - (2+9\alpha^2)v^2 - 2\alpha v + 3 + 4\alpha^2\} + v^4 + 2v^2 - 1 = 0 \quad \text{......(88).}$$

Now $\overset{x=1}{\sigma_y^2} = \frac{\sigma^2}{N}\cdot\frac{(1+\alpha)^2}{\gamma}$ is the maximum which we want to make as low as possible, hence we have for a certain α to find the v for which $\frac{\gamma}{(1+\alpha)^2}$ as given by (88) is a maximum.

We shall examine the cases $\alpha = \cdot 5$ and $\alpha = \cdot 9$.

(10) For $\alpha = \cdot 5$ (88) takes the form

$$\left[\frac{\gamma}{(1+\alpha)^2}\right]^2\{\cdot 625v^6 + 2\cdot 5v^5 + 5\cdot 125v^4 - 14v^3 + 1\cdot 125v^2 + 6\cdot 5v - 1\cdot 75\}$$
$$+ \frac{\gamma}{(1+\alpha)^2}\{-\cdot 25v^6 - v^5 - 4\cdot 5v^4 + 6v^3 - 4\cdot 25v^2 - v + 4\} + v^4 + 2v^2 - 1 = 0,$$

which differentiated with regard to v gives

$$\left[\frac{\gamma}{(1+\alpha)^2}\right]^2\{3\cdot 75v^6 + 12\cdot 5v^4 + 20\cdot 5v^3 - 42v^2 + 2\cdot 25v + 6\cdot 5\}$$
$$+ \frac{\gamma}{(1+\alpha)^2}\{-1\cdot 5v^5 - 5v^4 - 18v^3 + 18v^2 - 8\cdot 5v - 1\} + 4v(v^2+1) = 0.$$

We find that these two equations have for $v = -\cdot 190$ the root $\frac{\gamma}{(1+\alpha)^2} = \cdot 2936$ in common which represents a maximum.

The maximum of the curve is hence $\frac{\sigma^2}{N} . 3\cdot 405$, which value occurs for $x = \pm 1$ and for $x = \cdot 064$ determined by (87).

The distribution of observations is

$$\cdot 6607\, N \text{ at } 1,$$
$$\cdot 0734\, N \text{ at } -1,$$
and
$$\cdot 2659\, N \text{ at } -\cdot 190.$$

For comparison we shall consider what would result from taking for the $\phi(x)$ distribution three equally big groups of observations at -1, 0 and 1. This would for observations with the constant error σ make the maximum of the curve equal to $\frac{\sigma^2}{N} . 3$ and that multiplied by

$$1 + 2\alpha\mu_1 + \alpha^2\mu_2 = \frac{3\cdot 5}{3}$$

gives

$$\frac{\sigma^2}{N} . 3\cdot 5.$$

The actual distribution $\psi(x)$ would be

$$\cdot 6429\, N \text{ at } 1,$$
$$\cdot 0714\, N \text{ at } -1,$$
and
$$\cdot 2857\, N \text{ at } 0.$$

This last distribution only makes the maximum σ_{y}^2 about 3 per cent. greater than the value which we obtained by our special distribution and it will therefore for most practical cases be as useful.

(11) When $\alpha = \cdot 9$ we find for (87),

$$\left[\frac{\gamma}{(1+\alpha)^2}\right]^2 \{2\cdot 9322v^6 + 6\cdot 516v^5 + \cdot 4434v^4 - 33\cdot 264v^3 + 17\cdot 141v^2 + 13\cdot 716v - 7\cdot 4844\}$$
$$+ \frac{\gamma}{(1+\alpha)^2}\{-\cdot 81v^6 - 1\cdot 8v^5 - 3\cdot 38v^4 + 10\cdot 8v^3 - 9\cdot 29v^2 - 1\cdot 8v + 6\cdot 24\}$$
$$+ v^4 + 2v^2 - 1 = 0,$$

which differentiated with regard to v gives

$$\left[\frac{\gamma}{(1+\alpha)^2}\right]^2 \{17\cdot 5932v^5 + 32\cdot 58v^4 + 1\cdot 7736v^3 - 99\cdot 792v^2 + 34\cdot 282v + 13\cdot 716\}$$
$$+ \frac{\gamma}{(1+\alpha)^2}\{-4\cdot 86v^5 - 9v^4 - 13\cdot 52v^3 + 32\cdot 4v^2 - 18\cdot 58v - 1\cdot 8\} + 4v(v^2+1) = 0.$$

For $v = -\cdot 354$ these two equations have the root $\frac{\gamma}{(1+\alpha)^2} = \cdot 23214$ in common which is therefore the maximum of $\frac{\gamma}{(1+\alpha)^2}$.

The maximum of the corresponding σ_y^2 is hence

$$\frac{\sigma^2}{N} \cdot \frac{(1+\alpha)^2}{\gamma} = \frac{\sigma^2}{N} \cdot 4{\cdot}308.$$

From (87) we find that it occurs at $x = {\cdot}125$ as well as at $x = \pm 1$. The distribution of observations is then

$${\cdot}8380\,N \text{ at } 1,$$
$${\cdot}0023\,N \text{ at } -1,$$
and
$${\cdot}1597\,N \text{ at } -{\cdot}354.$$

Comparing again with a distribution consisting of three groups of observations at -1, 0 and 1 with frequencies proportional to the squared standard deviations at these places we find that the distribution would be

$${\cdot}7814\,N \text{ at } 1,$$
$${\cdot}0022\,N \text{ at } -1,$$
and
$${\cdot}2164\,N \text{ at } 0,$$

and the maximum of σ_y^2 would be

$$\frac{\sigma^2}{N} \cdot 3\,(1 + 2\alpha\mu_1 + \alpha^2\mu_2) = \frac{\sigma^2}{N} \cdot 4{\cdot}62.$$

We thus find that by our special distribution the maximum of σ_y^2 was 7 per cent. lower, the choice of that distribution would thus permit us to reduce the total number of observations at the same rate without raising the maximum of σ_y^2.

(12) The result of these investigations is that *the maximum σ_y obtained from the best three groups of observations differs so little from that obtained from three groups at* -1, 0 *and* 1 *that the first grouping only in quite exceptional practice would be preferred.*

We shall therefore in Table VI give the maximum σ_y arrived at from the following three distributions: (1) three groups of observations at -1, 0 and 1 in numbers proportional to the squared standard deviations at these places, (2) a distribution for which $\phi(x) = \frac{N}{2}$, and (3) a distribution for which $\phi(x) = {\cdot}2797\,N$ with additional clusters ${\cdot}2203\,N$ at ± 1 (see Table II, p. 50).

Both in Table V and in Table VI the difference between the two first maxima as a proportion of the first decreases with increasing α so that the distribution with uniform $\phi(x)$ is more profitable for $\alpha > 0$ than for observations with constant errors.

VIII. *Best distribution of observations for determining a single constant of the function.*

(1) Our choice of observations has hitherto aimed at giving within the working range of observations a determination of the function as accurate and uniform as possible. We shall now consider what is the best choice of observations for

TABLE VI.

α	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from three groups at 0 and ± 1	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x) = \frac{N}{2}$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x) = \cdot 2797N$ and clusters at ± 1	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from best three groups
·0	1·732	3·000	1·862	—
·1	1·738	3·005	1·868	—
·2	1·755	3·020	1·886	—
·3	1·783	3·045	1·914	—
·4	1·822	3·079	1·954	—
·5	1·871	3·122	2·003	1·845
·6	1·929	3·175	2·062	—
·7	1·995	3·236	2·129	—
·8	2·069	3·304	2·205	—
·9	2·149	3·381	2·287	2·076

determining a single constant of the function. The investigations will be carried out for functions of the first and of the second degree for which the standard deviations of the observations are

$$s_y = \sigma(1 + \alpha x^2), \quad \alpha > -1$$

or

$$s_y = \sigma(1 + \alpha x), \quad 1 > \alpha \geqq 0.$$

We have in (3) of Section I given the formula (8) for $\sigma^2_{a_p}$ and shall here give only the form to which it is transferred by putting

$$\psi(x) = k\phi(x) f(x),$$

$$\frac{1}{k} = \frac{1}{N} \int \phi(x) \,.\, f(x)\, dx.$$

The formula analogous to that given for σ^2_y (66) is

$$\begin{vmatrix} \sigma^2_{a_p} \cdot \frac{N}{\sigma^2} k & 0 & 0 & 0 & \cdots\cdots & 1 & \cdots\cdots & 0 \\ 0 & 1 & \mu_1 & \mu_2 & \cdots\cdots & \mu_p & \cdots\cdots & \mu_n \\ 0 & \mu_1 & \mu_2 & \mu_3 & \cdots\cdots & \mu_{p+1} & \cdots\cdots & \mu_{n+1} \\ 0 & \mu_2 & \mu_3 & \mu_4 & \cdots\cdots & \mu_{p+2} & \cdots\cdots & \mu_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & \mu_p & \mu_{p+1} & \mu_{p+2} & \cdots\cdots & \mu_{2p} & \cdots\cdots & \mu_{n+p} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots\cdots & \mu_{p+n} & \cdots\cdots & \mu_{2n} \end{vmatrix} = 0 \quad \ldots(89).$$

(2) For a *function of the first degree*

$$y = a_0 + a_1 x,$$

for which the standard deviation of an observation is

$$s_y = \sigma(1 + \alpha x^2), \quad \alpha > -1,$$

and therefore

$$\frac{1}{k} = 1 + 2\alpha\mu_2 + \alpha^2\mu_4,$$

we find, according to (89),

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\left(1 + \frac{\mu_1^2}{\mu_2 - \mu_1^2}\right) \quad \text{.....................(90)},$$

and

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\frac{1}{\mu_2 - \mu_1^2} \quad \text{..........................(91)}.$$

As for any skew distribution of observations we can find a corresponding symmetrical distribution with the same μ_2 and μ_4, both these expressions are a minimum for $\mu_1 = 0$.

We have already shown in (2) of Section VII that any possible values of μ_2 and μ_4 can be produced by three symmetrical groups of observations, so that by introducing the variables v and γ determined by

$$\mu_2 = v^2\gamma,$$

and

$$\mu_4 = v^4\gamma,$$

and limited by

$$v^2 \leqq 1,$$

$$0 \leqq \gamma \leqq 1,$$

we do not leave out any possibilities.

From (90) we then get

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}(1 + 2a\gamma v^2 + a^2\gamma v^4),$$

which for $a > 0$ is a minimum when $\gamma = v^2 = 0$, and for $a = 0$ is $\frac{\sigma^2}{N}$ for any γ and v^2.

For $a < 0$ we find, since

$$\frac{d\sigma^2_{a_0}}{dv^2} = \frac{\sigma^2}{N} 2a\gamma(1 + av^2) \text{ and } v^2 < -\frac{1}{a},$$

that for a constant γ, $\sigma^2_{a_0}$ has the least value when v^2 is as great as possible, that is for $v^2 = 1$.

The minimum of $\sigma^2_{a_0}$ is then

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}\{1 + (2 + a)a\gamma\},$$

which, since $a(2 + a) < 0$, is a minimum when γ takes its greatest possible value 1.

The minimum is thus

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}(1 + a)^2.$$

Hence we conclude that:

when $a > 0$, $\sigma^2_{a_0}$ is a minimum and equal to $\frac{\sigma^2}{N}$ for N observations at $x = 0$,

when $a = 0$, $\sigma^2_{a_0}$ is a minimum and equal to $\frac{\sigma^2}{N}$ for any distribution for which $\mu_1 = 0$, and

when $a < 0$, $\sigma^2_{a_0}$ is a minimum and equal to $\frac{\sigma^2}{N}(1 + a)^2$ for two equally big groups of observations at ± 1.

(3) When we introduce $\mu_1 = 0$, $\mu_2 = \gamma v^2$ and $\mu_4 = \gamma v^4$ in (91) we get

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}(1 + 2\alpha\gamma v^2 + \alpha^2\gamma v^4)\frac{1}{\gamma v^2}.$$

This for constant v^2 is a minimum when $\gamma = 1$ and then equal to

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}(1 + 2\alpha v^2 + \alpha^2 v^4)\frac{1}{v^2} \quad \text{..........................(92)}.$$

As

$$\frac{d\sigma^2_{a_1}}{dv^2} = \frac{\sigma^2}{N}\left(\alpha^2 - \frac{1}{v^4}\right),$$

$v^2 = \pm\frac{1}{\alpha}$ when possible, that is for $\alpha \geqq 1$ determines a minimum, while for $\alpha < 1$, $\sigma^2_{a_1}$ reaches its lowest value for $v^2 = 1$. From (92) we find for $\alpha \geqq 1$ the minimum

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}\,.\,4\alpha,$$

and for $\alpha < 1$ the minimum

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}(1 + \alpha)^2,$$

both formulae giving $\sigma^2_{a_1} = \frac{\sigma^2}{N}\,.\,4$ for $\alpha = 1$.

Our results are accordingly:

when $\alpha > 1$, $\sigma^2_{a_1}$ is a minimum and equal to $\frac{\sigma^2}{N}\,.\,4\alpha$ for two equally big groups of observations at $x = \pm\frac{1}{\alpha}$ or for any distribution with the same μ_2 and μ_4,

and when $\alpha \leqq 1$, $\sigma^2_{a_1}$ is a minimum and equal to $\frac{\sigma^2}{N}(1 + \alpha)^2$ for two equally big groups of observations at $x = \pm 1$.

We see that for $\alpha \leqq 0$ two equally big groups of observations at ± 1 make both $\sigma^2_{a_0}$ and $\sigma^2_{a_1}$ minima and these groups in addition form the distribution for which σ^2_y has the lowest maximum within the possible range of observations.

(4) For a *function of the second degree*

$$y = a_0 + a_1 x + a_2 x^2,$$

with the standard deviations of observations

$$s_y = \sigma(1 + \alpha x^2), \quad \alpha > -1,$$

and therefore

$$\frac{1}{k} = 1 + 2\alpha\mu_2 + \alpha^2\mu_4,$$

we find, from (89),

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}(1 + 2\alpha\mu_2 + \alpha^2\mu_4)\,.\,\frac{\mu_2\mu_4 - \mu_3^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \quad \text{.........(93)},$$

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}(1 + 2\alpha\mu_2 + \alpha^2\mu_4)\,.\,\frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \quad \text{.........(94)},$$

and

$$\sigma^2_{a_2} = \frac{\sigma^2}{N}(1 + 2\alpha\mu_2 + \alpha^2\mu_4)\,.\,\frac{\mu_2 - \mu_1^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \quad \text{.........(95)}.$$

We shall prove that the last factor of each of these formulae is a minimum for

$$\mu_1 = \mu_3 = 0.$$

To prove this for (93) we consider the difference

$$\frac{\mu_2\mu_4}{\mu_2\mu_4 - \mu_2^3} - \frac{\mu_3^2}{\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4} = \frac{\mu_2(\mu_2\mu_3 - \mu_1\mu_4)^2}{\mu_2(\mu_4 - \mu_2^2)\left[(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)\right]} > 0,$$

from which follows

$$\frac{\mu_2\mu_4 - \mu_3^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} > \frac{\mu_2\mu_4}{\mu_2\mu_4 - \mu_2^3} > \frac{\mu_3^2}{\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4}.$$

For (94) it is at once clear that

$$\frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3 - \left[(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)\right]} > \frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3}.$$

For the case of (95) we compare

$$\frac{\mu_2}{\mu_2\mu_4 - \mu_2^3} \text{ and } \frac{\mu_1^2}{\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4},$$

and we find the difference

$$\frac{1}{\mu_4 - \mu_2^2} - \frac{1}{\mu_4 - \mu_2^2 + \left(\frac{\mu_3}{\mu_1} - \mu_2\right)^2} > 0,$$

and hence

$$\frac{\mu_2 - \mu_1^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} > \frac{\mu_2}{\mu_2\mu_4 - \mu_2^3} > \frac{\mu_1^2}{\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4}.$$

It is thus proved for the three formulae that a distribution of observations for which $\mu_1 = \mu_3 = 0$ gives lower values than any distribution with the same μ_2 and μ_4 as the former and with $\mu_1 \lesseqgtr 0$, $\mu_3 \gtreqless 0$.

Hence our problem is reduced to finding the μ_2 and μ_4 which make the following expressions minima:

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\frac{\mu_4}{\mu_4 - \mu_2^2} \quad \text{....................(96)},$$

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\frac{1}{\mu_2} \quad \text{..........................(97)},$$

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N}(1 + 2a\mu_2 + a^2\mu_4)\frac{1}{\mu_4 - \mu_2^2} \quad \text{....................(98)}.$$

(5) Introducing $\mu_2 = \gamma v^2$ and $\mu_4 = \gamma v^4$ in (96) we get

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N}\left(1 + \frac{\gamma}{1-\gamma}(1 + av^2)^2\right),$$

which is seen to be $> \frac{\sigma^2}{N}$ except when $\gamma = 0$.

Hence *the minimum value of* $\sigma_{a_0}^2 = \frac{\sigma^2}{N}$ *can only be obtained by taking all the observations at* $x = 0$.

(97) is identical with (91) for $\mu_1 = 0$. The conditions for a minimum of $\sigma_{a_1}^2$ are

therefore the same for a function of the second degree as for a function of the first degree. That is, when $\alpha > 1$, *$\sigma^2_{a_1}$ is a minimum and equal to* $\frac{\sigma^2}{N} \cdot 4\alpha$ *for two equally big groups of observations at* $x = \pm \frac{1}{\alpha}$, or for any distribution with the same μ_2 and μ_4, and when $\alpha \leqq 1$, *$\sigma^2_{a_1}$ is a minimum and equal to* $\frac{\sigma^2}{N}(1+\alpha)^2$ *for two equally big groups of observations at* $x = \pm 1$.

With the variates γ and v (98) takes the form

$$\sigma^2_{a_2} = \frac{\sigma^2}{N}(1 + 2\alpha\gamma v^2 + \alpha^2\gamma v^4)\frac{1}{v^4\gamma(1-\gamma)}.$$

By differentiating with regard to v^2 we get

$$\frac{d\sigma^2_{a_2}}{dv^2} = \frac{\sigma^2}{N} \cdot \frac{2}{\gamma(1-\gamma)v^6}(-1 - \alpha\gamma v^2),$$

which is negative for any α, v and γ within our limits.

For constant γ, $\sigma^2_{a_2}$ is therefore least when $v^2 = 1$ and the minimum value is

$$\sigma^2_{a_2} = \frac{\sigma^2}{N}\left(\frac{1}{\gamma} + 2\alpha + \alpha^2\right)\frac{1}{1-\gamma} \qquad \text{.........................(99).}$$

This is again a minimum when

$$\frac{d\sigma^2_{a_2}}{d\gamma} = \frac{\sigma^2}{N} \cdot \frac{1}{\gamma^2(1-\gamma)^2}\{\alpha(2+\alpha)\gamma^2 + 2\gamma - 1\} = 0,$$

that is for $\gamma = \frac{1}{2+\alpha}$ which gives a minimum both for positive and negative α.

Thus the distribution that makes $\sigma^2_{a_2}$ a minimum has a $\phi(x)$-distribution consisting of $\frac{N}{2(2+\alpha)}$ observations at -1 and 1 and $\frac{1+\alpha}{2+\alpha}N$ observations at 0.

We have
$$\mu_2 = \mu_4 = \frac{1}{2+\alpha}$$
and
$$\frac{1}{k} = (1+\alpha).$$

The relation
$$\psi(x) = k\phi(x)f(x)$$
then gives us
$$\psi(0) = \frac{N}{2+\alpha}$$
and
$$\psi(\pm 1) = \frac{1+\alpha}{2(2+\alpha)} \cdot N.$$

From (99) we find the minimum value

$$\sigma^2_{a_2} = \frac{\sigma^2}{N}(2+\alpha)^2.$$

Our result is thus that *$\sigma^2_{a_2}$ is a minimum and equal to* $\frac{\sigma^2}{N}(2+\alpha)^2$ *for a distribution consisting of* $\frac{N}{2+\alpha}$ *observations at* 0 *and* $\frac{1+\alpha}{2(2+\alpha)}N$ *at* ± 1.

(6) When the standard deviation of an observation is

$$s_y = \sigma(1+\alpha x) \quad \text{and} \quad 0 \leqq \alpha < 1,$$

we have

$$\frac{1}{k} = 1 + 2\alpha\mu_1 + \alpha^2\mu_2,$$

and according to (89) we find for a *function of the first degree*

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}(1+2\alpha\mu_1+\alpha^2\mu_2)\frac{\mu_2}{\mu_2-\mu_1^2} \quad \text{.....................(100)}$$

and

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}(1+2\alpha\mu_1+\alpha^2\mu_2)\frac{1}{\mu_2-\mu_1^2} \quad \text{.....................(101).}$$

By differentiating (100) we find

$$\frac{d\sigma^2_{a_0}}{d\mu_1} = \frac{\sigma^2}{N}\frac{2\mu_2(1+\alpha\mu_1)(\mu_1+\alpha\mu_2)}{(\mu_2-\mu_1^2)^2}$$

and

$$\frac{d\sigma^2_{a_0}}{d\mu_2} = \frac{\sigma^2}{N}\frac{(\alpha\mu_2-2\alpha\mu_1^2-\mu_1)(\mu_1+\alpha\mu_2)}{(\mu_2-\mu_1^2)^2}.$$

Both of these can only be zero when

$$\mu_1 + \alpha\mu_2 = 0 \quad \text{................................(102),}$$

which is seen to determine a minimum of $\sigma^2_{a_0}$ the value of which is $\frac{\sigma^2}{N}$. The condition $\mu_2 = -\frac{\mu_1}{\alpha}$ *can be fulfilled by an infinity of different distributions.* From

$$0 \lesseqgtr \mu_2 \lesseqgtr 1$$

follows the condition

$$0 \gtreqless \mu_1 \gtreqless -\alpha.$$

We shall confine our attention to those distributions which consist of two groups of observations. Let there be $N\gamma$ observations at v_1 and $(1-\gamma)N$ at v_2, we then have

$$\mu_1 = v_2 + \gamma(v_1 - v_2),$$
$$\mu_2 = v_2^2 + \gamma(v_1^2 - v_2^2),$$

from which by means of (102) is found

$$\frac{\gamma}{-v_2(1+\alpha v_2)} = \frac{1-\gamma}{v_1(1+\alpha v_1)} = \frac{1}{(v_1-v_2)\{1+\alpha(v_1+v_2)\}}$$

and

$$\frac{1}{k} = 1 + \alpha\mu_1 = \frac{(1+\alpha v_1)(1+\alpha v_2)}{1+\alpha(v_1+v_2)}.$$

Thus we find that the $\phi(x)$-distribution consists of

$$\frac{-v_2(1+\alpha v_2)}{(v_1-v_2)\{1+\alpha(v_1+v_2)\}} N \text{ at } v_1$$

and

$$\frac{v_1(1+\alpha v_1)}{(v_1-v_2)\{1+\alpha(v_1+v_2)\}} N \text{ at } v_2,$$

while the actual distribution

$$\psi(x) = \frac{1+\alpha(v_1+v_2)}{(1+\alpha v_1)(1+\alpha v_2)}(1+\alpha x)^2\phi(x)$$

consists of and

$$\left.\begin{array}{l}\dfrac{-v_2(1+av_1)}{v_1-v_2}N \text{ at } v_1\\[2ex] \dfrac{v_1(1+av_2)}{v_1-v_2}N \text{ at } v_2\end{array}\right\} \quad \text{.........................(103).}$$

We thus see that for any two points v_1 and v_2 of which one is negative and the other positive we can choose the numbers of observations so as to make $\sigma^2_{a_0}=\dfrac{\sigma^2}{N}$ as it of course would be by taking a single group of observations at $x=0$.

(7) By differentiating (101) we get

$$\frac{d\sigma^2_{a_1}}{d\mu_1}=\frac{\sigma^2}{N}\frac{2}{(\mu_2-\mu_1^2)^2}(1+a\mu_1)(\mu_1+a\mu_2) \quad \text{.................(104)}$$

and

$$\frac{d\sigma^2_{a_1}}{d\mu_2}=-\frac{\sigma^2}{N}\frac{(1+a\mu_1)^2}{(\mu_2-\mu_1)^2}.$$

As the latter is always negative $\sigma^2_{a_1}$ is for constant μ_1 least when μ_2 has its greatest value, that is 1.

Introducing this in (104) we get as condition for a minimum,

$$\mu_1+a=0.$$

There is only one distribution for which $\mu_2=1$ and $\mu_1=-a$, and it is that consisting of two groups of observations at -1 and 1 included in the distributions examined in (6).

From (103) we find that the actual distribution consists of $\dfrac{1-a}{2}N$ observations at -1 and $\dfrac{1+a}{2}N$ at 1. The minimum of $\sigma^2_{a_1}$ is from (101) found to be $\dfrac{\sigma^2}{N}$.

The minimum $\dfrac{\sigma^2}{N}$ of $\sigma^2_{a_1}$ can thus only be obtained by taking two groups of observations at the limits of the range with numbers proportional to the standard deviation of observations at these places. This distribution makes also $\sigma^2_{a_0}$ a minimum, but it is not, except when $a=0$, the distribution which gives σ^2_y the lowest maximum value within the possible range of observations.

(8) For a *function of the second degree,*

$$y=a_0+a_1x+a_2x^2$$

with the standard deviation

$$\sigma_y=\sigma(1+ax),$$

where

$$0\leqq a<1$$

we have

$$\frac{1}{k}=1+2a\mu_1+a^2\mu_2,$$

and from (89),

$$\sigma^2_{a_0}=\frac{\sigma^2}{N}(1+2a\mu_1+a^2\mu_2)\frac{\mu_2\mu_4-\mu_3^2}{\mu_2\mu_4-\mu_2^3-\mu_3^2+2\mu_1\mu_2\mu_3-\mu_1^2\mu_4} \quad \text{...(105),}$$

$$\sigma^2_{a_1}=\frac{\sigma^2}{N}(1+2a\mu_1+a^2\mu_2)\frac{\mu_4-\mu_2^2}{\mu_2\mu_4-\mu_2^3-\mu_3^2+2\mu_1\mu_2\mu_3-\mu_1^2\mu_4} \quad \text{...(106),}$$

$$\sigma^2_{a_2}=\frac{\sigma^2}{N}(1+2a\mu_1+a^2\mu_2)\frac{\mu_2-\mu_1^2}{\mu_2\mu_4-\mu_2^3-\mu_3^2+2\mu_1\mu_2\mu_3-\mu_1^2\mu_4} \quad \text{...(107).}$$

(105) may be brought into the form

$$\sigma^2_{a_0} = \frac{\sigma^2}{N}\left\{1 + \frac{\mu_2 \,.\, (\mu_2\mu_4 - \mu_3^2)\left(a + \frac{\mu_1}{\mu_2}\right)^2 + \frac{1}{\mu_2}(\mu_2^2 - \mu_1\mu_3)^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4}\right\},$$

where the denominator and $\mu_2\mu_4 - \mu_3^2$ are always positive. Hence the condition for $\sigma^2_{a_0}$, taking its minimum value $\frac{\sigma^2}{N}$, is

$$\mu_1 + a\mu_2 = 0 \text{ and } \mu_2^2 - \mu_1\mu_3 = 0$$

or

$$\frac{\mu_3}{\mu_2} = \frac{\mu_2}{\mu_1} = -\frac{1}{a} \quad\text{.................................(108).}$$

We shall examine the possible distributions consisting of three groups of observations with the frequencies γ_1, γ_2 and γ_3 at v_1, v_2 and v_3. The conditions (108) require

$$\frac{\gamma_1 v_1^2 + \gamma_2 v_2^2 + \gamma_3 v_3^2}{\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3} = \frac{\gamma_1 v_1^3 + \gamma_2 v_2^3 + \gamma_3 v_3^3}{\gamma_1 v_1^2 + \gamma_2 v_2^2 + \gamma_3 v_3^2} = \frac{\gamma_2 v_2^2 (v_2 - v_1) + \gamma_3 v_3^2 (v_3 - v_1)}{\gamma_2 v_2 (v_2 - v_1) + \gamma_3 v_3 (v_3 - v_1)} = -\frac{1}{a}$$

or

$$\frac{\gamma_1 v_1 (1 + a v_1)}{v_2 - v_3} = \frac{\gamma_2 v_2 (1 + a v_2)}{v_3 - v_1} = \frac{\gamma_3 v_3 (1 + a v_3)}{v_1 - v_2} \quad\text{...........(109).}$$

Now $\frac{v_1}{v_2 - v_3}$, $\frac{v_2}{v_3 - v_1}$ and $\frac{v_3}{v_1 - v_2}$ can never all have the same sign and $(1 + av)$ is for any $v \geqq -1$ positive, from which it follows that (109) leads to negative frequencies. Nor can (109) be satisfied by two groups of observations as $\gamma_2 = 0$ requires $v_1 = v_3 = 0$, that is one group of observations at $x = 0$ which of course gives $\sigma^2_{a_0} = \frac{\sigma^2}{N}$.

(9) We may write (106)

$$\sigma^2_{a_1} = \frac{\sigma^2}{N}\left(\frac{1}{\mu_2} + \frac{1}{\mu_2} \,.\, \frac{(\mu_4 - \mu_2^2)(\mu_1 + a\mu_2)^2 + (\mu_3 - \mu_1\mu_2)^2}{(\mu_4 - \mu_2^2)(\mu_2 - \mu_1^2) - (\mu_3 - \mu_1\mu_2)^2}\right),$$

where the last ratio is seen to be positive unless

$$\mu_1 + a\mu_2 = 0 \text{ and } \mu_3 - \mu_1\mu_2 = 0 \quad\text{....................(110).}$$

If therefore any distribution of observations can give $\frac{1}{\mu_2}$ its minimum value 1 and at the same time fulfil those conditions it will make $\sigma^2_{a_1}$ a minimum and equal to $\frac{\sigma^2}{N}$. But $\mu_2 = 1$ together with (110) lead to

$$\mu_3 = \mu_1 = -a,$$

which require $\frac{1+a}{2}N$ observations at -1 and $\frac{1-a}{2}N$ at 1, whereas the actual distribution must consist of $\frac{1-a}{2}N$ observations at -1 and $\frac{1+a}{2}N$ at 1.

Thus *the only distribution which makes* $\sigma^2_{a_1}$ *a minimum and equal to* $\frac{\sigma^2}{N}$ *is that consisting of* $\frac{1-a}{2}N$ *observations at* -1 *and* $\frac{1+a}{2}N$ *at* 1.

(10) The general minimum conditions for σ_{a_2} cannot be found without more elaborate investigations into the possible variations of the moment coefficients than are at present available and we shall limit our research to the case of three groups of observations.

Let us suppose $\gamma_1 N$, $\gamma_2 N$ and $(1-\gamma_1-\gamma_2) N$ observations taken at x_1, x_2 and x_3, and let the corresponding means be $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$.

We then find, when

$$\Delta = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1),$$

$$a_2 = \frac{1}{\Delta}\{\bar{y}_1(x_3 - x_2) + \bar{y}_2(x_1 - x_3) + \bar{y}_3(x_2 - x_1)\},$$

and

$$\sigma_{a_2}^2 = \frac{\sigma^2}{\Delta^2 . N}\left\{\frac{(x_3-x_2)^2(1+\alpha x_1)^2}{\gamma_1} + \frac{(x_1-x_3)^2(1+\alpha x_2)^2}{\gamma_2} + \frac{(x_2-x_1)^2(1+\alpha x_3)^2}{1-\gamma_1-\gamma_2}\right\} \quad \text{............(111).}$$

Differentiations first with regard to γ_1 and then with regard to γ_2 give the minimum conditions

$$\frac{\gamma_1^2}{(x_3-x_2)^2(1+\alpha x_1)^2} = \frac{\gamma_2^2}{(x_1-x_3)^2(1+\alpha x_2)^2} = \frac{(1-\gamma_1-\gamma_2)^2}{(x_2-x_1)^2(1+\alpha x_3)^2},$$

or, when we suppose $x_1 < x_2 < x_3$,

$$\frac{\gamma_1}{(x_3-x_2)(1+\alpha x_1)} = \frac{-\gamma_2}{(x_1-x_3)(1+\alpha x_2)} = \frac{1-\gamma_1-\gamma_2}{(x_2-x_1)(1+\alpha x_3)} = \frac{1}{2(x_3-x_1)(1+\alpha x_2)} \quad \text{............(112).}$$

With these values for γ_1 and γ_2 we get from (111)

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N}\left\{\frac{2(x_3-x_1)(1+\alpha x_2)}{\Delta}\right\}^2 = \frac{\sigma^2}{N}\left\{\frac{2(1+\alpha x_2)}{(x_2-x_1)(x_3-x_2)}\right\}^2.$$

This for constant x_2 is obviously a minimum for $x_1 = -1$ and $x_3 = 1$ and is then equal to

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N}\left\{\frac{2(1+\alpha x_2)}{1-x_2^2}\right\}^2.$$

From this we find

$$\frac{d\sigma_{a_2}}{dx_2} = \frac{\sigma}{\sqrt{N}}\frac{2(\alpha x_2^2 + 2x_2 + \alpha)}{(1-x_2^2)^2},$$

which shows that

$$x_2 = \sqrt{\frac{1}{\alpha^2} - 1} - \frac{1}{\alpha}$$

determines a minimum.

The minimum value is

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N}(1 + \sqrt{1-\alpha^2})^2,$$

and the frequencies found from (112) are

$$\frac{1}{4\alpha}\sqrt{1-\alpha}\,(\sqrt{1+\alpha} - \sqrt{1-\alpha}) . N \text{ at } -1,$$

$$\frac{1}{4\alpha}\sqrt{1+\alpha}\,(\sqrt{1+\alpha}-\sqrt{1-\alpha})\,.\,N \text{ at } 1,$$

and $$\tfrac{1}{2}N \text{ at } -\frac{1}{\alpha}(1-\sqrt{1-\alpha^2}).$$

IX. *Adjustment with regard to both of two variates connected by a linear relation.*

(1) The case often occurs when both of the variates observed have errors of observations of the same order so that adjustment only of one of them is unsatisfactory. We shall therefore in this section consider adjustment with regard to both of the variates and give the adjusted relation between them and the standard deviations of the constants.

Let x' be observed with the standard deviation $\sqrt{\alpha}\sigma$ and y' with the standard deviation $\sqrt{\gamma}\sigma$, we shall then for the sake of greater perspicuity exchange the variates for $x = \frac{x'}{\sqrt{\alpha}}$ and $y = \frac{y'}{\sqrt{\gamma}}$ so that both of our variates have the same standard deviation σ. Let $\frac{1}{N}\Sigma\{x^r y^s\}$ taken over the N pairs of observations be denoted by $\mu_{r,s}$, we then find, by adjusting only the y's according to (3),

$$\begin{vmatrix} y & 1 & x \\ \mu_{01} & 1 & \mu_{10} \\ \mu_{11} & \mu_{10} & \mu_{20} \end{vmatrix} = 0,$$

or $$y - \mu_{01} = \frac{\mu_{11} - \mu_{01}\mu_{10}}{\mu_{20} - \mu_{10}^2}(x - \mu_{10}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(113).$$

By adjusting only the x's we get

$$y - \mu_{01} = \frac{\mu_{02} - \mu_{01}^2}{\mu_{11} - \mu_{01}\mu_{10}}(x - \mu_{10}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(114),$$

which only coincide with (113) when

$$(\mu_{20} - \mu_{10}^2)(\mu_{02} - \mu_{01}^2) = (\mu_{11} - \mu_{01}\mu_{10})^2,$$

that is when there is perfect correlation between x and y and no casual errors of observation.

(2) Adjusting at the same time with regard to x and y may be transformed to the problem of finding the straight line for which the sum of the squared distances of the observed points (x, y) is a minimum.

Let the line sought be

$$x\cos v + y\sin v + p = 0.$$

The sum which we want to make a minimum is then

$$S = \mu_{20}\cos^2 v + \mu_{02}\sin^2 v + 2\mu_{11}\cos v\sin v + 2p\mu_{10}\cos v + 2p\mu_{01}\sin v + p^2,$$

$\frac{dS}{dp} = 0$ requires $$p = -\mu_{10}\cos v - \mu_{01}\sin v,$$

indicating that the line passes through the mean (μ_{10}, μ_{01}); this determines a minimum for constant v.

The corresponding S is

$$S = (\mu_{20} - \mu_{10}^2)\cos^2 v + (\mu_{02} - \mu_{01}^2)\sin^2 v + 2(\mu_{11} - \mu_{01}\mu_{10})\cos v \sin v \quad \ldots\ldots(115),$$

which differentiated with regard to v gives

$$\frac{dS}{dv} = -\{\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)\}\sin 2v + 2(\mu_{11} - \mu_{01}\mu_{10})\cos 2v.$$

It thus follows that

$$\tan 2v = \frac{2(\mu_{11} - \mu_{01}\mu_{10})}{\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)} = \frac{2\tan v}{1 - \tan^2 v},$$

or

$$\tan v = \frac{\frac{1}{2}\{\mu_{02} - \mu_{01}^2 - (\mu_{20} - \mu_{10}^2) \pm \sqrt{[\mu_{02} - \mu_{01}^2 - (\mu_{20} - \mu_{10}^2)]^2 + 4[\mu_{11} - \mu_{10}\mu_{01}]^2}\}}{\mu_{11} - \mu_{01}\mu_{10}} \quad \ldots\ldots\ldots\ldots(116)$$

determine a maximum and a minimum of S.

Substituting in (115) we find

$$S = \tfrac{1}{2}\{\mu_{20} - \mu_{10}^2 + \mu_{02} - \mu_{01}^2 \pm \sqrt{[\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)]^2 + 4[\mu_{11} - \mu_{01}\mu_{10}]^2}\},$$

so that the minimum corresponds to the negative sign of the root in (116).

The adjusted function connecting x and y is hence a line through the general mean forming an angle u with the x-axis which is determined by

$$\tan u = -\cot v = \frac{\mu_{02} - \mu_{01}^2 - (\mu_{20} - \mu_{10}^2) + \sqrt{[\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)]^2 + 4[\mu_{11} - \mu_{01}\mu_{10}]^2}}{2(\mu_{11} - \mu_{01}\mu_{10})} \quad \ldots\ldots\ldots\ldots(117).$$

For the variates x' and y' there must to this value of the tangent be added the factor $\sqrt{\frac{\gamma}{\alpha}}$, expressed by the moment coefficients of x' and y' we therefore find

$$\tan u = \frac{\alpha(\mu_{02}' - \mu_{01}'^2) - \gamma(\mu_{20}' - \mu_{10}'^2) + \sqrt{[\gamma(\mu_{20}' - \mu_{10}'^2) - \alpha(\mu_{02}' - \mu_{01}'^2)]^2 + 4\alpha\gamma[\mu_{11}' - \mu_{01}'\mu_{10}']^2}}{2\alpha(\mu_{11}' - \mu_{01}'\mu_{10}')} \quad \ldots\ldots\ldots\ldots(118).$$

(3) We shall prove that the line is situated between the two regression curves (113) and (114).

Making (μ_{10}, μ_{01}) the zero point of the coordinates, the three tangents to be compared are

$$\frac{\mu_{11}}{\mu_{20}}, \quad \frac{\mu_{02}}{\mu_{11}} \quad \text{and} \quad \frac{1}{2\mu_{11}}\{\mu_{02} - \mu_{20} + \sqrt{(\mu_{02} - \mu_{20})^2 + 4\mu_{11}^2}\} = \tan u,$$

where the μ's now are the moment coefficients about the mean.

According to $\mu_{11} \gtrless 0$ we have

$$\frac{\mu_{11}}{\mu_{20}} \lessgtr \frac{\mu_{02}}{\mu_{11}},$$

since

$$\mu_{11}^2 < \mu_{20} \cdot \mu_{02}.$$

As

$$\sqrt{(\mu_{02} - \mu_{20})^2 + 4\mu_{11}^2} < \mu_{02} + \mu_{20},$$

we have

$$\tan u \lessgtr \frac{\mu_{02}}{\mu_{11}}.$$

It rests to compare $\tan u$ and $\frac{\mu_{11}}{\mu_{20}}$, we find

$$\tan u - \frac{\mu_{11}}{\mu_{20}} = \frac{1}{2\mu_{11}}\left\{\mu_{02} - \mu_{20} - \frac{2\mu_{11}^2}{\mu_{20}} + \sqrt{\left[\mu_{02} - \mu_{20} - \frac{2\mu_{11}^2}{\mu_{20}}\right]^2 + \frac{4\mu_{11}^2}{\mu_{20}^2}(\mu_{02}\mu_{20} - \mu_{11}^2)}\right\}.$$

The factor in curled brackets is hence positive and we have $\tan u >$ or $< \frac{\mu_{11}}{\mu_{20}}$ according as

$$\mu_{11} > \text{ or } < 0,$$

we have thus proved that

$$\frac{\mu_{11}}{\mu_{20}} \lesseqgtr \tan u \lesseqgtr \frac{\mu_{02}}{\mu_{11}}.$$

(4) In order to find the standard deviations of the constants of the line we shall express the observations, the standard deviations of which are $\sqrt{\alpha}\sigma$ and $\sqrt{\gamma}\sigma$, by a parameter r to get an equation for each observation.

Suppose
$$x_i = a + r_i \cos u,$$
$$y_i = b + r_i \sin u,$$

and suppose we have a good approximation for $a, b, u, r_1, r_2 \ldots\ldots r_N$ from which is calculated x and y corresponding to the observations. The differences between observed and calculated x and y can then be expressed by

$$\left.\begin{aligned}\Delta x_i &= \Delta a - r_i \sin u \,.\, \Delta u + \cos u \,.\, \Delta r_i\\ \Delta y_i &= \Delta b + r_i \cos u \,.\, \Delta u + \sin u \,.\, \Delta r_i\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(119),$$

and we can carry out an adjustment, Δa, Δb, Δu, Δr_1, $\Delta r_2 \ldots \Delta r_N$ being the elements.

The normal equations are:

$$\frac{1}{\alpha}\Sigma\{x_i\} = \frac{N}{\alpha}\Delta a + 0\,.\,\Delta b - \Sigma\{r_i\}\frac{\sin u}{\alpha}\Delta u + \frac{\cos u}{\alpha}\Delta r_1 + \ldots + \frac{\cos u}{\alpha}\Delta r_N,$$

$$\frac{1}{\gamma}\Sigma\{y_i\} = 0\,.\,\Delta a + \frac{N}{\gamma}\Delta b + \Sigma\{r_i\}\frac{\cos u}{\gamma}\Delta u + \frac{\sin u}{\gamma}\Delta r_1 + \ldots + \frac{\sin u}{\gamma}\Delta r_N,$$

$$\Sigma\left\{r_i\left[-\frac{\sin u}{\alpha}\Delta x_i + \frac{\cos u}{\gamma}\Delta y_i\right]\right\}$$
$$= -\Sigma\{r_i\}\frac{\sin u}{\alpha}\Delta a + \Sigma\{r_i\}\frac{\cos u}{\gamma}\Delta b + \Sigma\{r_i\}^2\left[\frac{\sin^2 u}{\alpha} + \frac{\cos^2 u}{\gamma}\right]\Delta u + r_1\left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)\cos u \sin u \Delta r_1 + \ldots$$
$$+ r_N\left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)\cos u \sin u \Delta r_N,$$

$$\frac{\cos u}{\alpha}\Delta x_1 + \frac{\sin u}{\gamma}\Delta y_1$$
$$= \frac{\cos u}{\alpha}\Delta a + \frac{\sin u}{\gamma}\Delta b + r_1\left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)\cos u \sin u \Delta u + \left(\frac{\cos^2 u}{\alpha} + \frac{\sin^2 u}{\gamma}\right)\Delta r_1 + \ldots + 0\,.\,\Delta r_N,$$

....

$$\frac{\cos u}{\alpha}\Delta x_N + \frac{\sin u}{\gamma}\Delta y_N$$
$$= \frac{\cos u}{\alpha}\Delta a + \frac{\sin u}{\gamma}\Delta b + r_N\left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)\cos u \sin u \Delta u + 0\,.\,\Delta r_1 + \ldots + \left(\frac{\cos^2 u}{\alpha} + \frac{\sin^2 u}{\gamma}\right)\Delta r_N$$

Eliminating $r_1, r_2 \ldots r_N$ from the first and the third of these equations by means of the last N equations, we obtain

$$\Sigma\{\sin u\Delta x_i - \cos u\Delta y_i\} = N\sin u\Delta a - N\cos u\Delta b - \Sigma\{r_i\}\Delta u \ldots\ldots(120),$$

and

$$\Sigma\{r_i[\sin u\Delta x_i - \cos u\Delta y_i]\} = \Sigma\{r_i\}\sin u\Delta a - \Sigma\{r_i\}\cos u\Delta b - \Sigma\{r_i^2\}\Delta u \ldots(121).$$

By eliminating the r's from the second of the normal equations we get an equation identical with (120), which shows that we have one more element than we can determine.

From (120) and (121) we are however able to find

$$(\sin u\Delta a - \cos u\Delta b) \text{ and } \Delta u\text{; we get}$$

$$\sin u\Delta a - \cos u\Delta b = \frac{1}{N(m_2 - m_1^2)} \Sigma \{(m_2 - m_1 r_i)(\sin u\Delta x_i - \cos u\Delta y_i)\}$$

and

$$\Delta u = \frac{1}{N(m_2 - m_1^2)} \Sigma \{(m_1 - r_i)(\sin u\Delta x_i - \cos u\Delta y_i)\},$$

where

$$m_1 = \frac{1}{N} \Sigma \{r_i\} \text{ and } m_2 = \frac{1}{N} \Sigma (r_i^2).$$

For a point of the adjusted line corresponding to r_q we find, according to (119),

$$p_q = \sin u\Delta x_q - \cos u\Delta y_q = \sin u\Delta a - \cos u\Delta b - r_p \Delta u.$$

The standard deviation of p_q is seen to be the standard deviation of the position of the adjusted point (x_p, y_p) in the direction at right angles to the line.

We find

$$p_q = \frac{1}{N(m_2 - m_1^2)} \Sigma \{[m_2 - m_1 r_i - r_p(m_1 - r_i)](\sin u\Delta x_i - \cos u\Delta y_i)\}$$

and

$$\sigma_{p_q}^2 = \frac{\sigma^2}{N}(a \sin^2 u + \gamma \cos^2 u)\left\{1 + \frac{(r_p - m_1)^2}{m_2 - m_1^2}\right\}.$$

This standard deviation is quite analogous to that obtained for an adjusted ordinate when the abscissa is errorless and gives the same indications for the distribution of the observations.

For σ_u we find

$$\sigma_u^2 = \frac{\sigma^2}{N} \frac{(a \sin^2 u + \gamma \cos^2 u)}{(m_2 - m_1^2)},$$

again emphasising that the standard deviation of the r's ought to be a maximum to give the best determination of the line.

In conclusion I should like to express my thanks to Miss H. Gertrude Jones for the care she has devoted to the preparation of the diagrams in this paper.