ON THE DISTRIBUTION OF THE CORRELATION COEFFICIENT IN SMALL SAMPLES. APPENDIX II TO THE PAPERS OF "STUDENT" AND R. A. FISHER.

A COOPERATIVE STUDY

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CONTENTS

(1) *Introductory.* In a paper of 1908* "Student" dealt experimentally with the distribution of the correlation coefficient of small samples, and gave empirical curves—in particular for the case of zero correlation in the sampled population—which have proved remarkably exact. The problem was next considered in 1913 by H. E. Soper† who obtained the mean correlation and the standard deviation of the distribution of correlations to second approximations. Of the

* *Biometrika*, Vol. VI. p. 302 *et seq.*

† *Biometrika*, Vol. IX. p. 91 *et seq.*

formulae he gives for $\bar{r}$ and σ_r of the distribution of the correlation r in samples of n from a population of correlation ρ, we have found in practice the most exact are*

$$\bar{r} = \rho \left\{1 - \frac{1}{2n}(1-\rho^2) - \frac{3}{8n^2}(1-\rho^2)(1+3\rho^2)\right\} \quad \text{.................(i)}$$

and

$$\sigma_r^2 = \frac{(1-\rho^2)^2}{n}\left(1 + \frac{1+5\cdot 5\rho^2}{n}\right) \quad \text{..................................(ii).}$$

Soper also by assuming a Pearson curve of limited range + 1 to − 1 of type

$$y = y_0 \left(1 - \frac{x}{a_1}\right)^{m_1} \left(1 + \frac{x}{a_2}\right)^{m_2}$$

deduces the modal value $\check{r}$ of r as approximately

$$\check{r} = \bar{r}\,\frac{\{1 - (\sigma_r^2 + \bar{r}^2)\}}{1 + 2\bar{r}^2 - 3(\sigma_r^2 + \bar{r}^2)} \quad \text{.........................(iii),}$$

so that $\check{r}$ would be determined from a knowledge of $\bar{r}$ and $\sigma_r^2 + \bar{r}^2$.

The next step was taken by R. A. Fisher who gave in 1915† the actual frequency distribution of r, namely the curve

$$y_n = f_n(r) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}(1-r^2)^{\frac{n-4}{2}} \frac{d^{n-2}}{d(r\rho)^{n-2}}\left(\frac{\cos^{-1}(-\rho r)}{\sqrt{1-\rho^2 r^2}}\right) \quad \text{......(iv).}$$

Except for very low values of n this expression for y_n does not provide a formula from which the ordinates of the frequency curve for r can be readily determined, and as the problem was left by Fisher there were no rapid means of numerically determining either $\bar{r}$ or $\check{r}$ or again σ_r^2.

Clearly in order to determine the approach to Soper's approximations, and ultimately to the normal curve as n increases we require expressions for the moment coefficients of (iv), and further for practical purposes we require to table the ordinates of (iv) in the region for which n is too small for Soper's formulae to provide adequate approximations. These are the aims of the present paper. It is only fair to state that the arithmetic involved has been of the most strenuous kind and has needed months of hard work on the part of the computers engaged‡. On the other hand the algebra has often been of a most interesting and suggestive character.

(2) *On Properties of the Function* $U = \cos^{-1}(-x)/\sqrt{1-x^2}$.

We have

$$\frac{dU}{dx} = \frac{1}{1-x^2} + \frac{x\cos^{-1}(-x)}{(1-x^2)^{\frac{3}{2}}},$$

or

$$(1-x^2)\frac{dU}{dx} = 1 + xU.$$

* See *loc. cit.* pp. 105 and 107.

† *Biometrika*, Vol. x. p. 507 *et seq.*

‡ Besides those whose names are given under the title, we have to thank I. Horwitz for some calculating aid, Ethel M. Elderton and D. Heron for occasional assistance, especially in the experimental part of the work, and lastly but very far from least we have to acknowledge the untiring work of H. Gertrude Jones and Adelaide G. Davin in the construction of the models the beauty and accuracy of which are not more than suggested in the plates.

Apply Leibnitz's Theorem and we have

$$(1-x^2)\frac{d^nU}{dx^n} - 2x(n-1)\frac{d^{n-1}U}{dx^{n-1}} - 2\frac{(n-1)(n-2)}{1\ \ 2}\frac{d^{n-2}U}{dx^{n-2}} = x\frac{d^{n-1}U}{dx^{n-1}} + (n-1)\frac{d^{n-2}U}{dx^{n-2}},$$

or,
$$(1-x^2)\frac{d^nU}{dx^n} - x(2n-1)\frac{d^{n-1}U}{dx^{n-1}} - (n-1)^2\frac{d^{n-2}U}{dx^{n-2}} = 0 \quad \text{.........(v).}$$

Put $x = 0$ and we have

$$\left(\frac{d^nU}{dx^n}\right)_0 = (n-1)^2\left(\frac{d^{n-2}U}{dx^{n-2}}\right)_0,$$

but clearly $U_0 = \frac{1}{2}\pi$ and $(dU/dx)_0 = 1$.

Hence by Maclaurin's Theorem

$$\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}} = \frac{\pi}{2}\left(1 + \frac{1^2}{2!}x^2 + \frac{3^2 \cdot 1^2}{4!}x^4 + \dots + \frac{(2s-1)^2(2s-3)^2\dots 1^2}{(2s)!}x^{2s} + \dots\right)$$
$$+ \left(x + \frac{2^2}{3!}x^3 + \frac{4^2 \cdot 2^2}{5!}x^5 + \dots + \frac{(2s)^2(2s-2)^2\dots 2^2}{(2s+1)!}x^{2s+1} + \dots\right) \text{...(vi).}$$

We are now in a position to give the successive differentials of U which may be either even or odd. We have for the two cases

$$\frac{d^{2s}}{dx^{2s}}\left\{\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}}\right\}$$
$$= \frac{\pi}{2}(2s-1)^2(2s-3)^2\dots 1^2\left\{1 + \frac{(2s+1)^2}{2!}x^2 + \frac{(2s+1)^2(2s+3)^2}{4!}x^4 + \dots\right\}$$
$$+ (2s)^2(2s-2)^2\dots 2^2\left\{x + \frac{(2s+2)^2}{3!}x^3 + \frac{(2s+2)^2(2s+4)^2}{5!}x^5 + \dots\right\} \text{...(vii),}$$

$$\frac{d^{2s-1}}{dx^{2s-1}}\left\{\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}}\right\}$$
$$= \frac{\pi}{2}(2s-1)^2(2s-3)^2\dots 1^2\left\{x + \frac{(2s+1)^2}{3!}x^3 + \frac{(2s+1)^2(2s+3)^2}{5!}x^5 + \dots\right\}$$
$$+ (2s-2)^2(2s-4)^2\dots 2^2\left\{1 + \frac{(2s)^2}{2!}x^2 + \frac{(2s)^2(2s+2)^2}{4!}x^4 + \dots\right\} \text{...(vii)}^{\text{bis}},$$

the development of the several series being clear.

For calculation of y_{2s} or y_{2s-1} the above series are idle, just as they are when substituted in the equation for $dy_n/dr = 0$ which gives $\check{r}$. They converge far too slowly to be of use for numerical evaluations. But as we shortly shall show, they are, after certain transformations, most valuable in determining the moment coefficients.

Now
$$y_n = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}(1-r^2)^{\frac{n-4}{2}}\frac{d^{n-2}U}{dx^{n-2}};$$

multiply (v) by
$$(1-\rho^2)^{\frac{n+1}{2}}(1-r^2)^{\frac{n-2}{2}}\Big/(\pi(n-1)!),$$

and we have

$$(1-\rho^2r^2)\,y_{n+2} - \frac{\rho r(2n-1)\sqrt{1-\rho^2}\sqrt{1-r^2}}{n-1}\,y_{n+1} - \frac{(1-\rho^2)(1-r^2)}{(n-1)(n-2)}(n-1)^2\,y_n = 0,$$

thus

$$y_{n+2} = \frac{2n-1}{n-1}\,\kappa_1 y_{n+1} + \frac{n-1}{n-2}\,\kappa_2 y_n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

Here

$$\kappa_1 = \frac{\rho r\sqrt{1-\rho^2}\sqrt{1-r^2}}{1-\rho^2r^2}, \qquad \kappa_2 = \frac{(1-\rho^2)(1-r^2)}{1-\rho^2r^2}$$

are constant for ρ and r given and thus (viii) enables us to deduce y_{n+2} from y_{n+1} and y_n for a given ρ and r. But by simple differentiation

$$\left.\begin{aligned} y_3 &= \frac{1-\rho^2}{\pi\sqrt{1-r^2}}\left(\frac{1}{1-\rho^2r^2} + \frac{\rho r\cos^{-1}(-\rho r)}{(1-\rho^2r^2)^{\frac{3}{2}}}\right) \\ y_4 &= \frac{(1-\rho^2)^{\frac{3}{2}}}{\pi}\left(\frac{3\rho r}{(1-\rho^2r^2)^2} + \frac{(1+2\rho^2r^2)\cos^{-1}(-\rho r)}{(1-\rho^2r^2)^{\frac{5}{2}}}\right)\end{aligned}\right\} \quad\ldots\ldots\ldots\text{(ix)}.$$

Hence if y_3 and y_4 be calculated for a series of values of r and ρ all higher values may be reached by a repeated use of (viii). The values chosen were: ρ proceeding by ·1 from 0 to 1 and r proceeding by ·05 from -1 to $+1$.

The disadvantage of this method of calculating y_n is that, except by independent computing, there is no means of checking accuracy until all the ordinates have been deduced, and any mistake in y_n for a low value of n is perpetuated throughout the series. When all the ordinates have been found, say for $n = 25$, then the smoothness of these ordinates and the fact that they give the correct total area with a suitable graduation-formula will be checks on the accuracy of the whole system of ordinates. In this manner Table A, p. 379, was calculated.

Another method of approaching the value of y_n is of some advantage. We may take

$$y_n = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}(1-r^2)^{\frac{n-4}{2}}\left(\frac{v_n}{(1-\rho^2r^2)^{n-2}} + \frac{u_n\cos^{-1}(-\rho r)}{(1-\rho^2r^2)^{\frac{2n-3}{2}}}\right)\ldots\ldots\text{(x)},$$

where v_n and u_n are functions of ρr in integer positive powers, and if we substitute in (viii) we obtain

$$\left.\begin{aligned} v_{n+2} &= (2n-1)\,\rho r v_{n+1} + (n-1)^2(1-\rho^2r^2)\,v_n \\ u_{n+2} &= (2n-1)\,\rho r u_{n+1} + (n-1)^2(1-\rho^2r^2)\,u_n\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(xi)}.$$

We may write y_n in the form

$$y_n = \frac{n-2}{(n-2)!}\,\frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi}(1-r^2)^{\frac{n-4}{2}}\frac{d^{n-2}}{d(\rho r)^{n-2}}\left\{\frac{\cos^{-1}(-\rho r)}{\sqrt{1-\rho^2r^2}}\right\},$$

or

$$y_2 = (n-2)_{n=2}\,\frac{\sqrt{1-\rho^2}}{\pi(1-r^2)}\left(\frac{0}{(1-\rho^2r^2)^0} + \frac{\cos^{-1}(-\rho r)}{(1-\rho^2r^2)^{\frac{1}{2}}}\right).$$

Hence $v_2 = 0$ and $u_2 = 1$, while y_2 will vanish for all values of r, except $r = \pm 1$ owing to the factor $(n-2)_{n=2}$. Thus (ix) gives us

$$v_3 = 1, \qquad u_3 = \rho r,$$

whence by (xi)

$$v_4 = 3\rho r, \qquad u_4 = 1 + 2\rho^2 r^2.$$
$$v_5 = 4 + 11\rho^2 r^2, \qquad u_5 = \rho r\,(9 + 6\rho^2 r^2),$$
$$v_6 = \rho r\,(55 + 50\rho^2 r^2), \qquad u_6 = 9 + 72\rho^2 r^2 + 24\rho^4 r^4,$$

and the successive values can be rapidly calculated, much faster than by actually differentiating out (iv). It is, however, shortest to insert the numerical values of ρ and r in (xi) and deduce the v_n's and u_n's numerically in succession. (Table A was, however, in the present case deduced from (viii). We did this by direct calculation of the values of

$$\left(\frac{y_n}{n-2}\right)_{n=2} = \frac{\sqrt{1-\rho^2}}{\pi(1-r^2)}\,\frac{\cos^{-1}(-\rho r)}{(1-\rho^2 r^2)^{\frac{1}{2}}},$$

and y_3 in equation (ix). Equation (viii) then gave us the numerical values of y_4, y_5, etc. in succession.)

We may write (x) in the form

$$y_n = \frac{(1-\rho^2)^{\frac{3}{2}}}{\pi}\,V_n\,(v_n + u_n U) \qquad \text{(xii)},$$

where
$$V_n = \frac{\{(1-\rho^2)(1-r^2)\}^{\frac{n-4}{2}}}{(n-3)!\,(1-\rho^2 r^2)^{n-2}} \quad \text{and} \quad U = \frac{\cos^{-1}(-\rho r)}{\sqrt{1-\rho^2 r^2}}$$

Here V_n, v_n, u_n and U are symmetrical in ρ and r and accordingly ρ and r can be interchanged. The problem approached this way involves:

(*a*) calculating $(1-\rho^2)^{\frac{3}{2}}/\pi$ for various values of ρ;

(*b*) U for various values of ρr;

(*c*) V_n for various values of ρ, r and n;

(*d*) determining u_n and v_n in succession from (xi) for various values of ρr and n.

Lastly we may use the series for y_n to be given later (see Eqn. (xliii)) which develops y_n in inverse powers of $(n-1)$. Actually we have adopted (viii) for tabling the ordinates of the first 25 curve-series, and the last expansion for verification and higher cases.

(3) *On the Determination of the Moment Coefficients.* We shall next determine the value of the moment coefficients about $r = 0$, as origin, and shall deal with the even and odd coefficients independently. Let them be μ'_{2p} and μ'_{2p+1}. Clearly, the total area having been taken as unity:

$$\mu'_{2p} = \int_{-1}^{+1} y_n r^{2p} dr$$

$$= \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!} \int_{-1}^{+1} (1-r^2)^{\frac{n-4}{2}} r^{2p} \frac{d^{n-2}U}{d(\rho r)^{n-2}} dr.$$

Now

$$r^{2p} = r^{2p} - (r^2-1)^p + (-1)^p (1-r^2)^p$$

$$= pr^{2p-2} - \frac{p(p-1)}{2!} r^{2p-4} + \frac{p(p-1)(p-2)}{3!} r^{2p-6} - \dots + (-1)^p (1-r^2)^p.$$

Hence

$$\mu'_{2p} = p\mu'_{2p-2} - \frac{p(p-1)}{2!} \mu'_{2p-4} + \frac{p(p-1)(p-2)}{3!} \mu'_{2p-6} - \dots + (-1)^p \chi_{2p} \quad \text{(xiii)},$$

where $$\chi_{2p} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!} \int_{-1}^{+1} (1-r^2)^{\frac{n-4+2p}{2}} \frac{d^{n-2}U}{d(\rho r)^{n-2}} dr.$$

Thus using (vii)^{bis} on the assumption that n is odd and remembering that odd powers of r will now disappear we reach:

$$\chi_{2p} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!} (n-3)^2 (n-5)^2 \dots 2^2 \left(i_0 + \frac{(n-1)^2}{2!} \rho^2 i_2 + \frac{(n+1)^2 (n-1)^2}{4!} \rho^4 i_4 + \dots \right),$$

where $$i_{2m} = \int_{-1}^{+1} (1-r^2)^{\frac{n-4+2p}{2}} r^{2m} dr.$$

Now we may write $r = \cos\phi$, so that

$$i_{2m} = 2 \int_0^{\frac{\pi}{2}} \sin^{n+2p-3}\phi \cos^{2m}\phi \, d\phi,$$

and we have $$i_{2m} = \frac{2m-1}{n+2p-3+2m} i_{2m-2} \quad \text{..........................(xiv)}.$$

Thus

$$\chi_{2p} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!} (n-3)^2 (n-5)^2 \dots 2^2 i_0 \left(1 + \frac{(n-1)^2}{2!} \rho^2 \frac{1}{n-1+2p} \right.$$

$$\left. + \frac{(n+1)^2 (n-1)^2}{4!} \rho^4 \frac{1}{n-1+2p} \frac{3}{n+1+2p} + \dots \right)$$

$$= \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!} (n-3)^2 (n-5)^2 \dots 2^2 i_0 F \left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} + p, \rho^2 \right),$$

where F as usual denotes the hypergeometrical series. But by a well-known transformation due to Euler

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x) \quad \text{.........(xv)},$$

and accordingly

$$F\left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}+p, \rho^2\right) = (1-\rho^2)^{p-\frac{n-1}{2}} F\left(p, p, \frac{n-1}{2}+p, \rho^2\right)$$

or $$\chi_{2p} = \frac{(1-\rho^2)^p}{\pi (n-3)!} (n-3)^2 (n-5)^2 \dots 2^2 i_0 . F\left(p, p, \frac{n-1}{2}+p, \rho^2\right)$$

Now $$i_0 = 2\int_0^{\frac{\pi}{2}} \sin^{n+2p-3}\phi\, d\phi = 2q_{n+2p-2}, \text{ say,}$$

and $$2q_n = 2\int_0^{\frac{\pi}{2}} \sin^{n-1}\phi\, d\phi$$

is known to be $$= \frac{(n-2)(n-4)\dots 1}{(n-1)(n-3)\dots 2}\pi \quad \text{.................................(xvi)},$$

if n be odd as supposed above. Thus finally we have

$$\chi_{2p} = (1-\rho^2)^p \frac{q_{n+2p-2}}{q_{n-2}} F\left(p, p, \frac{n-1}{2}+p, \rho^2\right) \quad \text{............(xvii)}.$$

A Table of $q_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\phi\, d\phi$ from $n = 1$ to $n = 105$ is given on p. 377 below.

Now (xvii) has only been proved for n odd. If n be even we must take the first series of (vii) and this gives

$$\chi_{2p} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi (n-3)!} \frac{\pi}{2} (n-3)^2 (n-5)^2 \dots\dots 1^2 i_0' \left(1 + \frac{(n-1)^2}{2!}\rho^2 \frac{1}{n+2p-1} + \frac{(n+1)^2 (n-1)^2}{4!}\rho^4 \frac{1}{n+2p-1}\, \frac{3}{n+2p+1} + \dots\right),$$

where $$i_0' = 2\int_0^{\frac{\pi}{2}} \sin^{n+2p-3}\phi\, d\phi = 2q_{n+2p-2}$$

and $$2q_n = \frac{(n-2)(n-4)\dots 2}{(n-1)(n-3)\dots 3} . 2 \quad \text{........................(xviii)},$$

since n is even. Thus

$$\chi_{2p} = (1-\rho^2)^{\frac{n-1}{2}} \frac{q_{n+2p-2}}{q_{n-2}} F\left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}+p, \rho^2\right)$$
$$= (1-\rho^2)^p \frac{q_{n+2p-2}}{q_{n-2}} F\left(p, p, \frac{n-1}{2}+p, \rho^2\right) \quad \text{.....................(xix)},$$

or (xvii) holds whether n be even or odd.

As particular cases we have for $p = 1$

$$\mu_2' = \bar{r}^2 + \sigma_r^2 = 1 - \chi_2$$
$$= 1 - \frac{q_n}{q_{n-2}}(1-\rho^2)\left(1 + \frac{2^2}{n+1}\frac{\rho^2}{2} + \frac{2^2 . 4^2}{1 . 2 (n+1)(n+3)}\frac{\rho^4}{4} + \frac{2^2 . 4^2 . 6^2}{1 . 2 . 3 . (n+1)(n+3)(n+5)}\frac{\rho^6}{8} + \dots\right)$$
$$\text{..................(xx)},$$

or
$$\sigma_r^2 = 1 - \bar{r}^2 - \frac{n-2}{n-1}(1-\rho^2)\left(1 + \frac{2^2}{n+1}\frac{\rho^2}{2} + \frac{2^2 . 4^2}{1 . 2 . (n+1)(n+3)}\frac{\rho^4}{4} + \frac{2^2 . 4^2 . 6^2}{1 . 2 . 3 . (n+1)(n+3)(n+5)}\frac{\rho^6}{8} + \dots\right) \quad \text{(xx)}^{\text{bis}},$$

and again for $p = 2$

$$\mu_4' = \mu_4 + 4\mu_3'\mu_1' - 6\mu_2'\mu_1'^2 + 3\mu_1'^4$$
$$= 2\mu_2' - 1 + \frac{n(n-2)}{(n+1)(n-1)}(1-\rho^2)^2\left\{1 + \frac{4^2}{n+3}\frac{\rho^2}{2} + \frac{4^2 . 6^2}{1 . 2 (n+3)(n+5)}\frac{\rho^4}{4} + \frac{4^2 . 6^2 . 8^2}{1 . 2 . 3 (n+3)(n+5)(n+7)}\frac{\rho^6}{8} + \dots\right\} \quad \text{(xxi)}.$$

The series in (xx) and (xxi) for $n = 25$ and upwards converge with sufficient rapidity to determine μ_2' and μ_4' rapidly and therefore with accurate values for μ_1' and μ_3' will give μ_2 or σ_r^2 and μ_4, and thus provide the determination of β_2.

We will now determine μ'_{2p+1} in like manner. We have

$$\mu'_{2p+1} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}\int_{-1}^{+1}(1-r^2)^{\frac{n-4}{2}} r^{2p+1}\frac{d^{n-2}U}{d(\rho r)^{n-2}}dr,$$

but

$$r^{2p+1} = r\{r^{2p} - (r^2-1)^p + (-1)^p(1-r^2)^p\}$$
$$= r\left\{pr^{2p-2} - \frac{p(p-1)}{2!}r^{2p-4} + \frac{p(p-1)(p-2)}{3!}r^{2p-6} - \dots + (-1)^p(1-r^2)^p\right\}.$$

Hence
$$\mu'_{2p+1} = p\mu'_{2p-1} - \frac{p(p-1)}{2!}\mu'_{2p-3} + \frac{p(p-1)(p-2)}{3!}\mu'_{2p-5} - \dots + (-1)^p\chi_{2p+1} \quad \text{(xxii)},$$

where

$$\chi_{2p+1} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}\int_{-1}^{+1}(1-r^2)^{\frac{n-4+2p}{2}} r\frac{d^{n-2}U}{d(r\rho)^{n-2}}dr$$
$$= \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}\frac{1}{n-2+2p}\int_{-1}^{+1}\frac{d^{n-2}U}{d(r\rho)^{n-2}}d\left(-(1-r^2)^{\frac{n-2+2p}{2}}\right),$$

or integrating by parts

$$= \frac{\rho(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!\,(n-2+2p)}\int_{-1}^{+1}(1-r^2)^{\frac{n-2+2p}{2}}\frac{d^{n-1}U}{d(r\rho)^{n-1}}dr.$$

Assuming n odd we must keep the first series in the value of $\frac{d^{n-1}U}{d\,(r\rho)^{n-1}}$, and we reach

$$\chi_{2p+1} = \frac{\rho\,(1-\rho^2)^{\frac{n-1}{2}}}{\pi\,(n-3)!\,(n-2+2p)}\frac{\pi}{2}(n-2)^2(n-4)^2\ldots 1^2 \,.\, 2q_{n+2p}$$
$$\times\left(1+\frac{n^2}{2!}\rho^2\frac{1}{n+1+2p}+\frac{n^2(n+2)^2}{4!}\frac{1}{n+1+2p}\frac{3}{n+3+2p}\rho^4+\text{etc.}\ldots\right)$$
$$=\frac{\rho\,(1-\rho^2)^{\frac{n-1}{2}}}{2\,(n-3)!\,(n-2+2p)}2q_{n+2p}(n-2)^2(n-4)^2\ldots 1^2$$
$$\times F\left(\frac{n}{2},\frac{n}{2},\frac{n+1}{2}+p,\rho^2\right)$$
$$=\frac{\rho\,(1-\rho^2)^p\,(n-2)}{n-2+2p}\frac{q_{n+2p}}{q_{n-1}}F\left(p+\frac{1}{2},\,p+\frac{1}{2},\,\frac{n+1}{2}+p,\,\rho^2\right)\quad\ldots\ldots\text{(xxiii)},$$

if we use Euler's reduction formula, and note that for n odd, or $n-1$ even

$$2q_{n-1}=\frac{(n-3)(n-5)\ldots 2}{(n-2)(n-4)\ldots 1}\cdot 2.$$

If we start with n even we reach an absolutely identical formula by a different route. Thus we have

$$\mu'_{2p+1}=p\mu'_{2p-1}-\frac{p\,(p-1)}{2!}\mu'_{2p-3}+\frac{p\,(p-1)(p-2)}{3!}\mu'_{2p-5}-\ldots$$
$$+(-1)^p\frac{\rho\,(1-\rho^2)^p\,(n-2)}{n-2+2p}\frac{q_{n+2p}}{q_{n-1}}F\left(p+\frac{1}{2},\,p+\frac{1}{2},\,\frac{n+1}{2}+p,\,\rho^2\right)$$
$$\ldots\ldots\ldots\ldots\ldots\text{(xxiv)}.$$

Taking p in succession equal to zero and to unity we find

$$\mu_1'=\bar{r}=\rho\frac{q_n}{q_{n-1}}\left(1+\frac{1^2}{n+1}\frac{\rho^2}{2}+\frac{1^2\,.\,3^2}{1\,.\,2\,.\,(n+1)(n+3)}\frac{\rho^4}{4}\right.$$
$$\left.+\frac{1^2\,.\,3^2\,.\,5^2}{3!\,(n+1)(n+3)(n+5)}\frac{\rho^6}{8}+\ldots\right)\ldots\ldots\text{(xxv)},$$

$$\mu_3'=\mu_3+3\mu_2'\bar{r}-2\bar{r}^3=\bar{r}-\rho\,(1-\rho^2)\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}$$
$$\times\left(1+\frac{3^2}{n+3}\frac{\rho^2}{2}+\frac{3^2\,.\,5^2}{1\,.\,2\,(n+3)(n+5)}\frac{\rho^4}{4}+\frac{3^2\,.\,5^2\,.\,7^2}{3!\,(n+3)(n+5)(n+7)}\frac{\rho^6}{8}+\ldots\right)$$
$$\ldots\ldots\ldots\ldots\ldots\text{(xxvi)}.$$

Equations (xxv) and (xxvi) provide the values of the odd moment coefficients about zero and this in fairly rapidly converging series. From them we can deduce the value about the mean μ_3 and thus find the fundamental β_1. Table X, p. 377, again gives the requisite values of q_n for the range $n=1$ to 105.

Illustration. Samples of 25 are taken out of a population in which two variates have the correlation $\rho = \cdot 6$. Determination of the nature of the distribution of r in these samples.

Here $n = 25$, and with $\rho = \cdot 6$ we find from (xx), (xxi), (xxv) and (xxvi) the values*

$$\mu_1' = \bar{r} = \cdot 591{,}825, \qquad \mu_2' = \cdot 368{,}739,$$
$$\mu_3' = \cdot 238{,}293, \qquad \mu_4' = \cdot 158{,}510.$$

Further
$$\mu_2 = \cdot 018{,}482, \qquad \sigma_r = \cdot 135{,}950,$$
$$\mu_3 = - \cdot 001{,}812{,}380, \qquad \mu_4 = \cdot 001{,}279{,}141,$$

giving
$$\beta_1 = \cdot 520{,}265, \qquad \beta_2 = 3\cdot 744{,}573.$$

The distribution is thus very far from normal.

Hence by the formula†:

$$\text{Distance from mean to mode} = \frac{\sigma_r \sqrt{\beta_1}\,(\beta_2 + 3)}{2\,(5\beta_2 - 6\beta_1 - 9)} \quad \text{.........(xxvii)},$$

we find
$$\check{r} - \bar{r} = \cdot 050{,}094,$$
$$\check{r} = \cdot 64192.$$

We shall see later that the actual value is

$$\check{r} = \cdot 64194,$$

or the approximation is very close.

The skewness is given by

$$Sk. = (\check{r} - \bar{r})/\sigma_r = \cdot 36847,$$

thus indicating that there is but little approach to normality.

Fig. 1, p. 338, shows the excellent fit of a Pearson curve of Type II to the distribution. The equation is

$$y = \cdot 31004 \left(1 - \frac{x}{\cdot 31075}\right)^{5\cdot 7536} \left(1 + \frac{x}{9\cdot 64157}\right)^{178\cdot 5135}$$

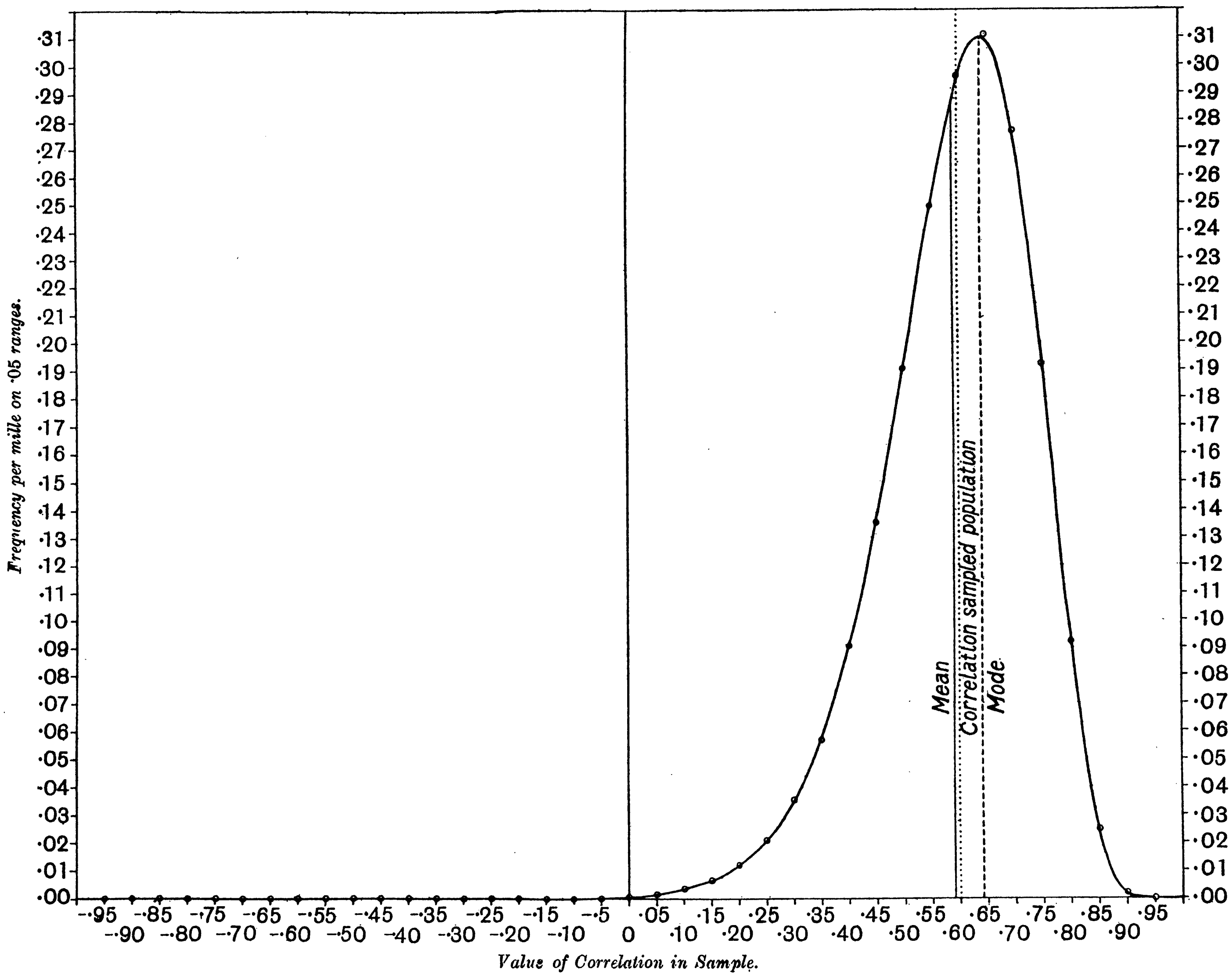
We see that when $n = 25$, Pearson's curves—fitted by moments not by range—adequately describe the frequencies, but there is still no real approach to a Gaussian distribution.

The series-expansions which have been given for the determination of the moments are of very little service when n is less than 25. We have therefore to consider formulae for deducing in succession the moments about $r = 0$ for $n = 5$ to $n = 25$.

* The values were in every case worked out to nine places of decimals.

† Pearson: Mathematical Contributions to the Theory of Evolution, XII, p. 7. *Drapers' Company Research Memoirs, Biometric Series*, Cambridge University Press.

FIG. 1. Comparison of Values of Frequency Ordinates for $n=25$, $\rho=0{\cdot}6$ as given by complete theory and by a Pearson Skew Curve of Frequency. The dots mark true ordinates.



By (xxv)

$$\bar{r}_{n+2} = \rho \frac{q_{n+2}}{q_{n+1}} \left(1 + \frac{1^2}{n+3}\frac{\rho^2}{2} + \dots + \frac{1 \quad 3^2 \dots (2s-1)^2}{s!\,(n+3)\,(n+5) - (n+2s+1)}\frac{\rho^{2s}}{2^s} + \dots\right),$$

$$\bar{r}_n = \rho \frac{q_n}{q_{n-1}} \left(1 + \frac{1^2}{n+1}\frac{\rho^2}{2} + \dots + \frac{1^2 . 3^2 \dots (2s-1)^2}{s!\,(n+1)\,(n+3) \dots (n+2s-1)}\frac{\rho^{2s}}{2^s} + \dots\right);$$

$$\bar{r}_{n+2} - \bar{r}_n = \rho \frac{q_{n+2}}{q_{n-1}} \left\{\frac{n}{n-1}\left(1 + \frac{1^2}{n+3}\frac{\rho^2}{2} + \dots + \frac{1^2 . 3^2 \dots (2s-1)^2}{s!\,(n+3)\,(n+5) \dots (n+2s+1)}\frac{\rho^{2s}}{2^s} + \dots\right) - \frac{n+1}{n}\left(1 + \frac{1^2}{n+1}\frac{\rho^2}{2} + \dots + \frac{1^2 . 3^2 \dots (2s-1)^2}{s!\,(n+1)\,(n+3) \dots (n+2s-1)}\frac{\rho^{2s}}{2^s} + \dots\right)\right\},$$

since
$$\frac{q_{n-1}}{q_{n+1}} = \frac{n}{n-1}.$$

The general term is therefore

$$\frac{1}{(n-1)\,(n-2)\,\rho}\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}\left(\frac{n^2\,(n+1)}{n+2s+1} - (n^2-1)\right) \times \frac{1^2 . 3^2 \dots (2s-1)^2}{s!\,(n+1)\,(n+3) \dots (n+2s-1)}\frac{\rho^{2s}}{2^s}.$$

Now in (xxvi) we have seen that

$$\bar{r}_n - \mu'_{3,n} = \rho\,(1-\rho^2)\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}\left\{1 + \frac{3^2}{n+3}\frac{\rho^2}{2} + \dots + \frac{3^2 . 5^2 \dots (2s+1)^2}{s!\,(n+3)\,(n+5) \dots (n+2s+1)}\frac{\rho^{2s}}{2^s} + \dots\right\},$$

and the general term is

$$\rho\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}\left(\frac{3^2 . 5^2 \dots (2s+1)^2}{s!\,(n+3)\,(n+5) \dots (n+2s+1)} - \frac{2 . 3^2 . 5^2 \dots (2s-1)^2}{(s-1)!\,(n+3)\,(n+5) \dots (n+2s-1)}\right)\frac{\rho^{2s}}{2^s}$$

$$= \rho\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}\left(\frac{(2s+1)^2}{n+2s+1} - 2s\right)\frac{1^2 . 3^2 . 5^2 \dots (2s-1)^2}{s!\,(n+3)(n+5) \dots (n+2s-1)}\frac{\rho^{2s}}{2^s}$$

$$= \rho\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}\frac{(n+1)\,(1-2s\,(n-1))}{n+2s+1}\frac{1^2 . 3^2 . 5^2 \dots (2s-1)^2}{s!\,(n+1)\,(n+3) \dots (n+2s-1)}\frac{\rho^{2s}}{2^s}$$

$$= \rho\frac{q_{n+2}}{q_{n-1}}\frac{n-2}{n}\left(\frac{n^2\,(n+1)}{n+2s+1} - (n^2-1)\right) \times \frac{1^2 . 3^2 . 5^2 \dots (2s-1)^2}{s!\,(n+1)\,(n+3)\,(n+5) \dots (n+2s+1)}\frac{\rho^{2s}}{2^s}$$

$$= (n-1)\,(n-2)\,(\bar{r}_{n+2} - \bar{r}_n).$$

Hence
$$\bar{r}_{n+2} - \bar{r}_n = \frac{\bar{r}_n - \mu'_{3,n}}{(n-1)\,(n-2)} \quad \text{..........................(xxviii).}$$

This result expresses the mean for samples of $n+2$ in terms of the mean for samples of n and the third moment of samples of n.

Next let

$$\chi_{2p,n} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi\,(n-3)\,!}\int_{-1}^{+1}(1-r^2)^{\frac{n-4+2p}{2}}\frac{d^{n-2}U}{d\,(\rho r)^{n-2}}\,dr$$
$$= \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi\,(n-3)\,!}f_{p,n}.$$

Now integrate $f_{p,n}$ by parts *twice*:

$$f_{p,n} = \frac{n-4+2p}{\rho}\int_{-1}^{+1} r\,(1-r^2)^{\frac{n-6+2p}{2}}\frac{d^{n-3}U}{d\,(\rho r)^{n-3}}\,dr$$
$$= \frac{n-4+2p}{\rho}\int_{-1}^{+1}\{(n-6+2p)\,(1-r^2)^{\frac{n-8+2p}{2}} - (n-6+2p+1)\,(1-r^2)\}^{\frac{n-6+2p}{2}}\times\frac{d^{n-4}U}{d\,(\rho r)^{n-4}}\,dr$$
$$= \frac{n-4+2p}{\rho^2}\{(n-6+2p)\,f_{p-1,n-2} - (n-6+2p+1)\,f_{p,n-2}\}.$$

Or returning to the $\chi_{2p,n}$ notation

$$\chi_{2p,n} = \frac{1-\rho^2}{\rho^2}\,\frac{n-4+2p}{(n-3)\,(n-4)}\{(n-6+2p)\,\chi_{2p-2,n-2} - (n-6+2p+1)\,\chi_{2p,n-2}\}.$$

As special cases put $p=1$ and 2, and change n to $n+2$. We have*

$$\chi_{2,n+2} = \frac{1-\rho^2}{\rho^2}\cdot\frac{n}{n-1}\cdot\left\{\chi_{0,n} - \frac{n-1}{n-2}\chi_{2,n}\right\} \quad\text{.....................(xxix)},$$

$$\chi_{4,n+2} = \frac{1-\rho^2}{\rho^2}\cdot\frac{n+2}{n-1}\cdot\left\{\frac{n}{n-2}\chi_{2,n} - \frac{n+1}{n-2}\chi_{4,n}\right\} \quad\text{..............(xxx)}.$$

But
$$\chi_{0,n} = 1 \quad\text{and}\quad \chi_{2,n} = 1 - \mu'_{2,n},$$
$$\chi_{4,n} = 1 - 2\mu'_{2,n} + \mu'_{4,n}.$$

Accordingly

$$\mu'_{2,n+2} = 1 - \frac{1-\rho^2}{\rho^2}\,\frac{n}{n-2}\left(\mu'_{2,n} - \frac{1}{n-1}\right) \quad\text{.................(xxxi)},$$

which can be verified directly from (xx) or $(\text{xx})^{\text{bis}}$. Again instead of working with the series for $\chi_{4,n+2}$ above (xxx), we can replace it by one involving the moments about $r=0$, directly:

$$\mu'_{4,n+2} = 1 - \frac{1-\rho^2}{\rho^2}\left\{\frac{(n+1)\,(n+2)}{(n-1)\,(n-2)}\mu'_{4,n} + \frac{n^2-6n-4}{(n-1)\,(n-2)}\mu'_{2,n} - \frac{1}{n-1}\right\}$$
$$\text{.........(xxxii)}.$$

* The process of integrating by parts shows that we must have $n > 2$.

It remains to determine the formula for $\mu'_{3,n+2}$. We have

$$\chi_{2p+1,n} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}\int_{-1}^{+1}(1-r^2)^{\frac{n-4+2p}{2}}\, r\,\frac{d^{n-2}U}{d(r\rho)^{n-2}}\,dr,$$

whence by double integration by parts to reduce the U differential coefficient we obtain

$$\chi_{2p+1,n} = \frac{1-\rho^2}{\rho^2}\frac{n-4+2p}{(n-3)(n-4)}\left\{(n-6+2p)\,\chi_{2p-1,n-2} - \frac{n-4+2p+1}{n-4}\chi_{2p+1,n-2}\right\}.$$

Putting $p = 1$ and changing n to $n+2$ we have

$$\chi_{3,n+2} = \frac{1-\rho^2}{\rho^2}\frac{n}{n-1}\left\{\chi_{1,n} - \frac{n+1}{n-2}\chi_{3,n}\right\} \quad \text{..............(xxxiii).}$$

This may again be read as a formula for $\mu'_{3,n+2}$:

$$\mu'_{3,n+2} = \bar{r}_n\left(1 - \frac{n}{n-1}\frac{1-\rho^2}{\rho^2}\right) + \frac{\bar{r}_n - \mu'_{3,n}}{(n-1)(n-2)}\left(1 + n(n+1)\frac{1-\rho^2}{\rho^2}\right) \quad \text{(xxxiv).}$$

Starting with the values of the μ's for $n = 3$, 4, 25 and 26, the moment coefficients about $r = 0$ have been determined for $n = 5$ to 25 in succession. As controls the values for $n = 20$ had already been determined and those for $n = 10$ were also obtained at a very considerable expenditure of labour from the very slowly converging series of Formulae (xx), (xxi), (xxv) and (xxvi). The initial values of the moment coefficients (i.e. those for $n = 3$, 4, 25 and 26) had to be calculated generally to 15 and sometimes to 20 significant figures, owing to the numerical factors in (xxviii), (xxxi), (xxxii) and (xxxiv) being frequently greater than unity, and thus errors in the last figure being repeatedly multiplied. According to the special value of ρ, it was found best sometimes to deduce moment coefficients of $n + 2$ from those for n, and sometimes those of n from those for $n + 2$, i.e. to work up from 3 and 4, or down from 25 and 26. It seems unnecessary to enter at length here into the many difficulties that arose in the course of these calculations. We think they have all been successfully surmounted and that our final values may be trusted to the figures actually recorded in the tables. We thus found the moment coefficients and from them the values of β_1 and β_2 for the ten values of ρ from 0 to ·9, and for the values of n, 2 to 25, 50, 100 and 400. Diagram I shows that our 270 frequency curves are fairly well distributed over the most frequently occurring portion of the β_1, β_2 plane. Now our view is that the constants β_1, β_2 describe adequately for statistical purposes the bulk of the usual frequencies distributions. But we have provided tables of the values of the ordinates for the above 270 curves. Hence *by interpolation* it will now be possible to determine rapidly ordinates which will graduate with reasonable accuracy any frequency distribution whatever quite apart from the idea of sampling normally correlated variates*.

* Francis Galton frequently insisted on the importance of forming Tables of frequency ordinates, which would graduate any frequency distribution in the β_1, β_2 plane. A scheme for covering this plane

The diagram referred to on p. 341 will appear with Part II of the paper.

In order to make use of our ordinates for graduating frequency curves we must express the distance from our origin to our mean (i.e. from $r = 0$ to $r = \bar{r}_n$) in terms of the standard deviation, and further the unit of argument of the abscissae, i.e. 05 in r, also in terms of the standard deviation. Our interpolated frequency ordinates (reduced of course, to the size of the actual population) will then have to be plotted to intervals of $\cdot 05\sigma_g/\sigma_r$, the origin being $\bar{r}_n\sigma_g/\sigma_r$ from the mean of the graduated data, where σ_g is the standard deviation of the graduated data. Care must be taken to so choose the axis of abscissae of the graduated data that the sign of μ_3 is the same in the graduated material and the graduating frequencies. Table C gives the distance from the mean to the origin of coordinates in each case and also the abscissal unit for plotting both in terms of the standard deviation.

(4) *On the Determination of the Mode.* Differentiating (iv) we have

$$\frac{dy_n}{dr} = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}\left\{(1-r^2)^{\frac{n-4}{2}}\rho\frac{d^{n-1}U}{d(r\rho)^{n-1}} - r(n-4)(1-r^2)^{\frac{n-6}{2}}\frac{d^{n-2}U}{d(r\rho)^{n-2}}\right\}$$

Hence the mode $\check{r}$ is given by

$$0 = (1-\check{r}^2)\rho^2\left(\frac{d^{n-1}U}{d(r\rho)^{n-1}}\right) - \check{r}\rho\frac{d^{n-2}U}{d(r\rho)^{n-2}}(n-4),$$

or writing $\check{\rho}^2 = \check{r}\rho$, we have

$$(\rho^2 - \check{\rho}^4)\frac{d^{n-1}\check{U}}{d(\check{\rho}^2)^{n-2}} = (n-4)\check{\rho}^2\frac{d^{n-2}\check{U}}{d(\check{\rho}^2)^{n-2}} \quad \text{..............(xxxv)},$$

where $\check{U}$ is U with $\check{\rho}$ put for $r\rho$.

Now (xxxv) is by no means easy to solve adequately, for if we solve it by approximation, $r = \rho$ and $\check{\rho} = \rho^2$ is not sufficiently close for an effective first approximation, especially when ρ differs considerably from zero. We have indeed from (v) the relation

$$(1-\check{\rho}^4)\frac{d^{n-1}\check{U}}{d(\check{\rho}^2)^{n-1}} - \check{\rho}^2(2n-3)\frac{d^{n-2}\check{U}}{d(\check{\rho}^2)^{n-2}} - (n-2)^2\frac{d^{n-3}\check{U}}{d(\check{\rho}^2)^{n-3}} = 0 \quad \text{(xxxvi)},$$

and this might be combined with (xxxv) to deduce in succession relations between lower pairs of differential coefficients, till we ultimately reach a relation between $d\check{U}/d\check{\rho}^2$ and $\check{U}$, but the process is too laborious except for very low values of n.

Fisher has outlined another method of approaching the mode*. It is easy to see that

$$\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}\left(\tan^{-1}\frac{1-x}{\sqrt{1-x^2}} - \tan^{-1}\frac{-x}{\sqrt{1-x^2}}\right)$$
$$= \frac{2}{\sqrt{1-x^2}}\left[\tan^{-1}\left(\frac{\xi - x}{\sqrt{1-x^2}}\right)\right]_0^1$$
$$= 2\int_0^1\frac{d\xi}{(\xi-x)^2+1-x^2} = 2\int_0^1\frac{d\xi}{\xi^2-2x\xi+1},$$

with a series of Pearson-curves has been long under consideration, but the immense labour of calculating the ordinates of 400 to 500 curves has so far prevented the actualisation of this idea. The present ordinate-tables go some way to supply the need Galton pointed out.

* *Biometrika*, Vol. x. p. 520.

or, if $\xi = e^{-z}$, $$= \int_0^\infty \frac{dz}{\cosh z - x} = \int_0^\infty \frac{dz}{\cosh z - \rho r}, \text{ if } x = \rho r.$$

But $$y_n = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi (n-3)!} (1-r^2)^{\frac{n-4}{2}} \frac{d^{n-2}}{d(\rho r)^{n-2}} \left(\frac{\cos^{-1}(-\rho r)}{\sqrt{1-\rho^2 r^2}}\right)$$

$$= \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi (n-3)!} (1-r^2)^{\frac{n-4}{2}} \frac{d^{n-2}}{d(\rho r)^{n-2}} \int_0 \frac{dz}{\cosh z - \rho r}$$

$$= (n-2) \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi} (1-r^2)^{\frac{n-4}{2}} \int_0^\infty \frac{dz}{(\cosh z - \rho r)^{n-1}} \quad \text{...(xxxvii)},$$

$$= \frac{(n-2)(1-\rho^2)^{\frac{n-1}{2}}}{\pi} (1-r^2)^{\frac{n-4}{2}} I_{n-1}, \text{ say............(xxxvii)}^{\text{bis}}.$$

Substituting in Eqn. (viii) we find

$$n(1-\rho^2 r^2) I_{n+1} = (2n-1)\rho r I_n + (n-1) I_{n-1} \quad \text{......(xxxviii)},$$

as the reduction formula for the I_n's.

Similarly, if $$I'_{n-1} = \int_0^\infty \frac{dz}{(\cosh z - \rho_0^2)^{n-1}},$$

then $$n(1-\rho_0^4) I'_{n+1} = (2n-1)\rho_0^2 I'_n + (n-1) I'_{n-1} \quad \text{...(xxxviii)}^{\text{bis}}.$$

Now using value (xxxvii)$^{\text{bis}}$ for y_n, the equation for the mode is

$$(n-4)\breve{r}\breve{I}_{n-1} = \rho(1-\breve{r}^2)(n-1)\breve{I}_n,$$

or, if as before, $\breve{\rho}^2 = \rho\breve{r}$, we have:

$$(n-4)\breve{\rho}^2 \breve{I}_{n-1} = (\rho^2 - \breve{\rho}^4)(n-1)\breve{I}_n \text{..................(xxxix)}.$$

This combined with

$$n(1-\breve{\rho}^4)\breve{I}_{n+1} = (2n-1)\breve{\rho}^2 \breve{I}_n + (n-1)\breve{I}_{n-1} \text{ ....(xl)},$$

should determine the mode.

Now assume $\breve{\rho}^2 = \rho_0^2 + \epsilon$, where ρ_0^2 is some first approximation to $\breve{\rho}^2$, then we find

$$\epsilon = -\frac{(n-4)\rho_0^2 I'_{n-1} - (n-1)(\rho^2 - \rho_0^4) I_n'}{(n-4) I'_{n-1} + (n-1)(n-2)\rho_0^2 I_n' - (n-1)(\rho^2 - \rho_0^4) I'_{n+1}} \text{ ....(xli)}.$$

If we had obtained an approximation ρ_0^2 to $\breve{\rho}^2$, we could start with

$$I_1' = \frac{\cos^{-1}(-\rho_0^2)}{\sqrt{1-\rho_0^4}} \quad \text{and} \quad I_2' = \frac{1}{1-\rho_0^4} + \frac{\rho_0^2 \cos^{-1}(-\rho_0^2)}{(1-\rho_0^4)^{\frac{3}{2}}} \quad \text{......(xlii)},$$

and by aid of (xxxviii)$^{\text{bis}}$ determine the I''s in succession. If $E_n = I_n/I_{n-1}$ we can put our results in the forms (xliii) and (xliv) below, and calculate successive E's:

$$\epsilon = -\frac{\left\{\frac{(n-4)\rho_0^2}{E_n} - (n-1)(\rho^2 - \rho_0^4)\right\}}{\frac{n-4}{E_n} + (n-1)(n-2)\rho_0^2 - n(n-1)(\rho^2 - \rho_0^4) E_{n+1}} \quad \text{...(xliii)},$$

and $$(n-1)(1-\rho_0^4)E_n = (2n-3)\rho_0^2 + \frac{n-2}{E_{n-1}} \quad \text{..............(xliv).}$$

But even this would be laborious had we to find successive values of E_n from (xliv). Actually, if n be moderately large, E_n and E_{n-1} tend to equality fairly rapidly. For example the following are the values of E_n for $\rho_0 = \cdot 6$:

$$I_1 = 2\cdot 078{,}4173,\ I_2 = 1\cdot 873{,}8688 \text{ and therefore } E_2 = \cdot 901{,}5845.$$

E_2	·901,5845	E_9	1·470,8511	E_{16}	1·511,0031	E_{23}	1·527,2571
E_3	1·257,5588	E_{10}	1·475,5703	E_{17}	1·514,1874	E_{24}	1·528,7778
E_4	1·183,5106	E_{11}	1·486,5966	E_{18}	1·516,9985	E_{25}	1·530,1728
E_5	1·451,8703	E_{12}	1·492,1848	E_{19}	1·519,5018	E_{26}	1·531,4570
E_6	1·377,5430	E_{13}	1·498,5199	E_{20}	1·521,7436	E'	1·529,7263
E_7	1·453,2879	E_{14}	1·503,1022	E_{21}	1·523,7636	E''	1·531,0459
E_8	1·445,7342	E_{15}	1·507,3770	E_{22}	1·525,5928	E'''	1·530,1488

Clearly E_n and E_{n-1} approach equality. Now put $E_{25} = E_{24}$ for $n = 25$ in (xliv) and we have for $\rho_0 = \cdot 6$

$$20\cdot 8896E'^2 - 16\cdot 92E' - 23 = 0,$$

which gives for the root required

$$E' = 1\cdot 529{,}7263.$$

But we might also have made $E_{25} = E_{26}$ and so reached

$$21\cdot 7600E''^2 - 17\cdot 64E'' - 24 = 0,$$

which gives $$E'' = 1\cdot 531{,}0459.$$

It is better therefore in finding E_n to equate E_n and E_{n-1} than E_n and E_{n+1}.

A still closer approximation may be found by noting that

$$E_n - E' = \epsilon = E_{n+1} - E'', \text{ nearly},$$

where ϵ is very small. Hence since

$$n(1-\rho_0^4)E_{n+1}E_n - (2n-1)\rho_0^2E_n - (n-1) = 0,$$

we have

$$\epsilon = \frac{(n-1) + (2n-1)\rho_0^2E' - n(1-\rho_0^4)E'E''}{n(1-\rho_0^4)(E'+E'') - (2n-1)\rho_0^2},$$

or, $$E''' = E' + \epsilon = \frac{(n-1) + n(1-\rho_0^4)E'^2}{n(1-\rho_0^4)(E'+E'') - (2n-1)\rho_0^2} \quad \text{...........(xlv).}$$

For the case of $\rho_0 = \cdot 6$ and $n = 25$ we find

$$E''' = 1\cdot 530{,}1488,$$

and $E_{25} - E''' = \cdot 000{,}0240$, a close agreement. As a matter of fact as we only use E_n in a small term the approximation E' is generally quite sufficient.

In the above method all turns on finding a good value of ρ_0^2, i.e. a first approximation to the value of the product of ρ and $\check{r}$. This may be obtained in either of the following ways:

First, choose the values of ρ_0^2 and E' to satisfy the simultaneous equations

$$\frac{(n-4)\rho_0^2}{E'} - (n-1)(\rho^2 - \rho_0^4) = 0,$$

and
$$(n-1)(1-\rho_0^4)E' = (2n-3)\rho_0^2 + \frac{n-2}{E'}.$$

Or, we have for ρ_0 the equation

$$\frac{(n-1)(n-4)(1-\rho_0^4)\rho_0^2}{(n-1)(\rho^2-\rho_0^4)} = (2n-3)\rho_0^2 + \frac{(n-2)(n-1)(\rho^2-\rho_0^4)}{(n-4)\rho_0^2},$$

which writing $\rho_0^4 = z$ gives us

$$(n-4)^2(1-z)z = (2n-3)(n-4)z(\rho^2-z) + (n-2)(n-1)(\rho^2-z)^2,$$

or
$$6z^2 - z\{(n-4)^2 + \rho^2(5n-8)\} + (n-2)(n-1)\rho^4 = 0 \quad \text{......(xlvi)}.$$

As illustration if $n = 25$ and $\rho = \cdot 6$

$$6z^2 - 483{\cdot}12z + 71{\cdot}5392 = 0,$$

giving $z = \cdot 148{,}351,$

or, $\rho_0^2 = \check{r}\rho = \cdot 385{,}164,$

and $\check{r} = \cdot 64194,$

a value* in excellent agreement with the results on p. 337, and needing no further approximation.

Again suppose $n = 5$, and $\rho = \cdot 6$, we have

$$6z^2 - 7{\cdot}12z + 1{\cdot}5552 = 0.$$

Hence $z = \cdot 288{,}6295$ and $\rho_0^2 = \cdot 537{,}2425$ leading to $\check{r} = \cdot 895{,}404$ as our approximation. We shall now use this value of ρ_0^2 to determine the true system of E's corresponding to this value.

We have
$$n(1-\rho_0^4)E_{n+1} = (2n-1)\rho_0^2 + \frac{n-1}{E_n},$$

while
$$E_2 = \rho_0^2 + \frac{1}{\sqrt{1-\rho_0^4}\cos^{-1}(-\rho_0^2)} \quad \text{....................(xlvii)}$$
$$= 1{\cdot}091{,}8073.$$

Substituting in

$$n \times \cdot 711{,}3705E_{n+1} = (2n-1) \times \cdot 537{,}2425 + \frac{n-1}{E_n},$$

we obtain the series

$E_2 = 1{\cdot}091{,}8073,\quad E_3 = 1{\cdot}776{,}5988,$

$E_4 = 1{\cdot}786{,}2042,\quad E_5 = 1{\cdot}911{,}8858,\quad E_6 = 1{\cdot}947{,}6088.$

The values show us that $E_5 = E_4$ was naturally much rougher in this case than that of $n = 25$. However we find $\epsilon = +\cdot 001{,}1177$, $\rho_0^2 = \cdot 538{,}3602$ and $\check{r} = \cdot 897{,}267$, as our next approximation, involving no very great change.

* Repeated use of Eqn. (xli) only modified this result to $\check{r} = 641{,}939$.

To confirm this value of $\check{r}$ we take as the first approximation to $\check{r}$ the value given in the method of the following section, i.e. $\check{r} = \cdot 91344$ giving $\rho_0{}^2 = \cdot 548{,}064$ and

$$n \times \cdot 699{,}6259 E_{n+1} = (2n-1) \times \cdot 548{,}0640 + \frac{n-1}{E_n}.$$

Using (xlvii) we find $E_2 = 1\cdot 103{,}9149$ and hence

$$E_3 = 1\cdot 822{,}4446, \qquad E_4 = 1\cdot 828{,}4768,$$
$$E_5 = 1\cdot 957{,}1738, \qquad E_6 = 1\cdot 994{,}3056,$$

leading to $\epsilon = -\cdot 008{,}8172,$

and $\rho_0{}^2 = \cdot 539{,}2468,$

or $\check{r} = \cdot 89875.$

It will be seen that our two methods of approaching the true value of $\check{r}$ still differ to some extent, although probably serviceable enough for practical purposes. Accordingly we will now make a further approximation starting from $\check{r} = \cdot 8980$ or $\rho_0{}^2 = \cdot 5388$, and we have

$$E_2 = 1\cdot 093{,}5399, \qquad E_3 = 1\cdot 783{,}0638, \qquad E_4 = 1\cdot 792{,}1629,$$
$$E_5 = 1\cdot 918{,}2742, \qquad E_6 = 1\cdot 954{,}1949.$$

These give $\epsilon = -\cdot 000{,}4924,$

and consequently $\rho_0{}^2 = \cdot 538{,}3036,$

with $\check{r} = \cdot 89717,$

a value no doubt correct to four figures.

It is clear that the process of finding the mode for n small is much more laborious than for $n = 25$ or over, because E_n is not nearly E_{n+1}. Actually the value given for E' by the simultaneous equation process from which we started is

$$E' = 1\cdot 881{,}8787,$$

which is only a rough approximation to the value $E_5 = 1\cdot 911{,}8858$. That method must therefore be followed by further approximations when n is much smaller than 25.

(5) *Determination of Ordinates and Mode by Expansions.*

Approximate Expression for the Ordinates. We may proceed to expand the Eqn. (xxxvii) in powers of $1/n$ or $1/(n-1)$. This will involve a knowledge of the expansion of

$$I_n = \int_0^\infty \frac{dz}{(\cosh z - \rho_0{}^2)^n}, \text{ where } \rho_0{}^2 = \rho\check{r},$$

and can be achieved by a process to which Pearson drew attention in 1902*.

Let $$\frac{1}{\cosh z - \rho_0{}^2} = \frac{1}{1-\rho_0{}^2} e^{-(a_2' z^2 + a_4' z^4 + a_6' z^6 + a_8' z^8 + \dots)} \quad \dots\dots\dots\text{(xlviii)}.$$

* *Biometrika*, Vol. I. p. 393.

Then, if

$$v = \log(\cosh z - \rho_0^2) = \log(1-\rho_0^2) + a_2' z^2 + a_4' z^4 + a_6' z^6 + \dots,$$

it follows that

$$a_2' = \frac{1}{2!}\left(\frac{d^2v}{dz^2}\right)_0, \qquad a_4' = \frac{1}{4!}\left(\frac{d^4v}{dz^4}\right)_0,$$

and so on.

Now $$\frac{dv}{dz} = \frac{\sinh z}{\cosh z - \rho_0^2}, \quad \text{or} \quad (\cosh z - \rho_0^2)\frac{dv}{dz} = \sinh z.$$

Apply Leibnitz's Theorem, differentiating $(2s-1)$ times, and we have

$$\sinh z \frac{dv}{dz} + (2s-1)\cosh z \frac{d^2v}{dz^2} + \frac{(2s-1)(2s-2)}{2!}\sinh z \frac{d^3v}{dz^3} + \dots$$
$$+ (\cosh z - \rho_0^2)\frac{d^{2s}v}{dz^{2s}} = \cosh z.$$

Hence when $z = 0$

$$(2s-1)\left(\frac{d^2v}{dz^2}\right)_0 + \frac{(2s-1)(2s-2)(2s-3)}{3!}\left(\frac{d^4v}{dz^4}\right)_0 + \dots + (1-\rho_0^2)\left(\frac{d^{2s}v}{dz^{2s}}\right)_0 = 1.$$

Now put s in succession 1, 2, 3, etc. and there results

$$(1-\rho_0^2)\left(\frac{d^2v}{dz^2}\right)_0 = 1, \qquad 3\left(\frac{d^2v}{dz^2}\right)_0 + (1-\rho_0^2)\left(\frac{d^4v}{dz^4}\right)_0 = 1,$$
$$5\left(\frac{d^2v}{dz^2}\right)_0 + 10\left(\frac{d^4v}{dz^4}\right)_0 + (1-\rho_0^2)\left(\frac{d^6v}{dz^6}\right)_0 = 1,$$
$$7\left(\frac{d^2v}{dz^2}\right)_0 + 35\left(\frac{d^4v}{dz^4}\right)_0 + 21\left(\frac{d^6v}{dz^6}\right)_0 + (1-\rho_0^2)\left(\frac{d^8v}{dz^8}\right)_0 = 1,$$
$$9\left(\frac{d^2v}{dz^2}\right)_0 + 84\left(\frac{d^4v}{dz^4}\right)_0 + 126\left(\frac{d^6v}{dz^6}\right)_0 + 36\left(\frac{d^8v}{dz^8}\right)_0 + (1-\rho_0^2)\left(\frac{d^{10}v}{dz^{10}}\right)_0 = 1,$$

etc., etc.

These lead to

$$a_2' = \frac{1}{2}\frac{1}{1-\rho_0^2}, \qquad a_4' = -\frac{2+\rho_0^2}{24(1-\rho_0^2)^2},$$
$$a_6' = \frac{16+13\rho_0^2+\rho_0^4}{720(1-\rho_0^2)^3}, \qquad a_8' = -\frac{(272+297\rho_0^2+60\rho_0^4+\rho_0^6)}{40320(1-\rho_0^2)^4},$$
$$a_{10}' = \frac{7936+10841\rho_0^2+3651\rho_0^4+251\rho_0^6+\rho_0^8}{3{,}628{,}800(1-\rho_0^2)^5}, \text{ etc.}$$

Accordingly we have, raising (xlviii) to the nth power and expanding the exponential after the term in $a_2'z^2$,

$$\frac{1}{(\cosh z - \rho_0^2)^n} = \frac{1}{(1-\rho_0^2)^n} e^{-na_2'z^2}\{1 - na_4'z^4 - na_6'z^6 - n(a_8' - \tfrac{1}{2}na_4'^2)z^8$$
$$- n(a_{10}' - na_4'a_6')z^{10} - n(a_{12}' - na_4'a_8' - \tfrac{1}{2}na_6'^2 + \tfrac{1}{6}n^2a_4'^3)z^{12} + \dots\}.$$

Remembering that

$$\int_{-\infty}^{+\infty} e^{-\frac{nz^2}{2(1-\rho_0^2)}} z^{2s} dz = \sqrt{2\pi}\sqrt{\frac{1-\rho_0^2}{n}}(2s-1)(2s-3)\dots 1 \times \frac{(1-\rho_0^2)^s}{n^s},$$

we find

$$I_n = \int_0^\infty \frac{dz}{(\cosh z - \rho_0^2)^n} = \frac{1}{2}\frac{\sqrt{2\pi}}{(1-\rho_0^2)^n}\sqrt{\frac{1-\rho_0^2}{n}}\left(1 + \frac{1}{8}\frac{\rho_0^2+2}{n} + \frac{1}{128}\frac{9\rho_0^4+12\rho_0^2+4}{n^2} + \frac{75\rho_0^6+90\rho_0^4-20\rho_0^2-40}{1024n^3} + \frac{3675\rho_0^8+4200\rho_0^6-2520\rho_0^4-3360\rho_0^2-336}{32768n^4} + \text{etc.}\right) \quad \text{...(xlix).}$$

But $\quad y_n = \frac{(n-2)(1-\rho^2)^{\frac{n-1}{2}}(1-r^2)^{\frac{n-4}{2}}}{\pi} I_{n-1}$, if $\rho_0^2 = \rho r$,

and thus we have

$$y_n = \frac{1}{\sqrt{2\pi}}\frac{n-2}{\sqrt{n-1}}(1-\rho^2)^{\frac{3}{2}}\chi_0(\rho, r) \times \left(1 + \frac{\phi_1(\rho r)}{(n-1)} + \frac{\phi_2(\rho r)}{(n-1)^2} + \frac{\phi_3(\rho r)}{(n-1)^3} + \frac{\phi_4(\rho r)}{(n-1)^4} + \dots\right) \quad \text{...(l),}$$

where $\qquad \chi_0(\rho, r) = \frac{(1-\rho^2)^{\frac{n-4}{2}}(1-r^2)^{\frac{n-4}{2}}}{(1-\rho r)^{n-\frac{3}{2}}}$

is symmetrical in ρ and r and

$$\left.\begin{aligned} &\phi_1(\rho r) = \frac{r\rho+2}{8}, \qquad \phi_2(\rho r) = \frac{(3r\rho+2)^2}{128}, \\ &\phi_3(\rho r) = \frac{5\{15(r\rho)^3 + 18(r\rho)^2 - 4(r\rho) - 8\}}{1024}, \\ &\phi_4(\rho r) = \frac{3675(r\rho)^4 + 4200(r\rho)^3 - 2520(r\rho)^2 - 3360(r\rho) - 336}{32768} \end{aligned}\right\} \quad \text{...(li),}$$

thus depend only on the product of ρ and r.

We may write

$$y_n = \frac{n-2}{\sqrt{n-1}}(1-\rho^2)^{\frac{3}{2}}\chi(\rho, r)\left\{1 + \frac{\phi_1(\rho r)}{(n-1)} + \frac{\phi_2(\rho r)}{(n-1)^2} + \frac{\phi_3(\rho r)}{(n-1)^3} + \frac{\phi_4(\rho r)}{(n-1)^4} + \dots\right\} \quad \text{.........(lii),}$$

where $\qquad \log\chi(\rho, r) = -(n-1)\log\chi_1 - \log\chi_2$,

and $\qquad \chi_1 = \frac{1-\rho r}{\{(1-\rho^2)(1-r^2)\}^{\frac{1}{2}}}$,

$$\chi_2 = \frac{\sqrt{2\pi}\{(1-\rho^2)(1-r^2)\}^{\frac{3}{2}}}{(1-\rho r)^{\frac{1}{2}}},$$

both being symmetrical in r and ρ

Table C in Appendix gives the values of $\log \frac{n-2}{\sqrt{n-1}}$, $\log (1-\rho^2)^{\frac{3}{2}}$, $\log \chi_1$, $\log \chi_2$, ϕ_1, ϕ_2, ϕ_3, ϕ_4, and enables the ordinates of the frequency curves to be calculated with considerable rapidity for $n = 25$ and upwards*

Approximate Expressions for the Mode.

Writing
$$I_n' = \int_0^\infty \frac{dz}{(\cosh z - \rho\breve{r})^n},$$
we find
$$I_n' = \frac{1}{2}\frac{\sqrt{2\pi}}{(1-\rho\breve{r})^n}\sqrt{\frac{1-\rho\breve{r}}{n}}\left(1 + \frac{\phi_1'}{n} + \frac{\phi_2'}{n^2} + \frac{\phi_3'}{n^3} + \dots\right),$$
where ϕ_s' stands for $\phi_s(\rho\breve{r})$. We now use Eqn. (xxxix) and find
$$\frac{\breve{r} - \rho\breve{r}^2}{\rho - \rho\breve{r}^2} = \frac{n-1}{n-4}\sqrt{\frac{n-1}{n}}\,R,$$
where
$$R = \left(1 + \frac{\phi_1'}{n} + \frac{\phi_2'}{n^2} + \frac{\phi_3'}{n^3} + \dots\right)\Big/\left(1 + \frac{\phi_1'}{(n-1)} + \frac{\phi_2'}{(n-1)^2} + \frac{\phi_3'}{(n-1)^3} + \dots\right) \quad \dots\dots\dots\text{(liii)}.$$

If we expand this in inverse powers of $\frac{1}{n-1}$, we deduce
$$R = 1 - \frac{\phi_1'}{(n-1)^2} + \frac{\phi_1'^2 - 2\phi_2' + \phi_1'}{(n-1)^3} - \frac{\phi_1'^3 + \phi_1'^2 + \phi_1' - 3\phi_1'\phi_2' - 3\phi_2' + 3\phi_3'}{(n-1)^4} + \dots.$$
Again
$$\frac{n-1}{n-4}\sqrt{\frac{n-1}{n}} = 1 + \frac{5}{2(n-1)} + \frac{63}{8(n-1)^2} + \frac{373}{16(n-1)^3} + \frac{8987}{128(n-1)^4} + \dots.$$
Thus we have
$$\frac{\breve{r} - \rho\breve{r}^2}{\rho - \rho\breve{r}^2} = 1 + \frac{5}{2(n-1)} + \frac{63 - \phi_1'}{8(n-1)^2} + \frac{373 - 24\phi_1' + 16\phi_1'^2 - 32\phi_2'}{16(n-1)^3}$$
$$+ \frac{8987 - 816\phi_1' + 192\phi_1'^2 - 128\phi_1'^3 + 384\phi_1'\phi_2' - 256\phi_2' - 384\phi_3'}{128(n-1)^4} + \dots.$$

Bringing the first term on the right to the left we reach after substituting for the ϕ''s
$$\breve{r} = \rho\left(1 + \frac{5}{2}\frac{(1-\breve{r}^2)}{(n-1)} + \frac{(61 - \rho\breve{r})(1-\breve{r}^2)}{8(n-1)^2} + \frac{(367 - 5\rho\breve{r} - 2\rho^2\breve{r}^2)(1-\breve{r}^2)}{16(n-1)^3}\right.$$
$$\left. + \frac{(17606 - 195\rho\breve{r} - 81\rho^2\breve{r}^2 - 50\rho^3\breve{r}^3)(1-\breve{r}^2)}{256(n-1)^4} + \dots\right) \dots\dots\text{(liv)}.$$

* The ordinates calculated by the rising difference formula were tested in this manner. For $n = 25$ the accordance was excellent, and quite good enough for practical purposes at $n = 10$. Below this (lii) becomes less reliable and needs more terms.

This series has now to be inverted and leads after considerable algebra to

$$\check{r} = \rho\left(1 + \frac{5}{2}\frac{(1-\rho^2)}{(n-1)} + \frac{(61-101\rho^2)(1-\rho^2)}{8(n-1)^2} + \frac{(367-1480\rho^2+1273\rho^4)(1-\rho^2)}{16(n-1)^3} + \frac{(17606-125727\rho^2+246783\rho^4-143782\rho^6)(1-\rho^2)}{256(n-1)^4} + \dots\right) \quad \text{......(lv).}$$

The above series is of very considerable interest from more than one standpoint*. In the first place it appears that Soper's approximation (*Biometrika*, Vol. IX. p. 108) was not valid. He obtained

$$\check{r} = \rho\left\{1 + \frac{3}{2}\frac{(1-\rho^2)}{(n-1)} + \frac{(41+23\rho^2)(1-\rho^2)}{8(n-1)^2} + \dots\right\}.$$

Thus for $n = 25$, $\rho = \cdot 6$, Soper's formula gives ·62811, and (lv) gives ·64205, while the exact value is ·64194. It is clear that the coefficient $\frac{3}{2}$ in Soper's second term of the series can never approach the $\frac{5}{2}$ of the more exact expression. At first the difference was found very perplexing, especially when the algebra had been verified; but the solution appears to lie in the consideration that the best fitting Pearson curve to the frequency is not one tied down to the range -1 to $+1$. That curve is fitted by two moments only, but if we fit a curve by the first four moments and use the general expression

$$\check{r} = \bar{r} + \frac{\sigma_r\sqrt{\beta_1}(\beta_2+3)}{2(5\beta_2 - 6\beta_1 - 9)} \quad \text{..........................(lvii),}$$

we obtain

$$\check{r} = \cdot 64192,$$

a value close to the true value. In other words the use of the third and fourth moments to find the mode is far more important than fixing down the range to the theoretically possible values; that process determines much more quickly the form of the frequency curve, but it does not give nearly such a good fit as allowing the Pearson curve freedom to adjust itself by means solely of the first four moments†. On the other hand a Pearson curve determined by the first four moments does describe fairly accurately the frequency distributions of r for $n = 25$ and upwards: see p. 337.

(6) *Equation for Modes and Antimodes* ($n = 3$).

Still another method of approaching the modal value has been found occasionally of service‡.

* We have used the expansion in terms of $1/(n-1)$ rather than $1/n$ as $(n-1)$ appears to arise more simply in all the formulae. The form in $1/n$ is

$$\check{r} = \left(1 + \frac{5(1-\rho^2)}{2n} + \frac{(81-101\rho^2)(1-\rho^2)}{8n^2} + \frac{(651-1884\rho^2+1273\rho^4)(1-\rho^2)}{16n^3} + \dots\right) \quad \text{...(lvi).}$$

† The Pearson curve determined from the range does not give good values of the frequency for $n = 25$, even when we use the true values of $\bar{r}$ and σ_r and not Soper's approximations to these constants.

‡ It was used successfully in calculating the antimode in the case of samples of three, when the correlation in the sampled population was low. It gave a fairly good "jumping off point" even for higher values of the correlation.

Starting from the equation (xxxix)

$$(n-4)\,\breve{\rho}^2 I_{n-1} = (\rho^2 - \breve{\rho}^4)\,(n-1)\,I_n,$$

where $I_n = \int_0^\infty \frac{dz}{(\cosh z - \breve{\rho}^2)^{n-1}}$ and $\breve{\rho}^2 = \rho\breve{r}$, $\breve{r}$ being the modal value of r, we may expand I_n and I_{n-1} in terms of powers of $\breve{\rho}^2$, the coefficients involving

$$\int_0^\infty \frac{dz}{(\cosh z)^{m-1}} = \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\, d\theta = q_m.$$

We find at once

$$(n-4)\,\breve{\rho}^2\left(q_{n-1} + (n-1)\,\breve{\rho}^2 q_n + \frac{(n-1)\,n}{1\,.\,2}\,\breve{\rho}^4 q_{n+1} + \frac{(n-1)\,n\,(n+1)}{1\,.\,2\,.\,3}\,\breve{\rho}^6 q_{n+2} + \ldots\ldots\right)$$

$$= (\rho^2 - \breve{\rho}^4)\,(n-1)\left(q_n + n\breve{\rho}^2 q_{n+1} + \frac{n\,(n+1)}{1\,.\,2}\,\breve{\rho}^4 q_{n+2} + \frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3}\,\breve{\rho}^6 q_{n+3} + \ldots\ldots\right).$$

Rearranging and substituting $\rho\breve{r}$ for $\breve{\rho}^2$ and noting that

$$q_{m+1} = \frac{m-1}{m}\,q_{m-1},$$

we have

$$\rho\,(n-1)\,q_n = \breve{r}q_{n-1}\,(n-4-\rho^2\,(n-1)^2) + \tfrac{1}{2}(n-1)\,\breve{r}^2\rho q_n\,(2\,(n-3) - \rho^2 n^2)$$
$$+ \frac{(n-1)\,n}{6}\,\breve{r}^3\rho^2 q_{n+1}\,(3\,(n-2) - \rho^2\,(n+1)^2) + \text{etc.} \;\ldots\ldots\text{(lviii)},$$

where the form of the successive terms is sufficiently obvious, and the series converges rapidly if ρ be small.

For the particular case in which we have chiefly used this equation to determine $\breve{r}$, namely samples of three, $\breve{r}$ corresponds to an antimode and the equation is for $n = 3$:

$$2\rho q_3 = -\,\breve{r}q_2\,(1+4\rho^2) - \breve{r}^2 q_3 9\rho^3 + \breve{r}^3 q_4\rho^2\,(3-16\rho^2)$$
$$+ \breve{r}^4 q_5\rho^3\,(8-25\rho^2) + \breve{r}^5 q_6\rho^4\,(15-36\rho^2)$$
$$+ \breve{r}^6 q_7\rho^5\,(24-49\rho^2) + \breve{r}^7 q_8\rho^6\,(35-64\rho^2)$$
$$+ \text{etc.} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(lix)}.$$

An equation which led to $\breve{r}$ with singular accuracy and comparative ease for small values of ρ by aid of Table X for q_n.

(7) *Tables and Models.*

Table A (p. 379) gives the values of the mean, of the mode, of the standard deviation, of β_1 and β_2 and thus of the skewness of the frequency distributions of r. It will be seen that long after we have reached the limit of what are usually treated as small samples, the skewness of the distribution of r is very considerable. The approach to the normal curve is very slow, and the "probable error of the correlation coefficient," i.e. $\cdot 67449\,(1-r^2)/\sqrt{n}$ as usually recorded, has very little worth. Models have been prepared to illustrate these points as follows:

Model A gives for $n=2$ to $n=25$, the distribution of r for $\rho=\cdot 6$.

Model B gives for $n=2$ to $n=25$, the distribution of r for $\rho=\cdot 8$.

Model C gives for $n=3$, the distribution of r for $\rho=0$ to $\cdot 9$.

Model D gives for $n=4$, the distribution of r for $\rho=0$ to $\cdot 9$.

Model E gives for $n=25$, the distribution of r for $\rho=0$ to $\cdot 9$.

Model F gives for $n=50$, the distribution of r for $\rho=0$ to $\cdot 9$.

(Further models are in process of construction for low values of n.)

Even the photographs of such models form a striking warning of the dangers which arise (i) from small samples, and (ii) from judging results from even repeated small samples; the modal value of the frequency distribution for the correlation of these will be very sensibly higher than the correlation of the sampled population.

(8) *On the Determination of the "most likely" Value of the Correlation in the Sampled Population,* i.e. $\hat{\rho}$.

We now turn to another point. Suppose we have found the value of the correlation in a small sample to be r, what is the most reasonable value $\hat{\rho}$ to give to the correlation ρ of the sampled population?

Now we know that

$$y_n=(n-2)\frac{(1-\rho^2)^{\frac{n-1}{2}}}{\pi}(1-r^2)^{\frac{n-4}{2}}\int_0^\infty\frac{dz}{(\cosh z-\rho r)^{n-1}},$$

and if $\phi(\rho)\,d\rho$ were the law of distribution of ρ's, we ought to make

$$\frac{n-2}{\pi}(1-\rho^2)^{\frac{n-1}{2}}\phi(\rho)\,(1-r^2)^{\frac{n-4}{2}}\int_0^\infty\frac{dz}{(\cosh z-\rho r)^{n-1}}\,d\rho$$

a maximum with ρ, or in other words deduce the value of ρ for a given r from

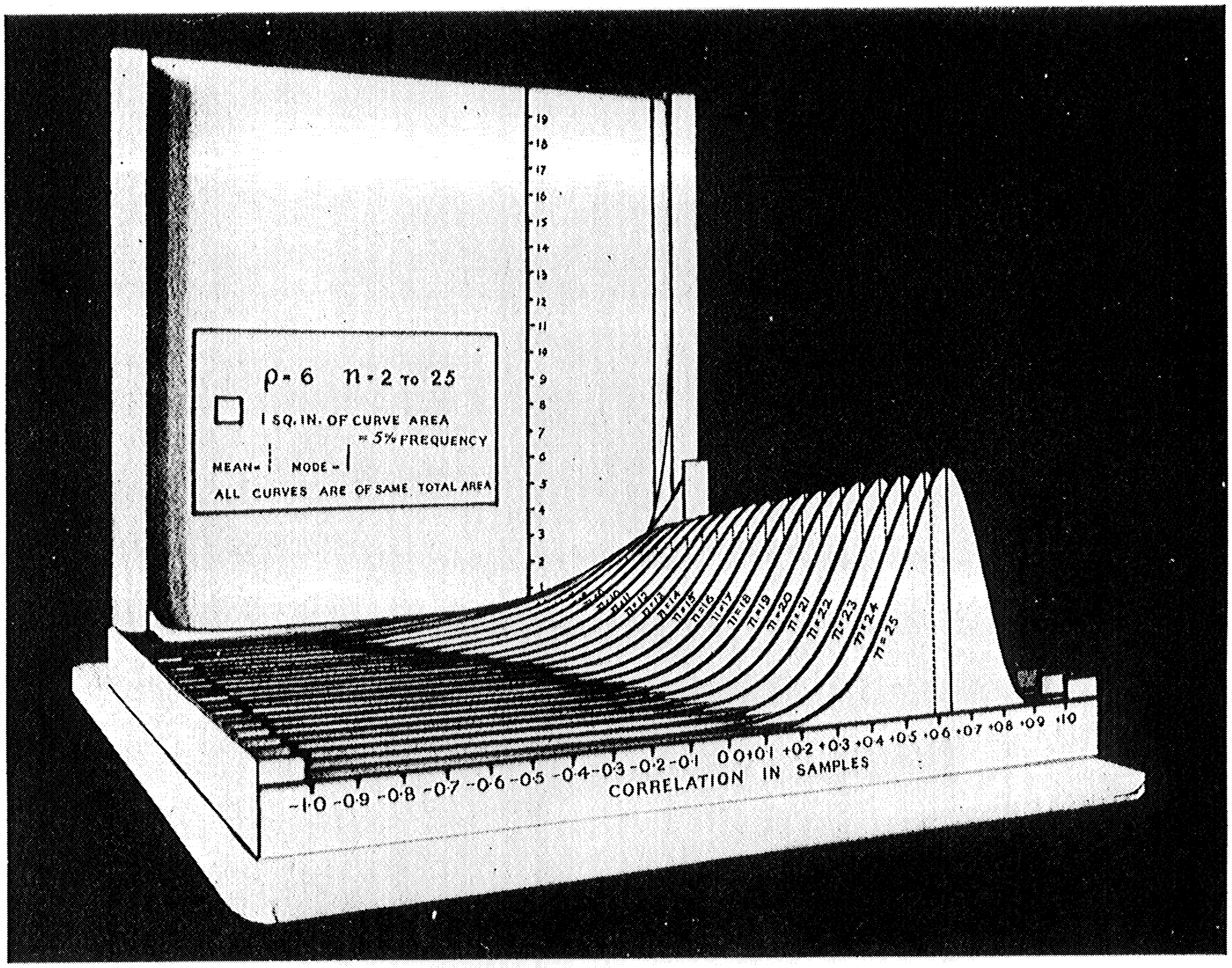
$$\frac{d}{d\rho}\left\{\int_0^\infty\frac{(1-\rho^2)^{\frac{n-1}{2}}\phi(\rho)\,dz}{(\cosh z-\rho r)^{n-1}}\right\}=0 \qquad \text{(lx)}.$$

Fisher puts $\phi(\rho)$ equal to a constant and then differentiating out reaches the equation

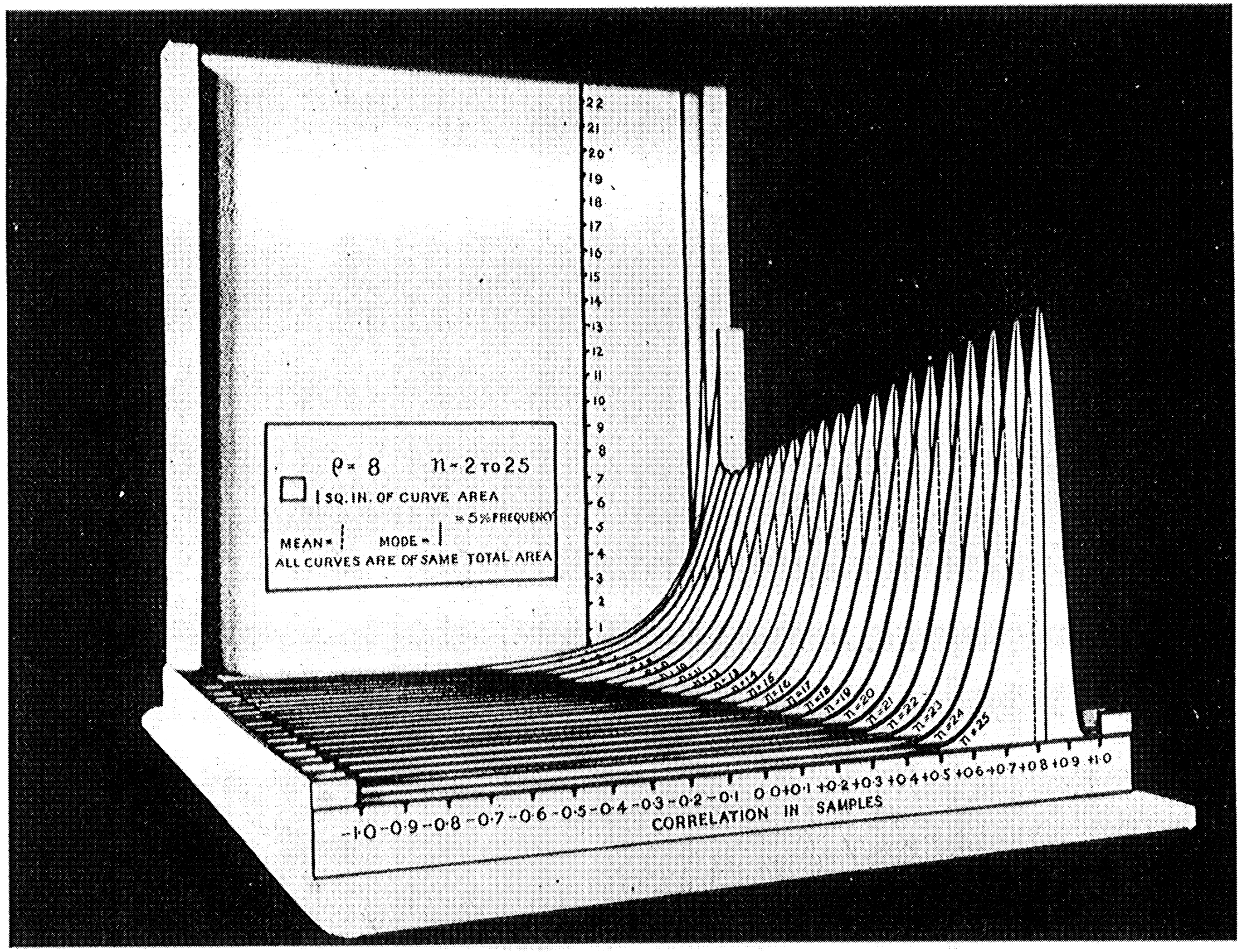
$$\int_0^\infty\frac{(r-\rho\cosh z)\,dz}{(\cosh z-\rho r)^n}=0 \qquad \text{(lxi)},$$

which should provide the value of ρ in terms of r. He solves this only to a first approximation, obtaining,

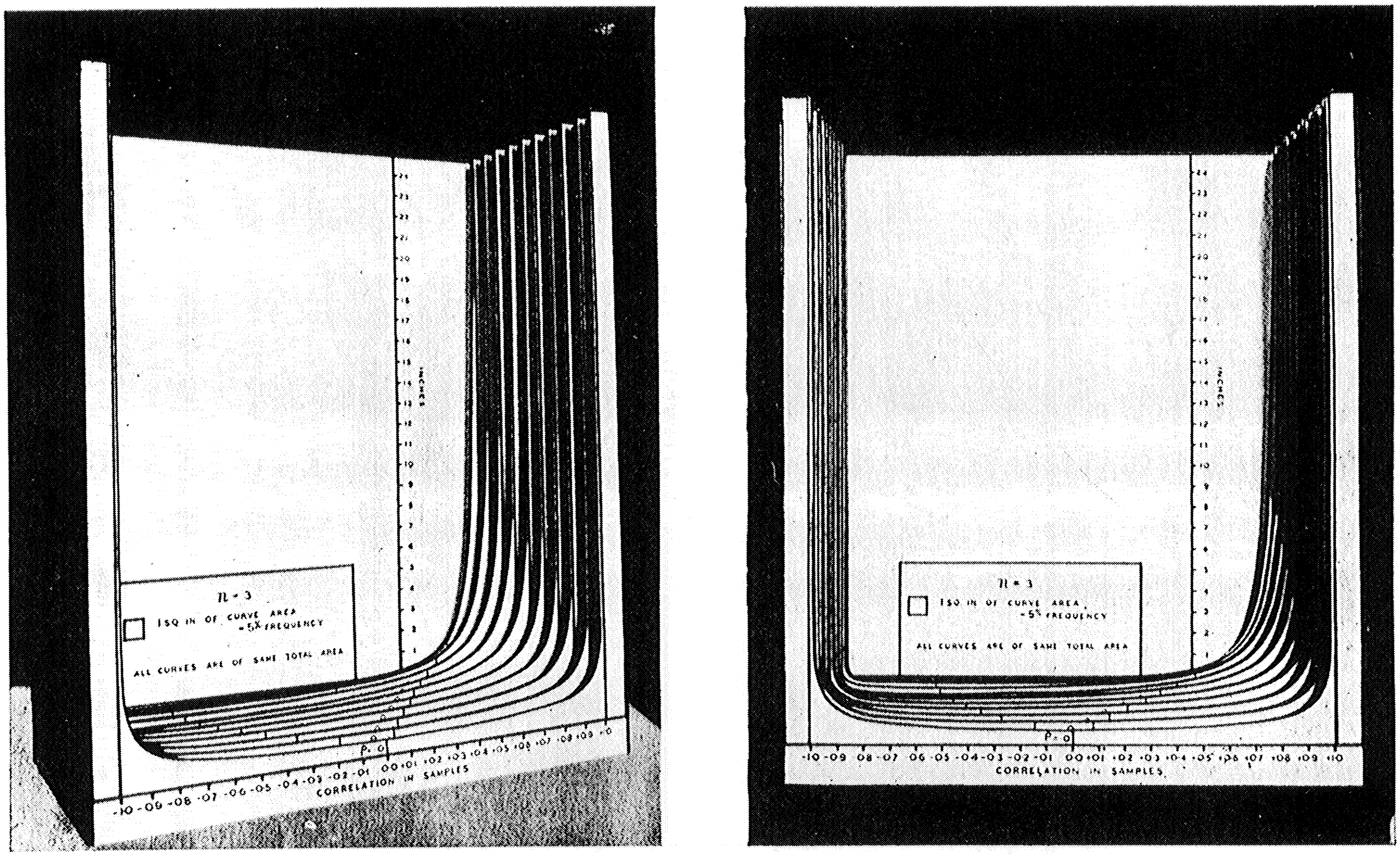
$$\rho=r\left(1-\frac{1-r^2}{2n}\right) \qquad \text{(lxii)}.$$



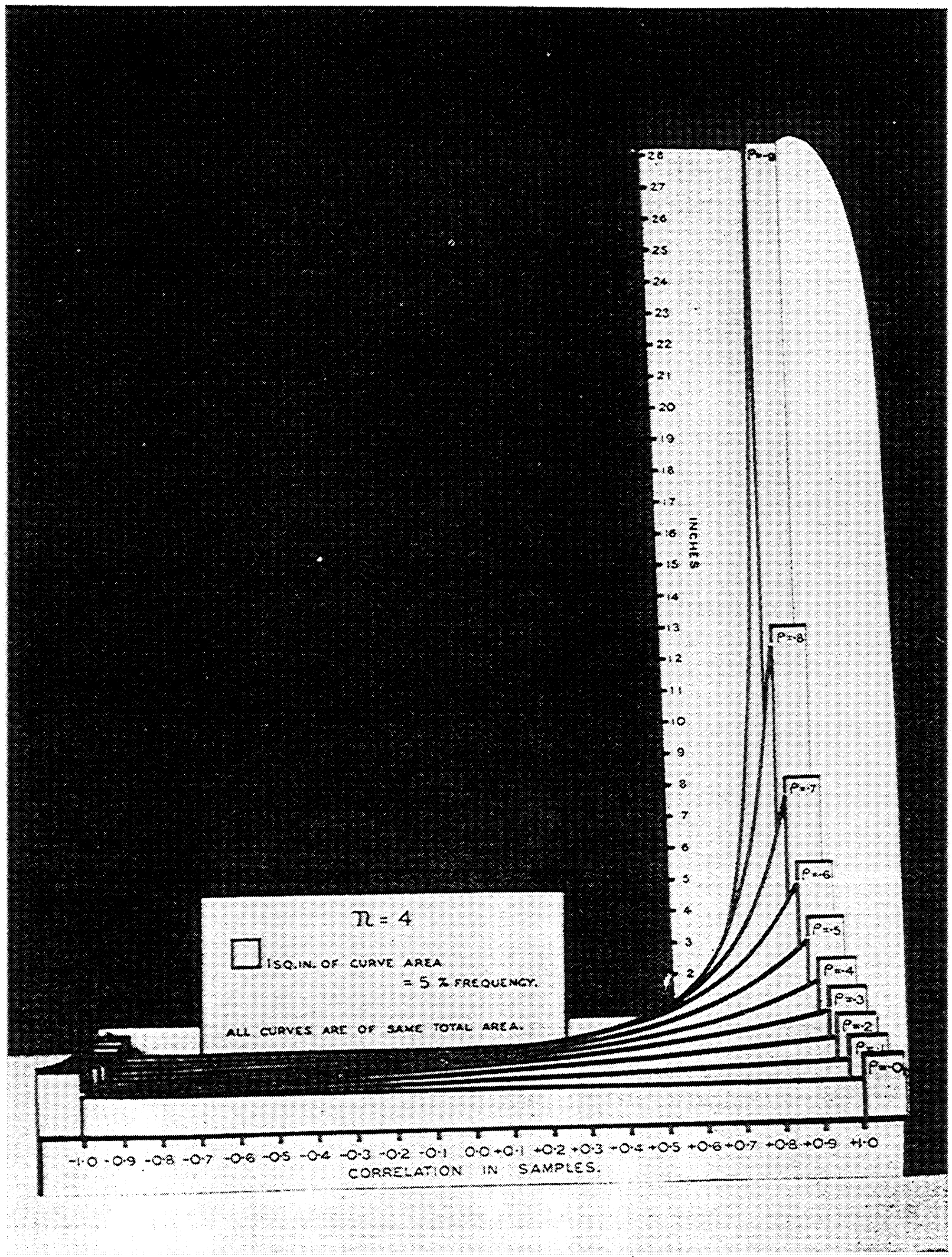
Correlation in Small Samples. $\rho=0{\cdot}6$. Frequency curves for samples of sizes two to twenty-five, showing the changes in type from a skew "cocked hat" to J- and U-forms. Model A.



Correlation in Small Samples. $\rho = 0{\cdot}8$. Frequency curves for samples of sizes two to twenty-five, showing the changes in type from a skew "cocked hat" to J- and U-forms. Model B.



Correlation in Small Samples, $\rho=0{\cdot}0$ to $\rho=0{\cdot}9$. Frequency curves for samples of size three. Model C in two aspects, illustrating the various U-forms of curves, which occur in this case.



Correlation in Small Samples, $\rho=0{\cdot}0$ to $\rho=0{\cdot}9$ for samples of four. Model D, illustrating forms of frequency passing from the rectangle to marked J-forms of curves, which occur in this case.

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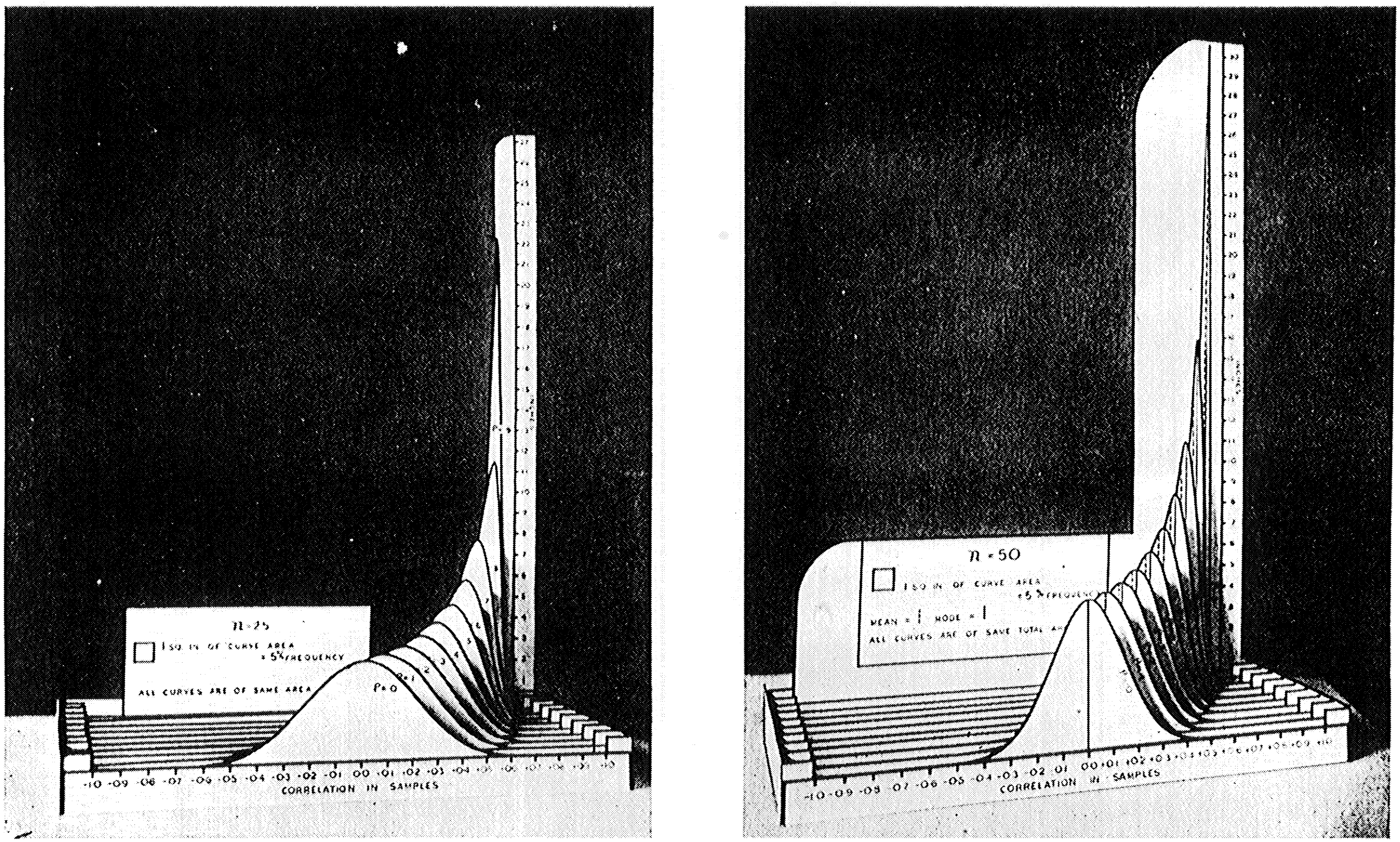
Plate XXVIII



Correlation in Samples of 25 and 50 for $\rho = 0{\cdot}0$ to $\rho = 0{\cdot}9$. Models E and F, illustrating forms still deviating very considerably from normality and increasing in skewness with increase of ρ.

This is a lower approximation than Soper's second approximation, in similar cases, and we know that even Soper's values are not sufficiently accurate when n is as large as 25. Hence no very great confidence can be put in (lxii).

But there is another point about (lx) which is of great importance. Fisher's equation, our (lxi), is deduced on the assumption that $\phi(\rho)$ is constant. In other words he assumes a horizontal frequency curve for ρ, or holds that *à priori* all values of ρ are equally likely to occur. This raises a number of philosophical points and difficulties. We ask:

When we are in absolute ignorance as to ρ, is it according to our experience that all values of the correlation are equally likely to occur? We think this question must probably if not certainly be answered in the negative. Very high correlations are relatively rare, and most biometricians would find it difficult to cite straight away a couple of cases of the correlation equal to $-\cdot 95$ although they could cite a score in which the correlation was sensibly zero, or again about $\cdot 5$. Every biometrician is seeking high correlations, for these are for him the all important data, but he knows how difficult and rare they are to find*. The equal distribution of ignorance which applies so well to many statistical ratios, does not seem valid in the case of correlations†. We generally know quite approximately

* We have recently had occasion to table (a) nearly 400 correlations between characters of the human femur, and (b) over 300 for characters of the human skull. The distributions were very far indeed from horizontal straight lines, and to suppose *à priori* such distributions horizontal could only lead to grave errors.

† A similar problem arises in the case of standard deviations. If Σ be the S.D. of the sample and σ of the sampled population, then the frequency curve for S.D.'s is (*Biometrika*, Vol. x. p. 523)

$$y = y_0 \frac{\Sigma^{n-2}}{\sigma^{n-1}} e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}}.$$

Now, if we make this a maximum for variation of σ, we obtain

$$\sigma = \sqrt{\frac{n}{n-1}}\,\Sigma \qquad\qquad (\alpha)$$

as the "best value" of σ.

This was pointed out to the Editor by "Student," and was a desirable criticism of the statement made (Vol. x. pp. 528–9) that the most reasonable value to give to Σ was the mode of the sampled population, i.e. to take the observed $\Sigma = \check{\Sigma} = \sqrt{\frac{n-2}{n}}\,\sigma$ or suppose

$$\sigma = \sqrt{\frac{n}{n-2}}\,\Sigma \qquad\qquad (\beta).$$

Equations (α) and (β) are not identical. But again (α) is based on the assumption that *all* values of the S.D. are *à priori* equally likely to occur. But surely this is not a result in accordance with our experience! Values of σ from 0 to ∞ are not equally within our experience, and there is almost an absurdity in talking about a standard deviation varying from 0 to ∞; are we to include all possible scales in this distribution? The S.D. of stature might certainly be anything from practically zero to infinity if we measured it first in "light years" and then in microns. Or, are we to measure our S.D.'s all in the same units, when we suppose the distribution of S.D.'s to be of equal probability from zero to infinity? How is this to be done in the case of an absolute length and an index? Given a definite problem, there is certainly no *à priori* likelihood that the S.D. will have every value from 0 to ∞, if we confine ourselves to one scale. It must practically be less than the mean value, and in most actual the correlation of the characters in the population samples, and desire to ascertain whether a small sample of some population *similar to a known population* confirms our experience.

For example, we may have twenty pairs of brothers recorded for some special character. Our *à priori* knowledge is certainly not that all correlations between pairs of brothers from -1 to $+1$ are equally likely to occur! On the contrary we anticipate a value which will not be very far from 0·5. And this *à priori* conviction is so great, that if the small sample did not give a value which considering the size of the sample was compatible with the correlation in the sampled population being near 0·5, we should suspect errors in the measurement or some form of disturbing selection. In such cases, and something like them appears to us most frequent in biometric practice, it is we think erroneous to apply Bayes' Theorem. All it seems possible to do is to assume that we have drawn a value near the mode of our distribution, for our sampled population is much more likely to have a single value, that of our *à priori* experience, than every possible value from -1 to $+1$. If Bayes' Theorem confirms this value—so much the better; if it does not, its fundamental hypothesis is usually so unjustified that it seems most unreasonable to assert that it must give the most likely value of the correlation in the sampled population.

The fuller solution of Eqn. (lxi) thus appears to have academic rather than practical value. Still certain points of theoretical interest arise in the discussion of both (lx) and (lxi). Let us suppose that our *à priori* knowledge consists in the distribution of ρ about a mean $\bar{\rho}$ with a standard deviation κ. It is convenient to take $\kappa^2 = m(1-\bar{\rho}^2)$, where m is an arbitrary constant. Probably $\kappa = 0$, whenever $\bar{\rho} = 1$, and this suggested this form; but since m is quite arbitrary we lay no stress on this point. The equation to determine the most likely value of ρ now becomes

$$\frac{d}{d\rho}\int_0^\infty \frac{(1-\rho^2)^{\frac{n-1}{2}} e^{-\frac{(\rho-\bar{\rho})^2}{2m(1-\bar{\rho})^2}}}{(\cosh z - \rho r)^{n-1}}\, dz = 0,$$

or

$$\frac{(\rho-\bar{\rho})(1-\rho^2)}{m(n-1)(1-\bar{\rho}^2)} I_{n-1} + \rho I_{n-1} = (1-\rho^2)\, r I_n \quad \text{.........(lxiii).}$$

Now this equation cannot in general be solved unless we know the order of the product $m(n-1)$. Certain cases, however, can be considered. If m be very large, i.e. if there be very considerable scatter in our past experience of ρ, then

$$\rho I_{n-1} = (1-\rho^2)\, r I_n \quad \text{..............................(lxiv),}$$

problems is very narrowly limited. For example we measure twenty individuals of a population for stature, and seek the best value of the variability of the sampled population from the result. Would it not be unreasonable to suppose that *à priori* this variability may be equally likely to have any value from 0 to ∞? Our *à priori* knowledge is that it is somewhere between 2″·5 and 3″·0 and very far from equally likely even between these values. To justify the equal distribution of our ignorance, we should have to assume that we neither knew the exact character measured, nor the unit in which it was measured, and such ignorance can only be very exceptional in the present state of our knowledge.

an equation identical with what we obtain by the "equal distribution of our ignorance." The same result is also reached if m be only moderately large and n very big. In other words "the equal distribution of our ignorance," even if we really have some knowledge of the frequency distribution of ρ, will not lead us badly astray in the case of big samples. The matter is quite otherwise, however, in the case of *small* samples; unless our knowledge is very limited (m very large) we have no right whatever to take (lxiv) as applying to such small samples. Indeed when m is fairly small ρ will not differ substantially from $\bar{\rho}$, and the solution of (lxiii) will differ widely from that of (lxiv). We may consider these cases in succession.

Case (i). *Very slight knowledge of* ρ, *or on the other hand a large sample.*

Here we are justified in using (lxiv). We can attempt its solution in two different ways as in the case of the mode.

Let $\hat{\rho}$ be the most likely value of ρ and let us write $\hat{\rho}r = \rho_1{}^2$, then

$$(1-\hat{\rho}^2)\, r\hat{I}_n = \hat{\rho}\hat{I}_{n-1},$$

or

$$(r^2-\rho_1{}^4)\,\hat{I}_n = \rho_1{}^2\hat{I}_{n-1}.$$

Now let $\rho_0{}^2$ be a first approximation to $\rho_1{}^2$ and suppose $\rho_1{}^2 = \rho_0{}^2 + \epsilon$, where ϵ is small. Then

$$(r^2-\rho_0{}^4-2\rho_0{}^2\epsilon)\,(I_n' + n\epsilon I'_{n+1}) = (\rho_0{}^2+\epsilon)\,(I'_{n-1} + \overline{n-1}\,\epsilon I_n'),$$

where

$$I_n' = \int_0^\infty \frac{dz}{(\cosh z - \rho_0{}^2)^n}.$$

Hence remembering that

$$n\,(1-\rho_0{}^4)\, I'_{n+1} = (2n-1)\,\rho_0{}^2 I_n' + (n-1)\, I'_{n-1},$$

we find

$$\epsilon = \frac{I_n'\,(r^2-\rho_0{}^4) - \rho_0{}^2 I'_{n-1}}{I'_{n-1} + \rho_0{}^2\,(n+1)\, I_n' - n\,(r^2-\rho_0{}^4)\, I'_{n+1}},$$

or

$$\epsilon = (1-\rho_0{}^4)\frac{(r^2-\rho_0{}^4)\,E_n - \rho_0{}^2}{(1-\rho_0{}^4)-(n-1)(r^2-\rho_0{}^4)+\rho_0{}^2\{(n+1)(1-\rho_0{}^4)-(2n-1)(r^2-\rho_0{}^4)\}E_n} \quad\text{..................(lxv)},$$

where

$$(1-\rho_0{}^4)\,(n-1)\,E_n = (2n-3)\,\rho_0{}^2 + \frac{n-2}{E_{n-1}} \quad\text{...............(lxvi)},$$

and

$$E_n = I_n'/I'_{n-1}.$$

Now (lxv) and (lxvi) may be treated exactly like the corresponding equations for the determination of the mode. If n be moderately large, we may put

$$E_n = E_{n-1} = E'$$

in (lxv), and if we know $\rho_0{}^2$ obtain the value of E', which value substituted for E_n in (lxvi) gives us ϵ and thus a new approximation. If we cannot guess a good value for $\rho_0{}^2$ (although $\rho_0{}^2 = \rho^2$ is in this case usually sufficient) we can treat the numerator of (lxv) equated to zero, and (lxvi) with E' for E_n and E_{n-1} as simultaneous equations to find E' and ρ_0^2, and so obtain a good approximation straight off, when n is of the order 25, or a fair first approximation when n is smaller.

Applying this we have from (lxv)

$$E_n = E_{n-1} = \rho_0^2/(r^2 - \rho_0^4).$$

Hence

$$(n-1)(1-\rho_0^4)\rho_0^4 - (2n-3)\rho_0^4(r^2-\rho_0^4) - (n-2)(r^2-\rho_0^4)^2 = 0,$$

or

$$\rho_0^2 = r^2\sqrt{\frac{n-2}{n-1-r^2}},$$

and therefore

$$\hat{\rho} = r \times \sqrt{\frac{n-2}{n-1-r^2}} \qquad \text{(lxvii)}.$$

Thus it will be seen that on the hypothesis of the equal distribution of ignorance for $n = 100$, the ratio of $\hat{\rho}$ to r will differ less than ·99402 from unity. On the other hand if n be 5, the ratio of $\hat{\rho}$ to r may differ from unity by as much as ·8660 does. For example, if $n = 5$, then $\hat{\rho} = \cdot 26278$ if $r = \cdot 3$. But for a sample of five the standard deviation of a value of ρ between ·2 and ·3 is of the order ·18 to ·20, so that there is little to be gained by treating the observed ·30 as corresponding to a sampled population of ·26.

We shall now proceed to determine an expansion for $\hat{\rho}$. If R be the ratio of Eqn. (liii), we find from (lxiv)

$$\hat{\rho} = r(1-\hat{\rho}^2)\hat{I}_n/\hat{I}_{n-1},$$

or

$$\hat{\rho} = \frac{r(1-\hat{\rho}^2)}{1-\hat{\rho}r}\sqrt{\frac{n-1}{n}}\,R$$

$$= \frac{r(1-\hat{\rho}^2)}{1-\hat{\rho}r}\left(1+\frac{1}{n-1}\right)^{-\frac{1}{2}}\left(1-\frac{\phi_1}{(n-1)^2}+\frac{\phi_1^2-2\phi_2+\phi_1}{(n-1)^3}+\dots\right),$$

whence on substituting

$$\hat{\rho} = r\left(1-\frac{1}{2}\frac{1-\hat{\rho}^2}{n-1}+\frac{1}{8}\frac{(1-r\hat{\rho})(1-\hat{\rho}^2)}{(n-1)^2}+\frac{1}{16}\frac{(1+r\hat{\rho}-2r^2\hat{\rho}^2)(1-\hat{\rho}^2)}{(n-1)^3}+\dots\right),$$

and after inversion

$$\hat{\rho} = r\left(1-\frac{1}{2}\frac{1-r^2}{n-1}+\frac{(1-5r^2)(1-r^2)}{8(n-1)^2}+\frac{(1+8r^2-17r^4)(1-r^2)}{16(n-1)^3}+\dots\right) \qquad \text{(lxviii)}.$$

This result gives us a measure of the correctness of (lxvii), for that equation may be written

$$\hat{\rho} = r\left(\frac{1-\dfrac{1}{n-1}}{1-\dfrac{r^2}{n-1}}\right)^{\frac{1}{2}}$$

$$= r\left\{1-\frac{(1-r^2)}{2(n-1)}-\frac{(1+3r^2)(1-r^2)}{8(n-1)^2}-\frac{(1+2r^2+5r^4)(1-r^2)}{16(n-1)^3}+\dots\right\} \text{(lxix)}.$$

Thus the divergence begins as early as the term in $1/(n-1)^2$ and (lxvii) can only be trusted for rough approximations to $\hat{\rho}$.

Illustration. Suppose $r = \cdot 6$, what is the "most likely" value of ρ, on the assumption of equal distribution of ignorance? Let $n = 25$, then we find from (lxviii)

$$\hat{\rho} = \cdot 59194,$$

while (lxvii) gives ·59182, an agreement adequate for most statistical purposes.

If $n = 5$, and $r = \cdot 6$, we find

$$\hat{\rho} = \cdot 55058 \text{ from (lxviii)},$$
$$= \cdot 56695 \text{ from (lxvii)}.$$

There is now considerable divergence in the two methods and another approximation is desirable. Let us take $\rho_0{}^2 = \hat{\rho} r = \cdot 33035$, then to find E_n we have

$$(n-1) \times \cdot 890{,}8689 E_n = (2n-3) \times \cdot 33035 + \frac{n-2}{E_{n-1}}$$

E_2 will be given by (xlvii) and equals ·885,5939, whence we determine

$$E_3 = 1\cdot 189{,}9819, \quad E_4 = 1\cdot 246{,}8946, \quad E_5 = 1\cdot 324{,}1082,$$

and accordingly from (lxv)

$$\epsilon = \cdot 001{,}3081,$$
$$r\hat{\rho} = \rho_0{}^2 + \epsilon = \cdot 331{,}8581,$$

and $\hat{\rho} = \cdot 553{,}097$, a value not far removed from that found by the first approximation. We conclude that even when n is small, quite good results will be found from (lxviii) and that it is probably better to use this rather than (lxvii) in such cases as the starting point for a second approximation.

Case (ii). *Close à priori Knowledge of* ρ.

We will now suppose m small, so that the first approximation to ρ may be taken as $\bar{\rho}$. We substitute in (lxiii) $\rho = \bar{\rho} + \psi$ and we find, neglecting squares of ψ,

$$\left(\frac{\psi}{m(n-1)} + \bar{\rho} + \psi\right)(\bar{I}_{n-1} + (n-1)\, r\psi \bar{I}_n) = (1 - \bar{\rho}^2 - 2\bar{\rho}\psi)(r\bar{I}_n + nr\psi \bar{I}_{n+1}),$$

where
$$\bar{I}_n = \int_0^\infty \frac{dz}{(\cosh z - \bar{\rho} r)^{n-1}}.$$

This leads us to

$$\psi = \frac{(1-\bar{\rho}^2)\, r\bar{I}_n - \bar{\rho}\bar{I}_{n-1}}{\bar{I}_{n-1}\left(1 + \dfrac{1}{m(n-1)}\right) + \bar{\rho}(n+1)\, r\bar{I}_n - (1-\bar{\rho}^2)\, nr^2 \bar{I}_{n+1}},$$

whence remembering that

$$n(1 - (r\bar{\rho})^2)\, \bar{I}_{n+1} = (2n-1)\, r\bar{\rho}\bar{I}_n + (n-1)\, \bar{I}_{n-1},$$

and writing $\bar{E}_n = \bar{I}_n/\bar{I}_{n-1}$, we have after some transformations

$$\begin{aligned}\hat{\rho} &= \bar{\rho} + \psi \\ &= \bar{\rho} + \frac{(1 - r^2\bar{\rho}^2)\,((1 - \bar{\rho}^2)\,r\bar{E}_n - \bar{\rho})}{(1 - r^2\bar{\rho}^2)\left(1 + \dfrac{1}{m\,(n-1)}\right) - (n-1)\,r^2\,(1 - \bar{\rho}^2) + r\bar{\rho}\,\{(n+1)(1 - r^2\bar{\rho}^2) - (2n-1)\,r^2\,(1 - \bar{\rho}^2)\}\,\bar{E}_n}\end{aligned} \quad \text{...........(lxx)},$$

where $$(n-1)\,(1 - r^2\bar{\rho}^2)\,\bar{E}_n = (rn - 3)\,r\bar{\rho} + \frac{n-2}{\bar{E}_{n-1}} \quad \text{..............(lxxi)}.$$

The method is now straightforward, at least for n moderately large. We put $\bar{E}_n = \bar{E}_{n-1} = \bar{E}'$ in (lxxi) and substitute the resulting value of $\bar{E}'$ for $\bar{E}_n$ in (lxx), and thus reach the small correction on $\bar{\rho}$.

Illustration. In a sample of 25 pairs only of parent and child the correlation for a certain character was found to be ·6. What is the most reasonable value to give to ρ in the sampled population?

If we distributed our ignorance equally the result would be that stated on p. 357, i.e.

$$\hat{\rho} = \cdot 59194.$$

But, in applying Bayes' Theorem to this case, to what result of experience do we appeal? Clearly the only result of experience by which we could justify this "equal distribution of ignorance" would be the accumulative experience that in past series the correlation of parent and child had taken with equal frequency of occurrence every value from -1 to $+1$. To appeal to such a result is absurd; Bayes' Theorem ought only to be used where we have in past experience, as for example in the case of probabilities and other statistical ratios, met with every admissible value with roughly equal frequency. There is no such experience in this case. On the contrary the mean value of ρ for very long series of frequencies of 1000 and upwards is known to be $+\cdot 46$ and the range is hardly more than ·40 to 52. We may accordingly take $\bar{\rho} = \cdot 46$ and $m\,(1 - \bar{\rho}^2) = \kappa^2 =$ about ·0004, whence

$$m = \cdot 0004/\cdot 7884 = \cdot 000{,}507 \text{ say.}$$

Thus $\dfrac{1}{m\,(n-1)} = 82\cdot 1828$ and the term containing it is the dominating term in Equation (lxiii). Thus $\hat{\rho}$ will differ little from $\bar{\rho}$. We find

$$\hat{\rho} = \bar{\rho} + \frac{\cdot 437{,}006\bar{E}_n - \cdot 424{,}959}{70\cdot 034{,}491 + 2\cdot 790{,}925\bar{E}_n}.$$

from (lxx).

We next determine $\bar{E}_n = \bar{E}'$ from (lxxi), i.e.

$$24 \times \cdot 923{,}824\bar{E}'^2 - 47 \times \cdot 276\bar{E}' - 23 = 0,$$

which gives us $\bar{E}' = 1\cdot 352{,}2185$, thus $\psi = \cdot 00225$ and

$$\hat{\rho} = \cdot 46225,$$

a totally different "most likely value" from that obtained by "equally distributing our ignorance."

Statistical workers cannot be too often reminded that there is no validity in a mathematical theory pure and simple. Bayes' Theorem must be based on experience, the experience that where we are *à priori* in ignorance all values are equally likely to occur. This is not the case in the present illustration, and we must use our past experience in the same way as we should use our past experience of equal frequency; the appeal to this experience has here absolutely the same validity as in Bayes' case and cannot be for a moment neglected. We see that our new experience scarcely modifies the old and this is what we should naturally conjecture would be the case. If we increase the size of the new sample, then ultimately $1/m(n-1)$ becomes very small, and we approach nearer the value ·59194 given by Bayes' Theorem. But past experience will bias the value obtained from the new material for a long time, and we see that according to the value of the past experience $\hat{\rho}$ may vary from ·46225 to ·59194. It will thus be evident that in problems like the present the indiscriminate use of Bayes' Theorem is to be deprecated. It has unfortunately been made into a fetish by certain purely mathematical writers on the theory of probability, who have not adequately appreciated the limits of Edgeworth's justification of the theorem by appeal to *general* experience.

Case (iii). *Past Experience a Factor, but not the Dominating Factor of Judgment.*

Cases can arise in which $\rho = \bar{\rho}$ is not a very close approximation, i.e. when we have some past experience, but not a very concentrated one of like correlations. In this case we must return to Equation (lxiii), and we shall assume $\hat{\rho}r = \rho_0^2 + \epsilon$, where ρ_0^2 is some fairly close approximation to $\hat{\rho}r$. We shall write $\bar{\rho}r = \bar{\rho}_0^2$. We find

$$\epsilon = \frac{(r^2-\rho_0^4)\, I_n - \left\{\rho_0^2 + \dfrac{(\rho_0^2-\bar{\rho}_0^2)(r^2-\rho_0^4)}{m(n-1)(r^2-\bar{\rho}_0^4)}\right\} I_{n-1}}{\left\{1 + \dfrac{r^2+2\rho_0^2\bar{\rho}_0^2-3\rho_0^4}{m(n-1)(r^2-\bar{\rho}_0^4)}\right\} I_{n-1} + \left\{(n+1)\rho_0^2 + \dfrac{(\rho_0^2-\bar{\rho}_0^2)(r^2-\rho_0^4)}{m(r^2-\bar{\rho}_0^4)}\right\} I_n - n(r^2-\rho_0^4)\, I_{n+1}} \quad \text{............(lxxii)},$$

where $$n(1-\rho_0^4)\, I_{n+1} = (2n-1)\,\rho_0^2 I_n + (n-1)\, I_{n-1} \quad \text{..............(lxxiii)},$$

equations which can be readily expressed in terms of E's.

Unfortunately the approximation obtained by equating the numerator of ϵ to zero and using (lxxiii) as simultaneous equations is not very rapidly obtained as the resulting equation is now of the eighth order. It is better from the data themselves to guess a reasonable value for ρ_0^2 and start the approximation from this.

Illustration. The correlation between the maximum length and breadth of crania is not very definitely known. Its mean is about ·30, but the values determined for it range from nearly zero to ·6. Assuming the standard deviation to be ·1, what is the "most likely value" to give to this correlation in the case of a sample of 25 skulls showing a correlation of ·50?

Here $\bar{\rho} = \cdot 30$, $m(1-\bar{\rho}^2) = \cdot 01$, and $n = 25$. $\bar{\rho}_0{}^2 = r\bar{\rho} = \cdot 15$. We will assume as a first approximation to $\hat{\rho}$, $\hat{\rho} = \cdot 40$, hence $\rho_0{}^2 = \cdot 20$. Equation (lxxiii) for $n = n - 1$ gives

$$(n-1) \times \cdot 96 E_n = (2n-3) \times \cdot 20 + (n-2)/E_{n-1}.$$

Put $n = 25$, and $E' = E_n = E_{n-1}$, and we have to find E',

$$23 \cdot 04 E'^2 - 9 \cdot 4 E' - 23 = 0,$$

which gives $E' = 1 \cdot 223{,}7367$; from this we deduce $E_{n+1} = 1 \cdot 225{,}5025$, and

$$\epsilon = - \cdot 029{,}9238,$$

leading to $r\hat{\rho} = \cdot 170{,}0762$, or $\hat{\rho} = \cdot 34015$.

Starting again with $\rho_0{}^2 = \cdot 17008$, we find

$$23 \cdot 305{,}7472 E'^2 - 7 \cdot 99376 E' - 23 = 0,$$

giving $\quad E' = 1 \cdot 179{,}6109 \quad$ and $\quad E_{n+1} = 1 \cdot 181{,}3579.$

Whence we deduce $\quad \epsilon = + \cdot 003{,}999,$

and accordingly $r\hat{\rho} = \cdot 174{,}079$ and $\hat{\rho} = \cdot 34816$, a close enough approximation.

But if we had "equally distributed our ignorance" we should have found*

$$\hat{\rho} = \cdot 49217.$$

These results seem extremely suggestive. If we were to observe the correlation of length and breadth of skull in a new sample of 25 skulls, then an observed value of ·50 would give a "most likely value" on the equal distribution of ignorance of ·4922.

But no biometrician would admit absolute ignorance in such a case; the correlation has been determined rather vaguely and not very adequately so that results range from something like zero to ·6. But this *à priori* knowledge leads on precisely the same basis as Bayes' Theorem to the value $\hat{\rho} = \cdot 3482$—a result very much closer to previous experience of the mean value, than to the observed result. And there are relatively few cases in which some such, if only vague, *à priori* experience does not exist.

In the light of the above illustrations we consider it justifiable to assert that the results deduced from the principle of the "equal distribution of ignorance" have academic rather than practical value, and we hold that to apply it without consideration of its basis to the problem of finding the most likely values of the statistical constants of a sampled population from the values observed in a small sample may lead to results very wide from the truth.

(9) *Special Cases of Frequency for n small.*

We shall now discuss individually the lowest sample sizes.

(i) *Samples of Two, n = 2.* Here

$$\bar{r} = \frac{\sin^{-1}\rho}{\frac{1}{2}\pi},$$

* Equation (lxvii) would give $\hat{\rho} = \cdot 49204$, nearly as good practically as (lxviii)

and the distribution consists of $\frac{\cos^{-1}(-\rho)}{\pi}$ at $r = +1$ and $\frac{\cos^{-1}\rho}{\pi}$ at $r = -1$*, or at what is the same thing $\frac{1}{2}(1 + \bar{r})$ and $\frac{1}{2}(1 - \bar{r})$.

The moments are

$$\mu_1' = \mu_3' = \bar{r}, \qquad \mu_2' = \mu_4' = 1,$$

and accordingly

$$\mu_2 = 1 - \bar{r}^2, \qquad \mu_3 = -2\bar{r}(1 - \bar{r}^2), \qquad \mu_4 = (1 - \bar{r}^2)(1 + 3\bar{r}^2) \ldots \text{(lxxiv)}.$$

Hence $\qquad \beta_1 = 4\bar{r}^2/(1 - \bar{r}^2), \qquad \beta_2 = (1 + 3\bar{r}^2)/(1 - \bar{r}^2),$

and accordingly

$$\beta_2 - \beta_1 - 1 = 0.$$

TABLE I. *Samples of Two.*

ρ, value of correlation in sampled population	$\bar{r}$, mean correlation of samples	σ_r	β_1	β_2	Number of positive correlations per 1000 samples	Number of negative correlations per 1000 samples
0·0	0	1	0	1	500·000	500·000
0·1	·063,7686	·997,965	·016,332	1·016,332	531·884	468·116
0·2	·128,1884	·991,750	·066,827	1·066,827	564·094	435·906
0·3	·193,9734	·981,007	·156,387	1·156,387	596·987	403·013
0·4	·261,9798	·965,007	·294,764	1·294,764	630·990	369·010
0·5	·333,3333	·942,809	·500,000	1·500,000	666·667	333·333
0·6	·409,6655	·912,236	·806,686	1·806,686	704·833	295·167
0·7	·493,6334	·869,670	1·288,724	2·288,724	746·817	253·183
0·8	·590,3345	·807,164	2·139,534	3·139,534	795·167	204·833
0·9	·712,8674	·701,299	4·133,056	5·133,056	856·434	143·566
1·0	1·000,0000	·000,000	∞	∞	1,000·000	·000,000

The distributions are two lumps given by the last two columns, and are accurately given by Pearson's skew frequency distributions for the relation $\beta_2 - \beta_1 - 1 = 0$ (see *Phil. Trans.* Vol. 216 A, p. 433).

(ii) *Samples of Three*, $n = 3$.

$$y_3 = N \frac{1 - \rho^2}{\pi} \frac{1}{\sqrt{1 - r^2}} \frac{dU}{dx},$$

where $\qquad x = r\rho$ and $U = \cos^{-1}(-x)/\sqrt{1 - x^2}$

as before. Hence

$$\mu_p' = \int_{-1}^{+1} r^p y_3 dr = \frac{1}{\rho^{p+1}} \int_{-\rho}^{+\rho} x^p y_3 dx$$

$$= \frac{\rho(1 - \rho^2)}{\pi \rho^{p+1}} \int_{-\rho}^{+\rho} \frac{x^p}{\sqrt{\rho^2 - x^2}} \frac{dU}{dx} dx.$$

* Since we can determine at sight whether any pair is positively or negatively correlated, this gives a method of determining ρ by simply counting the number of + 1 and − 1 correlations in the arrays, say m_p and m_n, then $\rho = \cos \pi \left(\frac{m_n}{m_n + m_p}\right)$ and the probable error of the determination is

$$\frac{\cdot 67449}{\sqrt{N}} \sqrt{1 - \rho^2} \sqrt{\frac{m_n}{N}\left(1 - \frac{m_n}{N}\right)},$$

N being the number of pairs used. Cf. "Student," *Biometrika*, Vol. VI. p. 304.

Now let $x = \rho \sin \phi$, then

$$\mu_p' = \frac{1-\rho^2}{\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin^p \phi \frac{dU}{dx} d\phi.$$

Now
$$\sin \phi \frac{dU}{dx} = \frac{dU}{d\rho},$$

hence
$$\mu_p' = \frac{1-\rho^2}{\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin^{p-1} \phi \frac{dU}{d\rho} d\phi$$

$$= \frac{1-\rho^2}{\pi} \frac{d}{d\rho} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin^{p-1} \phi U d\phi.$$

But
$$U = \int_0^\infty \frac{dz}{\cosh z - \rho \sin \phi},$$

thus
$$\mu_p' = \frac{1-\rho^2}{\pi} \frac{d}{d\rho} \int_0^\infty dz \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\sin^{p-1}\phi}{\cosh z - \rho \sin \phi} d\phi \ldots.$$

Write
$$\psi = \frac{\pi}{2} + \phi \text{ and } \cosh z = \eta,$$

we have
$$\mu_p' = \frac{1-\rho^2}{\pi} \frac{d}{d\rho} \int_0^\infty dz \int_0^\pi \frac{(-1)^{p-1} \cos^{p-1} \psi d\psi}{\eta + \rho \cos \psi} \quad \ldots\ldots\ldots\ldots\text{(lxxv)}.$$

Let $p = 1$, then we find

$$\mu_1' = \frac{2(1-\rho^2)}{\pi} \frac{d}{d\rho} \int_0^\infty dz \int_0^\pi \frac{d(\tan \frac{1}{2}\psi)}{(\eta+\rho) + (\eta-\rho) \tan^2 \frac{1}{2}\psi}.$$

But

$$\int_0^\pi \frac{d(\tan \frac{1}{2}\psi)}{(\eta+\rho) + (\eta-\rho)\tan^2 \frac{1}{2}\psi} = \left[\frac{1}{\sqrt{\eta^2-\rho^2}} \tan^{-1}\left(\frac{\tan \frac{1}{2}\psi}{\sqrt{\frac{\eta+\rho}{\eta-\rho}}}\right)\right]_0^\pi = \frac{\pi}{2} \frac{1}{\sqrt{\eta^2-\rho^2}}.$$

Thus
$$\mu_1' = (1-\rho^2) \frac{d}{d\rho} \int_0^\infty \frac{dz}{\sqrt{\eta^2-\rho^2}}.$$

Now take
$$\eta = \frac{1}{\sin \phi'} = \cosh z,$$

hence
$$-\frac{\cos \phi'}{\sin^2 \phi'} d\phi' = \sinh z dz,$$
$$= \sqrt{\eta^2 - 1}\, dz,$$

thus
$$dz = -\operatorname{cosec} \phi' d\phi'.$$

It follows that
$$\mu_1' = (1-\rho^2) \frac{d}{d\rho} \int_0^{\frac{\pi}{2}} \frac{d\phi'}{\sqrt{1-\rho^2 \sin^2 \phi'}}.$$

Let as usual $$F\left(k,\ \frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

and $$E\left(k,\ \frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\phi}\,d\phi.$$

Then we have $$\mu_1'=(1-\rho^2)\frac{dF\left(\rho,\ \frac{\pi}{2}\right)}{d\rho}.$$

But $$\frac{dF_1}{dk}=\frac{1}{kk'^2}(E_1-k'^2F_1),$$

where $$k'=\sqrt{1-k^2},$$

E_1 and F_1 denoting the complete elliptic integrals*.

Thus finally

$$\bar{r}=\frac{1}{\rho}\{E_1-(1-\rho^2)F_1\} \quad \text{..........(lxxvi)},$$

and $\bar{r}$ is known, as soon as ρ is given, from tables of the complete elliptic integrals.

Returning to Equation (xx) and putting $n=3$ we have

$$\begin{aligned}\mu_2'&=1-\frac{1}{2}(1-\rho^2)\left(1+\frac{\rho^2}{2}+\frac{\rho^4}{3}+\frac{\rho^6}{4}+\frac{\rho^8}{5}+\dots\right)\\&=1+\frac{1}{2}\frac{1-\rho^2}{\rho^2}\left\{-\rho^2-\frac{1}{2}(\rho^2)^2-\frac{1}{3}(\rho^2)^3-\frac{1}{4}(\rho^2)^4-\dots\right\}\\&=1+\frac{1}{2}\frac{1-\rho^2}{\rho^2}\log_e(1-\rho^2) \quad \text{..........(lxxvii)},\end{aligned}$$

and further

$$\sigma_r^2=\mu_2=1-\bar{r}^2+\frac{1}{2}\frac{1-\rho^2}{\rho^2}\log_e(1-\rho^2) \quad \text{..........(lxxviii)}.$$

We now turn to the third moment and may anticipate a recurrence of the elliptic integrals. We shall obtain our result on the whole most briefly by appealing to Equations (xxii) and (xxiii) on pp. 335 and 336. We have

$$\mu_3'=\chi_1-\chi_3=\frac{\rho(1-\rho^2)}{2}\left\{I_3\,.\,F\left(\frac{3}{2},\ \frac{3}{2},\ 2,\ \rho^2\right)-\frac{1}{3}I_5\,.\,F\left(\frac{3}{2},\ \frac{3}{2},\ 3,\ \rho^2\right)\right\},$$

or since $$I_3=\frac{\pi}{2},\quad I_5=\frac{3}{4}\frac{\pi}{2},$$

$$\begin{aligned}\mu_3'&=\frac{\rho(1-\rho^2)}{2}\frac{\pi}{2}\left\{F\left(\frac{3}{2},\ \frac{3}{2},\ 2,\ \rho^2\right)-\frac{1}{4}F\left(\frac{3}{2},\ \frac{3}{2},\ 3,\ \rho^2\right)\right\}\\&=\frac{\rho(1-\rho^2)}{2}\frac{\pi}{2}\left\{S\left[\frac{(3\,.\,5\dots 2s+1)^2}{2^{2s}(s)!\,(s+1)!}\rho^{2s}\right]-\frac{1}{2}S\left[\frac{(3\,.\,5\dots 2s+1)^2}{2^{2s}(s)!\,(s+2)!}\rho^{2s}\right]\right\}\\&=\frac{\rho(1-\rho^2)}{2}\frac{\pi}{2}\left\{S\left[\frac{(3\,.\,5\dots 2s+1)^2}{2^{2s}(s)!\,(s+1)!}\left(1-\frac{1}{2s+4}\right)\rho^{2s}\right]\right\}\end{aligned}$$

* Cayley, *Elementary Treatise on Elliptic Functions*, p. 48.

$$= \frac{\rho(1-\rho^2)}{2}\frac{\pi}{2}\left\{S\left[\frac{(3.5\ldots 2s+3)^2}{2.2^{2s}(s+2)!(s+2)!}\frac{(s+1)(s+2)}{2s+3}\rho^{2s}\right]\right\}$$

$$= \frac{(1-\rho^2)}{2\rho^2}\frac{\pi}{2}\frac{d}{d\rho}\left\{S\left[\frac{(3.5\ldots 2s+3)^2}{2^{2s}(s+2)!(s+2)!}\frac{s+1}{2s+3}\frac{\rho^{2s+4}}{2^2}\right]\right\}$$

$$= \frac{1-\rho^2}{\rho^2}\frac{d}{d\rho}\left\{\frac{\pi}{2}S\left[\frac{(3.5\ldots 2s+3)^2}{2^4.2^{2s}(s+2)!(s+2)!}\left(1-\frac{1}{2s+3}\right)\rho^{2s+4}\right]\right\}$$

$$= \frac{1-\rho^2}{\rho^2}\frac{d}{d\rho}\left\{\frac{\pi}{2}F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \rho^2\right)+\frac{\pi}{2}F\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ \rho^2\right)\right\}$$

$$= \frac{1-\rho^2}{\rho^2}\frac{d}{d\rho}\{F_1(\rho)+E_1(\rho)\}$$

$$= \frac{1-\rho^2}{\rho^2}\left\{\frac{E_1-(1-\rho^2)F_1}{\rho(1-\rho^2)}+\frac{E_1-F_1}{\rho}\right\},$$

or

$$\mu_3' = \frac{2-\rho^2}{\rho^3}E_1 - \frac{2(1-\rho^2)}{\rho^3}F_1 \quad\ldots\ldots\ldots\text{(lxxix)},$$

where E_1 and F_1 are as before the complete elliptic integrals.

In order to obtain the fourth moment coefficient about zero we will return to formulae (xx) and (xxi) of pp. 334–5 and write

$$\mu_2' = 1 - \frac{n-2}{n-1}(1-\rho^2)f_2$$

and
$$\mu_4' = 2\mu_2' - 1 + \frac{n(n-2)}{(n+1)(n-1)}(1-\rho^2)^2 f_4,$$

where f_2 and f_4 are the hypergeometrical series.

Now the general term of f_2 is

$$\frac{(2.4.6.8\ldots 2s+2)^2\rho^{2s+2}}{(s+1)!(n+1)(n+3)\ldots(n+2s+1)2^{s+1}},$$

and the general term of f_4 is

$$\frac{(4.6.8\ldots 2s+2)^2\rho^{2s}}{(s)!(n+3)(n+5)\ldots(n+2s+1)2^s}.$$

Hence it follows that
$$\frac{n+1}{4\rho}\frac{df_2}{d\rho} = f_4,$$

or we have

$$\mu_4' = 1 - \frac{2(n-2)}{n-1}(1-\rho^2)f_2 + \frac{n(n-2)}{4(n-1)}(1-\rho^2)^2\frac{df_2}{\rho d\rho} \quad\ldots\ldots\text{(lxxx)}.$$

Thus if we are able to sum f_2 algebraically, we can determine μ_4' algebraically*

* The corresponding formula for μ_3' and $\mu_1' = \bar{r}$ is

$$\mu_3' = \bar{r} - (1-\rho^2)(n-2)\frac{d}{d\rho}\left(\frac{\bar{r}}{\rho}\right)\ldots\ldots\ldots\text{(lxxxi)}.$$

If we put $n = 3$ and $\bar{r} = \frac{1}{\rho}(E_1 - (1-\rho^2)F_1)$ we find

$$\mu_3' = \frac{2-\rho^2}{\rho^3}E_1 - \frac{2(1-\rho^2)}{\rho^3}F_1$$

confirming the result in Eqn. (lxxix).

Putting $n=3$ and writing $f_2 = -\frac{1}{\rho^2}\log_e(1-\rho^2)$, we have

$$\mu_4' = 1 + \frac{1-\rho^2}{\rho^2}\log_e(1-\rho^2) - \frac{3}{8}\frac{(1-\rho^2)^2}{\rho}\frac{d}{d\rho}\left\{\frac{\log_e(1-\rho^2)}{\rho^2}\right\}$$

$$= 1 + \frac{1-\rho^2}{4\rho^4}\{3\rho^2 + (3+\rho^2)\log_e(1-\rho^2)\} \quad \text{.....................(lxxxii).}$$

This completes the moment coefficients for samples of three.

As ρ may be determined by considering the ratio of negative to positive correlations in samples of two, so it may be determined by considering the ratio of positive to negative correlations in samples of three. Let m_p be the number of positive and m_n the number of negative correlations, then since

$$y_3 = N\frac{1-\rho^2}{\pi}\frac{1}{\sqrt{1-r^2}}\frac{1}{r}\frac{dU}{d\rho},$$

$$m_p = \frac{1-\rho^2}{\pi}\frac{d}{d\rho}\int_0^{+1}\frac{U dr}{r\sqrt{1-r^2}}$$

$$= \frac{1-\rho^2}{\pi}\frac{d}{d\rho}\int_0^{+1}\frac{\cos^{-1}(-\rho r)\,dr}{r\sqrt{(1-r^2)(1-r^2\rho^2)}}$$

and

$$m_n = \frac{1-\rho^2}{\pi}\frac{d}{d\rho}\int_{-1}^{0}\frac{\cos^{-1}(-\rho r)\,dr}{r\sqrt{(1-r^2)(1-r^2\rho^2)}}.$$

Now put in the latter integral $r = -r$, then

$$m_n = \frac{1-\rho^2}{\pi}\frac{d}{d\rho}\int_0^{+1}\frac{-\cos^{-1}(\rho r)\,dr}{r\sqrt{(1-r^2)(1-r^2\rho^2)}}.$$

But $\cos^{-1}(-\rho r) + \cos^{-1}(\rho r) = \pi$, hence

$$\frac{m_p - m_n}{m_p + m_n} = (1-\rho^2)\frac{d}{d\rho}\int_0^{+1}\frac{dr}{r\sqrt{(1-r^2)(1-r^2\rho^2)}}$$

$$= (1-\rho^2)\int_0^1\frac{r^2\rho\,dr}{\sqrt{1-r^2}\,(1-r^2\rho^2)^{\frac{3}{2}}},$$

or if $r = \sin\phi$,

$$= (1-\rho^2)\int_0^{\frac{\pi}{2}}\frac{\rho\sin\phi\,d\phi}{(1-\rho^2\sin^2\phi)^{\frac{3}{2}}}$$

$$= -(1-\rho^2)\int_0^{\frac{\pi}{2}}\frac{d(\rho\cos\phi)}{(1-\rho^2+\rho^2\cos^2\phi)^{\frac{3}{2}}}$$

Or again if $\rho\cos\phi = \sqrt{1-\rho^2}\tan\theta$,

$$\frac{m_p - m_n}{m_p + m_n} = +(1-\rho^2)\int_0^{\sin^{-1}\rho}\frac{\sqrt{1-\rho^2}\cos^3\theta}{(1-\rho^2)^{\frac{3}{2}}}\frac{d\theta}{\cos^2\theta}$$

$$= \int_0^{\sin^{-1}\rho}\cos\theta\,d\theta = \rho,$$

or
$$\rho = (m_p - m_n)/(m_p + m_n) \quad \text{(lxxxiii)},$$
a very simple formula if m_p or m_n has been found.

Clearly
$$\rho = (2m_p - N)\, N,$$
and
$$\delta\rho = 2\delta m_p/N,$$
$$\sigma_\rho{}^2 = 4\sigma^2{}_{m_p} = 4\,\frac{m_p}{N^2}\left(1 - \frac{m_p}{N}\right)$$

Thus the probable error of ρ found in this manner is
$$\frac{\cdot 67449}{\sqrt{N}}\, 2\sqrt{\frac{m_p}{N}\left(1 - \frac{m_p}{N}\right)} \quad \text{(lxxxiv)},$$
and can easily be evaluated, for it gives:
$$\text{Probable Error of } \rho = \frac{\cdot 67449}{\sqrt{N}}\sqrt{1 - \rho^2} \quad \text{(lxxxv)}.$$

We see it is larger by the factor $\frac{1}{\sqrt{1 - \rho^2}}$, which is greater than unity, than the usual value for the product moment process of the correlation. But a new point arises: N is the number of triplets in the present process, and N the number of individuals in the product moment process. If we take M triplets and N individuals, we have to compare
$$\cdot 67449\sqrt{1 - \rho^2}/\sqrt{M} \quad \text{with} \quad \cdot 67449\,(1 - \rho^2)/\sqrt{N},$$
and these probable errors will be equal if
$$M = N/(1 - \rho^2).$$

If the number of triplets be $> N/(1 - \rho^2)$ the triplet process will be more accurate than the product moment method. The number of triplets required for equality of probable errors are for the various values of ρ:

$\rho = 0$	$M = N,$	$\rho = \cdot 5$	$M = 1{\cdot}333N,$
$\rho = \cdot 1$	$M = 1{\cdot}010N,$	$\rho = \cdot 6$	$M = 1{\cdot}563N,$
$\rho = \cdot 2$	$M = 1{\cdot}042N,$	$\rho = \cdot 7$	$M = 1{\cdot}923N,$
$\rho = \cdot 3$	$M = 1{\cdot}099N,$	$\rho = \cdot 8$	$M = 2{\cdot}778N,$
$\rho = \cdot 4$	$M = 1{\cdot}190N,$	$\rho = \cdot 9$	$M = 5{\cdot}263N.$

This series would seem to suggest, since a triplet contains three individuals, that to use the triplet process with equal exactness with the product moment process, in the case, say, of $\rho = \cdot 5$, we should need a population of $4N$. But this assumes that each triplet is based upon three independent individuals. Actually a population of N provides $\frac{1}{6}N(N-1)(N-2)$ triplets and if these could be considered as an *independent* sample of M triplets, we should have a less value of the probable error of ρ by the triplet process using all possible sets than by the product moment process, provided
$$\rho^2 < 1 - \frac{6}{(N-1)(N-2)}.$$

For example if $N = 10$, for all values of ρ between $+ \cdot 957$ and $- \cdot 957$, the 120 triplets will give a better result than the 10 individuals. Even 50 triplets would be better than 10 individuals for all values of ρ between $+ \cdot 894$ and $- \cdot 894$. But the question arises whether we can consider the 120 triplets from a sample of 10 individuals as much a random sample as 120 triplets from an indefinitely large population, and this can hardly be the case. It may be, however, that 50 triplets out of the 120 would be sufficiently independent to give a better result than 10 individuals. It is very desirable that a full study should be made of such restricted sampling, for without such study it is not possible to assert how far the probable errors of doublet or triplet procedure are greater than those of the product moment method.

Of course in such a case as that referred to, the labour of the triplet process will be considerably greater, for we have to determine the *sign* of the correlation in 50 or 120 cases, instead of applying the product moment process to 10 individuals, and the labour rapidly increases with increase in the size of the set (doublet, triplet, etc.) and the size of the sample. Still the labour may be worth while in the case of small populations, where the best result is of considerable importance. We have not endeavoured to extend the theory to quadruplets or quintettes, because the labour of determining the sign of the correlation in these cases is very considerable.

In the case of triplets, we require the sign of the product moment

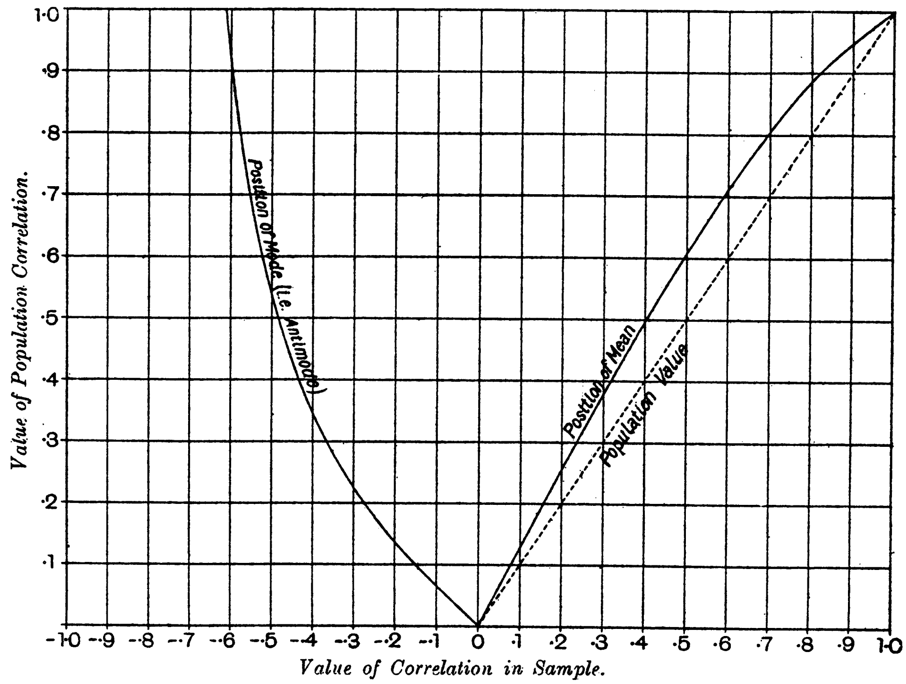
$$\frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{3} - \frac{(x_1 + x_2 + x_3)(y_1 + y_2 + y_3)}{9},$$

or the sign of $$(x_2 - x_1)(y_2 - \bar{y}) + (x_3 - x_1)(y_3 - \bar{y}).$$

Now suppose the triplet arranged according to the character x in ascending order x_1, x_2, x_3, then $x_2 - x_1$ and $x_3 - x_1$ will always be positive, and accordingly if either both $y_2 - \bar{y}$ and $y_3 - \bar{y}$ are positive, or both negative the sign of the correlation is obvious. On the other hand if $y_2 - \bar{y}$ and $y_3 - \bar{y}$ are of *opposite* sign, the matter has got to be a little more carefully considered. But if $\bar{y}$ has been found and the above differences determined, in most cases it is not needful to actually multiply out, in order to realise the sign. A graphic process depending on the plotting of the triplet triangle seemed on the whole more laborious than the above.

Of course in samples of three the U-shaped distributions give a minimum where $dy/dr = 0$, and we have therefore an *antimode*, not a mode. The values of this antimode are recorded in Table II, p. 368. It will be seen that all the antimodes are negative for positive correlations in the sampled population. The antimode asymptotes to the value $- \cdot 613{,}9616$, which it reaches when $\rho = + 1$. In this case the value actually fails at $\rho = + 1$, for the equation $dy/dr = 0$ is satisfied then for all values of r, owing to the presence of the factor $(1 - \rho^2)^{\frac{n-1}{2}}$: see Equation (xxxvii). Nevertheless the antimode curve goes right up to the point indicated above and this value must be used for the purpose of interpolation

FIG. 2. Mode (Antimode) and Mean Curves for $n = 3$, and for values of ρ from 0 to +1.



between $\rho = {\cdot}95$ and $\rho = 1{\cdot}00$. The curve is shown in Fig. 2, and both antimodal and mean lines on the photograph of the model of the frequency surface.

The distributions are after $\rho = {\cdot}2$ very skew U-shaped frequency curves, whose β_1, β_2 lie in the U-area of Pearson's skew curves, which, however, do not reproduce the antimode very closely. The ordinates are given in Table A (p. 379).

TABLE II. *Samples of Three.*

ρ, value of correlation in sampled population	$\bar{r}$, mean correlation of samples	$\check{r}$, antimode of samples	σ_r	β_1	β_2	Number of positive correlations per 1000 samples	Number of negative correlations per 1000 samples
0·0	·0000,0000	·000,0000	·707,1068	·000,0000	1·500,0000	500	500
0·1	·0786,3836	−·151,2541	·704,5029	·028,0136	1·532,3082	550	450
0·2	·1578,7706	−·275,5141	·696,5708	·115,2406	1·632,9424	600	400
0·3	·2383,6407	−·367,9037	·682,9273	·272,2374	1·814,1763	650	350
0·4	·3208,5431	−·435,4082	·662,8536	·520,5707	2·101,1480	700	300
0·5	·4062,9889	−·485,5321	·635,1363	·901,7817	2·542,3242	750	250
0·6	·4960,0160	−·523,8811	·597,7313	1·500,1840	3·236,2938	800	200
0·7	·5919,3885	−·553,8751	·546,9866	2·510,5375	4·411,4204	850	150
0·8	·6975,5118	−·577,9216	·475,4818	4·503,0667	6·738,8238	900	100
0·9	·8204,3635	−·597,5916	·363,4654	10·222,6204	13·467,2160	950	50
0·95	·8742,5455	−·606,1358	·268,4676	21·078,0376	26·340,9890	975	25
1·00	1·0000,0000	−·613,9616	·000,0000	∞	∞	1,000	0

(iii) *Samples of Four*, $n = 4$.

Here, if $x = r\rho$,

$$y_4 = \frac{N(1-\rho^2)^{\frac{3}{2}}}{\pi} \frac{d^2}{dx^2}\left(\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}}\right);$$

and if
$$U = \cos^{-1}(-x)/\sqrt{1-x^2},$$

then
$$\mu_q' = \frac{(1-\rho^2)^{\frac{3}{2}}}{\pi\rho^{q+1}}\left\{\left[x^q \frac{dU}{dx} - qx^{q-1}U\right]_{-\rho}^{+\rho} + q(q-1)\int_{-\rho}^{+\rho} x^{q-2}U\,dx\right\}.$$

To find the first four moments about $r = 0$, we have to determine

$$\int_{-\rho}^{+\rho} x^2 U dx, \qquad \int_{-\rho}^{+\rho} xU\,dx, \qquad \int_{-\rho}^{+\rho} U\,dx,$$

$$x^4\frac{dU}{dx} - 4x^3U, \quad x^3\frac{dU}{dx} - 3x^2U, \quad x^2\frac{dU}{dx} - 2xU \quad \text{and} \quad x\frac{dU}{dx} - U.$$

The results are most briefly expressed by using Fisher's notation $\rho = \sin\alpha$ and remembering that

$$\cos^{-1}(-\rho) + \cos^{-1}(\rho) = \pi, \quad \text{and} \quad \alpha = \cos^{-1}(-\rho) - \frac{\pi}{2}.$$

For the purposes of integration we put $x = -\cos\theta$ and integrate in terms of θ. We find

$$\int_{-\rho}^{+\rho} x^2 U\,dx = \tfrac{1}{2}\pi\{\alpha - \cos\alpha\sin\alpha\},$$

$$\int_{-\rho}^{+\rho} xU\,dx = 2\{\sin\alpha - \alpha\cos\alpha\},$$

$$\int_{-\rho}^{+\rho} U\,dx = \pi\alpha,$$

$$x^4\frac{dU}{dx} - 4x^3U = \pi\left\{\frac{\sin^5\alpha}{\cos^3\alpha} - 4\frac{\sin^3\alpha}{\cos\alpha}\right\},$$

$$x^3\frac{dU}{dx} - 3x^2U = 2\left\{\frac{\sin^3\alpha}{\cos^2\alpha} + \alpha\frac{\sin^4\alpha}{\cos^3\alpha} - 3\alpha\frac{\sin^2\alpha}{\cos\alpha}\right\},$$

$$x^2\frac{dU}{dx} - 2xU = \pi\left\{\frac{\sin^3\alpha}{\cos^3\alpha} - 2\frac{\sin\alpha}{\cos\alpha}\right\},$$

$$x\frac{dU}{dx} - U = 2\left\{\frac{\sin\alpha}{\cos^2\alpha} + \alpha\frac{\sin^2\alpha}{\cos^3\alpha} - \alpha\frac{1}{\cos\alpha}\right\}.$$

Hence substituting in the several values of μ_q', we obtain*

$$\bar{r} = \mu_1' = \frac{2}{\pi}\{\cot\alpha + \alpha(1 - \cot^2\alpha)\} \quad \text{..................(lxxxvi)},$$

* Tested by formulae (lxxx) and (lxxxi), p. 364, which can be put in the forms

$$\mu_3' = \mu_1'\{(n-2)/\rho^2 - (n-3)\} - \frac{(n-2)(1-\rho^2)}{\rho}\frac{d\mu_1'}{d\rho} \quad \text{.....................(xc)},$$

and
$$\mu_4' = \tfrac{1}{2}(n-2) - \tfrac{1}{2}(n-4)\mu_2' - \tfrac{1}{4}n\frac{1-\rho^2}{\rho}\frac{d\mu_2'}{d\rho} \quad \text{.............................(xci)}.$$

$$\sigma_r^2 + \bar{r}^2 = \mu_2' = 1 - 2\cot^2\alpha + 2\alpha\cot^3\alpha \quad \text{.........................(lxxxvii)},$$

$$\mu_3' = \frac{2}{\pi}\{\cot\alpha + 6\cot^3\alpha + \alpha(1 - 3\cot^2\alpha - 6\cot^4\alpha)\} \quad \text{.........(lxxxviii)},$$

$$\mu_4' = 1 - 4\cot^2\alpha - 6\cot^4\alpha + \alpha(6\cot^3\alpha + 6\cot^5\alpha)^* \quad \text{.........(lxxxix)}.$$

From these formulae were calculated the values given in the following table.

TABLE III. *Samples of Four.*

ρ, value of correlation of sampled population	$\bar{r}$, mean value of correlation in samples	μ_3, 3rd moment coefficient	σ_r	Usual value assumed for σ_r, i.e. $\frac{1-\rho^2}{\sqrt{n-1}}$	β_1	β_2	β_2/β_1
0·0	0	0	·577,3503	·577,3503	0	1·800,000	∞
0·1	·084,9678	−·033,6268	·574,5653	·571,5768	·031,429	1·839,929	58·5418
0·2	·170,4532	−·065,2863	·566,0965	·554,2563	·129,510	1·964,665	15·1700
0·3	·257,0089	−·092,9620	·551,5835	·525,3887	·306,862	2·190,708	7·1391
0·4	·345,2652	−·114,5383	·530,3576	·484,9742	·589,510	2·552,205	4·3294
0·5	·435,9911	−·127,7520	·501,3081	·433,0127	1·028,270	3·116,256	3·0306
0·6	·530,1976	−·130,1567	·462,6087	·369,5042	1·728,423	4·022,982	2·3275
0·7	·629,3378	−·119,1407	·411,1087	·294,4486	2·940,226	5·609,288	1·9078
0·8	·735,7362	−·092,1708	·340,7311	·207,8461	5·428,946	8·922,221	1·6435
0·9	·853,9806	−·048,1281	·236,6586	·109,6966	13·184,043	19·571,006	1·4844
0·95	·920,8889	−·021,4678	·157,7942	·056,2917	29·8558	43·4082	1·4539
0·98	·965,7599	−·006,4363	·088,2666	·022,8631	87·5994	130·1935	1·4862
0·99	·982,1321	−·002,4358	·055,4859	·011,4893	203·3250	311·7316	1·5332
1·00	1	0	0	0	∞	∞	1·8305†

* These expressions cannot be applied to the case of $\rho = 0$. We must return to Equation (ix) and put $\rho = 0$, finding $y_4 = \frac{1}{2}N$ or a horizontal line.

† The ratio β_2/β_1 equals in the limit $(27\pi^2 - 256)(3\pi^2 - 16)/(9\pi^2 - 80)^2 = 1{\cdot}8305$. This may be shown by putting $\alpha = \frac{1}{2}\pi - \epsilon$, or $\rho = 1 - \frac{1}{2}\epsilon^2$, or $\epsilon = \sqrt{1-\rho^2}$, where ϵ is small. We then find for values of ρ near unity

$$\bar{r} = \rho^2 - \frac{8}{3\pi}(1-\rho^2)^{\frac{3}{2}}, \qquad \mu_2 = \left(\pi - \frac{16}{3\pi}\right)(1-\rho^2)^{\frac{3}{2}},$$

$$\mu_3 = -\left(3\pi - \frac{80}{3\pi}\right)(1-\rho^2)^{\frac{3}{2}}, \quad \mu_4 = \left(9\pi - \frac{256}{3\pi}\right)(1-\rho^2)^{\frac{3}{2}}.$$

These lead to

$$\beta_1 = \frac{\left(3\pi - \frac{80}{3\pi}\right)^2}{\left(\pi - \frac{16}{3\pi}\right)^3(1-\rho^2)^{\frac{3}{2}}}, \quad \beta_2 = \frac{\left(9\pi - \frac{256}{3\pi}\right)}{\left(\pi - \frac{16}{3\pi}\right)^2(1-\rho^2)^{\frac{3}{2}}},$$

and give the above result.

(iv) *General Case of small Samples*, $n > 4$.

Equations (xxviii), (xxxi), (xxxii), and (xxxiv) enable us to express the moment-coefficients of a sample of $n + 2$ in terms of those of a sample of n. But we have found algebraic expressions for the moment-coefficients for $n = 3$ and $n = 4$ in terms of (a) the complete elliptic integrals and logarithmic functions, (b) trigonometrical functions of α and $\cot \alpha$. Hence all even samples can have their moment-coefficients expressed in terms of α and $\cot \alpha$, and all odd samples can have their odd moment-coefficients expressed in terms of the complete elliptic integrals and their even moments in terms of logarithmic functions. The former result has been already noticed by Fisher*. The arithmetical calculation of the successive moment-coefficients after $n = 4$ by the difference formulae is, however, shorter than obtaining the algebraical expressions and then substituting arithmetical values, and has been followed in our calculations.

(10) *Approach of the Distribution as n increases to a Normal Character.*

It is well known that for the "probable error" to have meaning the distribution must approach the Gaussian for which $\beta_1 = 0$, $\beta_2 = 3$. It is clear that these conditions are by no means fulfilled for samples of 25 or 50, whatever be the value of ρ. There is nearer approach in the *low* values of ρ in samples of 100, but there is considerable deviation for $\rho = 5$ and upwards.

TABLE IV. *Values of the Frequency Constants for the Correlation in Samples of* 25.

ρ	$\bar{r}$ mean	Actual $\check{r}$† mode	$\check{r}$ from Pearson's formula‡	Actual σ	$\frac{1-\rho^2}{\sqrt{n-1}}$	β_1	β_2
0·0	0	0	0	·2041,241	·2041,241	0	2·769,2305
0·1	·0979,577	·11173	·11127	·2022,954	·2020,829	·012,3106	2·791,6002
0·2	·1960,288	·22258	·22177	·1967,883	·1959,592	·049,8655	2·860,0511
0·3	·2943,287	·33172	·33090	·1875,386	·1857,530	·114,6242	2·978,8302
0·4	·3929,765	·43840	·43758	·1744,356	·1714,643	·210,1771	3·155,8537
0·5	·4920,974	·54197	·54149	·1573,152	·1530,931	·342,3386	3·404,2283
0·6	·5918,251	·64194	·64190	·1359,499	·1306,395	·520,2635	3·745,3432
0·7	·6923,054	·73792	·73826	·1100,322	·1041,033	·758,5549	4·214,8982
0·8	·7937,001	·82966	·83025	·0791,481	·0734,847	1·081,1286	4·869,2635
0·9	·8961,933	·91703	·91736	·0427,345	·0387,836	1·533,4124	5·858,3872
1·0	1	1	1	0	0	∞	∞

It will be realised that while the ordinary value for the standard deviation of r and the distribution of r by a normal curve is fairly close for samples of 400, there is still a quite sensible deviation from normality in the case of $\rho = \cdot 8$ or over. In fact it may be said that for the size of ordinary samples, there is always a sensible

* Fisher, *Biometrika*, Vol. x. p. 516.

† See Sections (4) and (5) above.

‡ $\check{r} = \bar{r} + \mu_3(\beta_2 + 3)/\{\mu_2(10\beta_2 - 12\beta_1 - 18)\}$.

TABLE V. *Values of the Frequency Constants for the Correlation in Samples of* 50.

ρ	$\bar{r}$ mean	Actual r mode	$\check{r}$ from Pearson's formula*	Actual σ	$\frac{1-\rho^2}{\sqrt{n-1}}$	β_1	β_2
0·0	0	0	0	·142,857	·142,857	0	2·88236
0·1	·098,995	·1054	·1053	·141,505	·141,429	·00666	2·89184
0·2	·198,047	·2104	·2102	·137,439	·137,143	·02683	2·93350
0·3	·297,218	·3147	·3144	·130,634	·130,000	·06107	2·99909
0·4	·396,565	·4180	·4177	·121,049	·120,000	·11041	3·09417
0·5	·496,150	·5198	·5196	·108,620	·107,143	·17635	3·22240
0·6	·596,038	·6201	·6199	·093,260	·091,429	·26110	3·38912
0·7	·696,295	·7184	·7183	·074,878	·072,857	·36774	3·60222
0·8	·796,989	·8146	·8146	·053,324	·051,429	·50037	3·87312
0·9	·898,198	·9085	·9085	·028,434	·027,143	·66608	4·22186
1·0	1	1	1	0	0	∞	∞

TABLE VI. *Values of the Frequency Constants for the Correlation in Samples of* 100.

ρ	$\bar{r}$ mean	Actual $\check{r}$ mode	$\check{r}$ from Pearson's formula*	Actual σ	$\frac{1-\rho^2}{\sqrt{n-1}}$	β_1	β_2
0·0	0	0	0	·100,5038	·100,5038	0	2·94060
0·1	·099,5016	·10258	·10255	·099,5260	·099,4987	·00346	2·94736
0·2	·199,0319	·20499	·20494	·096,5887	·096,4836	·01390	2·96774
0·3	·298,6219	·30708	·30701	·091,6832	·091,4584	·03147	3·00213
0·4	·398,3013	·40868	·40860	·084,7934	·084,4232	·05644	3·05118
0·5	·498,1002	·50964	·50957	·075,8968	·075,3778	·08919	3·11583
0·6	·598,0498	·60982	·60976	·064,9640	·064,3224	·13025	3·19739
0·7	·698,1815	·70907	·70903	·051,9577	·051,2569	·18031	3·29767
0·8	·798,5279	·80726	·80724	·036,8329	·036,1814	·24027	3·41896
0·9	·899,1225	·90427	·90423	·019,5352	·019,0957	·31148	3·57898
1·0	1	1	1	0	0	∞	∞

TABLE VII. *Values of the Frequency Constants for the Correlation in Samples of* 400.

ρ	$\bar{r}$ mean	Actual $\check{r}$ mode	$\check{r}$ from Pearson's formula*	Actual σ	$\frac{1-\rho^2}{\sqrt{n-1}}$	β_1	β_2
0·0	0	0	0	·0500,626	·0500,626	0	2·9850
0·1	·0998,760	·1006,250	·1006,232	·0495,654	·0495,620	·00089	2·9868
0·2	·1997,595	·2012,116	·2012,082	·0480,733	·0480,601	·00357	2·9921
0·3	·2996,579	·3017,217	·3017,171	·0455,851	·0455,570	·00804	3·0010
0·4	·3995,788	·4021,171	·4021,115	·0420,988	·0420,526	·01433	3·0138
0·5	·4995,297	·5023,602	·5023,584	·0376,115	·0375,470	·02250	3·0297
0·6	·5995,181	·6024,134	·6024,089	·0321,195	·0320,401	·03245	3·0498
0·7	·6995,517	·7022,401	·7022,386	·0256,183	·0255,319	·04435	3·0725
0·8	·7996,380	·8018,037	·8018,013	·0181,023	·0180,255	·05820	3·1017
0·9	·8997,849	·9010,725	·9010,668	·0095,653	·0095,119	·07402	3·1342
1·0	1	1	1	0	0	∞	∞

* $\check{r} = \bar{r} + \mu_3(\beta_2 + 3)/\{\mu_2(10\beta_2 - 12\beta_1 - 18)\}$.

deviation from normality for high values of ρ in the sampled population, and the usual "probable error of r" must be treated with caution in these cases. Pearson's curves would give better results, as is indicated by the agreement of $\check{r}$ to five figures.

(11) *Table for determining the Mode $\check{r}$ of the Frequency Distribution of considerable Size n when the Correlation in the sampled Population is known to be ρ.*

The required value of the mode is

$$\check{r} = \rho + \frac{\nu_1(\rho)}{n-1} + \frac{\nu_2(\rho)}{(n-1)^2} + \frac{\nu_3(\rho)}{(n-1)^3} + \frac{\nu_4(\rho)}{(n-1)^4},$$

when ρ is positive; if ρ be negative $\check{r}$ has the same value as for ρ positive, but with opposite sign.

TABLE VIII. *Functions required in determining the Mode of a large or fairly large Sample.*

ρ	$\nu_1(\rho)$	$\nu_2(\rho)$	$\nu_3(\rho)$	$\nu_4(\rho)$
·00	0	0	0	0
·05	·12468,75000	+ ·37872,26953	+1·13249,90199	+3·36913,85919
·10	·24750,00000	+ ·74237,62500	+2·18002,51688	+6·33154,83878
·15	·36656,25000	+1·07636,49609	+3·06395,34936	+8·53428,44284
·20	·48000,00000	+1·36704,00000	+3·71804,16000	+9·72192,80640
·25	·58593,75000	+1·60217,28516	+4·09383,77380	+9·77504,47690
·30	·68250,00000	+1·77142,87500	+4·16514,90563	+8·72821,01391
·35	·76781,25000	+1·86684,01172	+3·93125,67461	+6·77045,11170
·40	·84000,00000	+1·88328,00000	+3·41856,48000	+4·22414,17680
·45	·89718,75000	+1·81893,55078	+2·68036,91046	+1·50278,32653
·50	·93750,00000	+1·67578,12500	+1·79443,35938	− ·94981,38428
·55	·95906,25000	+1·46005,27734	+ ·85806,01815	−2·73364,05345
·60	·96000,00000	+1·18272,00000	− ·01966,08000	−3·57140,42880
·65	·93843,75000	+ ·85996,06641	− ·72873,29366	−3·38017,64160
·70	·89250,00000	+ ·51363,37500	−1·17258,21188	−2·31971,93665
·75	·82031,25000	+ ·17175,29297	−1·28614,42566	− ·79706,28440
·80	·72000,00000	− ·13104,00000	−1·05802,56000	+ ·59787,92160
·85	·58968,75000	− ·35300,16797	− ·55704,89418	+1·25603,36730
·90	·42750,00000	− ·44481,37500	+ ·03650,10188	+ ·84819,10191
·95	·23156,25000	− ·34910,94141	+ ·39461,98756	− ·19874,71356
1·00	0	0	0	0

The above Table will give the value of $\check{r}$ correctly to about the sixth figure if $n = 100$ or more, to about the fourth figure if $n = 25$ or more. Below 25 it can only serve as a "taking off point" for more accurate approximations, and these are fairly troublesome if n be very low. It will be found best to interpolate for the expression to be added to ρ.

Illustration. To find the modal $\check{r}$ for samples of 9 when $\rho = ·2852$.

$$\check{r} = \rho + ·106,272, \text{ for } \rho = ·25,$$

$$\check{r} = \rho + ·121,126, \text{ for } \rho = ·30.$$

Hence
$$\check{r} = \rho + \cdot 116{,}729, \text{ for } \rho = \cdot 2852$$
$$= \cdot 4019, \text{ say.}$$

We cannot, however, be certain that this is correct to more than two figures.

Equation (xlvi) gives us $\check{r} = \cdot 4038$.

We will therefore start with $\check{r} = \cdot 4030$, say, as the basis of a more elaborate approximation, or $\rho_0{}^2 = \cdot 1149372$ say.

Hence calculating I_1 and I_2 and using the difference formula we find

$$I_1 = 1\cdot 6972{,}3599, \qquad I_6 = 1\cdot 0465{,}6745,$$
$$I_2 = 1\cdot 2110{,}7453, \qquad I_7 = 1\cdot 0879{,}1327,$$
$$I_3 = 1\cdot 0715{,}7031, \qquad I_8 = 1\cdot 1443{,}9624,$$
$$I_4 = 1\cdot 0262{,}1201, \qquad I_9 = 1\cdot 2145{,}9528,$$
$$I_5 = 1\cdot 0236{,}1262, \qquad I_{10} = 1\cdot 2980{,}8251.$$

Thus the equality of I_9 and I_{10} has not been reached, so that we could hardly anticipate (xlvi) giving a very good result. Using (xli) we find

$$\epsilon = + \cdot 0006{,}0228,$$

a sufficiently small correction, leading to $\rho_0{}^2 = \cdot 1155{,}3948$, and $\check{r} = \cdot 40512$, correct to the fourth figure. Table VIII for $n = 9$ gives $\check{r}$ in error by about 0·8 %.

(12) *Table for determining the "most probable" value $\hat{\rho}$ of the correlation in a sampled population from the knowledge of the correlation r in a sample of size n, when n is considerable and it is legitimate to distribute ignorance equally.*

The required value is

$$\hat{\rho} = r - \frac{\lambda_1(r)}{n-1} - \frac{\lambda_2(r)}{(n-1)^2} - \frac{\lambda_3(r)}{(n-1)^3},$$

when r is positive; if r be negative, $\hat{\rho}$ has the same value as for r positive, but with opposite sign.

The above formula using Table IX will give $\hat{\rho}$ correct to five figures if $n = 25$ or over, and correct to four figures if $n = 10$ or over.

It appears best to interpolate not for the separate λ-functions, but for the total value to be subtracted from r to find $\hat{\rho}$. Thus, suppose we require to find $\hat{\rho}$ for $r = \cdot 6781$ and for $n = 16$. We have for $r = \cdot 65$

$$\hat{\rho} = \cdot 65 - \cdot 0127{,}3515,$$

and for $r = \cdot 70$
$$\hat{\rho} = \cdot 70 - \cdot 0121{,}8204.$$

Therefore a difference of $- \cdot 0005{,}5311$ corresponds to a rise of ·05 and accordingly one of $- \cdot 0003{,}1085$ to a rise of ·0281. Thus

$$\hat{\rho} = \cdot 6781 - \cdot 0124{,}2430$$
$$= \cdot 6657, \text{ accurately.}$$

TABLE IX. *Functions required in determining the "most probable" value $\hat{\rho}$ of the Correlation.*

r	$\lambda_1(r)$	$\lambda_2(r)$	$\lambda_3(r)$
·00	0	0	0
·05	·02493,75000	− ·00615,64453	− ·00317,92000
·10	·04950,00000	− ·01175,62500	− ·00667,19813
·15	·07331,25000	− ·01626,62109	− ·01073,47255
·20	·09600,00000	− ·01920,00000	− ·01551,36000
·25	·11718,75000	− ·02014,16016	− ·02099,99084
·30	·13650,00000	− ·01876,87500	− ·02699,79938
·35	·15356,25000	− ·01487,63672	− ·03310,98746
·40	·16800,00000	− ·00840,00000	− ·03874,08000
·45	·17943,75000	+ ·00056,07422	− ·04312,99059
·50	·18750,00000	+ ·01171,87500	− ·04541,01563
·55	·19181,25000	+ ·02457,59766	− ·04470,17533
·60	·19200,00000	+ ·03840,00000	− ·04024,32000
·65	·18768,75000	+ ·05220,05859	− ·03156,41987
·70	·17850,00000	+ ·06470,62500	− ·01870,45688
·75	·16406,25000	+ ·07434,08203	− ·00248,33679
·80	·14400,00000	+ ·07920,00000	+ ·01517,76000
·85	·11793,75000	+ ·07702,79297	+ ·03087,17070
·90	·08550,00000	+ ·06519,37500	+ ·03926,26688
·95	·04631,25000	+ ·04066,81641	+ ·03257,27752
1·00	0	0	0

(13) *On the Table for* $q_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\phi\, d\phi$.

While Table X, p. 377, gives the value of $q_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\phi\, d\phi$ to ten figures, and therefore will be of use in calculating the values of the moments, the reader may be compelled to deal with values of n greater than those tabled, or even may need more than ten significant figures. The present values were calculated to twelve figures, by means of the simple relations

$$q_{2p+2} = q_{2p} - \frac{1}{2p+1} q_{2p},$$

$$q_{2p+1} = q_{2p-1} - \frac{1}{2p} q_{2p-1},$$

with the control relation $q_{2p} \times q_{2p-1} = \frac{\pi}{2}\frac{1}{2p-1}$, and the occasional direct calculation of individual q_n's to control the accuracy by use of Degen's Tables* and Briggs's *Arithmetica Logarithmica*†.

Since $$q_{2p+1} = \frac{(2p)!}{2^{2p}\{(p)!\}^2}\frac{\pi}{2} \qquad \text{(xcii)},$$

and $$q_{2p} = \frac{2^{2p}\{(p)!\}^2}{2p\,(2p)!} \qquad \text{(xciii)},$$

* Tabularum ad faciliorem et breviorem probabilitatis computationem utilium Enneas. C. F. Degen, Havniae, 1824.

† Londini, W. Jones, 1624.

and Degen gives the logarithms of the factorials up to 1200! to eighteen mantissa figures, there is no difficulty in getting the logarithms of the q_n's to 18 figures. All we need is the value of either $\log \frac{\pi}{2}$ or $\frac{\pi}{2}$ to an adequate number of places* But for modern methods of machine calculation the logarithm is of small service and we need in this case to find also the antilogarithm. Let us illustrate the process in the determination of q_{104}:

$$\left.\begin{aligned}\log q_{104} = {}& 104 \log 2 \\ &+ 2 \log (52!) \\ &- \log 104 \\ &- \log (104!)\end{aligned}\right\} \begin{aligned}&= 31{\cdot}307119{,}549054{,}044280 \\ &+ 135{\cdot}813296{,}784409{,}541764 \\ &- 2{\cdot}017033{,}339298{,}780355 \\ &- 166{\cdot}012795{,}764264{,}301069\end{aligned}$$
$$= \bar{1}{\cdot}090587{,}229{,}900{,}504{,}620.$$

Thus far the work is very straightforward. But to obtain the antilogarithm to twelve figures is another matter. Tables like the original Vega (to 10 figures) or Mendizabel (to 8 figures) are not of service. We are thus compelled to use Briggs's 14 figure Table of Logarithms, but the fundamental defect of that magnificent piece of work is the largeness of the differences. The nearest logarithm to the above is log (·12319) = $\bar{1}$·090575,455222,21 with the remainder

$$r = {\cdot}000011{,}774678{,}29,$$

and the difference ·000035,252606,20. Mere linear interpolation gives

$${\cdot}1231{,}9334{,}0087{,}32,$$

which is wrong in the tenth figure.

We have therefore used the method of inverse interpolation given in the *Tables for Statisticians*† as (vii)$^{\text{bis}}$ on p. xiv. Unfortunately there are two misprints in the value given there (corrected in the *Errata*); it should run

$$\theta^2 \tfrac{1}{4} (- u_0 - u_1 + u_{-1} + u_2) + \theta \tfrac{1}{4} (5u_1 - 3u_0 - u_{-1} - u_2) + u_0 - u_0 (\theta) = 0 \quad \text{.........(xciv).}$$

But from Briggs's tables,

$$\begin{aligned} u_{-1} &= {\cdot}0905{,}4019{,}9754{,}24, \\ u_0 &= {\cdot}0905{,}7545{,}5222{,}21, \\ u_{+1} &= {\cdot}0906{,}1070{,}7828{,}41, \\ u_{+2} &= {\cdot}0906{,}4595{,}7573{,}32, \end{aligned}$$

whence

$${\cdot}0001{,}4101{,}614786\, \theta = {\cdot}0000{,}4709{,}8713{,}16 + \theta^2 \times {\cdot}0000{,}00005723{,}06,$$

or,
$$\theta = \frac{{\cdot}4709{,}8713{,}16}{1{\cdot}4101{,}6147{,}86} + \theta^2 \frac{{\cdot}0000{,}5723{,}06}{1{\cdot}4101{,}6147{,}86}.$$

* $\frac{\pi}{2} = 1{\cdot}570796{,}326794{,}8966$; $\log \frac{\pi}{2} = {\cdot}196119{,}877030{,}1527.$

† Cambridge University Press, 1914.

To a first approximation

$$\theta_1 = \frac{\cdot 4709{,}8713{,}16}{1\cdot 4101{,}6147}\left(1 - \frac{\cdot 0000{,}0000{,}86}{1\cdot 4101{,}6147}\right),$$

arranging it thus as most machines cannot divide by more than *nine* figures. Hence

$$\theta_1 = \cdot 3339{,}9517{,}625.$$

Substitute in the θ^2 term and we find

$$\theta_2 = \cdot 3339{,}9970{,}356,$$

or the value of $q_{104} = \cdot 1231{,}9333{,}9997$ to twelve figures. Found by the continuous process it was ·1231,9333,9996, so that the value tabled is correct to the tenth figure.

TABLE X. *Values of the Integral* $q_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\phi\, d\phi = \int_0^{\frac{\pi}{2}} \cos^{n-1}\phi\, d\phi$.

n	q_n	n	I_n	n	I_n
1	1·5707,9632,68	36	·2103,4114,55	71	·1492,6566,48
2	1·0000,0000,00	37	·2074,4030,47	72	·1482,1822,53
3	·7853,9816,34	38	·2046,5624,97	73	·1471,9253,06
4	·6666,6666,67	39	·2019,8134,93	74	·1461,8783,86
5	·5890,4862,25	40	·1994,0865,35	75	·1452,0344,23
6	·5333,3333,33	41	·1969,3181,56	76	·1442,3866,74
7	·4908,7385,21	42	·1945,4502,78	77	·1432,9287,07
8	·4571,4285,71	43	·1922,4296,29	78	·1423,6543,80
9	·4295,1462,06	44	·1900,2072,49	79	·1414,5578,26
10	·4063,4920,63	45	·1878,7380,46	80	·1405,6334,38
11	·3865,6315,85	46	·1857,9804,21	81	·1396,8758,53
12	·3694,0836,94	47	·1837,8959,15	82	·1388,2799,39
13	·3543,4956,20	48	·1818,4489,23	83	·1379,8407,82
14	·3409,9234,10	49	·1799,6064,16	84	·1371,5536,75
15	·3290,3887,90	50	·1781,3377,19	85	·1363,4141,06
16	·3182,5951,83	51	·1763,6142,88	86	·1355,4177,49
17	·3084,7394,91	52	·1746,4095,29	87	·1347,5604,54
18	·2995,3837,01	53	·1729,6986,29	88	·1339,8382,35
19	·2913,3650,74	54	·1713,4584,06	89	·1332,2472,67
20	·2837,7319,28	55	·1697,6671,73	90	·1324,7838,73
21	·2767,6968,21	56	·1682,3046,16	91	·1317,4445,19
22	·2702,6018,36	57	·1667,3516,87	92	·1310,2258,08
23	·2641,8924,20	58	·1652,7905,00	93	·1303,1244,70
24	·2585,0974,08	59	·1638,6042,44	94	·1296,1373,58
25	·2531,8135,69	60	·1624,7771,02	95	·1289,2614,44
26	·2481,6935,12	61	·1611,2941,74	96	·1282,4938,07
27	·2434,4361,24	62	·1598,1414,12	97	·1275,8316,37
28	·2389,7789,37	63	·1585,3055,58	98	·1269,2722,22
29	·2347,4919,77	64	·1572,7740,88	99	·1262,8129,47
30	·2307,3727,67	65	·1560,5351,59	100	·1256,4512,90
31	·2269,2422,44	66	·1548,5775,63	101	·1250,1848,17
32	·2232,9413,87	67	·1536,8906,87	102	·1244,0111,78
33	·2198,3284,24	68	·1525,4644,65	103	·1237,9281,04
34	·2165,2764,98	69	·1514,2893,53	104	·1231,9334,00
35	·2133,6717,06	70	·1503,3562,85	105	·1226,0249,49

Something corresponding to the above process must be used for values of q_n with $n > 105$, if we require the function correct to ten or twelve figures. The usual approximate formulae for the factorial, or for the Γ-function, do not converge rapidly enough and generally give the logarithm, so that they really involve the use of the logarithm tables to 14 places and the antilogarithm process.

For values of 100 and upwards the following formula will be found good:

For n even we deduce by Stirling's Theorem from Eqn. (xcii)

$$q_n = \frac{1{\cdot}2533{,}1413{,}7315}{\sqrt{n}}\left(1 + \frac{{\cdot}25}{n} + \frac{{\cdot}03125}{n^2} - \frac{{\cdot}039{,}0625}{n^3} - \frac{{\cdot}0102{,}5390{,}625}{n^4}\right) \quad \text{(xcv)}.$$

For example, $n = 100$

$$q_{100} = 1{\cdot}2533{,}1413{,}7315 \times {\cdot}1002{,}5031{,}64165,$$
$$= {\cdot}1256{,}4512{,}9017{,}89,$$

which is correct to twelve places.

For n odd we deduce by Stirling's Theorem from Eqn. (xcii) by somewhat more lengthy algebra precisely the same value.

[Owing to the growth of this memoir far beyond its original limits, it has been found impossible to include in this first portion the experimental work which accompanied the algebraical investigations, nor to give illustrations of the various uses which the tables serve. These matters are therefore reserved for a continuation of this memoir, which will appear later.]

APPENDIX.

CORRELATION IN SMALL SAMPLES.

TABLE A. *Ordinates and Constants of Frequency Curves**.

$n = 3$ ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
− *·95*	1019·41	874·99	744·21	624·87	515·25	413·97	319·92	232·19	150·04	72·82
− *·90*	730·25	631·33	540 26	456·01	377·73	304·70	236·31	172·05	111·49	54·25
− *·85*	604·25	526·19	453·08	384·48	319·97	259·17	201·74	147·37	95·78	46·73
− *·80*	530·52	465·35	403·21	344·03	287·69	234·02	182·85	134·03	87·38	42·75
− *·75*	481·24	425·22	370·79	318·13	267·34	218·43	171·35	126·04	82·44	40·45
− *·70*	445·72	396·74	348·18	300·44	253·76	208·27	164·05	121·12	79·49	39·12
− *·65*	418·87	375·59	331·78	287·95	244·48	201·60	159·47	118·20	77·84	38·44
− *·60*	397·89	359·43	319·60	279·03	238·18	197·36	156·81	116·70	77·14	38·22
− *·55*	381·13	346·87	310·50	272·73	234·08	194·94	155·60	116·29	77·18	38·38
− *·50*	367·55	337·02	303·74	268·44	231·71	193·97	155·57	116·79	77·82	38·85
− *·45*	356·44	329·29	298·82	265·77	230·74	194·20	156·54	118·06	79·02	39·61
− *·40*	347·30	323·28	295·41	264·44	230·97	195·48	158·40	120·05	80·72	40·63
− *·35*	339·80	318·70	293·29	264·28	232·26	197·72	161·09	122·71	82·91	41·93
− *·30*	333·68	315·35	292·30	265·17	234·53	200·85	164·58	126·05	85·60	43·51
− *·25*	328·75	313·08	292·30	267·01	237·71	204·86	168·86	130·08	88·82	45·38
− *·20*	324·87	311·77	293·24	269·76	241·79	209·74	173·97	134·82	92·59	47·57
− *·15*	321·95	311·36	295·05	273·39	246·77	215·51	179·94	140·33	96·98	50·12
− *·10*	319·91	311·80	297·71	277·91	252·66	222·22	186·82	146·69	102·04	53·08
− *·05*	318·71	313·06	301·21	283·32	259·51	229·93	194·72	153·99	107·88	56·51
·00	318·31	315·13	305·58	289·66	267·38	238·73	203·72	162·34	114·59	60·48
+ *·05*	318·71	318·02	310·83	296·99	276·35	248·73	213·97	171·89	122·33	65·09
+ *·10*	319·91	321·75	317·02	305·38	286·51	260·05	225·63	182·84	131·27	70·48
+ *·15*	321·95	326·39	324·22	314·94	298·02	272·89	238·91	195·41	141·63	76·79
+ *·20*	324·87	331·99	332·52	325·79	311·04	287·45	254·08	209·89	153·71	84·24
+ *·25*	328·75	338·66	342·06	338·10	325·78	304·00	271·45	226·66	167·88	93·11
+ *·30*	333·68	346·53	353·00	352·08	342·52	322·88	291·45	246·18	184·60	103·75
+ *·35*	339·80	355·76	365·56	368·00	361·58	344·53	314·60	269·08	204·53	116·66
+ *·40*	347·30	366·59	380·02	386·21	383·43	369·48	341·59	296·15	228·52	132·54
+ *·45*	356·44	379·33	396·75	407·18	408·63	398·48	373·30	328·48	257·73	152·34
+ *·50*	367·55	394·39	416·27	431·53	437·95	432·48	410·96	367·53	293·80	177·48
+ *·55*	381·13	412·36	439·27	460·12	472·45	472·82	456·23	415·36	339·09	210·05
+ *·60*	397·89	434·08	466·77	494·15	513·63	521·35	511·47	474·92	397·07	253·32
+ *·65*	418·87	460·81	500·25	535·44	563·68	580·81	580·15	550·63	473·12	312·61
+ *·70*	445·72	494·50	542·05	586·77	626·01	655·44	667·63	649·38	575·90	396·99
+ *·75*	481·24	538·43	596·05	652·77	706·23	752·17	782·72	782·63	720·25	523·20
+ *·80*	530·52	598·63	669·27	741·91	814·50	883·48	941·21	971·06	933·85	724·94
+ *·85*	604·25	687·68	776·82	871·73	971·83	1074·99	1175·29	1256·88	1274·71	1079·64
+ *·90*	730·25	838·25	956·77	1087·46	1232·06	1391·88	1566·09	1745·85	1890·70	1804·95
+ *·95*	1019·41	1180·31	1361·48	1568·00	1806·93	2088·24	2426·12	2839·55	3341·78	3802·31
+ *1·00*	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Mean	·0000	·0786	·1579	·2384	·3209	·4063	·4960	·5919	·6976	·8204
Antimode	− ·0000	− ·1513	− ·2755	− ·3679	− ·4354	− ·4855	− ·5239	− ·5539	− ·5779	− ·5976
σ	·7071	·7045	·6966	·6829	·6629	·6351	·5977	·5470	·4755	·3635
$(1-\rho^2)/\sqrt{n-1}$	·7071	·7000	·6788	·6435	·5940	·5303	·4525	·3606	·2546	·1344
β_1	·0000	·0280	·1152	·2722	·5206	·9018	1·5002	2·5105	4·5031	10·2226
β_2	1·5000	1·5323	1·6329	1·8142	2·1011	2·5423	3·2363	4·4114	6·7388	13·4672

* In all cases the total frequency of the curve is taken as 1000.

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 4.$ ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	500·00	386·33	295·47	222·28	163·11	115·34	77·06	46·93	24·03	8·02
− *·95*	500·00	390·84	301·85	228·97	169·22	120·41	80·89	49·50	25·46	8·53
− *·90*	500·00	395·43	308·42	235·94	175·66	125·79	84·99	52·27	27·01	9·09
− *·85*	500·00	400·09	315·19	243·21	182·44	131·52	89·38	55·26	28·69	9·69
− *·80*	500·00	404·83	322·17	250·79	189·59	137·61	94·10	58·49	30·52	10·36
− *·75*	500·00	409·64	329·36	258·71	197·13	144·10	99·16	61·99	32·51	11·08
− *·70*	500·00	414·54	336·78	266·97	205·09	151·02	104·61	65·79	34·69	11·89
− *·65*	500·00	419·52	344·44	275·61	213·51	158·41	110·48	69·92	37·08	12·77
− *·60*	500·00	424·58	352·34	284·64	222·42	166·31	116·82	74·41	39·70	13·75
− *·55*	500·00	429·73	360·49	294·09	231·85	174·77	123·68	79·32	42·58	14·83
− *·50*	500·00	434·96	368·90	303·97	241·84	183·84	131·10	84·68	45·76	16·04
− *·45*	500·00	440·28	377·59	314·32	252·44	193·57	139·16	90·56	49·28	17·39
− *·40*	500·00	445·69	386·57	325·17	263·69	204·03	147·92	97·03	53·19	18·90
− *·35*	500·00	451·20	395·84	336·54	275·65	215·29	157·46	104·15	57·54	20·59
− *·30*	500·00	456·80	405·43	348·48	288·38	227·44	167·88	112·01	62·40	22·51
− *·25*	500·00	462·50	415·34	361·01	301·94	240·55	179·28	120·72	67·85	24·68
− *·20*	500·00	468·30	425·59	374·17	316·40	254·74	191·79	130·41	73·98	27·16
− *·15*	500·00	474·19	436·20	388·01	331·84	270·11	205·53	141·20	80·91	29·99
− *·10*	500·00	480·20	447·17	402·57	348·34	286·81	220·69	153·27	88·77	33·25
− *·05*	500·00	486·30	458·53	417·89	366·00	304·96	237·43	166·83	97·73	37·02
·00	500·00	492·52	470·30	434·04	384·94	324·76	256·00	182·11	108·00	41·41
+ *·05*	500·00	498·85	482·49	451·07	405·26	346·38	276·64	199·40	119·83	46·56
+ *·10*	500·00	505·29	495·13	469·04	427·09	370·06	299·67	219·05	133·53	52·64
+ *·15*	500·00	511·84	508·23	488·02	450·60	396·04	325·45	241·50	149·52	59·89
+ *·20*	500·00	518·52	521·81	508·08	475·94	424·63	354·41	267·28	168·29	68·60
+ *·25*	500·00	525·31	535·90	529·31	503·31	456·16	387·08	297·05	190·49	79·17
+ *·30*	500·00	532·23	550·53	551·79	532·91	491·05	424·09	331·63	216·99	92·15
+ *·35*	500·00	539·28	565·72	575·62	564·99	529·75	466·20	372·07	248·90	108·29
+ *·40*	500·00	546·46	581·49	600·89	599·81	572·82	514·35	419·70	287·74	128·65
+ *·45*	500·00	553·77	597·89	627·74	637·68	620·92	569·71	476·24	335·53	154·74
+ *·50*	500·00	561·22	614·93	656·28	678·96	674·81	633·70	543·96	395·13	188·83
+ *·55*	500·00	568·81	632·65	686·66	724·05	735·44	708·15	625·86	470·54	234·34
+ *·60*	500·00	576·54	651·09	719·03	773·41	803·91	795·33	725·97	567·57	296·69
+ *·65*	500·00	584·41	670·29	753·55	827·57	881·57	898·20	849·83	694·88	384·78
+ *·70*	500·00	592·44	690·28	790·43	887·15	970·08	1020·55	1005·20	865·74	514·05
+ *·75*	500·00	600·62	711·11	829·86	952·87	1071·44	1167·38	1203·12	1101·28	712·86
+ *·80*	500·00	608·96	732·82	872·07	1025·55	1188·16	1345·36	1459·79	1436·61	1037·65
+ *·85*	500·00	617·46	755·47	917·33	1106·18	1323·35	1563·48	1799·62	1933·28	1612·67
+ *·90*	500·00	626·13	779·10	965·92	1195·91	1480·93	1834·16	2260·62	2706·28	2751·97
+ *·95*	500·00	634·97	803·76	1018·17	1296·08	1665·90	2174·80	2904·38	3989·04	5425·82
+ *1·00*	500·00	643·98	829·53	1074·43	1408·32	1884·66	2610·44	3835·42	6309·30	13781·45
Mean	·0000	·0850	·1705	·2570	·3453	·4360	·5302	·6293	·7357	·8540
Mode	Non-existent	·5746	·5661	·5516	·5304	·5013	·4626	·4111	·3407	·2367
σ	·5774	·5716	·5543	·5254	·4850	·4330	·3695	·2944	·2078	·1097
$(1-\rho^2)/\sqrt{n-1}$	·5774	·5716	·5543	·5254	·4850	·4330	·3695	·2944	·2078	·1097
β_1	·0000	·0314	·1295	·3069	·5895	1·0283	1·7284	2·9402	5·4289	13·1840
β_2	1·8000	1·8399	1·9647	2·1907	2·5522	3·1163	4·0230	5·6093	8·9222	19·5710

TABLE A—(*continued*).

$n = 5$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	198·78	141·35	99·03	67·80	44·87	28·25	16·49	8·50	3·48	·80
− *·90*	277·50	200·54	142·43	98·66	65·96	41·90	24·65	12·79	5·27	1·22
− *·85*	335·36	246·33	177·39	124·35	84·01	53·86	31·94	16·70	6·92	1·62
− *·80*	381·97	285·19	208·27	147·80	100·93	65·32	39·06	20·58	8·59	2·02
− *·75*	421·08	319·60	236·74	170·10	117·44	76·75	46·30	24·59	10·33	2·45
− *·70*	454·64	350·80	263·63	191·85	133·96	88·43	53·84	28·82	12·20	2·91
− *·65*	483·79	379·53	289·41	213·36	150·72	100·55	61·79	33·37	14·24	3·42
− *·60*	509·30	406·24	314·40	234·88	167·91	113·24	70·28	38·30	16·48	3·99
− *·55*	531·68	431·25	338·80	256·56	185·68	126·63	79·40	43·68	18·96	4·62
− *·50*	551·33	454·76	362·75	278·52	204·14	140·84	89·26	49·59	21·72	5·34
− *·45*	568·52	476·92	386·34	300·86	223·41	156·00	99·98	56·12	24·82	6·16
− *·40*	583·47	497·83	409·64	323·65	243·58	172·23	111·67	63·37	28·32	7·10
− *·35*	596·35	517·56	432·68	346·96	264·76	189·65	124·47	71·45	32·27	8·17
− *·30*	607·30	536·16	455·50	370·82	287·04	208·40	138·53	80·49	36·78	9·41
− *·25*	616·40	553·64	478·09	395·29	310·51	228·62	154·02	90·65	41·92	10·86
− *·20*	623·76	570·02	500·44	420·38	335·28	250·49	171·14	102·08	47·83	12·54
− *·15*	629·42	585·27	522·52	446·12	361·44	274·17	190·10	115·02	54·64	14·52
− *·10*	633·43	599·37	544·31	472·50	389·09	299·86	211·15	129·71	62·54	16·86
− *·05*	635·82	612·28	565·73	499·53	418·31	327·76	234·60	146·44	71·74	19·65
·00	636·62	623·95	586·71	527·18	449·20	358·10	260·76	165·58	82·51	22·98
+ *·05*	635·82	634·31	607·16	555·41	481·84	391·13	290·03	187·57	95·19	27·01
+ *·10*	633·43	643·27	626·97	584·15	516·30	427·11	322·86	212·91	110·22	31·91
+ *·15*	629·42	650·74	645·99	613·31	552·65	466·33	359·76	242·26	128·14	37·93
+ *·20*	623·76	656·59	664·06	642·76	590·91	509·09	401·33	276·40	149·65	45·39
+ *·25*	616·40	660·68	680·97	672·33	631·08	555·72	448·27	316·29	175·68	54·74
+ *·30*	607·30	662·86	696·46	701·77	673·11	606·51	501·37	363·13	207·42	66·60
+ *·35*	596·35	662·91	710·24	730·79	716·87	661·77	561·55	418·41	246·48	81·84
+ *·40*	583·47	660·60	721·95	758·98	762·14	721·75	629·83	483·98	295·00	101·75
+ *·45*	568·52	655·66	731·14	785·81	808·54	786·62	707·37	562·17	355·92	128·18
+ *·50*	551·33	647·74	737·25	810·60	855·49	856·39	795·39	655·88	433·30	163·98
+ *·55*	531·68	636·41	739·59	832·42	902·10	930·77	895·14	768·74	532·81	213·57
+ *·60*	509·30	621·15	737·29	850·06	947·05	1009·03	1007·71	905·20	662·54	284·13
+ *·65*	483·79	601·26	729·21	861·85	988·36	1089·62	1133·68	1070·61	834·18	387·69
+ *·70*	454·64	575·83	713·82	865·50	1023·03	1169·66	1272·51	1270·96	1064·70	545·45
+ *·75*	421·08	543·59	689·01	857·73	1046·50	1243·92	1421·02	1511·86	1378·85	796·89
+ *·80*	381·97	502·63	651·67	833·60	1051·45	1302·93	1570·47	1795·27	1811·55	1220·21
+ *·85*	335·36	449·87	596·86	785·20	1025·58	1328·96	1699·62	2109·71	2406·01	1981·91
+ *·90*	277·50	379·52	515·46	698·04	945·84	1286·02	1756·53	2398·74	3182·44	3457·80
+ *·95*	198·78	277·21	385·59	538·03	757·76	1085·21	1595·22	2436·67	3917·02	6392·40
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0884	·1773	·2671	·3584	·4517	·5480	·6482	·7541	·8687
Mode	·0000	·3264	·5520	·6936	·7863	·8500	·8965	·9318	·9595	·9817
σ	·5000	·4972	·4886	·4740	·4528	·4239	·3858	·3358	·2691	·1748
$(1-\rho^2)/\sqrt{n-1}$	·5000	·4950	·4800	·4550	·4200	·3750	·3200	·2550	·1800	·0950
β_1	·0000	·0315	·1299	·3077	·5909	1·0315	1·7297	2·9374	5·4065	13·0290
β_2	2·0000	2·0429	2·1769	2·4201	2·8097	3·4191	4·4027	6·1333	9·7830	21·7579

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 6$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	73·125	47·28	30·03	18·55	10·99	6·12	3·10	1·35	·44	·07
− *·90*	142·500	94·07	60·81	38·13	22·88	12·89	6·60	2·89	·95	·15
− *·85*	208·125	140·28	92·30	58·76	35·74	20·37	10·54	4·66	1·54	·25
− *·80*	270·000	185·83	124·49	80·50	49·64	28·64	14·97	6·68	2·23	·36
− *·75*	328·125	230·63	157·34	103·38	64·65	37·76	19·96	9·00	3·03	·50
− *·70*	382·500	274·58	190·82	127·43	80·85	47·84	25·59	11·66	3·96	·66
− *·65*	433·125	317·59	224·87	152·69	98·33	58·96	31·92	14·71	5·05	·85
− *·60*	480·000	359·54	259·44	179·18	117·16	71·24	39·06	18·20	6·32	1·07
− *·55*	523·125	400·33	294·48	206·94	137·45	84·78	47·09	22·22	7·79	1·33
− *·50*	562·500	439·82	329·89	235·97	159·29	99·73	56·16	26·83	9·52	1·64
− *·45*	598·125	477·89	365·60	266·29	182·79	116·21	66·38	32·13	11·55	2·02
− *·40*	630·000	514·41	401·49	297·90	208·03	134·39	77·92	38·25	13·93	2·46
− *·35*	658·125	549·23	437·46	330·80	235·13	154·44	90·95	45·31	16·73	3·00
− *·30*	682·500	582·20	473·37	364·95	264·20	176·55	105·68	53·47	20·03	3·64
− *·25*	703·125	613·16	509·07	400·33	295·32	200·93	122·35	62·92	23·94	4·41
− *·20*	720·000	641·93	544·37	436·88	328·62	227·80	141·22	73·89	28·59	5·35
− *·15*	733·125	668·33	579·08	474·49	364·17	257·41	162·61	86·65	34·12	6·50
− *·10*	742·500	692·18	612·97	513·07	402·05	290·01	186·88	101·53	40·75	7·91
− *·05*	748·125	713·27	645·79	552·46	442·32	325·89	214·44	118·92	48·71	9·65
·00	750·000	731·39	677·23	592·47	485·02	365·35	245·76	139·31	58·32	11·80
+ *·05*	748·125	746·31	706·99	632·84	530·14	408·70	281·39	163·28	69·98	14·50
+ *·10*	742·500	757·78	734·67	673·26	577·63	456·23	321·94	191·55	84·21	17·90
+ *·15*	733·125	765·56	759·87	713·34	627·35	508·26	368·14	224·98	101·67	22·24
+ *·20*	720·000	769·38	782·10	752·61	679·10	565·04	420·76	264·66	123·24	27·81
+ *·25*	703·125	768·94	800·84	790·47	732·53	626·81	480·72	311·90	150·07	35·06
+ *·30*	682·500	763·96	815·48	826·21	787·15	693·69	548·97	368·33	183·69	44·60
+ *·35*	658·125	754·11	825·36	858·93	842·24	765·62	626·56	435·94	226·18	57·33
+ *·40*	630·000	739·06	829·70	887·57	896·80	842·35	714·54	517·20	280·35	74·61
+ *·45*	598·125	718·45	827·66	910·82	949·48	923·20	813·87	615·10	350·06	98·48
+ *·50*	562·500	691·90	818·27	927·11	998·43	1006·98	925·29	733·19	440·67	132·11
+ *·55*	523·125	659·02	800·46	934·53	1041·18	1091·61	1048·92	875·62	559·69	180·65
+ *·60*	480·000	619·39	772·99	930·76	1074·42	1173·81	1183·82	1046·92	717·69	252·62
+ *·65*	433·125	572·56	734·52	913·01	1093·74	1248·41	1326·99	1251·36	929·54	362·77
+ *·70*	382·500	518·05	683·50	877·89	1093·27	1307·53	1471·73	1491·35	1215·82	537·74
+ *·75*	328·125	455·38	618·19	821·29	1065·22	1339·14	1604·80	1763·58	1603·56	828·01
+ *·80*	270·000	384·02	536·66	738·25	999·24	1325·11	1701·20	2050·08	2122·58	1334·30
+ *·85*	208·125	303·41	436·70	622·74	881·50	1237·97	1714·92	2297·17	2783·31	2265·98
+ *·90*	142·500	212·95	315·86	467·45	693·60	1036·09	1561·73	2364·83	3479·97	4043·96
+ *·95*	73·125	112·04	171·33	263·48	410·83	655·98	1086·58	1899·93	3578·10	7013·72
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0906	·1816	·2734	·3665	·4614	·5587	·6594	·7646	·8766
Mode	·0000	·2197	·4101	·5599	·6747	·7630	·8321	·8870	·9319	·9689
σ	·4472	·4444	·4360	·4216	·4007	·3725	·3356	·2878	·2253	·1397
$(1-\rho^2)/\sqrt{n-1}$	·4472	·4427	·4293	·4070	·3757	·3354	·2862	·2281	·1610	·0850
β_1	·0000	·0304	·1251	·2959	·5667	·9838	1·6418	2·7599	4·9848	11·4554
β_2	2·1429	2·1863	2·3222	2·5682	2·9615	3·5746	4·5585	6·2752	9·8417	21·1653

TABLE A—(*continued*).

$n = 7$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	25·84	15·19	8·75	4·87	2·58	1·27	·56	·20	·05	·01
− *·90*	70·30	42·38	24·93	14·14	7·62	3·81	1·69	·63	·16	·02
− *·85*	124·08	76·72	46·12	26·66	14·60	7·39	3·34	1·25	·33	·04
− *·80*	183·35	116·30	71·46	42·10	23·44	12·05	5·51	2·08	·56	·06
− *·75*	245·63	159·85	100·42	60·33	34·17	17·84	8·26	3·16	·85	·10
− *·70*	309·15	206·44	132·64	81·28	46·85	24·84	11·67	4·53	1·24	·14
− *·65*	372·52	255·27	167·80	104·93	61·59	33·20	15·83	6·22	1·72	·20
− *·60*	434·60	305·66	205·61	131·26	78·49	43·03	20·83	8·30	2·32	·27
− *·55*	494·46	356·96	245·82	160·28	97·70	54·50	26·81	10·85	3·08	·37
− *·50*	551·33	408·59	288·14	191·98	119·36	67·80	33·92	13·93	4·01	·49
− *·45*	604·53	459·99	332·29	226·34	143·61	83·12	42·31	17·66	5·16	·63
− *·40*	653·49	510·59	377·96	263·34	170·62	100·70	52·20	22·16	6·58	·82
− *·35*	697·73	559·87	424·83	302·91	200·54	120·78	63·81	27·58	8·32	1·05
− *·30*	736·85	607·28	472·53	344·98	233·54	143·64	77·42	34·10	10·48	1·35
− *·25*	770·51	652·32	520·67	389·43	269·77	169·60	93·33	41·94	13·13	1·72
− *·20*	798·41	694·44	568·81	436·10	309·36	198·97	111·92	51·36	16·41	2·19
− *·15*	820·34	733·15	616·47	484·78	352·43	232·12	133·60	62·69	20·47	2·80
− *·10*	836·13	767·91	663·12	535·18	399·06	269·42	158·87	76·33	25·50	3·56
− *·05*	845·65	798·24	708·17	586·95	449·29	311·28	188·29	92·77	31·77	4·55
·00	848·83	823·62	750·99	639·65	503·10	358·10	222·51	112·60	39·60	5·82
+ *·05*	845·65	843·56	790·87	692·72	560·37	410·29	262·29	136·56	49·43	7·48
+ *·10*	836·13	857·59	827·06	745·49	620·88	468·23	308·46	165·58	61·82	9·65
+ *·15*	820·34	865·26	858·73	797·15	684·25	532·27	361·98	200·77	77·52	12·53
+ *·20*	798·41	866·13	884·99	846·70	749·91	602·63	423·92	243·54	97·54	16·38
+ *·25*	770·51	859·80	904·89	893·00	817·07	679·42	495·43	295·61	123·21	21·58
+ *·30*	736·85	845·93	917·44	934·67	884·59	762·49	577·72	359·10	156·38	28·71
+ *·35*	697·73	824·21	921·57	970·10	950·96	851·33	671·98	436·64	199·53	38·62
+ *·40*	653·49	794·40	916·22	997·43	1014·18	944·93	779·26	531·38	256·18	52·62
+ *·45*	604·53	756·38	900·29	1014·55	1071·64	1041·52	900·25	647·10	331·09	72·76
+ *·50*	551·33	710·11	872·71	1019·07	1120·03	1138·25	1034·91	788·14	431·03	102·38
+ *·55*	494·46	655·69	832·49	1008·34	1155·12	1230·82	1181·85	959·18	565·53	147·00
+ *·60*	434·60	593·44	778·80	979·51	1171·74	1312·87	1337·35	1164·60	747·90	216·12
+ *·65*	372·52	523·87	711·02	929·65	1163·57	1375·33	1493·80	1406·95	996·60	326·69
+ *·70*	309·15	447·82	628·96	855·90	1123·23	1405·52	1637·17	1683·53	1336·03	510·28
+ *·75*	245·63	366·56	533·05	755·92	1042·49	1386·41	1743·33	1979·38	1794·82	828·27
+ *·80*	183·35	281·92	424·74	628·49	913·07	1296·13	1772·80	2252·77	2393·94	1404·91
+ *·85*	124·08	196·62	307·09	474·80	728·55	1109·20	1664·80	2407·27	3099·80	2495·12
+ *·90*	70·30	114·82	186·02	300·94	489·11	802·95	1336·07	2244·07	3664·14	4555·85
+ *·95*	25·84	43·51	73·17	124·05	214·20	381·46	712·23	1426·14	3147·79	7414·56
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0921	·1845	·2777	·3720	·4678	·5658	·6667	·7713	·8814
Mode	·0000	·1813	·3484	·4919	·6106	·7087	·7894	·8563	·9122	·9595
σ	·4082	·4055	·3973	·3832	·3629	·3356	·3001	·2545	·1958	·1175
$(1-\rho^2)/\sqrt{n-1}$	·4082	·4042	·3919	·3715	·3429	·3062	·2613	·2082	·1470	·0776
β_1	·0000	·0288	·1186	·2798	·5340	·9222	1·5263	2·5318	4·4611	9·6408
β_2	2·2500	2·2929	2·4267	2·6686	3·0535	3·6503	4·5968	6·2238	9·5129	19·3424

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 8$. ρ variate (correlation in population sampled).

r variate (correlation in sample)	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	8·91	4·76	2·49	1·25	·59	·26	·10	·03	·01	·00
− *·90*	33·84	18·63	9·97	5·12	2·47	1·10	·42	·13	·03	·00
− *·85*	72·19	40·95	22·48	11·80	5·82	2·62	1·03	·33	·07	·01
− *·80*	121·50	71·02	40·02	21·48	10·80	4·95	1·98	·63	·13	·01
− *·75*	179·44	108·11	62·53	34·34	17·61	8·22	3·33	1·08	·23	·02
− *·70*	243·84	151·45	89·96	50·58	26·48	12·58	5·19	1·71	·38	·03
− *·65*	312·66	200·21	122·17	70·35	37·63	18·23	7·66	2·57	·57	·05
− *·60*	384·00	253·56	158·99	93·82	51·30	25·35	10·84	3·69	·83	·07
− *·55*	456·10	310·59	200·22	121·13	67·75	34·18	14·89	5·16	1·18	·10
− *·50*	527·34	370·40	245·57	152·40	87·25	44·96	19·98	7·06	1·64	·14
− *·45*	596·26	432·05	294·69	187·72	110·08	58·00	26·31	9·47	2·25	·19
− *·40*	661·50	494·55	347·19	227·13	136·53	73·61	34·12	12·53	3·03	·27
− *·35*	721·88	556·91	402·57	270·64	166·89	92·15	43·68	16·38	4·04	·36
− *·30*	776·34	618·14	460·28	318·19	201·43	114·03	55·33	21·22	5·34	·49
− *·25*	823·97	677·22	519·66	369·65	240·45	139·67	69·47	27·27	7·03	·66
− *·20*	864·00	733·11	579·99	424·79	284·17	169·57	86·55	34·83	9·19	·88
− *·15*	895·79	784·83	640·42	483·31	332·81	204·24	107·11	44·26	11·98	1·17
− *·10*	918·84	831·37	700·04	544·75	386·52	244·24	131·79	56·00	15·57	1·57
− *·05*	932·82	871·77	757·84	608·54	445·36	290·13	161·34	70·61	20·22	2·09
·00	937·50	905·10	812·68	673·93	509·27	342·52	196·61	88·81	26·24	2·80
+ *·05*	932·82	930·49	863·38	740·00	578·06	401·96	238·59	111·47	34·07	3·77
+ *·10*	918·84	947·15	908·63	805·60	651·31	468·99	288·43	139·69	44·29	5·08
+ *·15*	895·79	954·37	947·08	869·36	728·36	544·03	347·39	174·87	57·69	6·89
+ *·20*	864·00	951·55	977·31	929·66	808·23	627·31	416·87	218·74	75·35	9·42
+ *·25*	823·97	938·24	997·87	984·60	889·50	718·81	498·39	273·48	98·76	12·97
+ *·30*	776·34	914·13	1007·32	1031·99	970·27	818·08	593·47	341·77	129·97	18·05
+ *·35*	721·88	879·12	1004·28	1069·38	1048·02	924·04	703·52	426·94	171·88	25·40
+ *·40*	661·50	833·34	987·47	1094·05	1119·51	1034·75	829·63	532·99	228·55	36·23
+ *·45*	596·26	777·15	955·78	1103·06	1180·66	1147·03	972·16	664·66	305·76	52·50
+ *·50*	527·34	711·26	908·44	1093·37	1226·48	1256·06	1130·10	827·21	411·69	77·48
+ *·55*	456·10	636·70	845·06	1061·99	1251·03	1354·86	1300·15	1025·98	558·02	116·83
+ *·60*	384·00	554·91	765·85	1006·21	1247·49	1433·64	1475·18	1265·10	761·16	180·59
+ *·65*	312·66	467·80	671·79	924·01	1208·46	1479·32	1642·00	1544·83	1043·60	287·38
+ *·70*	243·84	377·81	564·92	814·59	1126·64	1475·21	1778·44	1856·10	1434·02	473·04
+ *·75*	179·44	287·97	448·65	679·20	996·09	1401·54	1849·45	2169·84	1962·40	809·47
+ *·80*	121·50	202·00	328·13	522·33	814·61	1237·97	1804·24	2418·01	2637·69	1445·38
+ *·85*	72·19	124·36	210·79	353·40	587·92	970·50	1578·45	2464·22	3372·91	2684·78
+ *·90*	33·84	60·42	106·94	189·15	336·77	607·69	1116·43	2080·31	3769·68	5016·00
+ *·95*	8·91	16·49	30·50	57·02	109·05	216·63	456·02	1045·85	2706·04	7661·16
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0932	·1867	·2808	·3759	·4725	·5709	·6718	·7760	·8847
Mode	·0000	·1613	·3135	·4505	·5698	·6722	·7594	·8338	·8975	·9524
σ	·3780	·3753	·3673	·3536	·3339	·3074	·2733	·2299	·1746	·1024
$(1-\rho^2)/\sqrt{n-1}$	·3780	·3742	·3628	·3439	·3175	·2835	·2419	·1928	·1361	·0718
β_1	·0000	·0272	·1117	·2630	·5000	·8586	1·4088	2·3044	3·9581	8·0434
β_2	2·3333	2·3751	2·5051	2·7393	3·1101	3·6800	4·5751	6·0839	9·0368	17·2512

TABLE A—(*continued*).

$n = 9$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	3·02	1·47	·69	·31	·13	·05	·02	·00	·00	—
− *·90*	16·03	8·06	3·92	1·82	·79	·31	·10	·03	·00	—
− *·85*	41·32	21·50	10·78	5·14	2·28	·91	·31	·08	·01	—
− *·80*	79·21	42·66	22·04	10·78	4·89	2·00	·70	·19	·03	—
− *·75*	128·96	71·93	38·30	19·23	8·93	3·72	1·32	·37	·06	·00
− *·70*	189·20	109·29	60·01	30·95	14·72	6·27	2·27	·64	·11	·01
− *·65*	258·15	154·47	87·49	46·39	22·61	9·85	3·64	1·04	·19	·01
− *·60*	333·77	206·90	120·93	65·95	32·98	14·69	5·55	1·62	·29	·02
− *·55*	413·87	265·84	160·41	90·03	46·21	21·08	8·14	2·42	·45	·03
− *·50*	496·20	330·31	205·86	118·99	62·73	29·33	11·58	3·52	·66	·04
− *·45*	578·53	399·18	257·08	153·13	83·00	39·81	16·09	4·99	·96	·06
− *·40*	658·72	471·20	313·71	192·70	107·46	52·93	21·93	6·97	1·37	·09
− *·35*	734·71	544·95	375·25	237·86	136·60	69·16	29·41	9·57	1·93	·12
− *·30*	804·64	618·94	441·03	288·69	170·90	89·03	38·90	12·99	2·68	·17
− *·25*	866·82	691·61	510·20	345·14	210·81	113·15	50·86	17·45	3·70	·25
− *·20*	919·77	761·33	581·74	407·03	256·77	142·16	65·83	23·24	5·06	·35
− *·15*	962·26	826·48	654·46	473·98	309·16	176·78	84·46	30·73	6·89	·48
− *·10*	993·32	885·42	726·99	545·46	368·27	217·80	107·54	40·41	9·36	·68
− *·05*	1012·24	936·57	797·78	620·65	434·27	266·02	135·99	52·88	12·66	·95
·00	1018·59	978·46	865·14	698·50	507·13	322·29	170·89	68·91	17·11	1·33
+ *·05*	1012·24	1009·68	927·20	777·65	586·61	387·41	213·52	89·51	23·11	1·86
+ *·10*	993·32	1029·05	982·02	856·41	672·14	462·13	265·33	115·94	31·22	2·63
+ *·15*	962·26	1035·54	1027·55	932·73	762·75	547·03	327·98	149·85	42·24	3·73
+ *·20*	919·77	1028·41	1061·74	1004·19	856·97	642·43	403·32	193·30	57·27	5·33
+ *·25*	866·82	1007·20	1082·54	1068·00	952·68	748·20	493·28	248·93	77·88	7·67
+ *·30*	804·64	971·79	1088·08	1121·01	1047·05	863·56	599·83	320·04	106·28	11·16
+ *·35*	734·71	922·48	1076·68	1159·76	1136·34	986·80	724·70	410·75	145·67	16·43
+ *·40*	658·72	859·99	1047·01	1180·62	1215·85	1114·86	869·09	526·05	200·65	24·55
+ *·45*	578·53	785·54	998·27	1179·91	1279·81	1242·93	1032·99	671·79	277·86	37·28
+ *·50*	496·20	700·85	930·34	1154·15	1321·44	1363·83	1214·32	854·38	386·97	57·71
+ *·55*	413·87	608·22	843·95	1100·46	1333·12	1467·50	1407·47	1079·97	541·89	91·38
+ *·60*	333·77	510·46	740·94	1016·99	1306·81	1540·47	1601·27	1352·44	762·41	148·52
+ *·65*	258·15	410·96	624·47	903·63	1234·97	1565·77	1776·21	1669·36	1075·58	248·83
+ *·70*	189·20	313·58	499·21	762·81	1111·97	1523·66	1901·22	2014·01	1515·00	431·66
+ *·75*	128·96	222·57	371·51	600·46	936·53	1394·27	1930·95	2341·13	2111·97	778·77
+ *·80*	79·21	142·39	249·41	427·14	715·16	1163·62	1807·20	2554·55	2860·84	1463·91
+ *·85*	41·32	77·38	142·36	258·83	466·86	835·66	1472·97	2482·96	3612·89	2844·14
+ *·90*	16·03	31·28	60·49	116·98	228·19	452·62	918·20	1898·32	3818·02	5437·47
+ *·95*	3·02	6·15	12·51	25·79	54·64	121·08	287·39	755·00	2290·26	7794·37
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0940	·1883	·2832	·3789	·4760	·5747	·6756	·7793	·8869
Mode	·0000	·1492	·2920	·4238	·5420	·6463	·7374	·8168	·8861	·9467
σ	·3536	·3510	·3431	·3298	·3107	·2852	·2524	·2109	·1586	·0915
$(1-\rho^2)/\sqrt{n-1}$	·3536	·3500	·3394	·3217	·2670	·2652	·2263	·1803	·1273	·0672
β_1	·0000	·0256	·1051	·2468	·4677	·7983	1·2989	2·0963	3·5152	6·7561
β_2	2·4000	2·4403	2·5657	2·7907	3·1456	3·6857	4·5239	5·9099	8·5326	15·3136

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 10$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	1·01	·45	·19	·08	·03	·01	·00	·00	—	—
− *·90*	7·50	3·44	1·53	·64	·25	·09	·03	·01	—	—
− *·85*	23·37	11·15	5·11	2·21	·88	·31	·09	·02	·00	—
− *·80*	51·03	25·33	12·00	5·34	2·19	·80	·24	·06	·01	—
− *·75*	91·59	47·29	23·18	10·64	4·47	1·67	·52	·12	·02	—
− *·70*	145·09	77·94	39·56	18·72	8·09	3·09	·98	·23	·03	—
− *·65*	210·66	117·77	61·92	30·23	13·43	5·25	1·71	·42	·06	—
− *·60*	286·72	166·85	90·90	45·82	20·95	8·41	2·81	·70	·10	·00
− *·55*	371·15	224·86	127·01	66·13	31·14	12·85	4·39	1·12	·17	·01
− *·50*	461·43	291·10	170·55	91·82	44·57	18·91	6·63	1·73	·26	·01
− *·45*	554·77	364·50	221·63	123·45	61·84	27·00	9·73	2·60	·41	·02
− *·40*	648·27	443·69	280·14	161·56	83·58	37·60	13·93	3·83	·61	·03
− *·35*	739·03	526·99	345·68	206·59	110·50	51·29	19·56	5·52	·91	·04
− *·30*	824·22	612·48	417·62	258·84	143·28	68·70	27·02	7·85	1·33	·06
− *·25*	901·22	698·04	495·03	318·47	182·65	90·58	36·79	11·03	1·92	·09
− *·20*	967·68	781·38	576·65	385·42	229·29	117·77	49·48	15·32	2·76	·13
− *·15*	1021·57	860·14	660·97	459·39	283·82	151·22	65·82	21·09	3·92	·20
− *·10*	1061·26	931·94	746·13	539·76	346·77	191·94	86·73	28·82	5·55	·29
− *·05*	1085·57	994·42	830·01	625·59	418·49	241·06	113·28	39·13	7·83	·42
·00	1093·75	1045·39	910·20	715·49	499·09	299·70	146·80	52·84	11·02	·62
+ *·05*	1085·57	1082·81	984·11	807·66	588·33	369·02	188·84	71·03	15·49	·91
+ *·10*	1061·26	1104·96	1048·94	899·79	685·53	450·06	241·23	95·11	21·75	1·35
+ *·15*	1021·57	1110·49	1101·85	989·04	789·44	543·65	306·06	126·91	30·57	1·99
+ *·20*	967·68	1098·49	1139·99	1072·04	898·05	650·25	385·66	168·83	43·03	2·98
+ *·25*	901·22	1068·59	1160·70	1144·96	1008·47	769·73	482·55	223·95	60·70	4·48
+ *·30*	824·22	1021·01	1161·60	1203·51	1116·77	900·98	599·22	296·23	85·91	6·82
+ *·35*	739·03	956·66	1140·83	1243·14	1217·79	1041·59	737·88	390·62	122·04	10·51
+ *·40*	648·27	877·13	1097·21	1259·23	1305·16	1187·27	899·90	513·21	174·12	16·44
+ *·45*	554·77	784·74	1030·51	1247·45	1371·20	1331·27	1084·96	671·18	249·62	26·17
+ *·50*	461·43	682·54	941·67	1204·17	1407·27	1463·74	1289·77	872·31	359·56	42·50
+ *·55*	371·15	574·23	833·02	1127·09	1404·16	1571·18	1506·11	1123·76	520·21	70·66
+ *·60*	286·72	464·10	708·50	1015·97	1353·14	1636·20	1718·18	1429·27	754·96	120·77
+ *·65*	210·66	356·82	573·73	873·46	1247·48	1638·20	1899·37	1783·34	1095·94	213·01
+ *·70*	145·09	257·23	436·02	706·04	1084·83	1555·63	2009·23	2160·46	1582·39	389·45
+ *·75*	91·59	170·01	304·07	524·71	870·38	1371·13	1993·01	2497·22	2247·23	740·79
+ *·80*	51·03	99·20	187·37	345·25	620·62	1081·22	1789·53	2668·18	3067·85	1466·04
+ *·85*	23·37	47·59	95·03	187·37	366·47	711·33	1358·89	2473·52	3826·37	2979·24
+ *·90*	7·50	16·01	33·82	71·51	152·84	333·27	746·59	1712·69	3823·56	5828·60
+ *·95*	1·01	2·27	5·07	11·53	27·06	66·90	179·06	538·90	1916·67	7841·74
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0946	·1896	·2850	·3813	·4787	·5776	·6785	·7819	·8887
Mode	·0000	·1411	·2774	·4050	·5219	·6270	·7206	·8036	·8771	·9422
σ	·3333	·3308	·3232	·3103	·2917	·2671	·2355	·1958	·1461	·0832
$(1-\rho^2)/\sqrt{n-1}$	·3333	·3300	·3200	·3033	·2800	·2500	·2133	·1700	·1200	·0633
β_1	·0000	·0242	·0989	·2317	·4374	·7431	1·2002	1·9122	3·1377	5·7475
β_2	2·4545	2·4933	2·6137	2·8292	3·1669	3·6774	4·4598	5·7290	8·0534	13·6667

TABLE A—(*continued*).

$n = 11$. ρ variate (correlation in population sampled).

r variate (correlation in sample).

	0	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·34	·14	·05	·02	·01	·00	·00	—	—	—
− *·90*	3·48	1·46	·59	·22	·08	·02	·01	·00	—	—
− *·85*	13·10	5·73	2·40	·94	·34	·11	·03	·01	—	—
− *·80*	32·59	14·90	6·47	2·63	·97	·31	·08	·02	—	—
− *·75*	64·48	30·82	13·91	5·84	2·22	·74	·20	·04	·00	—
− *·70*	110·28	55·10	25·85	11·22	4·40	1·51	·42	·09	·01	—
− *·65*	170·38	89·00	43·43	19·52	7·90	2·78	·80	·17	·02	—
− *·60*	244·13	133·37	67·72	31·55	13·19	4·77	1·41	·30	·04	—
− *·55*	329·91	188·52	99·67	48·15	20·80	7·76	2·35	·51	·06	—
− *·50*	425·31	254·27	140·04	70·22	31·39	12·08	3·76	·84	·10	·00
− *·45*	527·29	329·89	189·38	98·64	45·66	18·15	5·83	1·34	·17	·01
− *·40*	632·37	414·10	247·94	134·25	64·43	26·48	8·77	2·08	·27	·01
− *·35*	736·81	505·13	315·62	177·84	88·59	37·70	12·90	3·16	·43	·01
− *·30*	836·83	600·74	391·97	230·03	119·07	52·54	18·61	4·71	·65	·02
− *·25*	928·73	698·31	476·06	291·27	156·85	71·87	26·38	6·91	·99	·03
− *·20*	1009·12	794·88	566·57	361·75	202·94	96·71	36·86	10·01	1·49	·05
− *·15*	1074·98	887·29	661·66	441·31	258·26	128·20	50·84	14·35	2·21	·08
− *·10*	1123·87	972·26	759·03	529·42	323·64	167·66	69·32	20·37	3·27	·12
− *·05*	1153·95	1046·54	855·92	625·00	399·73	216·51	93·53	28·70	4·80	·19
·00	1164·10	1107·05	949·18	726·44	486·84	276·25	124·99	40·16	7·04	·29
+ *·05*	1153·95	1150·99	1035·30	831·45	584·86	348·45	165·55	55·88	10·29	·44
+ *·10*	1123·87	1176·02	1110·55	937·05	693·05	434·44	217·39	77·34	15·02	·68
+ *·15*	1074·98	1180·37	1171·11	1039·52	809·89	535·54	283·09	106·54	21·93	1·06
+ *·20*	1009·12	1163·01	1213·24	1134·42	932·85	652·40	365·55	146·17	32·04	1·65
+ *·25*	928·73	1123·74	1233·55	1216·69	1058·17	784·94	467·92	199·73	46·90	2·60
+ *·30*	836·83	1063·29	1229·19	1280·75	1180·68	931·80	593·39	271·79	68·83	4·13
+ *·35*	736·81	983·38	1198·19	1320·82	1293·66	1089·82	744·74	368·23	101·35	6·67
+ *·40*	632·37	886·74	1139·72	1331·30	1388·77	1253·34	923·68	496·34	149·79	10·92
+ *·45*	527·29	777·05	1054·45	1307·30	1456·28	1413·46	1129·63	664·75	222·30	18·21
+ *·50*	425·31	658·86	944·78	1245·35	1485·58	1557·29	1358·02	882·89	331·21	31·02
+ *·55*	329·91	537·38	815·03	1144·27	1466·08	1667·54	1597·69	1159·22	495·09	54·17
+ *·60*	244·13	418·24	671·55	1006·07	1388·90	1722·77	1827·66	1497·43	741·14	97·35
+ *·65*	170·38	307·09	522·51	836·93	1249·15	1699·11	2013·49	1888·67	1107·09	180·79
+ *·70*	110·28	209·16	377·49	647·80	1049·15	1574·50	2105·04	2297·63	1638·62	348·37
+ *·75*	64·48	128·73	246·69	454·52	801·88	1336·70	2039·33	2640·85	2370·71	698·67
+ *·80*	32·59	68·50	139·53	276·64	533·90	995·96	1756·78	2762·99	3261·77	1455·70
+ *·85*	13·10	29·01	62·88	134·46	285·17	600·28	1242·87	2443·04	4017·99	3094·34
+ *·90*	3·48	8·12	18·74	43·33	101·48	243·28	601·85	1532·03	3796·61	6195·13
+ *·95*	·34	·83	2·04	5·11	13·29	36·65	110·61	381·37	1590·43	7823·00
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0952	·1906	·2865	·3831	·4808	·5799	·6807	·7839	·8900
Mode	·0000	·1352	·2667	·3912	·5066	·6121	·7074	·7930	·8697	·9384
σ	·3162	·3138	·3064	·2938	·2758	·2519	·2214	·1833	·1360	·0767
$(1-\rho^2)/\sqrt{n-1}$	·3162	·3131	·3036	·2878	·2656	·2372	·2024	·1613	·1138	·0601
β_1	·0000	·0228	·0932	·2179	·4101	·6933	1·1112	1·7516	2·8193	4·9565
β_2	2·5000	2·5372	2·6527	2·8587	3·1798	3·6616	4·3914	5·5540	7·6188	12·2879

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 12.$ ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·11	·04	·01	·00	·00	·00	—	—	—	—
− *·90*	1·60	·61	·22	·08	·02	·01	·00	—	—	—
− *·85*	7·30	2·93	1·12	·40	·13	·04	·01	—	—	—
− *·80*	20·67	8·71	3·47	1·28	·43	·12	·03	·00	—	—
− *·75*	45·08	19·94	8·29	3·18	1·10	·33	·08	·01	—	—
− *·70*	83·24	38·68	16·77	6·68	2·38	·73	·18	·03	·00	—
− *·65*	136·86	66·80	30·25	12·52	4·62	1·46	·37	·07	·01	—
− *·60*	206·44	105·87	50·10	21·57	8·24	2·69	·70	·13	·01	—
− *·55*	291·24	156·97	77·68	34·81	13·80	4·65	1·25	·23	·02	—
− *·50*	389·33	220·58	114·20	53·33	21·95	7·66	2·12	·41	·04	—
− *·45*	497·73	296·51	160·71	78·27	33·48	12·11	3·46	·69	·07	—
− *·40*	612·62	383·82	217·93	110·79	49·33	18·52	5·48	1·13	·12	—
− *·35*	729·56	480·84	286·20	152·04	70·53	27·52	8·45	1·80	·20	·00
− *·30*	843·79	585·17	365·35	203·01	98·26	39·90	12·72	2·80	·32	·01
− *·25*	950·51	693·78	454·68	264·55	133·77	56·63	18·79	4·30	·51	·01
− *·20*	1045·09	803·05	552·83	337·18	178·37	78·86	27·27	6·49	·80	·02
− *·15*	1123·41	908·99	657·79	421·03	233·38	107·94	39·00	9·69	1·24	·03
− *·10*	1181·98	1007·35	766·83	515·70	299·98	145·45	55·03	14·30	1·91	·05
− *·05*	1218·21	1093·82	876·57	620·13	379·19	193·12	76·70	20·91	2·93	·08
·00	1230·47	1164·30	983·02	732·48	471·64	252·88	105·70	30·32	4·46	·13
+ *·05*	1218·21	1215·06	1081·67	850·05	577·41	326·69	144·13	43·65	6·79	·21
+ *·10*	1181·98	1243·05	1167·70	969·15	695·83	416·49	194·57	62·45	10·30	·34
+ *·15*	1123·41	1246·03	1236·18	1085·08	825·16	523·93	260·05	88·83	15·62	·56
+ *·20*	1045·09	1222·86	1282·35	1192·20	962·35	650·07	344·11	125·68	23·70	·91
+ *·25*	950·51	1173·63	1301·99	1284·05	1102·71	794·98	450·63	176·90	35·99	1·49
+ *·30*	843·79	1099·72	1291·80	1353·61	1239·72	957·09	583·60	247·67	54·78	2·49
+ *·35*	729·56	1003·91	1249·80	1393·76	1364·85	1132·50	746·54	344·77	83·60	4·20
+ *·40*	612·62	890·31	1175·76	1397·87	1467·64	1314·07	941·64	476·75	127·99	7·20
+ *·45*	497·73	764·16	1071·56	1360·66	1536·09	1490·50	1168·15	653·92	196·63	12·58
+ *·50*	389·33	631·64	941·41	1279·15	1557·56	1645·55	1420·17	887·55	303·03	22·49
+ *·55*	291·24	499·45	791·98	1153·78	1520·32	1757·80	1683·36	1187·72	468·00	41·25
+ *·60*	206·44	374·33	632·17	989·48	1415·90	1801·62	1930·97	1558·25	722·68	77·95
+ *·65*	136·86	262·47	472·60	796·46	1242·33	1750·35	2120·06	1986·75	1110·84	152·41
+ *·70*	83·24	168·90	324·59	590·32	1007·76	1582·81	2190·55	2427·07	1685·48	309·54
+ *·75*	45·08	96·80	198·77	391·03	733·76	1294·33	2072·67	2774·00	2484·25	654·55
+ *·80*	20·67	46·98	103·20	220·15	456·19	911·23	1713·02	2842·00	3444·82	1435·84
+ *·85*	7·30	17·56	41·32	95·84	220·41	503·13	1129·13	2396·80	4191·11	3192·59
+ *·90*	1·60	4·09	10·31	26·08	66·93	176·39	481·91	1361·28	3744·80	6541·18
+ *·95*	·11	·30	·81	2·25	6·48	19·94	67·87	268·10	1310·98	7752·85
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0956	·1914	·2877	·3847	·4826	·5818	·6826	·7855	·8910
Mode	·0000	·1309	·2586	·3805	·4948	·6004	·6968	·7843	·8636	·9353
σ	·3015	·2991	·2919	·2797	·2622	·2390	·2096	·1729	·1276	·0714
$(1-\rho^2)/\sqrt{n-1}$	·3015	·2985	·2895	·2744	·2533	·2261	·1930	·1538	·1085	·0573
β_1	·0000	·0215	·0880	·2054	·3854	·6487	1·0329	1·6118	2·5509	4·3322
β_2	2·5385	2·5742	2·6848	2·8816	3·1870	3·6417	4·3231	5·3905	7·2330	11·1595

TABLE A—(*continued*).

$n = 13$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·04	·01	·00	·00	·00	—	—	—	—	—
− *·90*	·73	·26	·09	·03	·01	·00	—	—	—	—
− *·85*	4·04	1·49	·52	·17	·05	·01	·00	—	—	—
− *·80*	13·03	5·06	1·85	·62	·19	·05	·01	—	—	—
− *·75*	31·34	12·84	4·91	1·72	·54	·14	·03	·00	—	—
− *·70*	62·49	27·00	10·82	3·95	1·28	·35	·08	·01	—	—
− *·65*	109·33	49·86	20·96	7·99	2·69	·76	·17	·03	—	—
− *·60*	173·60	83·57	36·87	14·67	5·13	1·51	·35	·05	·00	—
− *·55*	255·68	129·97	60·20	25·03	9·10	2·78	·66	·11	·01	—
− *·50*	354·43	190·29	92·61	40·28	15·26	4·83	1·19	·20	·02	—
− *·45*	467·24	265·03	135·62	61·76	24·42	8·04	2·05	·35	·03	—
− *·40*	590·21	353·79	190·50	90·93	37·56	12·88	3·41	·61	·05	—
− *·35*	718·39	455·19	258·08	129·26	55·84	19·97	5·50	1·01	·09	—
− *·30*	846·13	566·86	338·66	178·18	80·64	30·14	8·65	1·66	·15	—
− *·25*	967·43	685·47	431·85	238·96	113·45	44·38	13·30	2·66	·26	·00
− *·20*	1076·39	806·84	536·45	312·55	155·92	63·95	20·07	4·19	·42	·01
− *·15*	1167·55	926·10	650·33	399·46	209·73	90·38	29·75	6·51	·69	·01
− *·10*	1236·25	1037·95	770·44	499·56	276·51	125·48	43·44	9·98	1·11	·02
− *·05*	1278·96	1136·93	892·78	611·89	357·72	171·31	62·54	15·15	1·77	·04
·00	1293·45	1217·76	1012·46	734·51	454·39	230·21	88·89	22·76	2·82	·06
+ *·05*	1278·96	1275·64	1123·90	864·29	566·93	304·67	124·80	33·91	4·45	·10
+ *·10*	1236·25	1306·66	1221·04	996·83	694·79	397·09	173·18	50·15	7·03	·17
+ *·15*	1167·55	1308·10	1297·68	1126·41	836·11	509·76	237·58	73·66	11·07	·29
+ *·20*	1076·39	1278·72	1347·93	1246·04	987·33	644·20	322·16	107·48	17·43	·50
+ *·25*	967·43	1218·98	1366·66	1347·70	1142·82	800·73	431·61	155·83	27·47	·86
+ *·30*	846·13	1131·13	1350·13	1422·76	1294·58	977·68	570·83	224·46	43·35	1·49
+ *·35*	718·39	1019·23	1296·48	1462·65	1432·08	1170·41	744·26	321·04	68·58	2·63
+ *·40*	590·21	888·97	1206·29	1459·72	1542·51	1370·21	954·71	455·44	108·77	4·72
+ *·45*	467·24	747·35	1082·96	1408·44	1611·41	1563·16	1201·40	639·76	172·98	8·65
+ *·50*	354·43	602·21	932·90	1306·67	1624·11	1729·32	1477·09	887·39	275·75	16·22
+ *·55*	255·68	461·64	765·35	1157·00	1567·94	1842·84	1763·97	1210·31	440·01	31·24
+ *·60*	173·60	333·18	591·84	967·83	1435·55	1873·82	2029·03	1612·76	700·88	62·08
+ *·65*	109·33	223·11	425·11	753·80	1228·80	1793·32	2220·16	2078·63	1108·60	127·80
+ *·70*	62·49	135·64	277·57	534·99	962·72	1582·52	2267·17	2549·96	1724·34	273·57
+ *·75*	31·34	72·39	159·28	334·58	667·77	1246·49	2095·16	2898·15	2589·24	609·94
+ *·80*	13·03	32·04	75·91	174·24	387·67	829·19	1661·34	2907·54	3618·62	1408·68
+ *·85*	4·04	10·58	27·01	67·94	169·42	419·43	1020·26	2338·80	4348·30	3276·40
+ *·90*	·73	2·05	5·65	15·61	43·90	127·20	383·80	1203·06	3673·97	6869·84
+ *·95*	·04	·11	·32	·98	3·14	10·79	41·42	187·46	1074·87	7642·57
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0960	·1921	·2887	·3859	·4841	·5834	·6841	·7868	·8919
Mode	·0000	·1274	·2522	·3721	·4853	·5908	·6880	·7772	·8585	·9326
σ	·2887	·2863	·2793	·2674	·2504	·2279	·1994	·1640	·1206	·0671
$(1-\rho^2)/\sqrt{n-1}$	·2887	·2858	·2771	·2627	·2425	·2165	·1848	·1472	·1039	·0548
β_1	·0000	·0204	·0833	·1941	·3631	·6087	·9636	1·4902	2·3236	3·8337
β_2	2·5714	2·6057	2·7117	2·8999	3·1904	3·6201	4·2575	5·2404	6·8937	10·2454

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 14$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·01	·00	·00	·00	—	—	—	—	—	—
− *·90*	·34	·11	·03	·01	·00	—	—	—	—	—
− *·85*	2·23	·75	·24	·07	·02	·00	—	—	—	—
− *·80*	8·18	2·93	·98	·30	·08	·02	·00	—	—	—
− *·75*	21·69	8·22	2·90	·93	·26	·06	·01	—	—	—
− *·70*	46·70	18·77	6·95	2·33	·69	·17	·03	·00	—	—
− *·65*	86·94	37·04	14·45	5·07	1·55	·40	·08	·01	—	—
− *·60*	145·33	65·68	27·00	9·93	3·17	·84	·17	·02	—	—
− *·55*	223·45	107·13	46·44	17·91	5·98	1·65	·35	·05	·00	—
− *·50*	321·20	163·42	74·76	30·28	10·57	3·04	·66	·09	·01	—
− *·45*	436·63	235·83	113·94	48·51	17·73	5·31	1·21	·18	·01	—
− *·40*	566·06	324·64	165·76	74·28	28·46	8·92	2·11	·32	·02	—
− *·35*	704·20	428·97	231·67	109·39	44·02	14·43	3·57	·57	·04	—
− *·30*	844·64	546·64	312·50	155·68	65·88	22·66	5·86	·98	·07	—
− *·25*	980·21	674·21	408·32	214·86	95·78	34·62	9·38	1·64	·13	—
− *·20*	1103·62	806·97	518·19	288·41	135·67	51·63	14·70	2·69	·23	·00
− *·15*	1207·94	939·26	640·05	377·28	187·62	75·34	22·59	4·35	·38	·01
− *·10*	1287·18	1064·65	770·57	481·74	253·73	107·76	34·14	6·94	·64	·01
− *·05*	1336·68	1176·41	905·17	601·04	335·93	151·28	50·77	10·92	1·07	·02
·00	1353·52	1267·92	1038·07	733·22	435·79	208·62	74·41	17·01	1·77	·03
+ *·05*	1336·68	1333·18	1162·50	874·80	554·12	282·80	107·57	26·23	2·91	·05
+ *·10*	1287·18	1367·32	1271·05	1020·68	690·62	376·89	153·45	40·10	4·77	·09
+ *·15*	1207·94	1367·06	1356·11	1164·04	843·38	493·74	216·07	60·80	7·81	·15
+ *·20*	1103·62	1331·09	1410·48	1296·43	1008·40	635·51	300·25	91·50	12·76	·27
+ *·25*	980·21	1260·37	1428·09	1408·13	1179·07	802·90	411·53	136·65	20·87	·49
+ *·30*	844·64	1158·20	1404·74	1488·72	1345·78	994·23	555·84	202·51	34·16	·89
+ *·35*	704·20	1030·12	1338·84	1528·05	1495·87	1204·16	738·66	297·60	56·01	1·64
+ *·40*	566·06	883·63	1232·03	1517·46	1613·92	1422·35	963·63	433·15	92·02	3·08
+ *·45*	436·63	727·61	1089·56	1451·35	1682·84	1632·02	1230·08	623·12	151·50	5·92
+ *·50*	321·20	571·57	920·31	1328·79	1685·90	1809·23	1529·42	883·27	249·81	11·64
+ *·55*	223·45	424·77	736·29	1155·02	1609·82	1923·36	1840·21	1227·85	411·85	23·56
+ *·60*	145·33	295·23	551·59	942·42	1448·96	1940·22	2122·58	1661·76	676·71	49·22
+ *·65*	86·94	188·79	380·68	710·22	1209·98	1829·15	2314·64	2165·10	1101·47	106·69
+ *·70*	46·70	108·45	236·30	482·68	915·58	1575·19	2336·05	2667·21	1756·31	240·71
+ *·75*	21·69	53·89	127·07	284·99	605·00	1195·09	2108·50	3014·47	2686·77	565·87
+ *·80*	8·18	21·76	55·58	137·29	327·97	751·18	1604·06	2961·44	3784·47	1375·97
+ *·85*	2·23	6·34	17·57	47·94	129·65	348·10	917·81	2272·14	4491·55	3347·70
+ *·90*	·34	1·02	3·08	9·30	28·67	91·32	304·31	1058·56	3588·66	7183·43
+ *·95*	·01	·04	·13	·43	1·52	5·81	25·17	130·49	877·42	7501·01
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0963	·1927	·2896	·3870	·4853	·5847	·6854	·7879	·8926
Mode	·0000	·1246	·2470	·3652	·4775	·5828	·6808	·7711	·8541	·9303
σ	·2774	·2751	·2682	·2566	·2401	·2182	·1906	·1564	·1145	·0634
$(1-\rho^2)/\sqrt{n-1}$	·2774	·2746	·2663	·2524	·2330	·2080	·1775	·1414	·0998	·0527
β_1	·0000	·0194	·0790	·1838	·3430	·5729	·9020	1·3838	2·1298	3·4290
β_2	2·6000	2·6329	2·7346	2·9145	3·1912	3·5979	4·1955	5·1038	6·5961	9·4886

TABLE A—(*continued*).

$n = 15$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	·00	·00	—	—	—	—	—	—	—
− *·90*	·15	·04	·01	·00	·00	—	—	—	—	—
− *·85*	1·22	·38	·11	·03	·01	·00	—	—	—	—
− *·80*	5·12	1·69	·52	·14	·04	·01	—	—	—	—
− *·75*	14·96	5·25	1·70	·50	·13	·03	·00	—	—	—
− *·70*	34·77	12·99	4·45	1·37	·37	·08	·01	—	—	—
− *·65*	68·88	27·42	9·93	3·21	·90	·21	·04	·00	—	—
− *·60*	121·21	51·42	19·70	6·69	1·96	·47	·08	·01	—	—
− *·55*	194·55	87·97	35·70	12·77	3·91	·97	·18	·02	—	—
− *·50*	289·98	139·82	60·13	22·68	7·29	1·90	·37	·05	·00	—
− *·45*	406·50	209·05	95·35	37·96	12·82	3·50	·71	·09	·01	—
− *·40*	540·85	296·76	143·69	60·46	21·49	6·15	1·30	·17	·01	—
− *·35*	687·70	402·73	207·17	92·23	34·56	10·39	2·30	·32	·02	—
− *·30*	839·97	525·16	287·27	135·50	53·62	16·97	3·95	·57	·04	—
− *·25*	989·42	660·62	384·61	192·47	80·56	26·90	6·59	1·00	·07	—
− *·20*	1127·28	804·07	498·68	265·12	117·61	41·52	10·72	1·72	·12	—
− *·15*	1245·03	949·02	627·56	354·99	167·21	62·56	17·09	2·90	·21	—
− *·10*	1335·15	1087·93	767·80	462·81	231·94	92·19	26·73	4·80	·37	·00
− *·05*	1391·74	1212·67	914·28	588·16	314·29	133·08	41·06	7·85	·64	·01
·00	1411·04	1315·18	1060·32	729·17	416·38	188·35	62·06	12·66	1·11	·01
+ *·05*	1391·74	1388·08	1197·90	882·11	539·56	261·56	92·37	20·21	1·89	·02
+ *·10*	1335·15	1425·42	1318·13	1041·17	683·89	356·36	135·46	31·94	3·23	·04
+ *·15*	1245·03	1423·30	1411·83	1198·41	847·52	476·43	195·77	50·00	5·49	·08
+ *·20*	1127·28	1380·40	1470·37	1343·80	1026·06	624·58	278·78	77·60	9·31	·15
+ *·25*	989·42	1298·27	1486·66	1465·74	1211·89	802·05	390·92	119·38	15·80	·28
+ *·30*	839·97	1181·45	1456·06	1551·88	1393·76	1007·27	539·22	182·03	26·81	·53
+ *·35*	687·70	1037·21	1377·39	1590·37	1556·64	1234·25	730·36	274·85	45·57	1·02
+ *·40*	540·85	875·02	1253·60	1571·56	1682·30	1470·94	968·99	410·40	77·56	2·01
+ *·45*	406·50	705·73	1092·09	1489·95	1750·86	1697·55	1254·73	604·65	132·19	4·04
+ *·50*	289·98	540·45	904·49	1346·21	1743·50	1885·77	1577·71	875·91	225·46	8·33
+ *·55*	194·55	389·38	705·68	1148·73	1646·64	1999·91	1912·58	1241·01	384·06	17·70
+ *·60*	121·21	260·61	512·15	914·23	1457·04	2001·47	2212·18	1705·89	650·97	38·88
+ *·65*	68·88	159·16	339·61	666·66	1187·01	1858·75	2404·17	2246·82	1090·34	88·74
+ *·70*	34·77	86·37	200·40	433·86	867·51	1562·05	2398·10	2779·52	1782·27	211·01
+ *·75*	14·96	39·97	100·99	241·85	546·09	1141·54	2114·05	3123·86	2777·70	523·05
+ *·80*	5·12	14·72	40·55	107·77	276·43	677·98	1543·04	3005·21	3943·34	1339·09
+ *·85*	1·22	3·79	11·39	33·71	98·85	287·83	822·59	2199·24	4622·48	3408·02
+ *·90*	·15	·51	1·67	5·52	18·65	65·32	240·39	927·98	3492·48	7483·96
+ *·95*	·00	·01	·05	·19	·73	3·12	15·23	90·51	713·62	7335·23
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0965	·1932	·2903	·3879	·4864	·5858	·6865	·7888	·8932
Mode	·0000	·1224	·2429	·3595	·4710	·5761	·6745	·7659	·8504	·9283
σ	·2673	·2650	·2583	·2470	·2309	·2096	·1828	·1496	·1093	·0602
$(1-\rho^2)/\sqrt{n-1}$	·2673	·2646	·2566	·2423	·2245	·2004	·1710	·1363	·0962	·0508
β_1	·0000	·0184	·0751	·1745	·3248	·5407	·8473	1·2904	1·9635	3·0956
β_2	2·6250	2·6566	2·7542	2·9265	3·1904	3·5759	4·1375	4·9799	6·3347	8·8548

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 16$. ρ variate (correlation in population sampled).

r variate (correlation in sample)	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	—	—	—	—	—	—	—	—	—
− *·90*	·07	·02	·00	·00	—	—	—	—	—	—
− *·85*	·67	·19	·05	·01	·00	—	—	—	—	—
− *·80*	3·19	·97	·27	·07	·02	·00	—	—	—	—
− *·75*	10·28	3·34	1·00	·27	·06	·01	·00	—	—	—
− *·70*	25·80	8·97	2·84	·80	·19	·04	·01	—	—	—
− *·65*	54·39	20·23	6·80	2·02	·51	·11	·02	—	—	—
− *·60*	100·76	40·12	14·33	4·50	1·20	·26	·04	·00	—	—
− *·55*	168·85	72·01	27·35	9·08	2·55	·57	·09	·01	—	—
− *·50*	260·97	119·24	48·20	16·93	5·01	1·18	·20	·02	—	—
− *·45*	377·23	184·72	79·55	29·61	9·24	2·29	·41	·05	—	—
− *·40*	515·11	270·42	124·16	49·05	16·17	4·23	·80	·09	·00	—
− *·35*	669·43	376·88	184·68	77·51	27·05	7·46	1·48	·18	·01	—
− *·30*	832·67	502·91	263·24	117·56	43·50	12·67	2·65	·34	·02	—
− *·25*	995·53	645·25	361·12	171·85	67·54	20·84	4·61	·61	·03	—
− *·20*	1147·76	798·62	478·36	242·94	101·63	33·28	7·80	1·10	·06	—
− *·15*	1279·16	955·82	613·35	332·95	148·55	51·78	12·89	1·93	·12	—
− *·10*	1380·50	1108·16	762·59	443·20	211·35	78·63	20·86	3·32	·21	—
− *·05*	1444·45	1246·06	920·53	573·72	293·10	116·70	33·10	5·62	·38	—
·00	1466·31	1359·84	1079·59	722·83	396·57	169·51	51·59	9·40	·69	·00
+ *·05*	1444·45	1440·62	1230·45	886·63	523·71	241·10	79·06	15·52	1·23	·01
+ *·10*	1380·50	1481·25	1362·60	1058·69	675·07	335·89	119·19	25·36	2·18	·02
+ *·15*	1279·16	1477·14	1465·15	1229·86	848·97	458·26	176·81	40·99	3·84	·04
+ *·20*	1147·76	1426·97	1527·93	1388·46	1040·70	611·89	258·02	65·60	6·77	·08
+ *·25*	995·53	1333·04	1542·70	1520·86	1241·67	798·66	370·16	103·97	11·92	·16
+ *·30*	832·67	1201·32	1504·45	1612·58	1438·86	1017·24	521·43	163·10	20·98	·31
+ *·35*	669·43	1041·01	1412·54	1649·98	1614·73	1261·08	719·87	253·03	36·96	·63
+ *·40*	515·11	863·74	1271·49	1622·42	1748·02	1516·38	971·31	387·62	65·16	1·30
+ *·45*	377·23	682·33	1091·14	1524·73	1815·85	1760·11	1275·84	584·87	114·98	2·74
+ *·50*	260·97	509·39	886·11	1359·54	1797·36	1959·32	1622·38	865·86	202·85	5·94
+ *·55*	168·85	355·79	674·19	1138·85	1678·96	2072·94	1981·54	1250·36	357·03	13·25
+ *·60*	100·76	229·32	474·01	884·08	1460·52	2058·14	2298·30	1745·69	624·23	30·62
+ *·65*	54·39	133·74	302·02	623·79	1160·78	1882·87	2489·31	2324·29	1075·93	73·57
+ *·70*	25·80	68·58	169·43	388·74	819·36	1544·13	2454·04	2887·46	1802·95	184·41
+ *·75*	10·28	29·55	80·00	204·59	491·36	1086·96	2112·96	3227·08	2862·73	481·97
+ *·80*	3·19	9·92	29·49	84·33	232·25	609·99	1479·67	3040·08	4096·07	1299·15
+ *·85*	·67	2·25	7·36	23·62	75·13	237·24	734·93	2121·99	4742·40	3458·65
+ *·90*	·07	·25	·90	3·27	12·09	46·58	189·31	810·97	3388·31	7772·88
+ *·95*	·00	·00	·02	·08	·35	1·67	9·19	62·58	578·60	7150·91
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0968	·1937	·2909	·3888	·4873	·5868	·6875	·7896	·8937
Mode	·0000	·1206	·2394	·3548	·4655	·5705	·6692	·7614	·8471	·9265
σ	·2582	·2560	·2495	·2384	·2227	·2020	·1759	·1437	·1047	·0575
$(1-\rho^2)/\sqrt{n-1}$	·2582	·2556	·2479	·2350	·2169	·1936	·1652	·1310	·0930	·0491
β_1	·0000	·0176	·0716	·1660	·3083	·5117	·7983	1·2080	1·8195	2·8181
β_2	2·6471	2·6775	2·7712	2·9363	3·1883	3·5545	4·0836	4·8677	6·1046	8·3239

TABLE A—(*continued*).

$n = 17$. ρ variate (correlation in population sampled).

r variate (correlation in sample).

	0	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	·00	—	—	—	—	—	—	—	—
− *·90*	·03	·01	·00	—	—	—	—	—	—	—
− *·85*	·37	·10	·02	·01	·00	—	—	—	—	—
− *·80*	1·98	·56	·14	·03	·01	·00	—	—	—	—
− *·75*	7·05	2·12	·58	·14	·03	·01	—	—	—	—
− *·70*	19·10	6·17	1·81	·47	·10	·02	·00	—	—	—
− *·65*	42·84	14·88	4·64	1·27	·29	·05	·01	—	—	—
− *·60*	83·54	31·23	10·39	3·02	·74	·14	·02	—	—	—
− *·55*	146·13	58·78	20·89	6·43	1·66	·34	·05	·00	—	—
− *·50*	234·22	101·41	38·54	12·61	3·43	·74	·11	·01	—	—
− *·45*	349·12	162·78	66·18	23·04	6·64	1·50	·24	·02	—	—
− *·40*	489·26	245·73	106·99	39·68	12·14	2·90	·49	·05	—	—
− *·35*	649·87	351·73	164·17	64·96	21·11	5·33	95	·10	·00	—
− *·30*	823·17	480·28	240·55	101·72	35·19	9·43	1·78	·20	·01	—
− *·25*	998·93	628·51	338·14	153·03	56·47	16·10	3·22	·37	·02	—
− *·20*	1165·43	791·03	457·61	222·01	87·57	26·61	5·66	·70	·03	—
− *·15*	1310·64	960·04	597·82	311·43	131·60	42·74	9·69	1·28	·06	—
− *·10*	1423·48	1125·69	755·35	423·25	192·06	66·87	16·23	2·28	·12	—
− *·05*	1495·05	1276·87	924·29	558·10	272·60	102·06	26·61	4·01	·23	—
·00	1519·58	1402·18	1096·21	714·59	376·67	152·13	42·77	6·95	·43	·00
+ *·05*	1495·05	1491·07	1260·42	888·75	506·94	221·67	67·49	11·89	·79	·01
+ *·10*	1423·48	1535·05	1404·71	1073·56	664·54	315·72	104·60	20·08	1·46	·01
+ *·15*	1310·64	1528·81	1516·33	1258·69	848·09	439·58	159·26	33·51	2·68	·02
+ *·20*	1165·43	1471·07	1583·40	1430·69	1052·66	597·82	238·16	55·31	4·91	·04
+ *·25*	998·93	1365·00	1596·49	1573·74	1268·71	793·12	349·55	90·30	8·97	·09
+ *·30*	823·17	1218·19	1550·21	1671·08	1481·38	1024·51	502·86	145·74	16·37	·18
+ *·35*	649·87	1041·98	1444·64	1707·15	1670·43	1284·98	707·60	232·31	29·89	·39
+ *·40*	489·26	850·26	1286·11	1670·36	1811·36	1558·97	970·98	365·12	54·60	·84
+ *·45*	349·12	657·90	1087·22	1556·06	1878·12	1820·03	1293·78	564·21	99·74	1·86
+ *·50*	234·22	478·81	865·73	1369·26	1847·84	2030·22	1663·80	853·62	182·02	4·22
+ *·55*	146·13	324·22	642·35	1125·98	1707·27	2142·80	2047·42	1256·38	331·00	9·90
+ *·60*	83·54	201·23	437·52	852·59	1460·03	2110·68	2381·31	1781·59	596·99	24·05
+ *·65*	42·84	112·08	267·85	582·09	1132·06	1902·13	2570·49	2397·95	1058·86	60·84
+ *·70*	19·10	54·30	142·84	347·36	771·78	1522·30	2504·51	2991·52	1818·97	160·72
+ *·75*	7·05	21·79	63·21	172·59	440·91	1032·18	2106·17	3324·73	2942·45	442·92
+ *·80*	1·98	6·67	21·38	65·80	194·61	547·33	1415·08	3067·10	4243·31	1257·03
+ *·85*	·37	1·34	4·74	16·51	56·94	195·02	654·85	2042·00	4852·41	3500·67
+ *·90*	·03	·12	·49	1·93	7·82	33·12	148·67	706·81	3278·45	8051·44
+ *·95*	·00	·00	·01	·03	·17	·89	5·53	43·15	467·87	6952·68
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0970	·1941	·2915	·3895	·4881	·5876	·6883	·7903	·8941
Mode	·0000	·1190	·2364	·3507	·4608	·5656	·6646	·7575	·8442	·9250
σ	·2500	·2479	·2415	·2307	·2153	·1951	·1696	·1384	·1006	·0551
$(1-\rho^2)/\sqrt{n-1}$	·2500	·2475	·2400	·2275	·2100	·1875	·1600	·1275	·0900	·0475
β_1	·0000	·0168	·0683	·1582	·2934	·4855	·7543	1·1348	1·6940	2·5832
β_2	2·6667	2·6960	2·7861	2·9446	3·1855	3·5340	4·0337	4·7661	5·9012	7·8748

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 18$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	—	—	—	—	—	—	—	—	—
− *·90*	·01	·00	·00	—	—	—	—	—	—	—
− *·85*	·20	·05	·01	·00	—	—	—	—	—	—
− *·80*	1·23	·32	·08	·02	·00	—	—	—	—	—
− *·75*	4·82	1·34	·34	·08	·01	·00	—	—	—	—
− *·70*	14·10	4·24	1·15	·27	·05	·01	—	—	—	—
− *·65*	33·66	10·93	3·16	·80	·17	·03	—	—	—	—
− *·60*	69·10	24·24	7·52	2·02	·45	·08	·01	—	—	—
− *·55*	126·18	47·87	15·93	4·55	1·08	·20	·03	—	—	—
− *·50*	209·71	86·05	30·73	9·36	2·35	·46	·06	—	—	—
− *·45*	322·33	143·10	54·93	17·88	4·76	·98	·14	·01	—	—
− *·40*	463·60	222·77	91·98	32·03	9·09	1·98	·30	·03	—	—
− *·35*	629·39	327·48	145·60	54·32	16·44	3·80	·61	·06	—	—
− *·30*	811·85	457·58	219·30	87·80	28·40	7·01	1·19	·11	—	—
− *·25*	999·97	610·75	315·87	135·94	47·10	12·40	2·24	·23	·01	—
− *·20*	1180·56	781·66	436·73	202·39	75·28	21·22	4·09	·44	·02	—
− *·15*	1339·70	961·98	581·30	290·60	116·32	35·20	7·27	·84	·03	—
− *·10*	1464·32	1140·77	746·40	403·25	174·11	56·74	12·60	1·57	·07	—
− *·05*	1543·76	1305·33	925·87	541·62	252·92	89·04	21·34	2·86	·14	—
·00	1571·04	1442·41	1110·44	704·76	356·91	136·21	35·38	5·13	·27	—
+ *·05*	1543·76	1539·62	1288·06	888·76	489·54	203·29	57·47	9·09	·51	·00
+ *·10*	1464·32	1587·04	1444·69	1086·05	652·63	296·06	91·57	15·86	·98	·01
+ *·15*	1339·70	1578·54	1565·59	1285·14	845·22	420·66	143·10	27·33	1·87	·01
+ *·20*	1180·56	1512·94	1637·00	1470·71	1062·25	582·69	219·31	46·52	3·55	·02
+ *·25*	999·97	1394·42	1648·23	1624·60	1293·27	785·76	329·31	78·24	6·73	·05
+ *·30*	811·85	1232·37	1593·57	1727·60	1521·54	1029·39	483·81	129·92	12·75	·11
+ *·35*	629·39	1040·48	1473·96	1762·12	1723·97	1306·25	693·90	212·79	24·12	·24
+ *·40*	463·60	835·02	1297·82	1715·64	1872·57	1598·97	968·37	343·11	45·64	·54
+ *·45*	322·33	632·84	1080·74	1584·29	1937·95	1877·54	1308·88	543·00	86·32	1·26
+ *·50*	209·71	449·00	843·82	1375·79	1895·25	2098·72	1702·25	839·57	162·94	3·00
+ *·55*	126·18	294·75	610·56	1110·63	1731·96	2209·80	2110·51	1259·46	306·15	7·37
+ *·60*	69·10	176·17	402·90	820·29	1456·10	2159·46	2461·52	1813·97	569·59	18·84
+ *·65*	33·66	93·71	236·98	541·90	1101·45	1917·07	2648·10	2468·15	1039·63	50·19
+ *·70*	14·10	42·89	120·15	309·66	725·26	1497·24	2550·03	3092·09	1830·85	139·75
+ *·75*	4·82	16·03	49·82	145·26	394·72	977·87	2094·49	3417·34	3017·33	406·10
+ *·80*	1·23	4·48	15·47	51·23	162·68	489·96	1350·14	3087·13	4385·61	1213·45
+ *·85*	·20	·79	3·05	11·51	43·06	159·94	582·14	1960·44	4953·44	3534·99
+ *·90*	·01	·05	·26	1·14	5·05	23·50	116·49	614·60	3164·79	8320·69
+ *·95*	·00	·00	·00	·01	·08	·47	3·32	29·69	377·45	6744·32
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0971	·1944	·2920	·3901	·4888	·5884	·6890	·7909	·8945
Mode	·0000	·1176	·2338	·3472	·4567	·5613	·6605	·7540	·8417	·9236
σ	·2425	·2405	·2342	·2237	·2086	·1889	·1641	·1337	·0970	·0530
$(1-\rho^2)/\sqrt{n-1}$	·2425	·2401	·2328	·2207	·2037	·1819	·1552	·1237	·0873	·0461
β_1	·0000	·0161	·0653	·1511	·2797	·4617	·7147	1·0695	1·5839	2·3830
β_2	2·6842	2·7124	2·7992	2·9515	3·1823	3·5144	3·9873	4·6737	5·7207	7·4908

TABLE A—(*continued*).

$n = 19$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	—	—	—	—	—	—	—	—	—
− *·90*	·01	·00	—	—	—	—	—	—	—	—
− *·85*	·11	·02	·00	·00	—	—	—	—	—	—
− *·80*	·76	·18	·04	·01	·00	—	—	—	—	—
− *·75*	3·29	·85	·20	·04	·01	—	—	—	—	—
− *·70*	10·39	2·90	·73	·16	·03	·00	—	—	—	—
− *·65*	26·39	8·00	2·15	·50	·10	·01	—	—	—	—
− *·60*	57·03	18·78	5·43	1·35	·27	·04	·00	—	—	—
− *·55*	108·73	38·90	12·11	3·21	·70	·12	·01	—	—	—
− *·50*	187·37	72·86	24·46	6·94	1·60	·28	·03	·00	—	—
− *·45*	296·98	125·54	45·49	13·84	3·41	·64	·08	·01	—	—
− *·40*	438·38	201·54	78·91	25·79	6·79	1·35	·18	·01	—	—
− *·35*	608·28	304·27	128·86	45·32	12·77	2·71	·39	·03	—	—
− *·30*	799·03	435·05	199·51	75·63	22·88	5·19	·79	·07	—	—
− *·25*	998·93	592·25	294·45	120·51	39·21	9·54	1·56	·14	·00	—
− *·20*	1193·40	770·79	415·93	184·13	64·59	16·89	2·96	·28	·01	—
− *·15*	1366·56	961·93	564·06	270·60	102·59	28·93	5·45	·56	·02	—
− *·10*	1503·20	1153·66	736·02	383·39	157·51	48·04	9·76	1·07	·04	—
− *·05*	1590·74	1331·66	925·52	524·53	234·18	77·52	17·08	2·03	·08	—
·00	1620·88	1480·70	1122·52	693·62	337·49	121·70	29·20	3·78	·16	—
+ *·05*	1590·74	1586·45	1313·57	886·92	471·76	186·07	48·84	6·93	·33	—
+ *·10*	1503·20	1637·37	1482·73	1096·42	639·60	277·05	80·00	12·50	·66	·00
+ *·15*	1366·56	1626·51	1613·08	1309·42	840·60	401·73	128·32	22·24	1·30	·01
+ *·20*	1193·40	1552·77	1688·90	1508·71	1069·70	566·77	201·53	39·05	2·57	·01
+ *·25*	998·93	1421·51	1698·12	1673·64	1315·58	776·85	309·60	67·65	5·04	·03
+ *·30*	799·03	1244·13	1634·75	1782·33	1559·56	1032·16	464·51	115·58	9·90	·06
+ *·35*	608·28	1036·82	1500·76	1815·10	1775·54	1325·12	679·05	194·50	19·43	·15
+ *·40*	438·38	818·35	1306·92	1758·51	1931·83	1636·61	963·78	321·76	38·08	·35
+ *·45*	296·98	607·48	1072·08	1609·70	1995·55	1932·88	1321·43	521·51	74·55	·85
+ *·50*	187·37	420·16	820·76	1379·49	1939·87	2165·06	1738·01	824·05	145·56	2·12
+ *·55*	108·73	267·40	579·15	1093·22	1753·38	2274·19	2171·08	1259·96	282·59	5·48
+ *·60*	57·03	153·91	370·23	787·57	1449·19	2204·82	2539·21	1843·14	542·34	14·73
+ *·65*	26·39	78·18	209·24	503·43	1069·45	1928·16	2722·44	2535·20	1018·65	41·32
+ *·70*	10·39	33·81	100·85	275·47	680·14	1469·57	2591·04	3189·48	1839·04	121·27
+ *·75*	3·29	11·76	39·19	122·01	352·63	924·51	2078·60	3505·34	3087·80	371·57
+ *·80*	·76	3·00	11·17	39·80	135·72	437·70	1285·55	3100·93	4523·43	1169·00
+ *·85*	·11	·47	1·96	8·01	32·49	130·90	516·43	1878·29	5046·26	3562·39
+ *·90*	·01	·02	·14	·67	3·25	16·63	91·09	533·33	3048·86	8581·50
+ *·95*	·00	·00	·00	·01	·04	·25	1·99	20·38	303·89	6528·96
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0973	·1947	·2924	·3906	·4894	·5890	·6896	·7915	·8948
Mode	·0000	·1165	·2316	·3442	·4531	·5576	·6569	·7509	·8394	·9224
σ	·2357	·2337	·2275	·2172	·2025	·1832	·1590	·1294	·0937	·0511
$(1-\rho^2)/\sqrt{n-1}$	·2357	·2333	·2263	·2145	·1980	·1768	·1508	·1202	·0849	·0448
β_1	·0000	·0154	·0626	·1446	·2672	·4400	·6789	1·0110	1·4866	2·2105
β_2	2·7000	2·7272	2·8109	2·9573	3·1787	3·4958	3·9447	4·5897	5·5597	7·1586

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 20$. ρ variate (correlation in population sampled).

r variate (correlation in sample)	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	—	—	—	—	—	—	—	—	—	—
− *·90*	·00	·00	—	—	—	—	—	—	—	—
− *·85*	06	·01	·00	—	—	—	—	—	—	—
− *·80*	·47	·10	·02	·00	—	—	—	—	—	—
− *·75*	2·24	·53	·11	·02	·00	—	—	—	—	—
− *·70*	7·64	1·99	·46	·10	·02	·00	—	—	—	—
− *·65*	20·65	5·85	1·46	·31	·05	·01	—	—	—	—
− *·60*	46·98	14·53	3·91	·90	·17	·02	·00	—	—	—
− *·55*	93·51	31·56	9·20	2·26	·45	·07	·01	—	—	—
− *·50*	167·11	61·57	19·43	5·14	1·09	·17	·02	—	—	—
− *·45*	273·13	109·93	37·61	10·70	2·44	·42	·05	·00	—	—
− *·40*	413·76	181·99	67·57	20·74	5·06	·92	·11	·01	—	—
− *·35*	586·81	282·18	113·83	37·75	9·91	1·93	·25	·02	—	—
− *·30*	784·96	412·86	181·17	65·03	18·39	3·84	·53	·04	—	—
− *·25*	996·07	573·27	273·98	106·64	32·57	7·32	1·08	·08	—	—
− *·20*	1204·17	758·68	395·39	167·20	55·31	13·42	2·13	·18	·00	—
− *·15*	1391·40	960·11	546·33	251·51	90·32	23·73	4·07	·37	·01	—
− *·10*	1540·28	1164·55	724·46	363·84	142·24	40·60	7·55	·73	·02	—
− *·05*	1636·14	1356·03	923·47	507·05	216·43	67·37	13·64	1·44	·05	—
·00	1669·24	1517·23	1132·65	681·41	318·54	108·54	24·06	2·78	·10	—
+ *·05*	1636·14	1631·71	1337·14	883·46	453·78	169·98	41·43	5·27	·21	—
+ *·10*	1540·28	1686·21	1518·98	1104·85	625·68	258·79	69·76	9·84	·44	—
+ *·15*	1391·40	1672·86	1658·98	1331·72	834·48	382·94	114·86	18·07	·90	·00
+ *·20*	1204·17	1590·74	1739·26	1544·87	1075·23	550·27	184·86	32·72	1·85	·01
+ *·25*	996·07	1446·47	1746·32	1720·99	1335·82	766·63	290·54	58·39	3·77	·02
+ *·30*	784·96	1253·70	1673·93	1835·44	1595·60	1033·05	445·18	102·63	7·68	·04
+ *·35*	586·81	1031·28	1525·25	1866·25	1825·32	1341·82	663·32	177·47	15·62	·09
+ *·40*	413·76	800·54	1313·68	1799·15	1989·34	1672·08	957·45	301·19	31·71	·22
+ *·45*	273·13	582·06	1061·54	1632·51	2051·11	1986·22	1331·67	499·96	64·26	·57
+ *·50*	167·11	392·47	796·88	1380·68	1981·92	2229·43	1771·29	807·35	129·80	1·50
+ *·55*	93·51	242·15	548·35	1074·11	1771·83	2336·21	2229·32	1258·17	260·36	4·07
+ *·60*	46·98	134·22	339·60	754·79	1439·68	2247·04	2614·58	1869·39	515·46	11·50
+ *·65*	20·65	65·11	184·41	466·85	1036·50	1935·78	2793·79	2599·35	996·30	33·96
+ *·70*	7·64	26·60	84·50	244·62	636·66	1439·79	2627·94	3283·99	1843·93	105·05
+ *·75*	2·24	8·62	30·77	102·29	314·46	872·48	2059·10	3589·11	3154·22	339·37
+ *·80*	·47	2·00	8·05	30·87	113·01	390·31	1221·83	3109·18	4657·18	1124·15
+ *·85*	·06	·28	1·25	5·56	24·47	106·93	457·32	1796·33	5131·57	3583·56
+ *·90*	·00	·00	·08	·39	2·09	11·76	71·10	461·97	2931·90	8834·61
+ *·95*	—	—	·00	·00	·02	·13	1·19	13·97	244·23	6309·15
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0974	·1950	·2928	·3911	·4900	·5896	·6902	·7919	·8951
Mode	·0000	·1154	·2297	·3415	·4500	·5543	·6538	·7482	·8374	·9213
σ	·2294	·2274	·2214	·2113	·1969	·1780	·1543	·1254	·0907	·0493
$(1-\rho^2)/\sqrt{n-1}$	·2294	·2271	·2202	·2088	·1927	·1721	·1468	·1170	·0826	·0436
β_1	·0000	·0148	·0600	·1386	·2557	·4202	·6464	·9584	1·4001	2·0603
β_2	2·7143	2·7406	2·8213	2·9623	3·1749	3·4783	3·9055	4·5131	5·4154	6·8681

TABLE A—(*continued*).

$n = 21$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	—	—	—	—	—	—	—	—	—
− *·90*	·00	·00	—	—	—	—	—	—	—	—
− *·85*	·03	·01	·00	—	—	—	—	—	—	—
− *·80*	·29	·06	·01	·00	—	—	—	—	—	—
− *·75*	1·52	·34	·07	·01	·00	—	—	—	—	—
− *·70*	5·61	1·36	·29	·05	·01	—	—	—	—	—
− *·65*	16·13	4·27	·99	·19	·03	·00	—	—	—	—
− *·60*	38·65	11·22	2·82	·60	·10	·01	—	—	—	—
− *·55*	80·30	25·56	6·97	1·59	·29	·04	·00	—	—	—
− *·50*	148·80	51·95	15·41	3·79	·74	·11	·01	—	—	—
− *·45*	250·78	96·10	31·04	8·26	1·74	·27	·03	—	—	—
− *·40*	389·90	164·07	57·77	16·64	3·77	·63	·07	·00	—	—
− *·35*	565·17	261·27	100·39	31·39	7·67	1·37	·16	·01	—	—
− *·30*	769·89	391·17	164·25	55·82	14·76	2·84	·35	·02	—	—
− *·25*	991·59	553·98	254·51	94·20	27·02	5·61	·75	·05	—	—
− *·20*	1213·06	745·68	375·26	151·59	47·28	10·64	1·53	·11	·00	—
− *·15*	1414·38	956·73	528·29	233·39	79·38	19·43	3·04	·24	·01	—
− *·10*	1575·70	1173·63	711·91	344·73	128·23	34·26	5·83	·50	·01	—
− *·05*	1680·10	1378·59	919·93	489·35	199·70	58·45	10·88	1·02	·03	—
·00	1716·23	1552·13	1141·00	668·33	300·17	96·65	19·79	2·04	·06	—
+ *·05*	1680·10	1675·52	1358·90	878·59	435·79	155·05	35·09	4·01	·14	—
+ *·10*	1575·70	1733·68	1553·58	1111·53	611·07	241·33	60·74	7·73	·29	—
+ *·15*	1414·38	1717·73	1703·39	1352·20	827·05	364·44	102·64	14·66	·63	—
+ *·20*	1213·06	1626·97	1788·20	1579·31	1079·03	533·39	169·29	27·37	1·33	·00
+ *·25*	991·59	1469·48	1792·97	1766·81	1354·17	755·33	272·20	50·31	2·82	·01
+ *·30*	769·89	1261·29	1711·25	1887·05	1629·83	1032·26	425·95	90·99	5·95	·02
+ *·35*	565·17	1024·10	1547·62	1915·72	1873·44	1356·52	646·90	161·66	12·53	·06
+ *·40*	389·90	781·85	1318·32	1837·75	2045·23	1705·56	949·63	281·48	26·36	·14
+ *·45*	250·78	556·80	1049·40	1652·96	2104·80	2037·72	1339·82	478·52	55·31	·38
+ *·50*	148·80	366·00	772·43	1379·62	2021·59	2291·99	1802·29	789·71	115·56	1·06
+ *·55*	80·30	218·92	518·34	1053·63	1787·57	2396·03	2285·43	1254·35	239·50	3·02
+ *·60*	38·65	116·85	311·00	722·19	1427·92	2286·37	2687·85	1892·96	489·12	8·96
+ *·65*	16·13	54·13	162·26	432·22	1002·93	1940·29	2862·39	2660·83	972·87	27·86
+ *·70*	5·61	20·90	70·68	216·87	595·00	1408·33	2661·07	3375·86	1845·86	90·85
+ *·75*	1·52	6·31	24·12	85·61	279·97	822·04	2036·49	3668·97	3216·90	309·47
+ *·80*	·29	1·34	5·79	23·90	93·95	347·48	1159·40	3112·44	4787·20	1079·30
+ *·85*	·03	·16	·80	3·86	18·41	87·22	404·32	1715·20	5209·97	3599·09
+ *·90*	·00	·00	·04	·23	1·34	8·29	55·40	399·51	2814·91	9080·68
+ *·95*	—	—	·00	·00	·01	·07	·71	9·56	195·96	6087·03
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0976	·1952	·2932	·3916	·4905	·5902	·6907	·7924	·8954
Mode	·0000	·1145	·2279	·3391	·4472	·5515	·6509	·7457	·8354	·9203
σ	·2236	·2216	·2157	·2058	·1917	·1732	·1500	·1218	·0880	·0478
$(1-\rho^2)/\sqrt{n-1}$	·2236	·2214	·2147	·2035	·1878	·1677	·1431	·1140	·0805	·0425
β_1	·0000	·0142	·0577	·1331	·2451	·4020	·6166	·9107	1·3227	1·9288
β_2	2·7273	2·7527	2·8306	2·9666	3·1711	3·4617	3·8683	4·4432	5·2858	6·6169

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 22$. ρ variate (correlation in population sampled).

r, variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	—	—	—	—	—	—	—	—	—
− *·90*	·01	—	—	—	—	—	—	—	—	—
− *·85*	·02	·00	·00	—	—	—	—	—	—	—
− *·80*	·18	·03	·01	·00	—	—	—	—	—	—
− *·75*	1·03	·21	·04	·01	—	—	—	—	—	—
− *·70*	4·11	·92	·18	·03	·00	—	—	—	—	—
− *·65*	12·59	3·11	·67	·12	·02	·00	—	—	—	—
− *·60*	31·74	8·65	2·02	·40	·06	·01	—	—	—	—
− *·55*	68·85	20·67	5·28	1·12	·19	·02	·00	—	—	—
− *·50*	132·30	43·77	12·21	2·80	·50	·07	·01	—	—	—
− *·45*	229·92	83·90	25·58	6·36	1·24	·17	·02	—	—	—
− *·40*	366·87	147·70	49·31	13·34	2·80	·43	·04	·00	—	—
− *·35*	543·53	241·56	88·41	26·06	5·93	·97	·10	·01	—	—
− *·30*	754·00	370·08	148·69	47·84	11·83	2·09	·23	·01	—	—
− *·25*	985·69	534·57	236·09	83·10	22·38	4·29	·52	·03	—	—
− *·20*	1220·22	731·56	355·63	137·23	40·36	8·43	1·10	·07	—	—
− *·15*	1435·65	951·98	510·10	216·26	69·67	15·89	2·26	·16	·00	—
− *·10*	1609·59	1181·06	698·57	326·14	115·44	28·86	4·49	·34	·01	—
− *·05*	1722·72	1399·48	915·06	471·58	183·99	50·64	8·67	·72	·02	—
·00	1761·97	1585·51	1147·75	654·54	282·44	85·93	16·25	1·50	·04	—
+ *·05*	1722·72	1718·00	1379·01	872·46	417·89	141·20	29·67	3·04	·09	—
+ *·10*	1609·59	1779·88	1586·65	1116·63	595·93	224·73	52·80	6·06	·20	—
+ *·15*	1435·65	1761·23	1746·45	1370·99	818·50	346·33	91·59	11·87	·43	—
+ *·20*	1220·22	1661·60	1835·83	1612·18	1081·27	516·27	154·80	22·86	·96	—
+ *·25*	985·69	1490·67	1838·17	1811·21	1370·78	743·10	254·65	43·29	2·10	·00
+ *·30*	754·00	1267·07	1746·86	1937·29	1662·37	1029·97	406·96	80·55	4·60	·01
+ *·35*	543·53	1015·49	1568·03	1963·65	1920·03	1369·38	629·96	147·04	10·04	·03
+ *·40*	366·87	762·48	1321·06	1874·43	2099·64	1737·17	940·51	262·68	21·88	·09
+ *·45*	229·92	531·86	1035·89	1671·23	2156·76	2087·52	1346·06	457·35	47·54	·26
+ *·50*	132·30	340·82	747·64	1376·56	2059·06	2352·90	1831·17	771·34	102·73	·75
+ *·55*	68·85	197·63	489·26	1032·04	1800·84	2453·82	2339·55	1248·74	219·99	2·23
+ *·60*	31·74	101·58	284·38	690·00	1414·20	2323·01	2759·17	1914·05	463·46	6·97
+ *·65*	12·59	44·94	142·57	399·58	969·04	1942·00	2928·43	2719·83	948·62	22·83
+ *·70*	4·11	16·39	59·04	191·99	555·25	1375·57	2690·72	3465·29	1845·14	78·45
+ *·75*	1·03	4·60	18·88	71·56	248·90	773·40	2011·22	3745·20	3276·10	281·79
+ *·80*	·18	·89	4·16	18·48	78·00	308·91	1098·57	3111·21	4913·78	1034·74
+ *·85*	·02	·10	·51	2·67	13·82	71·03	356·95	1635·37	5281·97	3609·52
+ *·90*	·01	·00	·02	·13	·86	5·84	43·11	345·00	2698·71	9320·25
+ *·95*	·00	—	·00	·00	·00	·04	·42	6·53	157·01	5864·34
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0977	·1955	·2935	·3920	·4910	·5906	·6912	·7928	·8956
Mode	·0000	·1137	·2264	·3369	·4447	·5486	·6484	·7435	·8339	·9194
σ	·2182	·2162	·2105	·2007	·1869	·1688	·1461	·1185	·0855	·0464
$(1-\rho^2)/\sqrt{n-1}$	·2182	·2160	·2095	·1986	·1833	·1637	·1396	·1113	·0786	·0415
β_1	·0000	·0137	·0555	·1279	·2354	·3853	·5893	·8674	1·2532	1·8125
β_2	2·7391	2·7630	2·8390	2·9703	3·1672	3·4461	3·8328	4·3790	5·1687	6·3926

TABLE A—(*continued*).

$n = 23$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	·00	—	—	—	—	—	—	—	—	—
− *·90*	·00	—	—	—	—	—	—	—	—	—
− *·85*	·01	·00	—	—	—	—	—	—	—	—
− *·80*	·11	·02	—	—	—	—	—	—	—	—
− *·75*	·70	·13	·00	—	—	—	—	—	—	—
− *·70*	3·01	·63	·11	·00	·00	—	—	—	—	—
− *·65*	9·81	2·26	·45	·08	·01	—	—	—	—	—
− *·60*	26·04	6·66	1·45	·26	·04	·00	—	—	—	—
− *·55*	58·96	16·69	3·99	·78	·12	·01	—	—	—	—
− *·50*	117·47	36·83	9·65	2·06	·34	·04	·00	—	—	—
− *·45*	210·52	73·14	21·06	4·90	·88	·11	·01	—	—	—
− *·40*	344·75	132·79	42·04	10·68	2·08	·29	·02	—	—	—
− *·35*	522·03	223·04	77·76	21·61	4·58	·69	·06	·00	—	—
− *·30*	737·47	349·67	134·43	40·95	9·47	1·54	·15	·01	—	—
− *·25*	978·54	515·16	218·71	73·21	18·51	3·28	·36	·02	—	—
− *·20*	1225·82	716·90	336·58	124·07	34·41	6·67	·79	·04	—	—
− *·15*	1455·33	946·00	491·89	200·13	61·07	12·98	1·68	·10	—	—
− *·10*	1642·05	1186·97	684·57	308·15	103·79	24·29	3·46	·23	·00	—
− *·05*	1764·10	1418·82	909·02	453·86	169·29	43·81	6·89	·51	·01	—
·00	1806·56	1617·48	1153·02	640·19	265·41	76·30	13·33	1·10	·02	—
+ *·05*	1764·10	1759·25	1397·58	865·25	400·21	128·44	25·06	2·30	·06	—
+ *·10*	1642·05	1824·91	1618·30	1120·27	580·40	208·99	45·84	4·75	·13	—
+ *·15*	1455·33	1803·46	1788·25	1388·22	808·98	328·68	81·62	9·60	·30	—
+ *·20*	1225·82	1694·75	1882·27	1643·56	1082·09	499·05	141·37	19·07	·69	—
+ *·25*	978·54	1510·18	1882·05	1854·28	1385·76	730·12	237·92	37·20	1·56	·00
+ *·30*	737·47	1271·20	1780·86	1986·26	1693·34	1026·34	388·31	71·21	3·55	·01
+ *·35*	522·03	1005·63	1586·63	2010·13	1965·21	1380·56	612·67	133·58	8·04	·02
+ *·40*	344·75	742·61	1322·06	1909·35	2152·67	1767·06	930·25	244·82	18·14	·06
+ *·45*	210·52	507·37	1021·21	1687·50	2207·11	2135·75	1350·57	436·53	40·80	·17
+ *·50*	117·47	316·95	722·70	1371·71	2094·49	2412·26	1858·09	752·42	91·21	·52
+ *·55*	58·96	178·18	461·21	1009·57	1811·83	2509·73	2391·84	1241·53	201·81	1·65
+ *·60*	26·04	88·20	259·71	658·38	1398·78	2357·16	2828·70	1932·86	438·57	5·42
+ *·65*	9·81	37·26	125·10	368·92	935·07	1941·17	2992·09	2776·53	923·77	18·68
+ *·70*	3·01	12·84	49·25	169·74	517·49	1341·82	2717·16	3552·48	1842·02	67·66
+ *·75*	·70	3·36	14·76	59·73	220·99	726·69	1983·68	3818·06	3332·08	256·26
+ *·80*	·11	·59	2·99	14·27	64·66	274·26	1039·58	3105·96	5037·18	990·75
+ *·85*	·01	·06	·33	1·85	10·37	57·78	314·71	1557·24	5348·05	3615·31
+ *·90*	·00	·00	·01	·08	·55	4·11	33·50	297·55	2583·97	9553·82
+ *·95*	·00	·00	·00	·00	·00	·02	·25	4·46	125·64	5642·53
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0978	·1957	·2938	·3923	·4914	·5911	·6916	·7931	·8958
Mode	·0000	·1130	·2250	·3351	·4424	·5462	·6461	·7415	·8324	·9185
σ	·2132	·2113	·2056	·1960	·1825	·1647	·1425	·1155	·0832	·0450
$(1-\rho^2)/\sqrt{n-1}$	·2132	·2111	·2047	·1940	·1791	·1599	·1364	·1087	·0768	·0405
β_1	·0000	·0132	·0535	·1232	·2264	·3698	·5645	·8279	1·1905	1·7092
β_2	2·7500	2·7738	2·8467	2·9735	3·1633	3·4313	3·8024	4·3197	5·0623	6·1951

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 24$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	—	—	—	—	—	—	—	—	—	—
− *·90*	·00	—	—	—	—	—	—	—	—	—
− *·85*	·01	—	—	—	—	—	—	—	—	—
− *·80*	·07	·00	·00	—	—	—	—	—	—	—
− *·75*	·48	·08	·01	—	—	—	—	—	—	—
− *·70*	2·20	·43	·07	·00	—	—	—	—	—	—
− *·65*	7·63	1·65	·30	·05	·00	—	—	—	—	—
− *·60*	21·33	5·12	1·04	·17	·02	·00	—	—	—	—
− *·55*	50·42	13·46	3·01	·55	·08	·01	—	—	—	—
− *·50*	104·18	30·95	7·63	1·52	·23	·03	·00	—	—	—
− *·45*	192·53	63·69	17·31	3·76	·63	·08	·01	—	—	—
− *·40*	323·58	119·24	35·80	8·54	1·54	·20	·02	—	—	—
− *·35*	500·79	205·69	68·31	17·90	3·53	·49	·04	—	—	—
− *·30*	720·45	329·99	121·39	35·02	7·57	1·13	·10	·00	—	—
− *·25*	970·29	495·86	202·37	64·42	15·29	2·51	·25	·01	—	—
− *·20*	1229·99	701·70	318·18	112·03	29·30	5·27	·57	·03	—	—
− *·15*	1473·52	938·94	473·77	184·97	53·46	10·58	1·25	·07	—	—
− *·10*	1673·17	1191·49	670·05	290·81	93·20	20·41	2·66	·16	·00	—
− *·05*	1804·33	1436·72	901·95	436·29	155·59	37·86	5·48	·36	·01	—
·00	1850·07	1648·13	1156·93	625·41	249·11	67·67	10·92	·80	·01	—
+ *·05*	1804·33	1799·35	1414·71	857·07	382·82	116·68	21·14	1·74	·04	—
+ *·10*	1673·17	1868·86	1648·62	1122·60	564·61	194·13	39·75	3·71	·09	—
+ *·15*	1473·52	1844·51	1828·88	1403·99	798·61	311·57	72·65	7·76	·21	—
+ *·20*	1229·99	1726·50	1927·58	1673·57	1081·63	481·83	128·95	15·89	·49	—
+ *·25*	970·29	1528·14	1924·68	1896·13	1399·26	716·52	222·03	31·93	1·16	—
+ *·30*	720·45	1273·84	1813·38	2034·06	1722·83	1021·51	370·08	62·89	2·74	·00
+ *·35*	500·79	994·68	1603·54	2055·28	2009·07	1390·19	595·15	121·20	6·42	·01
+ *·40*	323·58	722·40	1321·50	1942·61	2204·43	1795·33	919·02	227·90	15·02	·04
+ *·45*	192·53	483·43	1005·54	1701·90	2255·95	2182·51	1353·49	416·18	34·98	·12
+ *·50*	104·18	294·40	697·76	1365·25	2128·00	2470·20	1883·18	733·09	80·88	·37
+ *·55*	50·42	160·45	434·25	986·41	1820·73	2563·87	2442·41	1232·91	184·91	1·22
+ *·60*	21·33	76·48	236·89	627·46	1381·90	2388·99	2896·56	1949·56	414·53	4·21
+ *·65*	7·63	30·86	109·64	340·21	901·23	1938·06	3053·53	2831·08	898·52	15·27
+ *·70*	2·20	10·05	41·03	149·89	481·72	1307·36	2740·64	3637·57	1836·75	58·29
+ *·75*	·48	2·45	11·52	49·80	195·98	682·00	1954·22	3887·76	3385·03	232·77
+ *·80*	·07	·39	2·14	11·00	53·55	243·21	982·60	3097·07	5157·63	947·52
+ *·85*	·01	·03	·21	1·28	7·76	46·94	277·15	1481·10	5408·62	3616·87
+ *·90*	·00	·00	·01	·04	·35	2·89	26·01	256·32	2471·22	9781·81
+ *·95*	·00	·00	·00	·00	·00	·01	·15	3·04	100·42	5422·78
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0979	·1959	·2941	·3927	·4918	·5915	·6920	·7934	·8960
Mode	·0000	·1124	·2238	·3334	·4404	·5441	·6440	·7397	·8310	·9178
σ	·2085	·2067	·2011	·1916	·1783	·1609	·1391	·1127	·0811	·0438
$(1-\rho^2)/\sqrt{n-1}$	·2085	·2064	·2002	·1897	·1752	·1564	·1334	·1063	·0751	·0396
β_1	·0000	·0127	·0516	·1187	·2180	·3557	·5419	·7918	1·1335	1·6167
β_2	2·7600	2·7826	2·8537	2·9764	3·1596	3·4174	3·7774	4·2653	4·9654	6·0161

TABLE A—(*continued*).

$n = 25$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*	*·9*
− *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	—	—	—	—	—	—	—	—	—	—
− *·90*	—	—	—	—	—	—	—	—	—	—
− *·85*	·00	—	—	—	—	—	—	—	—	—
− *·80*	·04	·00	—	—	—	—	—	—	—	—
− *·75*	·32	·05	·00	—	—	—	—	—	—	—
− *·70*	1·61	·29	·05	·00	—	—	—	—	—	—
− *·65*	5·93	1·20	·21	·03	·00	—	—	—	—	—
− *·60*	17·46	3·93	·75	·12	·01	—	—	—	—	—
− *·55*	43·08	10·85	2·27	·38	·05	·00	—	—	—	—
− *·50*	92·30	25·99	6·01	1·11	·16	·01	—	—	—	—
− *·45*	175·88	55·40	14·22	2·89	·44	·05	·00	—	—	—
− *·40*	303·38	106·96	30·45	6·82	1·14	·13	·01	—	—	—
− *·35*	479·90	189·49	59·94	14·80	2·72	·34	·03	—	—	—
− *·30*	703·06	311·08	109·50	29·91	6·05	·83	·07	·00	—	—
− *·25*	961·07	476·77	187·04	56·62	12·62	1·80	·17	·01	—	—
− *·20*	1232·83	686·08	300·46	101·06	24·92	4·15	·41	·02	—	—
− *·15*	1490·32	930·93	455·82	170·78	46·75	8·62	·93	·04	—	—
− *·10*	1703·04	1194·73	655·13	274·14	83·60	17·14	2·04	·11	—	—
− *·05*	1843·49	1453·28	893·96	418·94	142·84	32·68	4·35	·25	·00	—
·00	1892·58	1677·56	1159·60	610·31	233·56	59·95	8·94	·59	·01	—
+ *·05*	1843·49	1838·37	1430·51	848·05	365·78	105·89	17·81	1·32	·02	—
+ *·10*	1703·04	1911·80	1677·69	1123·71	548·65	180·12	34·44	2·90	·06	—
+ *·15*	1490·32	1884·46	1868·40	1418·41	787·53	295·02	64·59	6·26	·14	—
+ *·20*	1232·83	1756·95	1971·86	1702·29	1080·00	464·70	117·49	13·22	·35	—
+ *·25*	961·07	1544·63	1966·16	1936·84	1411·36	702·41	206·98	27·37	·86	—
+ *·30*	703·06	1275·10	1844·50	2080·76	1750·96	1015·61	352·32	55·47	2·11	·00
+ *·35*	479·90	982·79	1618·89	2099·17	2051·69	1398·37	577·51	109·85	5·13	·01
+ *·40*	303·38	701·99	1319·51	1974·33	2255·00	1822·10	906·95	211·92	12·43	·02
+ *·45*	175·88	460·12	989·05	1714·58	2303·40	2227·89	1354·96	396·34	29·95	·08
+ *·50*	92·30	273·17	672·95	1357·37	2159·72	2526·88	1906·56	713·49	71·65	·26
+ *·55*	43·08	144·33	408·43	962·75	1827·71	2616·37	2491·37	1223·04	169·24	·90
+ *·60*	17·46	66·25	215·85	597·36	1363·75	2418·65	2962·87	1964·30	391·39	3·26
+ *·65*	5·93	25·53	95·99	313·40	867·68	1932·87	3112·90	2883·60	873·02	12·46
+ *·70*	1·61	7·85	34·15	132·22	447·95	1272·41	2761·35	3720·72	1829·54	50·16
+ *·75*	·32	1·78	8·99	41·47	173·61	639·37	1923·13	3954·50	3435·16	211·20
+ *·80*	·04	·26	1·53	8·48	44·30	215·45	927·75	3084·92	5275·33	905·21
+ *·85*	·00	·02	·13	·88	5·81	38·09	243·81	1407·18	5464·05	3614·59
+ *·90*	—	·00	·00	·00	·22	2·03	20·16	220·57	2360·87	10004·61
+ *·95*	—	—	—	—	·00	·01	·09	2·07	80·18	5206·06
+ *1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00	·00
Mean	·0000	·0980	·1960	·2943	·3930	·4921	·5918	·6923	·7937	·8962
Mode	·0000	·1118	·2227	·3318	·4385	·5421	·6420	·7380	·8297	·9170
σ	·2041	·2023	·1968	·1875	·1744	·1573	·1359	·1100	·0791	·0427
$(1-\rho^2)/\sqrt{n-1}$	·2041	·2021	·1960	·1858	·1715	·1531	·1306	·1041	·0735	·0388
β_1	·0000	·0123	·0499	·1146	·2102	·3423	·5203	·7586	1·0816	1·5334
β_2	2·7692	2·7916	2·8601	2·9788	3·1559	3·4042	3·7453	4·2149	4·8769	5·8584

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 50$. ρ variate (correlation in population sampled).

r variate (correlation in sample)	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*
−*1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	—	—	—	—	—	—	—	—	—
− *·90*	—	—	—	—	—	—	—	—	—
− *·85*	—	—	—	—	—	—	—	—	—
− *·80*	—	—	—	—	—	—	—	—	—
− *·75*	—	—	—	—	—	—	—	—	—
− *·70*	·00	—	—	—	—	—	—	—	—
− *·65*	·01	—	—	—	—	—	—	—	—
− *·60*	·10	·00	—	—	—	—	—	—	—
− *·55*	·69	·04	·00	—	—	—	—	—	—
− *·50*	3·68	·27	·01	—	—	—	—	—	—
− *·45*	15·10	1·40	·08	·00	—	—	—	—	—
− *·40*	49·85	5·82	·44	·02	—	—	—	—	—
− *·35*	136·13	20·06	1·88	·11	·00	—	—	—	—
− *·30*	314·21	58·57	6·85	·48	·02	—	—	—	—
− *·25*	623·17	147·07	21·50	1·85	·09	—	—	—	—
− *·20*	1075·24	321·70	59·02	6·32	·36	·00	—	—	—
− *·15*	1629·13	618·58	142·88	19·11	1·35	·04	—	—	—
− *·10*	2182·12	1052·79	307·18	51·61	4·55	·18	·00	—	—
− *·05*	2595·77	1593·19	589·27	125·03	13·87	·68	·01	—	—
·00	2749·60	2149·47	1011·38	272·76	38·38	2·39	·05	—	—
+ *·05*	2595·77	2587·70	1554·59	535·97	96·53	7·70	·20	·00	—
+ *·10*	2182·12	2777·44	2138·36	948·41	220·60	22·82	·78	·01	—
+ *·15*	1629·13	2650·80	2625·45	1507·84	457·26	62·11	2·82	·02	—
+ *·20*	1075·24	2239·38	2864·46	2144·80	856·50	154·86	9·46	·11	—
+ *·25*	623·17	1663·32	2758·76	2712·27	1441·46	351·85	29·47	·48	—
+ *·30*	314·21	1076·19	2323·98	3022·24	2161·66	723·72	84·89	1·99	·00
+ *·35*	136·13	599·07	1691·43	2932·02	2856·13	1333·93	224·22	7·75	·02
+ *·40*	49·85	282·26	1046·45	2437·43	3274·68	2172·97	536·92	28·37	·09
+ *·45*	15·10	110·16	538·66	1700·04	3192·62	3070·40	1147·17	96·36	·51
+ *·50*	3·68	34·61	224·22	967·26	2575·61	3668·50	2138·79	298·71	2·85
+ *·55*	·69	8·42	72·63	432·21	1656·86	3578·51	3372·39	824·60	15·26
+ *·60*	·10	1·51	17·37	143·93	807·35	2713·26	4300·25	1954·75	76·77
+ *·65*	·01	·18	2·84	33·16	276·92	1489·70	4152·55	3764·57	350·56
+ *·70*	·00	·01	·29	4·73	59·97	532·70	2747·94	5398·21	1367·63
+ *·75*	—	·00	·01	·35	6·91	104·96	1061·91	4990·96	4092·05
+ *·80*	—	—	·00	·01	·32	8·58	181·98	2306·06	7650·44
+ *·85*	—	—	—	·00	·00	·17	8·16	322·77	5817·30
+ *·90*	—	—	—	—	—	·00	·03	4·26	621·81
+ *·95*	—	—	—	—	—	—	·00	·00	·24
+*1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00

r variate (correlation in sample)	*·9*
·67	·2
·68	·3
·69	·6
·70	1·0
·71	1·7
·72	2·9
·73	5·1
·74	8·8
·75	15·3
·76	26·7
·77	46·4
·78	80·4
·79	138·9
·80	238·4
·81	406·0
·82	683·7
·83	1134·8
·84	1848·2
·85	2936·7
·86	4519·5
·87	6673·8
·88	9340·1
·89	12187·5
·90	14502·0
·91	15261·3
·92	13599·9
·93	9630·6
·94	4918·7
·95	1550·4
·96	230·3
·97	9·6
·98	·3
·99	·0

For the constants of the curves for $n=50$: see p. 372 above.

TABLE A—(*continued*).

$n = 100$. ρ variate (correlation in population sampled).

r variate (correlation in sample).	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*	*·7*	*·8*
−*1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00
− *·95*	—	—	—	—	—	—	—	—	—
− *·90*	—	—	—	—	—	—	—	—	—
− *·85*	—	—	—	—	—	—	—	—	—
− *·80*	—	—	—	—	—	—	—	—	—
− *·75*	—	—	—	—	—	—	—	—	—
− *·70*	—	—	—	—	—	—	—	—	—
− *·65*	—	—	—	—	—	—	—	—	—
− *·60*	—	—	—	—	—	—	—	—	—
− *·55*	—	—	—	—	—	—	—	—	—
− *·50*	·00	—	—	—	—	—	—	—	—
− *·45*	·08	·00	—	—	—	—	—	—	—
− *·40*	·91	·01	—	—	—	—	—	—	—
− *·35*	7·43	·15	·00	—	—	—	—	—	—
− *·30*	42·60	1·41	·02	—	—	—	—	—	—
− *·25*	177·84	9·50	·18	·00	—	—	—	—	—
− *·20*	555·18	48·00	1·55	·02	—	—	—	—	—
− *·15*	1321·36	185·37	9·53	·16	·00	—	—	—	—
− *·10*	2431·67	554·86	45·84	1·24	·01	—	—	—	—
− *·05*	3493·29	1299·62	173·78	7·57	·09	—	—	—	—
·00	3939·27	2395·29	522·21	36·98	·70	·00	—	—	—
+ *·05*	3493·29	3480·22	1246·23	145·32	4·56	·03	—	—	—
+ *·10*	2431·67	3979·11	2358·16	458·62	24·21	·25	—	—	—
+ *·15*	1321·36	3560·42	3519·22	1156·81	104·66	1·87	·00	—	—
+ *·20*	555·18	2469·56	4103·61	2311·65	365·76	11·67	·04	—	—
+ *·25*	177·84	1309·30	3687·36	3611·32	1021·01	59·96	·41	—	—
+ *·30*	42·60	520·41	2504·77	4329·36	2237·43	249·60	3·35	·00	—
+ *·35*	7·43	151·11	1253·64	3884·32	3759·08	824·50	22·96	·03	—
+ *·40*	·91	30·98	446·88	2522·86	4690·55	2099·42	127·86	·35	—
+ *·45*	·08	4·29	108·49	1135·08	4166·27	3962·12	558·78	3·87	—
+ *·50*	·00	·38	16·90	333·60	2488·44	5254·02	1829·25	35·59	·00
+ *·55*	—	·02	1·56	59·16	925·05	4549·13	4200·10	254·84	·02
+ *·60*	—	·00	·08	5·68	192·25	2320·57	6157·92	1316·29	2·01
+ *·65*	—	—	·00	·25	19·17	601·45	5023·96	4363·45	38·45
+ *·70*	—	—	—	·00	·73	63·47	1850·41	7728·87	519·97
+ *·75*	—	—	—	—	·01	1·92	220·19	5408·23	3951·47
+ *·80*	—	—	—	—	·00	·01	4·76	876·76	10951·32
+ *·85*	—	—	—	—	—	·00	·01	11·56	4488·65
+ *·90*	—	—	—	—	—	—	·00	·00	29·37
+ *·95*	—	—	—	—	—	—	—	—	·00
+*1·00*	·00	·00	·00	·00	·00	·00	·00	·00	·00

r variate (correlation in sample).	*·9*	*·9* (normal curve)*
·75	·1	·0
·76	·2	—
·77	·5	—
·78	1·4	—
·79	4·0	·0
·80	11·3	·2
·81	31·3	·3
·82	84·7	3·2
·83	221·8	25·3
·84	556·2	150·2
·85	1320·0	678·6
·86	2919·0	2330·1
·87	5895·9	6082·6
·88	10597·3	11944·8
·89	16373·8	18211·3
·90	20754·4	20887·0
·91	20233·5	18211·3
·92	13848·4	11944·8
·93	5823·1	6082·6
·94	1227·8	2330·1
·95	93·7	678·6
·96	1·5	150·2
·97	—	25·3
·98	—	3·2
·99	—	·3
1·00	—	·2

* These ordinates indicate how poor is the approximation of a normal curve to the frequency of r, where n is 100, but ρ is large.

For the constants of the curves for $n = 100$: see p. 372 above.

TABLE A. *Ordinates and Constants of Frequency Curves.*

$n = 400$. ρ variate (correlation in population sampled).

r variate (correlation in sample)	*0*	*·1*	*·2*	*·3*	*·4*	*·5*	*·6*
− *·40*	·00	·00	·00	·00	·00	·00	·00
− *·35*	—	—	—	—	—	—	—
− *·30*	—	—	—	—	—	—	—
− *·25*	·00	—	—	—	—	—	—
− *·20*	2·46	·00	—	—	—	—	—
− *·15*	87·84	·03	—	—	—	—	—
− *·10*	1087·30	6·72	·00	—	—	—	—
− *·05*	4845·66	89·41	·03	—	—	—	—
·00	7953·88	1071·03	2·31	·00	—	—	—
+ *·05*	4845·66	4809·08	77·24	·01	—	—	—
+ *·10*	1087·30	8034·24	990·50	1·37	—	—	—
+ *·15*	87·84	4878·55	4767·51	55·10	·00	—	—
+ *·20*	2·46	1037·29	8285·39	846·48	·52	—	—
+ *·25*	·00	72·74	4910·62	4701·88	30·13	·00	—
+ *·30*	—	1·55	917·19	8740·76	642·95	·01	—
+ *·35*	—	·01	48·55	4906·74	4567·01	10·75	·00
+ *·40*	—	·00	·63	724·97	9469·37	399·28	·01
+ *·45*	—	—	·00	23·50	4810·20	4278·49	1·75
+ *·50*	—	—	—	·13	473·26	10606·00	167·53
+ *·55*	—	—	—	·00	6·55	4491·43	3668·41
+ *·60*	—	—	—	—	·08	212·53	12429·40
+ *·65*	—	—	—	—	·00	·16	3688·70
+ *·70*	—	—	—	—	—	·00	40·04
+ *·75*	—	—	—	—	—	—	·00
+ *·80*	—	—	—	—	—	—	—
+ *·85*	—	—	—	—	—	—	—
+ *·90*	—	—	—	—	—	—	—
+ *·95*	—	—	—	—	—	—	—
+*1·00*	·00	·00	·00	·00	·00	·00	·00

r variate (correlation in sample)	*·7*	*·7* normal*	*·8*	*·8* normal*	*·9*	*·9* normal*
·54	·0	·0	·0	·0	·0	·0
·55	·1	—	—	—	—	—
·56	·2	—	—	—	—	—
·57	·8	·0	—	—	—	—
·58	2·8	·2	—	—	—	—
·59	9·3	1·4	—	—	—	—
·60	28·7	7·3	—	—	—	—
·61	82·9	31·3	—	—	—	—
·62	221·2	115·3	—	—	—	—
·63	542·5	364·4	—	—	—	—
·64	1212·0	987·7	—	—	—	—
·65	2478·7	2296·4	—	—	—	—
·66	4553·8	4579·8	—	—	—	—
·67	7487·9	7834·8	—	—	—	—
·68	10922·9	11497·0	·0	—	—	—
·69	13999·7	14471·6	·1	—	—	—
·70	15598·3	15625·2	·2	·0	—	—
·71	14931·6	14471·6	2·2	·1	—	—
·72	12106·9	11497·0	11·2	1·2	—	—
·73	8212·0	7834·8	52·2	11·7	—	—
·74	4572·5	4579·8	213·0	86·8	—	—
·75	2052·1	2296·4	751·2	471·8	—	—
·76	726·6	987·7	2248·2	1885·6	—	—
·77	198·1	364·4	5595·6	5538·9	—	—
·78	40·5	115·3	11311·7	11958·6	—	—
·79	6·0	31·3	18071·0	18977·4	—	—
·80	·0	7·3	22139·3	22135·7	—	—
·81	—	1·4	19920·7	18977·4	—	—
·82	—	·2	12665·6	11958·6	—	—
·83	—	·0	5383·5	5538·9	·0	—
·84	—	—	1435·4	1885·6	·1	—
·85	—	—	222·2	471·8	2·6	·0
·86	—	—	18·2	86·8	49·7	·6
·87	—	—	·7	11·7	658·1	290·2
·88	—	—	·0	1·2	5308·0	4598·7
·89	—	—	—	·1	22606·9	24134·5
·90	—	—	—	·0	41873·1	41941·4
·91	—	—	—	—	25806·1	24134·5
·92	—	—	—	—	3626·0	4598·7
·93	—	—	—	—	66·8	290·2
·94	—	—	—	—	·1	·6
·95	—	—	—	—	·0	·0
·96	—	—	—	—	—	—
·97	—	—	—	—	—	—
·98	—	—	—	—	—	—
·99	—	—	—	—	—	—
1·00	·0	·0	·0	·0	·0	·0

* The columns headed 'normal' mark how roughly the Gaussian frequency describes the distribution of the correlation even for large samples, if the correlation be not small in the sampled population.

For the constants of the curves for $n=400$: see p. 372 above.

TABLE B*. *To assist the calculation of the Ordinates of the Correlation Frequency Curves from Expansion Formulae.*

$\log \frac{n-2}{\sqrt{n-1}}$.

$\rho = 0.$

$\log (1-\rho^2)^{\frac{3}{2}} = 0.$

n	$\log \frac{n-2}{\sqrt{n-1}}$	n	$\log \frac{n-2}{\sqrt{n-1}}$	n	$\log \frac{n-2}{\sqrt{n-1}}$	r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
3	$\bar{1}$·8494850	42	·7956681	81	·9460821	−·95	·5054977	$\bar{2}$·8825969				
4	·0624694	43	·8011592	82	·9488475	−·90	·3606232	$\bar{1}$·3172203				
5	·1760913	44	·8065151	83	·9515781	−·85	·2783685	$\bar{1}$·5639844				
6	·2525750	45	·8117421	84	·9542748	−·80	·2218487	$\bar{1}$·7335437				
7	·3098944	46	·8168464	85	·9569384	−·75	·1795110	$\bar{1}$·8605570				
8	·3556022	47	·8218336	86	·9595698	−·70	·1462149	$\bar{1}$·9604452				
9	·3935530	48	·8267089	87	·9621697	−·65	·1192240	·0414179				
10	·4259687	49	·8314772	88	·9647388	−·60	·0969100	·1083599				
11	·4542425	50	·8361432	89	·9672779	−·55	·0782279	·1644063				
12	·4793037	51	·8407111	90	·9697877	−·50	·0624694	·2116818				
13	·5018021	52	·8451849	91	·9722688	−·45	·0491347	·2516860				
14	·5222096	53	·8495685	92	·9747218	−·40	·0378604	·2855089				
15	·5408793	54	·8538654	93	·9771475	−·35	·0283764	·3139606				
16	·5580824	55	·8580790	94	·9795464	−·30	·0204793	·3376520				
17	·5740313	56	·8622124	95	·9819190	−·25	·0140144	·3570469				
18	·5888955	57	·8662687	96	·9842661	−·20	·0088644	·3724968				
19	·6028127	58	·8702506	97	·9865880	−·15	·0049416	·3842651				
20	·6158957	59	·8741609	98	·9888854	−·10	·0021824	·3925427				
21	·6282386	60	·8780020	99	·9911587	−·05	·0005435	·3974593				
22	·6399203	61	·8817764	100	·9934085	·00	·0	·3990899	ϕ_1 = ·25	ϕ_2 = ·03125	ϕ_3 = − ·0390625	ϕ_4 = − ·0102539
23	·6510080	62	·8854863	400	1·2993966	+·05	·0005435	·3974593				
24	·6615588	63	·8891340			+·10	·0021824	·3925427				
25	·6716222	64	·8927214			+·15	·0049416	·3842651				
26	·6812412	65	·8962506			+·20	·0088644	·3724968				
27	·6904533	66	·8997233			+·25	·0140144	·3570469				
28	·6992915	67	·9031414			+·30	·0204793	·3376520				
29	·7077847	68	·9065065			+·35	·0283764	·3139606				
30	·7159590	69	·9098203			+·40	·0378604	·2855089				
31	·7238374	70	·9130844			+·45	·0491347	·2516860				
32	·7314404	71	·9163001			+·50	·0624694	·2116818				
33	·7387867	72	·9194689			+·55	·0782279	·1644063				
34	·7458930	73	·9225921			+·60	·0969100	·1083599				
35	·7527745	74	·9256711			+·65	·1192240	·0414179				
36	·7594449	75	·9287070			+·70	·1462149	$\bar{1}$·9604452				
37	·7659168	76	·9317011			+·75	·1795110	$\bar{1}$·8605570				
38	·7722016	77	·9346545			+·80	·2218487	$\bar{1}$·7335437				
39	·7783099	78	·9375682			+·85	·2783685	$\bar{1}$·5639844				
40	·7842513	79	·9404434			+·90	·3606232	$\bar{1}$·3172203				
41	·7900346	80	·9432811			+·95	·5054977	$\bar{2}$·8825969				

* If the ordinate at r be y, then (see p. 348)

$$y = Y\left(1 + \frac{\phi_1}{n-1} + \frac{\phi_2}{(n-1)^2} + \frac{\phi_3}{(n-1)^3} + \frac{\phi_4}{(n-1)^4}\right),$$

where
$$\log Y = \log \frac{n-2}{\sqrt{n-1}} + \log (1-\rho^2)^{\frac{3}{2}} - (n-1) \log \chi_1 - \log \chi_2.$$

All these quantities are given for $r = -·95$ to $+·95$ and $\rho = 0·0$ to $0·9$ in this Table B.

TABLE B. *To assist the calculation of the Ordinates of the Correlation Frequency Curves from Expansion Formulae.*

$\rho = \cdot 1.$

$\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 9934528.$

r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
−·95	·5470942	$\bar{1}$·8563426	·238125	·0229783	−·0364766	−·0013075
−·90	·4002321	$\bar{1}$·2919599	·23875	·0233820	−·0366462	−·0017344
−·85	·3159806	$\bar{1}$·5397223	·239375	·0237893	−·0368123	−·0021666
−·80	·2574549	$\bar{1}$·7102846	·24	·0242	−·0369750	−·0026040
−·75	·2131018	$\bar{1}$·8383056	·240625	·0246143	−·0371342	−·0030466
−·70	·1777811	$\bar{1}$·9392061	·24125	·0250320	−·0372898	−·0034943
−·65	·1487560	·0211959	·241875	·0254533	−·0374417	−·0039470
−·60	·1243983	·0891598	·2425	·0258781	−·0375900	−·0044047
−·55	·1036628	·1462328	·243125	·0263064	−·0377346	−·0048672
−·50	·0858411	·1945400	·24375	·0267383	−·0378754	−·0053345
−·45	·0704333	·2355806	·244375	·0271736	−·0380123	−·0058066
−·40	·0570761	·2704450	·245	·0276125	−·0381453	−·0062833
−·35	·0454992	·2999432	·245625	·0280549	−·0382744	−·0067646
−·30	·0354989	·3246862	·24625	·0285008	−·0383994	−·0072503
−·25	·0269206	·3451377	·246875	·0289502	−·0385204	−·0077405
−·20	·0196470	·3616495	·2475	·0294031	−·0386373	−·0082349
−·15	·0135901	·3744849	·248125	·0298596	−·0387500	−·0087336
−·10	·0086862	·3838348	·24875	·0303195	−·0388585	−·0092363
−·05	·0048920	·3898291	·249375	·0307830	−·0389627	−·0097431
·00	·0021824	·3925427	·25	·03125	−·0390625	−·0102539
+·05	·0005490	·3920005	·250625	·0317205	−·0391579	−·0107685
+·10	·0	·3881779	·25125	·0321945	−·0392490	−·0112869
+·15	·0005603	·3809998	·251875	·0326721	−·0393354	−·0118089
+·20	·0022729	·3703365	·2525	·0331531	−·0394174	−·0123344
+·25	·0052014	·3559973	·253125	·0336377	−·0394947	−·0128634
+·30	·0094334	·3377189	·25375	·0341258	−·0395674	−·0133957
+·35	·0150862	·3151498	·254375	·0346174	−·0396353	−·0139313
+·40	·0223140	·2878260	·255	·0351125	−·0396984	−·0144700
+·45	·0313204	·2551371	·255625	·0356111	−·0397568	−·0150118
+·50	·0423754	·2162728	·25625	·0361133	−·0398102	−·0155564
+·55	·0558421	·1701431	·256875	·0366189	−·0398587	−·0161038
+·60	·0722203	·1152488	·2575	·0371281	−·0399021	−·0166540
+·65	·0922180	·0494649	·258125	·0376408	−·0399406	−·0172067
+·70	·1168803	$\bar{1}$·9696565	·25875	·0381570	−·0399739	−·0177618
+·75	·1478351	$\bar{1}$·8709390	·259375	·0386768	−·0400021	−·0183193
+·80	·1878190	$\bar{1}$·7451026	·26	·0392	−·0400250	−·0188790
+·85	·2419720	$\bar{1}$·5767267	·260625	·0397268	−·0400427	−·0194408
+·90	·3218470	$\bar{1}$·3311524	·26125	·0402570	−·0400550	−·0200045
+·95	·4643287	$\bar{2}$·8977254	·261875	·0407908	−·0400620	−·0205702

$\rho = \cdot 2.$

$\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 9734068.$

r	$\log \chi_1$	$\log \chi_2$	ϕ_1
−·95	·5899090	$\bar{1}$·8182302	·22625
−·90	·4413696	$\bar{1}$·2546862	·2275
−·85	·3554188	$\bar{1}$·5032983	·22875
−·80	·2951711	$\bar{1}$·6747215	·23
−·75	·2490732	$\bar{1}$·8036150	·23125
−·70	·2119841	$\bar{1}$·9053996	·2325
−·65	·1811668	$\bar{1}$·9882856	·23375
−·60	·1549924	·0571577	·235
−·55	·1324153	·1151516	·23625
−·50	·1127264	·1643923	·2375
−·45	·0954255	·2063796	·23875
−·40	·0801485	·2422038	·24
−·35	·0666246	·2726756	·24125
−·30	·0546496	·2984059	·2425
−·25	·0440680	·3198590	·24375
−·20	·0347621	·3373870	·245
−·15	·0266432	·3512533	·24625
−·10	·0196470	·3616495	·2475
−·05	·0137293	·3687055	·24875
·00	·0088644	·3724968	·25
+·05	·0050431	·3730485	·25125
+·10	·0022729	·3703365	·2525
+·15	·0005777	·3642861	·25375
+·20	·0	·3547680	·255
+·25	·0006024	·3415919	·25625
+·30	·0024715	·3244949	·2575
+·35	·0057238	·3031260	·25875
+·40	·0105126	·2770218	·26
+·45	·0170404	·2455721	·26125
+·50	·0255763	·2079674	·2625
+·55	·0364823	·1631181	·26375
+·60	·0502571	·1095254	·265
+·65	·0676076	·0450651	·26625
+·70	·0895777	$\bar{1}$·9666028	·2675
+·75	·1177943	$\bar{1}$·8692544	·26875
+·80	·1549924	$\bar{1}$·7448109	·27
+·85	·2063110	$\bar{1}$·5778522	·27125
+·90	·2833014	$\bar{1}$·3337203	·2725
+·95	·4228471	$\bar{2}$·9017612	·27375

TABLE B—(*continued*).

$\rho = \cdot 2$. $\log(1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 9734068$.				$\rho = \cdot 3$. $\log(1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 9385621$.					
ϕ_2	ϕ_3	ϕ_4	r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
·0159758	−·0326811	+·0057193	−·95	·6348801	$\bar{2}$·7667074	·214375	·0102424	−·0280527	+·0104960
·0166531	−·0331264	+·0050816	−·90	·4849062	$\bar{1}$·2038806	·21625	·0110633	−·0288235	+·0098985
·0173445	−·0335620	+·0044192	−·85	·3974915	$\bar{1}$·4532246	·218125	·0119158	−·0295824	+·0092418
·01805	−·0339875	+·0037321	−·80	·3357497	$\bar{1}$·6253949	·22	·0128	−·0303250	+·0085260
·0187695	−·0344025	+·0030208	−·75	·2881264	$\bar{1}$·7550511	·221875	·0137158	−·0310528	+·0077516
·0195031	−·0348065	+·0022856	−·70	·2494796	$\bar{1}$·8576146	·22375	·0146633	−·0317633	+·0069189
·0202508	−·0351990	+·0015269	−·65	·2170712	$\bar{1}$·9412961	·225625	·0156424	−·0324549	+·0060287
·0210125	−·0355797	+·0007451	−·60	·1892713	·0109810	·2275	·0166531	−·0331264	+·0050816
·0217883	−·0359481	−·0000593	−·55	·1650331	·0698054	·229375	·0176955	−·0337760	+·0040787
·0225781	−·0363037	−·0008860	−·50	·1436465	·1198950	·23125	·0187695	−·0344025	+·0030208
·0233820	−·0366462	−·0017344	−·45	·1246098	·1627501	·233125	·0198752	−·0350042	+·0019092
·0242	−·0369750	−·0026040	−·40	·1075577	·1994619	·235	·0210125	−·0355797	+·0007451
·0250320	−·0372898	−·0034943	−·35	·0922180	·2308416	·236875	·0221814	−·0361275	−·0004699
·0258781	−·0375900	−·0044047	−·30	·0783851	·2575009	·23875	·0233820	−·0366462	−·0017344
·0267383	−·0378754	−·0053345	−·25	·0659021	·2799047	·240625	·0246143	−·0371342	−·0030466
·0276125	−·0381453	−·0062833	−·20	·0546496	·2984059	·2425	·0258781	−·0375900	−·0044047
·0285008	−·0383994	−·0072503	−·15	·0445372	·3132690	·244375	·0271736	−·0380123	−·0058066
·0294031	−·0386373	−·0082349	−·10	·0354989	·3246862	·24625	·0285008	−·0383994	−·0072503
·0303195	−·0388585	−·0092363	−·05	·0274889	·3327884	·248125	·0298596	−·0387500	−·0087336
·03125	−·0390625	−·0102539	·00	·0204793	·3376520	·25	·03125	−·0390625	−·0102539
·0321945	−·0392490	−·0112869	+·05	·0144591	·3393033	·251875	·0326721	−·0393354	−·0118089
·0331531	−·0394170	−·0123344	+·10	·0094334	·3377189	·25375	·0341258	−·0395674	−·0133957
·0341258	−·0395674	−·0133957	+·15	·0054243	·3328255	·255625	·0356111	−·0397568	−·0150118
·0351125	−·0396984	−·0144700	+·20	·0024715	·3244949	·2575	·0371281	−·0399021	−·0166540
·0361133	−·0398102	−·0155564	+·25	·0006354	·3125381	·259375	·0386768	−·0400021	−·0183193
·0371281	−·0399021	−·0166540	+·30	·0	·2966934	·26125	·0402570	−·0400550	−·0200045
·0381570	−·0399739	−·0177618	+·35	·0006788	·2766112	·263125	·0418689	−·0400595	−·0217064
·0392	−·0400250	−·0188790	+·40	·0028223	·2518296	·265	·0435125	−·0400141	−·0234213
·0402570	−·0400550	−·0200045	+·45	·0066301	·2217400	·266875	·0451877	−·0399172	−·0251457
·0413281	−·0400635	−·0211375	+·50	·0123676	·1855345	·26875	·0468945	−·0397675	−·0268757
·0424133	−·0400500	−·0222767	+·55	·0203937	·1421251	·270625	·0486330	−·0395633	−·0286077
·0435125	−·0400141	−·0234213	+·60	·0312032	·0900151	·2725	·0504031	−·0393033	−·0303374
·0446258	−·0399553	−·0245700	+·65	·0454992	·0270821	·274375	·0522049	−·0389860	−·0320608
·0457531	−·0398732	−·0257219	+·70	·0643213	$\bar{1}$·9501937	·27625	·0540383	−·0386098	−·0337735
·0468945	−·0397675	−·0268757	+·75	·0892920	$\bar{1}$·8544683	·278125	·0559033	−·0381733	−·0354711
·04805	−·0396375	−·0280304	+·80	·1231416	$\bar{1}$·7316990	·28	·0578	−·0376750	−·0371490
·0492195	−·0394829	−·0291847	+·85	·1710041	$\bar{1}$·5664684	·281875	·0597283	−·0371134	−·0388026
·0504031	−·0393033	−·0303374	+·90	·2444254	$\bar{1}$·3241210	·28375	·0616883	−·0364871	−·0404269
·0516008	−·0390982	−·0314873	+·95	·3802830	1·8940059	·285625	·0636799	−·0357945	−·0420171

TABLE B. *To assist the calculation of the Ordinates of the Correlation Frequency Curves from Expansion Formulae.*

$\rho = \cdot 4$.
$\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 8864189$.

r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
−·95	·6832371	$\bar{2}$·6990762	·2025	·0057781	−·0229682	+·0129113
−·90	·5320225	$\bar{1}$·1368698	·205	·0066125	−·0240578	+·0125970
−·85	·4433337	$\bar{1}$·3868509	·2075	·0075031	−·0251404	+·0121802
−·80	·3802830	$\bar{1}$·5596756	·21	·00845	−·0262125	+·0116596
−·75	·3313147	$\bar{1}$·6900043	·2125	·0094531	−·0272705	+·0110342
−·70	·2912852	$\bar{1}$·7932591	·215	·0105125	−·0283109	+·0103034
−·65	·2574549	$\bar{1}$·8776516	·2175	·0116281	−·0293303	+·0094672
−·60	·2281921	$\bar{1}$·9480680	·22	·0128	−·0303250	+·0085260
−·55	·2024481	·0076453	·2225	·0140281	−·0312916	+·0074804
−·50	·1795110	·0585101	·225	·0153125	−·0322266	+·0063318
−·45	·1588770	·1021639	·2275	·0166531	−·0331264	+·0050816
−·40	·1401787	·1396988	·23	·01805	−·0339875	+·0037321
−·35	·1231416	·1719271	·2325	·0195031	−·0348065	+·0022856
−·30	·1075577	·1994619	·235	·0210125	−·0355797	+·0007451
−·25	·0932674	·2227694	·2375	·0225781	−·0363037	−·0008860
−·20	·0801485	·2422038	·24	·0242	−·0369750	−·0026040
−·15	·0681078	·2580318	·2425	·0258781	−·0375900	−·0044047
−·10	·0570761	·2704450	·245	·0276125	−·0381453	−·0062833
−·05	·0470041	·2795781	·2475	·0294031	−·0386373	−·0082349
·00	·0378604	·2855089	·25	·03125	−·0390625	−·0102539
+·05	·0296300	·2882652	·2525	·0331531	−·0394174	−·0123344
+·10	·0223140	·2878260	·255	·0351125	−·0396984	−·0144700
+·15	·0159298	·2841201	·2575	·0371281	−·0399021	−·0166540
+·20	·0105126	·2770218	·26	·0392	−·0400250	−·0188790
+·25	·0061172	·2663445	·2625	·0413281	−·0400635	−·0211375
+·30	·0028223	·2518296	·265	·0435125	−·0400141	−·0234213
+·35	·0007352	·2331303	·2675	·0457531	−·0398732	−·0257219
+·40	·0	·2097881	·27	·04805	−·0396375	−·0280304
+·45	·0008089	·1811980	·2725	·0504031	−·0393033	−·0303374
+·50	·0034197	·1465558	·275	·0528125	−·0388672	−·0326331
+·55	·0081829	·1047779	·2775	·0552781	−·0383256	−·0349072
+·60	·0155840	·0543720	·28	·0578	−·0376750	−·0371490
+·65	·0263161	$\bar{1}$·9932210	·2825	·0603781	−·0369119	−·0393475
+·70	·0414078	$\bar{1}$·9181979	·285	·0630125	−·0360328	−·0414911
+·75	·0624694	$\bar{1}$·8244269	·2875	·0657031	−·0350342	−·0435679
+·80	·0922180	$\bar{1}$·7037082	·29	·06845	−·0339125	−·0455654
+·85	·1357728	$\bar{1}$·5406314	·2925	·0712531	−·0326643	−·0474709
+·90	·2046635	$\bar{1}$·3005493	·295	·0741125	−·0312859	−·0492710
+·95	·3357497	$\bar{2}$·8728199	·2975	·0770281	−·0297740	−·0509521

$\rho = \cdot 5$.
$\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 8125919$.

r	$\log \chi_1$	$\log \chi_2$	ϕ_1
−·95	·7367591	$\bar{2}$·6107927	·190625
−·90	·5844606	$\bar{1}$·0491282	·19375
−·85	·4946527	$\bar{1}$·2996689	·196875
−·80	·4304462	$\bar{1}$·4730716	·2
−·75	·3802830	$\bar{1}$·6039976	·203125
−·70	·3390180	$\bar{1}$·7078702	·20625
−·65	·3039093	$\bar{1}$·7929019	·209375
−·60	·2733227	$\bar{1}$·8639801	·2125
−·55	·2462074	$\bar{1}$·9242431	·215625
−·50	·2218487	$\bar{1}$·9758187	·21875
−·45	·1997401	·0202098	·221875
−·40	·1795110	·0585101	·225
−·35	·1608837	·0915336	·228125
−·30	·1436465	·1198950	·23125
−·25	·1276363	·1440625	·234375
−·20	·1127264	·1643923	·2375
−·15	·0988194	·1811527	·240625
−·10	·0858411	·1945400	·24375
−·05	·0737368	·2046893	·246875
·00	·0624694	·2116818	·25
+·05	·0520175	·2155489	·253125
+·10	·0423754	·2162728	·25625
+·15	·0335527	·2137861	·259375
+·20	·0255763	·2079674	·2625
+·25	·0184918	·1986347	·265625
+·30	·0123676	·1855345	·26875
+·35	·0072998	·1683255	·271875
+·40	·0034197	·1465558	·275
+·45	·0009057	·1196270	·278125
+·50	·0	·0867431	·28125
+·55	·0010353	·0468291	·284375
+·60	·0044774	$\bar{1}$·9984028	·2875
+·65	·0109971	$\bar{1}$·9393579	·290625
+·70	·0215976	$\bar{1}$·8665804	·29375
+·75	·0378604	$\bar{1}$·7752089	·296875
+·80	·0624694	$\bar{1}$·6570600	·3
+·85	·1005057	$\bar{1}$·4967424	·303125
+·90	·1634553	$\bar{1}$·2596309	30625
+·95	·2881264	$\bar{2}$·8351091	·309375

TABLE B—(*continued*).

$\rho = \cdot 5$. $\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 8125919$. | $\rho = \cdot 6$. $\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 7092700$.

ϕ_2	ϕ_3	ϕ_4	r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
·0025830	− ·0178043	+ ·0130733	− ·95	·7983074	$\bar{2}$·4939170	·17875	·0006570	− ·0129380	+ ·0113090
·0033008	− ·0191498	+ ·0132346	− ·90	·6450539	$\bar{2}$·9327299	·1825	·0011281	− ·0148103	+ ·0120455
·0041064	− ·0205090	+ ·0132540	− ·85	·5542555	$\bar{1}$·1837659	·18625	·0017258	− ·0159569	+ ·0126231
·005	− ·0218750	+ ·0131250	− ·80	·4890205	$\bar{1}$·3576828	·19	·00245	− ·0175375	+ ·0130246
·0059814	− ·0232410	+ ·0128423	− ·75	·4377890	$\bar{1}$·4891430	·19375	·0033008	− ·0191498	+ ·0132346
·0070508	− ·0246002	+ ·0124015	− ·70	·3954124	$\bar{1}$·5935710	·1975	·0042781	− ·0207818	+ ·0132403
·0082080	− ·0259457	+ ·0117995	− ·65	·3591488	$\bar{1}$·6791805	·20125	·0053820	− ·0224218	+ ·0130306
·0094531	− ·0272705	+ ·0110342	− ·60	·3273589	$\bar{1}$·7508604	·205	·0066125	− ·0240578	+ ·0125970
·0107861	− ·0285679	+ ·0101042	− ·55	·2989895	$\bar{1}$·8117504	·20875	·0079695	− ·0256780	+ ·0119330
·0122070	− ·0298309	+ ·0090097	− ·50	·2733227	$\bar{1}$·8639801	·2125	·0094531	− ·0272705	+ ·0110342
·0137158	− ·0310528	+ ·0077516	− ·45	·2498484	$\bar{1}$·9090541	·21625	·0110633	− ·0288235	+ ·0098985
·0153125	− ·0322266	+ ·0063318	− ·40	·2281921	$\bar{1}$·9480680	·22	·0128	− ·0303250	+ ·0085260
·0169971	− ·0333454	+ ·0047535	− ·35	·2080718	$\bar{1}$·9818379	·22375	·0146633	− ·0317633	+ ·0069189
·0187695	− ·0344025	+ ·0030208	− ·30	·1892713	·0109810	·2275	·0166531	− ·0331264	+ ·0050816
·0206299	− ·0353909	+ ·0011389	− ·25	·1716222	·0359679	·23125	·0187695	− ·0344025	+ ·0030208
·0225781	− ·0363037	− ·0008860	− ·20	·1549924	·0571577	·235	·0210125	− ·0355797	+ ·0007451
·0246143	− ·0371342	− ·0030466	− ·15	·1392781	·0748218	·23875	·0233820	− ·0366462	− ·0017344
·0267383	− ·0378754	− ·0053345	− ·10	·1243983	·0891598	·2425	·0258781	− ·0375900	− ·0044047
·0289502	− ·0385204	− ·0077405	− ·05	·1102908	·1003106	·24625	·0285008	− ·0383994	− ·0072503
·03125	− ·0390625	− ·0102539	·00	·0969100	·1083599	·25	·03125	− ·0390625	− ·0102539
·0336377	− ·0394947	− ·0128634	+ ·05	·0842253	·1133434	·25375	·0341258	− ·0395674	− ·0133957
·0361133	− ·0398102	− ·0155564	+ ·10	·0722203	·1152488	·2575	·0371281	− ·0399021	− ·0166540
·0386768	− ·0400021	− ·0183193	+ ·15	·0608930	·1140143	·26125	·0402570	− ·0400550	− ·0200045
·0413281	− ·0400635	− ·0211375	+ ·20	·0502571	·1095254	·265	·0435125	− ·0400141	− ·0234213
·0440674	− ·0399876	− ·0239952	+ ·25	·0403433	·1016073	·26875	·0468945	− ·0397675	− ·0268757
·0468945	− ·0397675	− ·0268757	+ ·30	·0312032	·0900151	·2725	·0504031	− ·0393033	− ·0303374
·0498096	− ·0393963	− ·0297613	+ ·35	·0229135	·0744170	·27625	·0540383	− ·0386098	− ·0337735
·0528125	− ·0388672	− ·0326331	+ ·40	·0155840	·0543720	·28	·0578	− ·0376750	− ·0371490
·0559033	− ·0381733	− ·0354711	+ ·45	·0093675	·0292945	·28375	·0616883	− ·0364871	− ·0404269
·0590820	− ·0373077	− ·0382544	+ ·50	·0044774	$\bar{1}$·9984028	·2875	·0657031	− ·0350342	− ·0435679
·0623486	− ·0362637	− ·0409610	+ ·55	·0012127	$\bar{1}$·9606388	·29125	·0698445	− ·0333044	− ·0465305
·0657031	− ·0350342	− ·0435679	+ ·60	·0	$\bar{1}$·9145399	·295	·0741125	− ·0312859	− ·0492710
·0691455	− ·0336124	− ·0460509	+ ·65	·0014639	$\bar{1}$·8580230	·29875	·0785070	− ·0289669	− ·0517436
·0726758	− ·0319916	− ·0483849	+ ·70	·0065529	$\bar{1}$·7880012	·3025	·0830281	− ·0263353	− ·0539003
·0762939	− ·0301647	− ·0505437	+ ·75	·0167837	$\bar{1}$·6996456	·30625	·0876758	− ·0233795	− ·0556908
·08	− ·0281250	− ·0525000	+ ·80	·0347621	$\bar{1}$·5848120	·31	·09245	− ·0200875	− ·0570629
·0837939	− ·0258656	− ·0542255	+ ·85	·0654746	$\bar{1}$·4281563	·31375	·0973508	− ·0164474	− ·0579619
·0876758	− ·0233795	− ·0556908	+ ·90	·1202910	$\bar{1}$·1951114	·3175	·1023781	− ·0120568	− ·0583312
·0916455	− ·0206600	− ·0568656	+ ·95	·2358762	$\bar{2}$·7751326	·32125	·1075320	− ·0080757	− ·0581117

TABLE B. *To assist the calculation of the Ordinates of the Correlation Frequency Curves from Expansion Formulae.*

$\rho = \cdot 7$.

$\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 5613553$.

r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
−·95	·8731268	$\bar{2}$·3332450	·166875	·0000002	−·0087458	+·0081651
−·90	·7190257	$\bar{2}$·7724818	·17125	·0000945	−·0101880	+·0094402
−·85	·6273441	$\bar{1}$·0239593	·175625	·0003611	−·0117540	+·0105881
−·80	·5611883	$\bar{1}$·1983367	·18	·0008	−·0134250	+·0115710
−·75	·5089957	$\bar{1}$·3302774	·184375	·0014111	−·0151821	+·0123553
−·70	·4656161	$\bar{1}$·4352073	·18875	·0021945	−·0170065	+·0129113
−·65	·4283019	$\bar{1}$·5213417	·193125	·0031502	−·0188794	+·0132135
−·60	·3954124	$\bar{1}$·5935710	·1975	·0042781	−·0207818	+·0132403
−·55	·3658926	$\bar{1}$·6550366	·201875	·0055783	−·0226951	+·0129741
−·50	·3390180	$\bar{1}$·7078702	·20625	·0070508	−·0246002	+·0124015
−·45	·3142753	$\bar{1}$·7535784	·210625	·0086955	−·0264785	+·0115131
−·40	·2912852	$\bar{1}$·7932591	·215	·0105125	−·0283109	+·0103034
−·35	·2697607	$\bar{1}$·8277312	·219375	·0125018	−·0300788	+·0087711
−·30	·2494796	$\bar{1}$·8576146	·22375	·0146633	−·0317633	+·0069189
−·25	·2302671	$\bar{1}$·8833832	·228125	·0169971	−·0333454	+·0047535
−·20	·2119841	$\bar{1}$·9053996	·2325	·0195031	−·0348065	+·0022856
−·15	·1945188	$\bar{1}$·9239392	·236875	·0221814	−·0361275	−·0004699
−·10	·1777811	$\bar{1}$·9392061	·24125	·0250320	−·0372898	−·0034943
−·05	·1616988	$\bar{1}$·9513444	·245625	·0280549	−·0382744	−·0067646
·00	·1462149	$\bar{1}$·9604452	·25	·03125	−·0390625	−·0102539
+·05	·1312858	$\bar{1}$·9665509	·254375	·0346174	−·0396353	−·0139313
+·10	·1168803	$\bar{1}$·9696565	·25875	·0381570	−·0399739	−·0177618
+·15	·1029796	$\bar{1}$·9697088	·263125	·0418689	−·0400595	−·0217064
+·20	·0895777	$\bar{1}$·9666028	·2675	·0457531	−·0398732	−·0257219
+·25	·0766832	$\bar{1}$·9601751	·271875	·0498096	−·0393963	−·0297613
+·30	·0643213	$\bar{1}$·9501937	·27625	·0540383	−·0386098	−·0337735
+·35	·0525383	$\bar{1}$·9363424	·280625	·0584393	−·0374949	−·0377031
+·40	·0414078	$\bar{1}$·9181979	·285	·0630125	−·0360328	−·0414911
+·45	·0310401	$\bar{1}$·8951960	·289375	·0677580	−·0342047	−·0450741
+·50	·0215976	$\bar{1}$·8665804	·29375	·0726758	−·0319916	−·0483849
+·55	·0133179	$\bar{1}$·8313240	·298125	·0777658	−·0293748	−·0513521
+·60	·0065529	$\bar{1}$·7880012	·3025	·0830281	−·0263353	−·0539003
+·65	·0018354	$\bar{1}$·7345749	·306875	·0884627	−·0228545	−·0559500
+·70	·0	$\bar{1}$·6680154	·31125	·0940695	−·0189134	−·0574179
+·75	·0024195	$\bar{1}$·5835655	·315625	·0998486	−·0144932	−·0582164
+·80	·0115163	$\bar{1}$·4731726	·32	·1058	−·0095750	−·0582540
+·85	·0320384	$\bar{1}$·3216122	·324375	·1119236	−·0041400	−·0574351
+·90	·0750398	$\bar{1}$·0944747	·32875	·1182195	+·0018306	−·0556601
+·95	·1767574	$\bar{2}$·6814297	·333125	·1246877	+·0083557	−·0528253

$\rho = \cdot 8$.

$\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 3344538$.

r	$\log \chi_1$	$\log \chi_2$	ϕ_1
−·95	·9728591	$\bar{2}$·0942943	·155
−·90	·8180004	$\bar{2}$·5339099	·16
−·85	·7255265	$\bar{2}$·7857835	·165
−·80	·6585413	$\bar{2}$·9605755	·17
−·75	·6054797	$\bar{1}$·0929508	·175
−·70	·5611883	$\bar{1}$·1983367	·18
−·65	·5229163	$\bar{1}$·2849499	·185
−·60	·4890205	$\bar{1}$·3576828	·19
−·55	·4584391	$\bar{1}$·4196788	·195
−·50	·4304462	$\bar{1}$·4730716	·2
−·45	·4045223	$\bar{1}$·5193703	·205
−·40	·3802830	$\bar{1}$·5596756	·21
−·35	·3574352	$\bar{1}$·5948094	·215
−·30	·3357497	$\bar{1}$·6253949	·22
−·25	·3150444	$\bar{1}$·6519100	·225
−·20	·2951711	$\bar{1}$·6747215	·23
−·15	·2760084	$\bar{1}$·6941098	·235
−·10	·2574549	$\bar{1}$·7102846	·24
−·05	·2394256	$\bar{1}$·7233964	·245
·00	·2218487	$\bar{1}$·7335437	·25
+·05	·2046635	$\bar{1}$·7407774	·255
+·10	·1878190	$\bar{1}$·7451026	·26
+·15	·1712730	$\bar{1}$·7464775	·265
+·20	·1549924	$\bar{1}$·7448109	·27
+·25	·1389531	$\bar{1}$·7399556	·275
+·30	·1231416	$\bar{1}$·7316990	·28
+·35	·1075577	$\bar{1}$·7197481	·285
+·40	·0922180	$\bar{1}$·7037082	·29
+·45	·0771634	$\bar{1}$·6830497	·295
+·50	·0624694	$\bar{1}$·6570600	·30
+·55	·0482647	$\bar{1}$·6247660	·305
+·60	·0347621	$\bar{1}$·5848120	·31
+·65	·0223140	$\bar{1}$·5352510	·315
+·70	·0115163	$\bar{1}$·4731726	·32
+·75	·0034197	$\bar{1}$·3939808	·325
+·80	·0	$\bar{1}$·2898462	·33
+·85	·0053672	$\bar{1}$·1458632	·335
+·90	·0296300	$\bar{2}$·9280951	·34
+·95	·1075577	$\bar{2}$·5269450	·345

TABLE B—(*continued*).

$\rho = \cdot 8$. $\log (1-\rho^2)^{\frac{3}{2}} = \bar{1}\cdot 3344538$. | $\rho = \cdot 9$. $\log (1-\rho^2)^{\frac{3}{2}} = \bar{2}\cdot 9181304$.

ϕ_2	ϕ_3	ϕ_4	r	$\log \chi_1$	$\log \chi_2$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
·0006125	−·0056047	+·0044070	−·95	1·1344648	$\bar{3}$·6665553	·143125	·0024939	−·0038913	+·0010198
·0002	−·0067750	+·0060060	−·90	·9789250	$\bar{2}$·1065114	·14875	·0014445	−·0045010	+·0025068
·0000125	−·0081703	+·0075898	−·85	·8857364	$\bar{2}$·3587425	·154375	·0006799	−·0054756	+·0042099
·00005	−·0097625	+·0090871	−·80	·8180004	$\bar{2}$·5339099	·16	·0002	−·0067750	+·0060060
·0003125	−·0115234	+·0104333	−·75	·7641490	$\bar{2}$·6666800	·165625	·0000049	−·0083591	+·0077831
·0008	−·0134250	+·0115710	−·70	·7190257	$\bar{2}$·7724818	·17125	·0000945	−·0101880	+·0094402
·0015125	−·0154391	+·0124493	−·65	·6798765	$\bar{2}$·8595337	·176875	·0004689	−·0122216	+·0108873
·00245	−·0175375	+·0130246	−·60	·6450539	$\bar{2}$·9327299	·1825	·0011281	−·0148103	+·0120455
·0036125	−·0196922	+·0132598	−·55	·6134923	$\bar{2}$·9952161	·188125	·0020721	−·0167425	+·0128469
·005	−·0218750	+·0131250	−·50	·5844606	$\bar{1}$·0491282	·19375	·0033008	−·0191498	+·0132346
·0066125	−·0240578	+·0125970	−·45	·5574342	$\bar{1}$·0959782	·199375	·0048143	−·0216015	+·0131629
·00845	−·0262125	+·0116596	−·40	·5320225	$\bar{1}$·1368698	·205	·0066125	−·0240578	+·0125970
·0105125	−·0283109	+·0103034	−·35	·5079254	$\bar{1}$·1726281	·210625	·0086955	−·0264785	+·0115131
·0128	−·0303250	+·0085260	−·30	·4849062	$\bar{1}$·2038806	·21625	·0110633	−·0288235	+·0098985
·0153125	−·0322266	+·0063318	−·25	·4627737	$\bar{1}$·2311092	·221875	·0137158	−·0310528	+·0077516
·01805	−·0339875	+·0037321	−·20	·4413696	$\bar{1}$·2546862	·2275	·0166531	−·0331264	+·0050816
·0210125	−·0355797	+·0007451	−·15	·4205607	$\bar{1}$·2748976	·233125	·0198752	−·0350042	+·0019092
·0242	−·0369750	−·0026040	−·10	·4002321	$\bar{1}$·2919599	·23875	·0233820	−·0366462	−·0017344
·0276125	−·0381453	−·0062833	−·05	·3802830	$\bar{1}$·3060315	·244375	·0271736	−·0380123	−·0058066
·03125	−·0390625	−·0102539	·00	·3606232	$\bar{1}$·3172203	·25	·03125	−·0390625	−·0102539
·0351125	−·0396984	−·0144700	+·05	·3411701	$\bar{1}$·3255880	·255625	·0356111	−·0397568	−·0150118
·0392	−·0400250	−·0188790	+·10	·3218470	$\bar{1}$·3311524	·26125	·0402570	−·0400550	−·0200045
·0435125	−·0400141	−·0234213	+·15	·3025809	$\bar{1}$·3338874	·266875	·0451877	−·0399172	−·0251457
·04805	−·0396375	−·0280304	+·20	·2833014	$\bar{1}$·3337203	·2725	·0504031	−·0393033	−·0303374
·0528125	−·0388672	−·0326331	+·25	·2639393	$\bar{1}$·3305264	·278125	·0559033	−·0381733	−·0354711
·0578	−·0376750	−·0371490	+·30	·2444254	$\bar{1}$·3241210	·28375	·0616883	−·0364871	−·0404269
·0630125	−·0360328	−·0414911	+·35	·2246902	$\bar{1}$·3142457	·289375	·0677580	−·0342047	−·0450741
·06845	−·0339125	−·0455654	+·40	·2046635	$\bar{1}$·3005493	·295	·0741125	−·0312859	−·0492710
·0741125	−·0312859	−·0492710	+·45	·1842748	$\bar{1}$·2825579	·300625	·0807518	−·0276909	−·0528645
·08	−·0281250	−·0525000	+·50	·1634553	$\bar{1}$·2596309	·30625	·0876758	−·0233795	−·0556908
·0861125	−·0244016	−·0551378	+·55	·1421425	$\bar{1}$·2308910	·311875	·0948846	−·0183117	−·0575750
·09245	−·0200875	−·0570629	+·60	·1202910	$\bar{1}$·1951114	·3175	·1023781	−·0120568	−·0583312
·0990125	−·0151547	−·0581467	+·65	·0978953	$\bar{1}$·1505243	·323125	·1101564	−·0057467	−·0577622
·1058	−·0095750	−·0582540	+·70	·0750398	$\bar{1}$·0944747	·32875	·1182195	+·0018306	−·0556601
·1128125	−·0033203	−·0572424	+·75	·0520175	$\bar{1}$·0227457	·334375	·1265674	+·0103245	−·0518057
·12005	+·0036375	−·0549629	+·80	·0296300	$\bar{2}$·9280951	·34	·1352	+·0197750	−·0459690
·1275125	+·0113266	−·0512594	+·85	·0100596	$\bar{2}$·7965809	·345625	·1441174	+·0302222	−·0379088
·1352	+·0197750	−·0459690	+·90	·0	$\bar{2}$·5959739	·35125	·1533195	+·0417061	−·0273728
·1431125	+·0290109	−·0389219	+·95	·0274889	$\bar{2}$·2200433	·356875	·1628064	+·0542668	−·0140979

TABLE C. *Position of Origin and Abscissal Unit in terms of Standard Deviation. The first Number gives the Position of the Origin, the second the Abscissal Unit.*

n	$\rho=0$	$\rho=\cdot 1$	$\rho=\cdot 2$	$\rho=\cdot 3$	$\rho=\cdot 4$
3	0	·1116225	·2266490	·3490329	·4840501
	·0707107	·0709720	·0717802	·0732142	·0754314
4	0	·1478819	·3011028	·4659475	·6510044
	·0866025	·0870223	·0883242	·0906481	·0942760
5	0	·1778638	·3643600	·5634956	·7915154
	·1000000	·1005663	·1023258	·1054792	·1104330
6	0	·2038711	·4165411	·6485234	·9145610
	·1118034	·1125038	·1146829	·1185985	·1247759
7	0	·2271065	·4645401	·7246924	1·0250662
	·1224745	·1232981	·1258629	·1304802	·1377861
8	0	·2482790	·5083017	·7942029	1·1260506
	·1322876	·1332252	·1361468	·1414136	·1497650
9	0	·2678441	·5487533	·8584888	1·2195137
	·1414214	·1424651	·1457192	·1515911	·1609168
10	0	·2861133	·5865316	·9185423	1·3068548
	·1500000	·1511433	·1547090	·1611481	·1713870
11	0	·3033102	·6220953	·9750815	1·3890961
	·1581139	·1593510	·1632105	·1701841	·1812835
12	0	·3196014	·6557863	1·0286451	1·4670090
	·1658312	·1671572	·1712949	·1787749	·1906890
13	0	·3351148	·6878684	1·0796486	1·5411920
	·1732051	·1746157	·1790180	·1869796	·1996684
14	0	·3499533	·7185489	1·1284203	1·6121199
	·1802776	·1817701	·1864240	·1948454	·2082735
15	0	·3641907	·7479946	1·1752248	1·6801759
	·1870829	·1886516	·1935489	·2024107	·2165466
16	0	·3779001	·7763419	1·2202786	1·7456751
	·1936492	·1952922	·2004222	2097070	·2245228
17	0	·3911338	·8037041	1·2637617	1·8088795
	·2000000	·2017147	·2070688	·2167611	·2322314
18	0	·4039379	·8301762	1·3052186	1·8700099
	·2061553	·2079391	·2135096	·2234913	·2396972
19	0	·4163514	·8558391	1·3465992	1·9292543
	·2121320	·2139828	·2197626	·2302287	·2469416
20	0	·4284077	·8807623	1·3861929	1·9867744
	·2179449	·2198606	·2258433	·2366781	·2539828
21	0	·4401360	·9050057	1·4247026	2·0427101
	·2236068	·2255854	·2317651	·2429578	·2608369
22	0	·4515616	·9286218	1·4622122	2·0971840
	·2291288	·2311687	·2375398	·2490805	·2675179
23	0	·4627064	·9516565	1·4987939	2·1503024
	·2345208	·2366202	·2431779	·2550572	·2740380
24	0	·4735903	·9741508	1·5345139	2·2021616
	·2397916	·2419491	·2486887	·2608980	·2804084
25	0	·4842308	·9961404	1·5694295	2·2528457
	·2449490	·2471633	·2540801	·2666117	·2866387
50	0	·6995857	1·4409868	2·2751863	3·2760792
	·3500000	·3533454	·3637983	·3827476	·4130572
100	0	·9997548	2·0606127	3·2571060	4·6973149
	·4974937	·5023813	·5176589	·5453562	·5896685
400	0	2·0150354	4·1553109	6·5735909	9·4914497
	·9987492	1·0087684	1·0400784	1·0968491	1·1876817

TABLE C—(*continued*).

n	$\rho=\cdot 5$	$\rho=\cdot 6$	$\rho=\cdot 7$	$\rho=\cdot 8$	$\rho=\cdot 9$
3	·6397035	·8298069	1·0821815	1·4670408	2·2572614
	·0787233	·0836496	·0914099	·1051565	·1375647
4	·8697069	1·1461037	1·5308211	2·1592865	3·6084918
	·0997391	·1080827	·1216223	·1467433	·2112748
5	1·0657049	1·4204694	1·9303003	2·8019830	4·9700548
	·1179621	·1296152	·1488972	·1857778	·2860585
6	1·2384801	1·6646016	2·2906234	3·3941363	6·2758785
	·1342232	·1489755	·1737018	·2219447	·3579595
7	1·3941989	1·8857159	2·6192262	3·9396570	7·4992321
	·1490074	·1666372	·1964454	·2553750	·4253943
8	1·5367713	2·0886726	2·9218412	4·4441908	8·6363301
	·1626318	·1829343	·2174712	·2863637	·4881142
9	1·6688524	2·2769195	3·2029089	4·9133944	9·6932562
	·1753174	·1981116	·2370570	·3152309	·5464383
10	1·7923354	2·4529927	3·4658872	5·3522889	10·6791012
	·1872248	·2123532	·2554247	·3422692	·6008602
11	1·9086231	2·6188112	3·7134918	5·7651162	11·6030258
	·1984751	·2258005	·2727512	·3677294	·6518808
12	2·0191978	2·7758616	3·9478808	6·1553948	12·4732325
	·2092046	·2385646	·2891787	·3918213	·6999496
13	2·1236943	2·9253194	4·1707880	6·5260261	13·2967143
	·2193627	·2507341	·3048225	·4147185	·7454499
14	2·2238984	3·0681290	4·3836241	6·8794080	14·0793562
	·2291249	·2623809	·3197770	·4365653	·7887049
15	2·3200595	3·2050605	4·5875494	7·2175324	14·8260439
	·2385095	·2735636	·3341195	·4574817	·8299826
16	2·4125823	3·3367521	4·7835297	7·5420659	15·5412908
	·2475526	·2843315	·3479156	·4775684	·8695312
17	2·5018385	3·4637416	4·9723756	7·8544126	16·2273352
	·2562879	·2947261	·3612200	·4969104	·9074724
18	2·5881420	3·5864801	5·1547751	8·1557645	16·8882836
	·2647443	·3047823	·3740794	·5155798	·9440329
19	2·6717601	3·7053440	5·3313166	8·4471433	17·5262030
	·2729462	·3145295	·3865338	·5336387	·9793257
20	2·7529232	3·8206677	5·5025071	8·7294265	18·1432348
	·2809148	·3239936	·3986177	·5511403	1·0134685
21	2·8318307	3·9341863	5·6687868	9·0033775	18·7411717
	·2886686	·3333199	·4103611	·5681311	1·0465598
22	2·9086578	4·0418750	5·8305409	9·2696624	19·3216440
	·2962240	·3421658	·4217904	·5846516	1·0786891
23	2·9835558	4·1481734	5·9881082	9·5288659	19·8860017
	·3035949	·3509062	·4329287	·6007373	1·1099306
24	3·0566623	4·2518695	6·1417876	9·7815042	20·4355030
	·3107942	·3594371	·4437967	·6164198	1·1403534
25	3·1280980	4·3532587	6·2918434	10·0281639	20·9711894
	·3178324	·3677825	·4544124	·6317351	1·1700149
50	4·5677713	6·3911451	9·2990780	14·9462808	31·5884211
	·4603212	·5361355	·6677547	·9376711	1·7584334
100	6·5628599	9·2058648	13·4374982	21·6797456	46·0257637
	·6587893	·7696570	·9623213	1·3574820	2·5594824
400	13·2812941	18·6652227	27·3067557	44·1732879	94·0676645
	1·3293798	1·5566854	1·9517326	2·7620803	5·2272309