SETS OF INDEPENDENT POSTULATES FOR THE ALGEBRA OF LOGIC*

BY

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The algebra of symbolic logic, as developed by LEIBNIZ, BOOLE, C. S. PEIRCE, E. SCHRÖDER, and others,† is described by WHITEHEAD as "the only known member of the non-numerical genus of universal algebra." ‡ This algebra, although originally studied merely as a means of handling certain problems in the logic of classes and the logic of propositions, has recently assumed some importance as an independent calculus; it may therefore be not without interest to consider it from a purely mathematical or abstract point of view, and to show how the whole algebra, in its abstract form, may be developed from a selected set of fundamental propositions, or postulates, which shall be independent of each other, and from which all the other propositions of the algebra can be deduced by purely formal processes.

In other words, we are to consider the construction of a purely *deductive theory*, without regard to its possible applications.

Introductory remarks on deductive theories in general.§ The first step in such a discussion is to decide on the *fundamental concepts* or *undefined symbols*, concerning which the statements of the algebra are to be made.

One such concept, common to every mathematical theory, is the notion of

1) a *class* (K) of *elements* $(a, b, c, \cdots)$. ||

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† For an extensive bibliography, see SCHRÖDER's *Algebra der Logik*, vol. 1 (1890).

‡ A. N. WHITEHEAD, *Universal Algebra*, vol. 1 (1898), p. 35.

§ Cf. papers by A. PADOA, cited in Transactions, vol. 4 (1903), p. 358.

|| A *class* is determined by stating some condition which every entity in the universe must either satisfy or not satisfy; every entity which satisfies the condition is said to *belong* to the class. (If the condition is such that no entity can satisfy it, the class is called a "null" class.) Every entity which belongs to the class in question is called an *element* (cf. H. WEBER, *Algebra*, vol. 2 (1899), p. 3).

No further analysis of this concept *class*, or of similar concepts introduced below, is here attempted. For an elaborate discussion of the logical processes which underlie all mathematical thinking, see B. RUSSELL's work on *The Principles of Mathematics*, vol. 1, 1903.

If two elements a and b are, for the purposes of the discussion in hand, *equivalent*, that is, if either may replace the other in every proposition of the algebra in question, we write $a = b$; otherwise, $a \neq b$.*

In regard to the other fundamental concepts, one has usually a considerable freedom of choice; several different sets of undefined symbols may serve as the basis of the same algebra; the only logical requirement is that the symbols of every such set must be definable in terms of the symbols of every other set. †

Thus for the algebra of logic the fundamental concepts (besides the notion of *class*) may be selected at pleasure from the following: ‡

2) a *rule of combination*, § denoted, say, by $\oplus$ (read, for convenience, "plus"; see remark on these symbols below);

3) another *rule of combination*, denoted, say, by $\odot$ (read, "times");

4) a *dyadic relation*, || denoted, say, by ⧀ (read, "within").

Any two of these symbols can be defined, as we shall see, in terms of the third. ¶

In the present paper, I choose the fundamental concepts as follows: In § 1;

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* Concerning the symbol $=$ we have the following obvious theorems: 1) $a = a$, 2) if $a = b$, then $b = a$; and 3) if $a = b$ and $b = c$, then $a = c$; which are taken by many writers as the properties by which the symbol $=$ is to be defined. But cf. O. HÖLDER, *Die Axiome der Quantität und die Lehre vom Mass*, Leipziger Berichte, Math.-Phys. Classe, vol. 53 (1901), p. 4, footnote.

† Cf. remarks by M. PIERI, in his article called: *Nuovo modo di svolgere deduttivamente la geometria projettiva*, Reale Istituto Lombardo di scienze e lettere (Milano), Rendiconti, ser. 2, vol. 31 (1898), especially p. 797.

‡ For a quite different point of departure, see A. B. KEMPE, *On the relation between the logical theory of classes and the geometrical theory of points*, Proceedings of the London Mathematical Society, vol. 21, pp. 147–182, January, 1890, and *The subject-matter of exact thought*, Nature, vol. 43. pp. 156–162, December, 1890.

§ A *rule of combination* $\circ$, in the given class, is a convention according to which every two elements a and b (whether $a = b$ or $a \neq b$), in a definite order, determine uniquely an entity

$$a \circ b \text{ (read ``}a\text{ with }b\text{'')},$$

which is, however, not necessarily an element of the class. In the class of quantities or numbers, familiar examples of rules of combination are $+$, $-$, $\times$, $\div$, etc.

|| A *dyadic relation*, R, in the given class, is determined when, if any two elements a and b are given in a definite order, we can decide whether a stands in the relation R to b or not; if it does, we write

aRb, or, equally well, bЯa.

In the class of quantities or numbers, familiar examples of dyadic relations are $=$, $<$, $>$, $\leqq$, etc. Relationships among human beings furnish other examples. [If R is such that aRa for every element a, then R is called a *reflexive* relation; if R is such that aRb and bRc together always imply aRc, then R is called a *transitive* relation; if R is such that aRb always implies bRa, then R is called a *symmetric* relation. Thus the relation $\leqq$ is reflexive and transitive, but not symmetric; the relation of equivalence is reflexive, transitive and symmetric.]

¶ In all discussions like the present, *definitions* are purely *nominal definitions*, introducing a new symbol as an abbreviation for an old concept. Cf. papers by PEANO and BURALI-FORTI in Bibliothèque du congrès international de philosophie, Paris, 1900, vol. 3 (published in 1901).

the two rules of combination, $\oplus$ and $\odot$; in § 2, the relation $\otimes$; in § 3, a single rule of combination, $\oplus$. The three sections form properly three separate papers.

Having chosen the fundamental concepts, the next step is to decide on the *fundamental propositions*, or *postulates*, which are to stand at the basis of the algebra. These postulates are simply *conditions* arbitrarily imposed on the fundamental concepts and must not, of course, be inconsistent among themselves. Any set of *consistent* * postulates would give rise to a corresponding algebra,—namely the totality of propositions which follow from these postulates by logical deduction.† For the sake of elegance, every set of postulates should be free from redundancies; in other words, the postulates of every set should be *independent*, no one of them deducible from the rest.‡ For, if any one of the postulates were a consequence of the others, it should be counted among the derived, not among the fundamental propositions. Furthermore, each postulate should be as nearly as possible a *simple statement*, not decomposable into two or more parts; but the idea of a simple statement is a very elusive one, which has not yet been satisfactorily defined, much less attained.§

In selecting a set of consistent, independent postulates for any particular algebra, one has usually a considerable freedom of choice; several different sets of independent postulates (on a given set of fundamental concepts) may serve as the basis of the same algebra; || the only logical requirement is that every such set of postulates must be deducible from every other.¶

Thus, for the algebra of logic, several different sets of postulates might be given on each of the three sets of fundamental concepts which we have selected. In the present paper a single set of postulates is chosen for each of the three sections.

Object of the present paper. The object of the paper can now be stated as follows: Having chosen a set of fundamental concepts and a set of fundamental propositions for each of the three sections, I show, first, that the fundamental

* On the consistency (Widerspruchslosigkeit) of a set of postulates, see a problem of HILBERT's cited in Transactions, vol. 4 (1903), p. 361, and an article by A. PADOA, *Le problème no.* 2 *de M. David Hilbert*, L'Enseignement Mathématique, vol. 5 (1903), pp. 85–91.

† The processes involved in "logical deduction" have been subjected in recent years to a very searching analysis; see especially the work of G. PEANO and others in the Revue de Mathématiques, and B RUSSELL's *Principles of Mathematics*.

‡ The method of proving the independence of a postulate used and explained below, has been made familiar especially by the works of PEANO, PADOA, PIERI, and HILBERT.

§ Compare remarks by E. H. MOORE, in his paper on *A definition of abstract groups*, Transactions, vol. 3 (1902), especially pp. 488–489.

|| For a striking example, see the postulates for a field in recent articles by L. E. DICKSON and E. V. HUNTINGTON, Transactions, vol. 4 (1903), p. 13 and p. 31.

¶ Cf. M. PIERI, *loc. cit.* Even if the postulates could be made strictly simple statements, I see no reason why several different sets of consistent, independent, and simple postulates might not be possible for the same algebra. (Cf. SCHRÖDER, *loc. cit.*, vol. 3, p. 19.)

propositions of each set are consistent (and independent); and secondly, that the fundamental concepts of each section can be defined in terms of the fundamental concepts of each of the other sections, while the fundamental propositions of each section can be deduced from the fundamental propositions of each of the other sections. Then we may say, first, that each section determines a definite algebra, and secondly, that the three algebras are equivalent.

Finally, in order to justify the name "algebra of logic" for the algebra thus established, I show that the fundamental theorems of that algebra, as set forth in standard treatises like those of SCHRÖDER and WHITEHEAD, can be derived from either of my three sections. And the development of the theory in the present paper is carried only so far as is necessary for this object.

In working out the set of postulates in § 1, I have followed WHITEHEAD closely. The postulates I*a*–V are substantially the same as the fundamental propositions given in his *Universal Algebra*, Book II; except that the associative laws for addition and multiplication, which are there admitted as fundamental, are here deduced as theorems.

In § 2, postulates 1–10 are substantially the same as the fundamental prositions (called by various names*) in SCHRÖDER's *Algebra der Logik;* except that postulate 9 here replaces a much less simple postulate of SCHRÖDER's which I cite for reference as 9_2. For the possibility of this simplification I am especially indebted to Mr. C. S. PEIRCE, who has kindly communicated to me a proof of the second part of the distributive law ($22a$, b) on the basis of this postulate 9. (See footnote below.) A further problem in regard to postulate 9 is proposed at the end of § 2.

The third set of postulates (§ 3) is a fairly obvious modification of the second.

The only part of the paper for which I can claim any originality (except possibly the proofs of XIII*a*, *b* in § 1 and $20a$, b in § 2) is the establishment of the *complete independence* of all the postulates of each set. There has been no discussion of this question, as far as I know, except an only partially successful attempt of SCHRÖDER's to prove the independence of 9_2.†

A simple interpretation of the algebra. Although the algebra is necessarily treated here solely in its abstract form, without reference to its possible applications—that is, without reference to the possible interpretations of the symbols K, $\oplus$, $\odot$, and $\otimes$—nevertheless it may be well to mention at once one of the simplest of these applications, so that the reader may give a concrete interpre-

* "Prinzipien," "Postulaten," "Definitionen." See loc. cit., pp. 168, 170, 184, 188, 196, 293, 303.

† He succeeded in showing, by a very complicated method, that 9_2 is independent of postulates 1–7, omitting postulate 8. (Loc. cit., pp. 286–288, 617-628, 633–640, 642–643.) But the question whether 9_2 is independent of the full list of postulates 1–8 was left undecided ; see loc. cit. p. 310, bottom.

tation, if he so desires, to all the propositions of the algebra. Any system (K, $\oplus$, $\odot$, $\otimes$) which satisfies the postulates and definitions of § 1, § 2, or § 3 will answer the purpose. One of the simplest of such systems is the following: *

K = the class of regions in the plane including the "null region" [$= \wedge$; read "nothing"] and the whole plane [$= \vee$; read "everything"];

$a \oplus b$ = the smallest region which includes both a and b, called the "logical sum" of a and b;

$a \odot b$ = the largest region which lies within both a and b, called the "logical product" of a and b;

$\otimes$ = the relation of inclusion; that is, $a \otimes b$ signifies that the region a lies within or coincides with the region b.

Remarks on the symbols $\oplus$, $\odot$, *etc.* The symbols $\oplus$, $\odot$, and $\otimes$ are chosen with a double object in view. On account of the circles around them they are sufficiently unfamiliar to remind us of their true character as undefined symbols which have no properties not expressly stated in the postulates; while the $+$, $\cdot$, and $<$ within the circles enable us to adopt, with the least mental effort, the interpretation which is likely to be the most useful. The symbol $\oplus$ was used by LEIBNIZ for the same purpose about 1700. †

The symbols $\wedge$ and $\vee$, which occur below, I take from PEANO's Formulaire de Mathématiques, vol. 4 (1903), pp. 27–28. The resemblance which these symbols bear to an empty glass and a full glass will facilitate the interpretation of them as "nothing" and "everything" respectively.

§ 1. THE FIRST SET OF POSTULATES.

In § 1 we take as the fundamental concepts a class, K, with two rules of combination, $\oplus$ and $\odot$; and as the fundamental propositions, the following ten postulates:

I*a*. *$a \oplus b$ is in the class whenever a and b are in the class.*

I*b*. *$a \odot b$ is in the class whenever a and b are in the class.*

II*a*. *There is an element $\wedge$ such that $a \oplus \wedge = a$ for every element a.*

II*b*. *There is an element $\vee$ such that $a \odot \vee = a$ for every element a.*

III*a*. *$a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ are in the class.*

III*b*. *$a \odot b = b \odot a$ whenever a, b, $a \odot b$, and $b \odot a$ are in the class.*

IV*a*. *$a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$ whenever $a, b, c, a \oplus b, a \oplus c, b \odot c, a \oplus (b \odot c)$, and $(a \oplus b) \odot (a \oplus c)$ are in the class.*

* Compare EULER's diagrams, in works on logic.

† LEIBNIZ, *Philosophische Schriften, herausgegeben von* GERHARDT, vol. 7 (1890), p. 237; cf. Formulaire de Mathématiques, vol. 3 (1901), p. 19. On the use of the circles around these symbols, see also CHRISTINE LADD [Mrs. FRANKLIN], *On the algebra of logic*, in *Studies in Logic by members of Johns Hopkins University*, 1883, p. 18.

IVb. *$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ whenever $a, b, c, a \odot b, a \odot c, b \oplus c$, $a \odot (b \oplus c)$ and $(a \odot b) \oplus (a \odot c)$ are in the class.*

V. *If the elements $\wedge$ and $\vee$ in postulates* IIa *and* IIb *exist and are unique, then for every element a there is an element $\bar{a}$ such that $a \oplus \bar{a} = \vee$ and $a \odot \bar{a} = \wedge$.*

VI. *There are at least two elements, x and y, in the class such that $x \neq y$.*

Consistency of the postulates of the first set.

To show the consistency of the postulates, we have only to exhibit some system $(K, \oplus, \odot)$ in which K, $\oplus$, and $\odot$ are so interpreted that all the postulates are satisfied. For then the postulates themselves, and all their consequences, will be simply expressions of the properties of this system, and therefore cannot involve contradiction (since no system which really exists can have contradictory properties).

One such system is the following: $K =$ the class of regions in the plane including the "null region" and the whole plane; $a \oplus b =$ the "logical sum" of a and b (that is, the smallest region which includes them both); $a \odot b =$ the "logical product" of a and b (that is, the largest region which lies within them both).

Another such system, in fact the simplest possible one, is this: $K =$ a class comprising only two elements, say 0 and 1, with $\oplus$ and $\odot$ defined by the tables

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

For other such systems, see the appendix.—The existence of any one of these systems is sufficient to prove the consistency of the postulates.

Deductions from the postulates of the first set.

The following theorems follow readily from the postulates Ia–VI; the proofs are given in the next paragraph.

VIIa. The element $\wedge$ in IIa is unique: $a \oplus \wedge = a$.

VIIb. The element $\vee$ in IIb is unique: $a \odot \vee = a$.

VIIIa. $a \oplus a = a$.

VIIIb. $a \odot a = a$.

IXa. $a \oplus \vee = \vee$.

IXb. $a \odot \wedge = \wedge$.

Xa. $a \oplus (a \odot b) = a$. (The "law of absorption.")

Xb. $a \odot (a \oplus b) = a$.

XI. The element $\bar{a}$ in V is uniquely determined by a:

$$a \oplus \bar{a} = \bigvee \quad \text{and} \quad a \odot \bar{a} = \bigwedge.$$

DEFINITION. The element $\bar{a}$ is called non-a, or the *supplement* of a. By IIIa, IIIb, if b is the supplement of a then a is the supplement of b; that is, if $b = \bar{a}$, then $a = \bar{b}$, or, $\bar{\bar{a}} = a$.

XIIa. $a \oplus b = \overline{\bar{a} \odot \bar{b}}$, and

XIIb. $a \odot b = \overline{\bar{a} \oplus \bar{b}}$;

that is, $a \oplus b$ and $\bar{a} \odot \bar{b}$ are supplementary elements as are also $a \odot b$ and $\bar{a} \oplus \bar{b}$. These theorems establish the *principle of duality* between $\oplus$ and $\odot$, which is a characteristic feature of the algebra. They also enable us to define either multiplication or addition in terms of the other and negation.

XIIIa. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. (Associative law for addition.)

XIIIb. $(a \odot b) \odot c = a \odot (b \odot c)$. (Associative law for multiplication.)

These theorems are sufficient to make the connection between the postulates here adopted and the usual treatment of the subject. See, for example, WHITEHEAD's *Universal Algebra*, vol. 1, book II, where two lists of fundamental propositions for the algebra of logic are given; the first list (p. 35) comprises (besides Ia, b) II$a, b$, IIIa, b, IVa, V, VIIIa, b, Xa, and XIIIa, b; the second list (p. 37), which is more symmetric, includes II$a, b$, IIIa, b, IV$a, b$, IX$a, b$, Xa, b, XI, and XIIIa, b.

The further development of the subject is based on the definition of $\otimes$, which may be given in various forms, thus,

DEFINITION. *If* $a \oplus b = b$; *or, if* $a \odot b = a$; *or, if* $\bar{a} \oplus b = \bigvee$; *or, if* $a \odot \bar{b} = \bigwedge$; *then we write* $a \otimes b$ (*or* $b \otimes a$).

It is easily seen that these definitions are all equivalent, and that the properties of $\otimes$ used as postulates in § 2 can be readily deduced.

Proofs of theorems in the preceding paragraph.

In the following proofs we write, for brevity, $a \odot b = ab$. The proofs for the theorems "b" may be obtained from the proofs for the corresponding theorems "a" by interchanging $\oplus$ with $\odot$ and $\bigwedge$ with $\bigvee$.

Proof of VIIa. Suppose there were two elements, $\bigwedge_1$ and $\bigwedge_2$, such that $a \oplus \bigwedge_1 = a$ and $a \oplus \bigwedge_2 = a$ for every element a. Then, putting $a = \bigwedge_2$ in the first equation and $a = \bigwedge_1$ in the second, we should have $\bigwedge_2 \oplus \bigwedge_1 = \bigwedge_2$ and $\bigwedge_1 \oplus \bigwedge_2 = \bigwedge_1$; whence, by III$a$, $\bigwedge_1 = \bigwedge_2$.

Proof of VIIIa. By V (in view of VIIa, b) take $\bar{a}$ so that $a \oplus \bar{a} = \bigvee$ and $a\bar{a} = \bigwedge$. Then by Ia, II$a, b$, and IVa we have

$$a \oplus a = (a \oplus a) \bigvee = (a \oplus a)(a \oplus \bar{a}) = a \oplus (a\bar{a}) = a \oplus \bigwedge = a.$$

Proof of IXa. By V (in view of VIIa, b) take $\bar{a}$ so that $a \oplus \bar{a} = \vee$. Then by Ia, IIb, IIIb, and IVa we have

$$a \oplus \vee = (a \oplus \vee) \vee = \vee (a \oplus \vee) = (a \oplus \bar{a})(a \oplus \vee) = a \oplus (\bar{a} \vee) = a \oplus \bar{a} = \vee .$$

Proof of Xa. By Ia, b, IIb, IVb, IIIa, and IXa we have

$$a \oplus (ab) = (a \vee) \oplus (ab) = a(\vee \oplus b) = a(b \oplus \vee) = a \vee = a.$$

Proof of XI. Suppose that for a given element a there were two elements, $\bar{a}_1$ and $\bar{a}_2$, such that $a \oplus \bar{a}_1 = a \oplus \bar{a}_2 = \vee$ and $a\bar{a}_1 = a\bar{a}_2 = \wedge$; then using I$a$, b, IIa, b, IIIb, IVb, and V we should have

$$\bar{a}_2 = \vee \bar{a}_2 = (a \oplus \bar{a}_1)\bar{a}_2 = (a\bar{a}_2) \oplus (\bar{a}_1 \bar{a}_2) = \wedge \oplus (\bar{a}_1 \bar{a}_2)$$

$$= (\bar{a}_1 a) \oplus (\bar{a}_1 \bar{a}_2) = \bar{a}_1 (a \oplus \bar{a}_2) = \bar{a}_1 \vee = \bar{a}_1 .$$

Proof of XIIa. We notice first that

$$a \oplus (\bar{a} \oplus c) = \vee \qquad \text{and} \qquad a(\bar{a}c) = \wedge ;$$

for, by Ia, b, IIb, IIIb, IVa, V, and Xb,

$$a \oplus (\bar{a} \oplus c) = \vee [a \oplus (\bar{a} \oplus c)] = (a \oplus \bar{a})[a \oplus (\bar{a} \oplus c)]$$

$$= a \oplus [\bar{a}(\bar{a} \oplus c)] = a \oplus \bar{a} = \vee ,$$

and similarly for the reciprocal proposition.

Then, using IVa, b, IIIa, b, and XI,

$$(a \oplus b) \oplus (\bar{a} \cdot \bar{b}) = [(a \oplus b) \oplus \bar{a}] [(a \oplus b) \oplus \bar{b}] = \vee \vee = \vee ,$$

and

$$(a \oplus b)(\bar{a} \cdot \bar{b}) = [a(\bar{a} \cdot \bar{b})] \oplus [b(\bar{a} \cdot \bar{b})] = \wedge \wedge = \wedge ,$$

whence, by XI, $a \oplus b$ and $\bar{a} \cdot \bar{b}$ are supplementary elements.

Proof of XIIIa. Let $(a \oplus b) \oplus c = x$ and $a \oplus (b \oplus c) = y$; then $(\bar{a} \cdot \bar{b})\bar{c} = \bar{x}$, by XII$a$, and in order to prove that $x = y$ it is sufficient (by XI) to show that $\bar{x}$ and y are supplementary elements.

Now

(1) $$y \oplus \bar{a} = y \oplus \bar{b} = y \oplus \bar{c} = \vee .$$

For, first, $y \oplus \bar{a} = \bar{a} \oplus [a \oplus (b \oplus c)] = \vee$ as in the proof of XIIa; secondly,

$$y \oplus \bar{b} = \vee (\bar{b} \oplus y) = (\bar{b} \oplus b)(\bar{b} \oplus y) = \bar{b} \oplus (by) \text{ by IV}a,$$

while

$$by = b[a \oplus (b \oplus c)] = (ba) \oplus [b(b \oplus b)] = (ba) \oplus b = b, \text{ by IV}b \text{ and X}a, b,$$

so that $y \oplus \bar{b} = \bar{b} \oplus b = \vee$; and similarly, $y \oplus \bar{c} = \vee$.

Also, by a similar method,

(2) $$\bar{x}a = \bar{x}b = \bar{x}c = \wedge .$$

Therefore, by IVa,

$$y \oplus \bar{x} = y \oplus [(\bar{a}\cdot\bar{b})\bar{c}] = [(y \oplus \bar{a})(y \oplus \bar{b})](y \oplus \bar{c}) = (\vee \vee)\vee = \vee,$$

and, by IVb,

$$\bar{x}y = \bar{x}[a \oplus (b \oplus c)] = (\bar{x}a) \oplus [(\bar{x}b) \oplus (\bar{x}c)] = \wedge \oplus (\wedge \oplus \wedge) = \wedge ;$$

whence, by XI, $\bar{x}$ and y are supplementary elements. Therefore $x = y$.

Theorem XIIIb follows at once from XIIIa by XIIa, b.

Independence of the postulates of the first set.

The ten postulates of the first set are *independent;* that is, no one of them can be deduced from the other nine. To show this, we exhibit, in the case of each postulate, a system $(K, \oplus, \odot)$ which satisfies all the other postulates, but not the one in question. This postulate, then, cannot be a consequence of the others; for if it were, every system which had the other properties would have this property also, which is not the case.

For postulate VI take $K =$ the class comprising a single element, a, with $a \oplus a = a$ and $a \odot a = a$.

For the other postulates, take $K =$ a class containing two elements, say 0 and 1, with $\oplus$ and $\odot$ defined appropriately for each case, as indicated in the following scheme:

	$0 \oplus 0$	$0 \oplus 1$	$1 \oplus 0$	$1 \oplus 1$	$0 \odot 0$	$0 \odot 1$	$1 \odot 0$	$1 \odot 1$
Ia)	0	1	1	x	0	0	0	1
Ib)	0	1	1	1	x	0	0	1
IIa)	0	0	0	0	0	0	0	1
IIb)	0	1	1	1	1	1	1	1
IIIa)	0	0	1	1	0	0	0	1
IIIb)	0	1	1	1	0	0	1	1
IVa)	0	1	1	0	0	0	0	1
IVb)	0	1	1	1	1	0	0	1
V)	0	1	1	1	0	1	1	1

In verifying these results, notice that the system for IIa (or IIb) satisfies postulate V "vacuously," since no element having the properties of $\wedge$ (or $\vee$) exists; while the system for IIIa (or IIIb) also satisfies V vacuously, since the element $\wedge$ (or $\vee$) is not uniquely determined. In the other systems, $\wedge = 0$ and $\vee = 1$, except in the system for V, where $\wedge = 0$ and $\vee = 0$.

§ 2. The Second Set of Postulates.

In § 2 we take as the fundamental concepts a class, K, with a dyadic relation, $⧀$; and as the fundamental propositions, the following ten postulates. (Note that $a ⧀ b$ and $b ⧁ a$ mean the same thing.)

1. *$a ⧀ a$ whenever a belongs to the class.*
2. *If $a ⧀ b$ and also $a ⧁ b$, then $a = b$.*
3. *If $a ⧀ b$ and $b ⧀ c$, then $a ⧀ c$.*
4. *There is an element $\wedge$ such that $\wedge ⧀ a$ for every element $a \neq \wedge$.*
5. *There is an element $\vee$ such that $\vee ⧁ a$ for every element $a \neq \vee$.*
6. *If $a \neq b$, and neither $a ⧀ b$ nor $a ⧁ b$, there is an element s such that*

   1°) *$s ⧁ a$;* 2°) *$s ⧁ b$;* *and*

   3°) *if y, $\neq s$, is such that $y ⧁ a$ and $y ⧁ b$, then $y ⧁ s$.*
7. *If $a \neq b$, and neither $a ⧀ b$ nor $a ⧁ b$, there is an element p such that*

   1°) *$p ⧀ a$;* 2°) *$p ⧀ b$;* *and*

   3°) *if x, $\neq p$, is such that $x ⧀ a$ and $x ⧀ b$, then $x ⧀ p$.*
8. *If the elements $\wedge$ and $\vee$ in 4 and 5 exist and are unique, then for every element a there is an element $\bar{a}$ such that*

   1°) *if $x ⧀ a$ and $x ⧀ \bar{a}$, then $x = \wedge$;* *and*

   2°) *if $y ⧁ a$ and $y ⧁ \bar{a}$, then $y = \vee$.*
9. *If postulates 1, 4, 5, and 8 hold, and if $a ⧀ \bar{b}$ is false, then there is an element $x \neq \wedge$ such that $x ⧀ a$ and $x ⧀ b$.*
10. *There are at least two elements, x and y, such that $x \neq y$.*

In this list, postulates 1–7 are independent among themselves, and postulates 8 and 9 are independent of the first seven (ordinally independent). Taking the whole list together, however, either 6 or 7 can be deduced from the rest, as shown in 25 below. Both postulates 6 and 7 are allowed to stand in the list for reasons of symmetry; but if a set of absolutely (not merely ordinally) independent postulates is desired, either one or the other must be omitted.

Consistency of the postulates of the second set.

To show the consistency of the postulates, we have only to exhibit some system (K, $⧀$) in which K and $⧀$ are so interpreted that all the postulates are satisfied.

One such system is the following: $K =$ the class of regions in the plane (including the null-region and the whole plane); $a ⧀ b$ signifying that the region a lies within (or coincides with) the region b.

Another such system is the class composed of two elements, 0 and 1 with $0 \olessthan 0$, $0 \olessthan 1$, and $1 \olessthan 1$, but not $1 \olessthan 0$.

For other such systems, see the appendix.

Deductions from the postulates of the second set.

The following theorems are deduced from the postulates of the second set, and are sufficient to connect these postulates with the usual presentation of the theory; the proofs wherever needed are given in the next paragraph. The postulates on which each theorem depends are indicated at the right.

11*a*. The element $\wedge$ in 4 is unique. (2, 4)

Hence $\wedge \olessthan a$ for every element a; and if $x \olessthan \wedge$, then $x = \wedge$. (1, 2, 4)

11*b*. The element $\vee$ in 5 is unique. (2, 5)

Hence $\vee \ogreaterthan a$ for every element a; and if $y \ogreaterthan \vee$, then $y = \vee$. (1, 2, 4)

12*a*. The element s in 6 is uniquely determined by a and b; hence we may define $a \oplus b$ as follows:

Definition. If $a \olessthan b$, $a \oplus b = b$; if $a \ogreaterthan b$, $a \oplus b = a$; if $a = b$, $a \oplus a = a$; otherwise, $a \oplus b = s$ (in 6). Hence $a \oplus b \ogreaterthan a$ and $a \oplus b \ogreaterthan b$; and if $y \ogreaterthan a$ and $y \ogreaterthan b$, then $y \ogreaterthan a \oplus b$. Obviously, $a \oplus b = b \oplus a$. (1, 2, 6)

12*b*. The element p in 7 is uniquely determined by a and b; hence we may define $a \odot b$, or ab, as follows:

Definition. If $a \olessthan b$, $ab = a$; if $a \ogreaterthan b$, $ab = b$; if $a = b$, $aa = a$; otherwise, $ab = p$ (in 7). Hence $ab \olessthan a$ and $ab \olessthan b$; and if $x \olessthan a$ and $x \olessthan b$ then $x \olessthan ab$. Obviously $ab = ba$. (1, 2, 7)

13*a*. $a \oplus \wedge = a$ and $a \oplus \vee = \vee$. (1, 2, 4, 5, 6)

13*b*. $a\vee = a$ and $a\wedge = \wedge$. (1, 2, 4, 5, 7)

14*a*. If $y \ogreaterthan a \oplus b$, then $y \ogreaterthan a$ and $y \ogreaterthan b$. (1, 2, 3, 6)

14*b*. If $x \olessthan ab$, then $x \olessthan a$ and $x \olessthan b$. (1, 2, 3, 7)

15*a*. If $a \ogreaterthan b$ and $x \ogreaterthan y$, then $a \oplus x \ogreaterthan b \oplus y$. (1, 2, 3, 6)

In particular, if $x \ogreaterthan y$, then $a \oplus x \ogreaterthan a \oplus y$.

15*b*. If $a \olessthan b$ and $x \olessthan y$, then $ax \olessthan by$. (1, 2, 3, 7)

In particular, if $x \olessthan y$, then $ax \olessthan ay$.

16*a*. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. (1, 2, 3, 6)

16*b*. $(ab)c = a(bc)$. (1, 2, 3, 7)

17. The element $\bar{a}$ in 8 is uniquely determined by a; hence

Definition. *The element $\bar{a}$ (in 8) is called non-a, or the supplement of a.* Hence, $a \oplus \bar{a} = \vee$ and $a\bar{a} = \wedge$. Obviously, $\bar{\wedge} = \vee$ and $\bar{\vee} = \wedge$.
(1, 2, 4, 5, 8, 9)

18. If $\bar{a} = b$, then $a = \bar{b}$; hence, $\bar{\bar{a}} = a$, by 17. (1, 2, 4, 5, 8, 9)

19. If $a \olessthan b$ then, inversely, $\bar{b} \olessthan \bar{a}$. (1, 2, 3, 4, 5, 8, 9)

20a. $a \oplus b = \overline{\bar{a} \odot \bar{b}}$. (1, 2, 3, 4, 5, 6, 7, 8, 9)

20b. $a \odot b = \overline{\bar{a} \oplus \bar{b}}$. (1, 2, 3, 4, 5, 6, 7, 8, 9)

These last theorems, 20a and 20b, establish the duality between $a \oplus b$ and $a \odot b$.

21a. $a(b \oplus c) \ogreaterthan ab \oplus ac$. (1, 2, 3, 6, 7)

21b. $a \oplus bc \olessthan (a \oplus b)(a \oplus c)$. (1, 2, 3, 6, 7)

22a. $a(b \oplus c) \olessthan ab \oplus ac$. (1, 2, 3, 4, 5, 6, 7, 8, 9)

22b. $a \oplus bc \ogreaterthan (a \oplus b)(a \oplus c)$. (1, 2, 3, 4, 5, 6, 7, 8, 9)

From theorems 21 and 22 the distributive laws follow at once, by 2, namely,

23a. $a(b \oplus c) = ab \oplus ac$. (1, 2, 3, 4, 5, 6, 7, 8, 9)

23b. $a \oplus bc = (a \oplus b)(a \oplus c)$. (1, 2, 3, 4, 5, 6, 7, 8, 9)

All these theorems would hold for a class containing only a single element a, with $a \olessthan a$. This trivial case is excluded by postulate 10, however, and we have:

24. $\bar{a} \neq a$; in particular, $\wedge \neq \vee$. (1, 2, 4, 5, 6, 7, 8, 10)

25a. Postulate 6 is a consequence of postulates 1, 2, 3, 4, 5, 7, 8, and 9; the required element s being $s = \overline{\bar{a} \odot \bar{b}}$.

25b. Postulate 7 is a consequence of postulates 1, 2, 3, 4, 5, 6, 8, and 9; the required element p being $p = \overline{\bar{a} \oplus \bar{b}}$.

Proofs of theorems in the preceding paragraph.

The theorems 11a, b and 12a, b follow immediately from the postulates indicated. In theorems 13a, b it is sufficient to notice that the sum or product given has the properties stated in 12a, b. Theorems 14a and 14b follow from 12a and 12b by 3. The remaining theorems may be proved as follows, the proof for any theorem "b" being in each case readily supplied from the proof for the corresponding theorem "a" by interchanging $\olessthan$ with $\ogreaterthan$, $\oplus$ with $\odot$, and $\wedge$ with $\vee$.*

Proof of 15a. By 12a, $a \oplus x \ogreaterthan a$ and $a \oplus x \ogreaterthan x$. From $a \oplus x \ogreaterthan a$ and $a \ogreaterthan b$, by 3, $a \oplus x \ogreaterthan b$; from $a \oplus x \ogreaterthan x$ and $x \ogreaterthan y$, by 3, $a \oplus x \ogreaterthan y$; therefore, by 12$a$, $a \oplus x \ogreaterthan b \oplus y$.

Proof of 16a. By 12a, $(a \oplus b) \oplus c \ogreaterthan a \oplus b$, and $a \oplus b \ogreaterthan a$; hence, by 3, $(a \oplus b) \oplus c \ogreaterthan a$. But also, $a \oplus b \ogreaterthan b$, whence $(a \oplus b) \oplus c \ogreaterthan b$; and further, $(a \oplus b) \oplus c \ogreaterthan c$, by 12$a$; hence, by 12$a$, $(a \oplus b) \oplus c \ogreaterthan (b \oplus c)$. Therefore, by 12$a$, $(a \oplus b) \oplus c \ogreaterthan a \oplus (b \oplus c)$.

Similarly, $a \oplus (b \oplus c) \ogreaterthan (a \oplus b) \oplus c$. Hence the theorem, by 2.

* Theorems 21b, 22b, and 23b may also be inferred directly from 21a, 22a, and 23a, by the aid of the principle of duality established in 20a and 20b.

Proof of 17. Let $\bar{a}_1$ and $\bar{a}_2$ be two elements having the properties of $\bar{a}$ in 8. Then $\bar{a}_1 \otimes \bar{a}_2$; for if not, we should have, by 9, an element $x \neq \wedge$ such that $x \otimes \bar{a}_1$ and $x \otimes a$, whence, by 8 and 11a, $x = \wedge$.

Again, $\bar{a}_2 \otimes \bar{a}_1$; for, if not, we should have, by 9, an element $y \neq \wedge$ such that $y \otimes \bar{a}_2$ and $y \otimes a$, whence, by 8 and 11a, $y = \wedge$.

Therefore $\bar{a}_1 = \bar{a}_2$, by 2.

Proof of 18. From $\bar{a} = b$ we have: if $x \otimes a$ and $x \otimes b$, then $x = \wedge$; and if $y \otimes a$ and $y \otimes b$, then $y = \vee$. But these are precisely the conditions under which $a = \bar{b}$, by 8.

Proof of 19. If $\bar{b} \otimes \bar{a}$ were false, we should have, by 9, an element $x \neq \wedge$ such that $x \otimes \bar{b}$ and $x \otimes a$. But from $x \otimes a$ and $a \otimes b$ follows $x \otimes b$, by 3; and from $x \otimes b$ and $x \otimes \bar{b}$ follows $x = \wedge$, by 8 and 11a, which contradicts the condition $x \neq \wedge$.

Proof of 20a. Let $a \oplus b = s$ and $\bar{a} \cdot \bar{b} = \bar{p}$; it is required to prove (see 18) that $s = p$.

By 12a, $s \otimes a$ and $s \otimes b$; hence by 19, $\bar{s} \otimes \bar{a}$ and $\bar{s} \otimes \bar{b}$, or, by 12b, $\bar{s} \otimes \bar{a} \cdot \bar{b}$; that is $\bar{s} \otimes \bar{p}$, or by 19, $p \otimes s$.

Again, by 12b, $\bar{p} \otimes \bar{a}$ and $\bar{p} \otimes \bar{b}$; hence, by 19, $p \otimes a$ and $p \otimes b$, or, by 12a, $p \otimes a \oplus b$; that is, $p \otimes s$.

Therefore $s = p$, by 2.

Proof of 21a. By 12b, $ab \otimes a$ and $ac \otimes a$, whence, by 12a,

$$ab \oplus ac \otimes a.$$

Again, $ab \otimes b$ and $ac \otimes c$, by 12b; hence, by 15a,

$$ab \oplus ac \otimes b \oplus c.$$

Therefore, $ab \oplus ac \otimes a(b \oplus c)$, by 12$b$.

Proof of 22a.* In order to facilitate the proof of this theorem, we first establish the following

LEMMA: $a(b \oplus c) \otimes b \oplus ac$.

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* This demonstration is borrowed, almost verbatim, from a letter of Mr. C. S. PEIRCE'S, dated December 24, 1903. Mr. PEIRCE uses the symbol $\prec$ where I have used $\otimes$, and in a slightly different sense; so that he is enabled to state that the principle here called postulate 9 "follows from the definition of $\overline{P_i \prec C_i}$ on page 18" of his article of 1880. The demonstration was originally worked out for that article (American Journal of Mathematics, vol. 3 (1880), p. 33), but is now published for the first time (compare *ibid.*, vol. 7, p. 190, footnote, 1885 [wrongly cited as 1884 in SCHRÖDER'S bibliography], and SCHRÖDER, *loc. cit.*, p. 291).

Under the date February 14, 1904, Mr. PEIRCE writes as follows:

"Dear Mr. HUNTINGTON: Should you decide to print the proof of the distributive principle (and this would not only relieve me from a long procrastinated duty, but would have a certain value for exact logic, as removing the eclipse under which the method of developing the subject followed in my paper in vol. 3 of the American Journal of Mathematics has been obscured) I should feel that it was incumbent upon me, in decency, to explain its having been so long withheld. The truth is that the paper aforesaid was written during leisure hours gained to me by

Suppose the lemma to be false. Then, by 9 and 18, there is an element $x \neq \wedge$ such that

$$x \otimes a(b \oplus c), \qquad (1°)$$

and $$x \otimes \overline{b \oplus ac},$$

whence, by 19, $$\bar{x} \oslash b \oplus ac. \qquad (2°)$$

From (1°), by 14b, $$x \otimes a, \qquad (3°)$$

and $$x \otimes b \oplus c. \qquad (4°)$$

From (2°), by 14a, $\bar{x} \oslash b$ and $\bar{x} \oslash ac$, whence, by 19,

$$x \otimes \bar{b}, \qquad (5°)$$

and $$x \otimes \overline{ac}. \qquad (6°)$$

From (6°) and (3°) it follows that $x \otimes c$ must be false; for if $x \otimes c$ and $x \otimes a$, then $x \otimes ac$ by 12b, whence $x = \wedge$, by 17, which contradicts the condition $x \neq \wedge$.

Therefore, by 9 and 18, there is an element $y \neq \wedge$ such that

$$y \otimes x, \qquad (7°)$$

and $$y \otimes \bar{c},$$

whence, by 19, $$\bar{y} \oslash c. \qquad (8°)$$

From (7°) and (5°), by 3, $y \otimes \bar{b}$, whence by 19,

$$\bar{y} \oslash b. \qquad (9°)$$

From (8°) and (9°) by 12a, $\bar{y} \oslash b \oplus c$, whence, by 19,

$$y \otimes \overline{b \oplus c}. \qquad (10°)$$

But from (7°) and (4°), by 3, we have

$$y \otimes b \oplus c, \qquad (11°)$$

and from (10°) and (11°), by 17, $y = \wedge$, which contradicts the condition $y \neq \wedge$.

my being shut up with a severe influenza. In writing it, I omitted the proof, as there said, because it was 'too tedious' and because it seemed to me very obvious. Nevertheless, when Dr. SCHRÖDER questioned its possibility, I found myself unable to reproduce it, and so concluded that it was to be added to the list of blunders, due to the grippe, with which that paper abounds, —a conclusion that was strengthened when SCHRÖDER thought he demonstrated the indemonstrability of the law of distributiveness. (I must confess that I never carefully examined his proof, having my table loaded with logical books for the perusal of which life was not long enough.) It was not until many years afterwards that, looking over my papers of 1880 for a diffeıent purpose, I stumbled upon this proof written out in full for the press, though it was eventually cut out, and, at first, I was inclined to think that it employed the principle that *all* existence is individual, which my method, in the paper in question, did not permit me to employ at that stage. I venture to opine that it fully vindicates my characterization of it as 'tedious.' But this is how I have a new apology to make to exact logicians."

Therefore the supposition from which we started is impossible, and the lemma is established.

The proof of the main theorem then proceeds as follows: By the lemma,

$$a(b \oplus c) \olessthan b \oplus ac.$$

Therefore, by 15b, $$a\,[\,a(b \oplus c)\,] \olessthan a\,[\,b \oplus ac\,]. \qquad (12°)$$

But, by 16b and 12b,

$$a\,[\,a(b \oplus c)\,] = (aa)(b \oplus c) = a(b \oplus c), \qquad (13°)$$

and by 12a and the lemma again,

$$a\,[\,b \oplus ac\,] = a(ac \oplus b) \olessthan ac \oplus ab = ab \oplus ac. \qquad (14°)$$

Therefore, by 3,

$$a(b \oplus c) \olessthan ab \oplus ac.$$

Proof of 23a. This theorem follows at once from 21a and 22a by 2.

Proof of 24. If $\bar{a} = a$ for any particular element a, then $a = \wedge$, by 8, 1, and 11a. But if $\overline{\wedge} = \wedge$, then $\wedge = \vee$, by 8, 1, and 11b; whence, every element coincides with $\wedge$, by 2—a result which is impossible by 10. Therefore $\bar{a} \neq a$, for every element a.

Proof of 25a. Let $s = \overline{\bar{a} \odot \bar{b}}$; we have then to show that this element s has the properties 1°), 2°), and 3°) demanded by postulate 6. By 18 and 12b, $\bar{a} \odot \bar{b} \olessthan \bar{a}$ and $\bar{a} \odot \bar{b} \olessthan \bar{b}$; hence, by 19, $s \olessthan a$ and $s \olessthan b$. Further, if $y \olessthan a$ and $y \olessthan b$, then, by 19, $\bar{y} \olessthan \bar{a}$ and $\bar{y} \olessthan \bar{b}$; whence, by 12$b$, $\bar{y} \olessthan \bar{a} \odot \bar{b}$, or, by 19, $y \olessthan s$.

Independence of the postulates of the second set.

The independence of the nine postulates of the second set (either 6 or 7 being omitted) is shown by the following systems (K, $\olessthan$), each of which satisfies all the other postulates, but not the one for which it is numbered.

(1) K = a class of four elements, say 0, 1, 2, 3, with $\olessthan$ defined by the accompanying "relation table." In a table of this kind,* a dot standing to

$\olessthan$	2	0	1	3
2	●		●	
0	●		●	●
1				
3			●	●

*Schröder makes extensive use of tables of this kind in his *Algebra der Logik;* see vol. 3 (1895), p. 44.

the right of a and underneath b indicates that the relation $a ⧀ b$ is true; the absence of such a dot indicates that $a ⧀ b$ is false. Here postulate 1 is not satisfied, for $0 ⧀ 0$ and $1 ⧀ 1$ are not true. In postulates 4 and 5, take $\wedge = 0$ and $\vee = 1$. In postulate 8, take $\bar{0} = 1$, $\bar{2} = 3$, and conversely. Postulate 9 is satisfied vacuously.

(2) $K =$ a class of two elements, α and β, with ⧀ interpreted as "equal to or different from," so that $a ⧀ b$ is always true.

Here postulate 2 is clearly false. Postulate 9 is satisfied vacuously, since $a ⧀ \bar{b}$ is never false.

(3) $K =$ a class of six elements, $0, 1, 2, \cdots, 5$, with ⧀ defined as in the accompanying table.

⧀	4	2	0	1	3	5
4	●			●	●	
2	●	●		●		
0	●	●	●	●	●	●
1				●		
3				●	●	●
5		●		●		●

Here postulate 3 is false, since $2 ⧀ 4$ and $4 ⧀ 3$, but not $2 ⧀ 3$. In postulates 4 and 5, take $\wedge = 0$, $\vee = 1$. In postulate 8, take $\bar{0} = 1$, $\bar{2} = 3$, $\bar{4} = 5$, and conversely.

(4) $K =$ the class of all the finite sets of integers which include the integer 1; with ⧀ interpreted as "the same as or includes." (Thus, $a ⧀ b$ means that every integer in the set b is also in the set a.)

Postulate 4 fails, since there is no set which includes all the other sets. In 5, take $\vee = 1$. In 6 and 7, let s be the set of integers common to the sets a and b, and p the sets composed of a and b together. Postulates 8 and 9 are satisfied vacuously.

(5) $K =$ the class of all the finite sets of integers which include the integer 0; with ⧀ interpreted as "the same as or part of." (Thus, $a ⧀ b$ means that every integer in the set a is also in the set b.)

Postulate 5 fails, since there is no set of which every other set is a part. In 4, take $\wedge = 0$. In 6 and 7, let s be the set composed of sets a and b together, and p the set of integers common to the sets a and b.

(6 *and* 7). $K =$ a class of fourteen elements, denoted, say, by u_0; $u_{01}, u_{02}, u_{03}, u_{04}$; $u_{012}, u_{013}, u_{024}, u_{034}$; $u_{0123}, u_{0124}, u_{0134}, u_{0234}$; u_{01234}; with ⧀ defined as follows: $u_A ⧀ u_B$ when and only when the digits in the subscript A are all included among the digits in the subscript B. (Notice that u_{014} and u_{023} are not included in the class.)

Here postulates 1–5 clearly hold. Postulate 6 fails when $a = u_{01}$, $b = u_{04}$, and also when $a = u_{02}$, $b = u_{03}$. Postulate 7 fails when $a = u_{0124}$, $b = u_{0134}$, and also when $a = u_{0123}$, $b = u_{0234}$. Postulates 8 and 9 hold.

(8) $K =$ a class of three elements, 0, 1, 2, with ⧀ defined by the accompanying table.

⧀	0	2	1
0	•	•	•
2		•	•
1			•

Here postulates 4 and 5 hold, and $\wedge$ and $\vee$ are unique: $\wedge = 0$, $\vee = 1$ But postulate 8 is false for the case $a = 2$.

(9) $K =$ a class of five elements, 0, 1, 2, 3, 4, with ⧀ defined by the accompanying table.

⧀	0	2	3	4	1
0	•	•	•	•	•
2		•		•	•
3			•		•
4				•	•
1					•

Here postulates 4 and 5 are satisfied: $\wedge = 0$, $\vee = 1$; and also postulate 8, although the element $\bar{a}$ is not always uniquely determined by a: thus, $\bar{0} = 1$, $\bar{2} = 3$, $\bar{3} = 2$ or 4, $\bar{4} = 3$, $\bar{1} = 0$. Postulates 9 fails, since 4 ⧀ 2 is false, while $x = 0$ is the only element x such that x ⧀ 4 and x ⧀ 3.

I have not been able to find a system for (9) in which $\bar{a}$ is always uniquely determined by a; see the unsolved problem proposed below.

(10) $K =$ a class comprising a single element a, with a ⧀ a. (Postulate 9 is satisfied vacuously.)

Thus the postulates of § 2, *omitting either* 6 *or* 7, are independent, as was to be proved.

It is interesting to notice also that, if we confine ourselves to the first seven postulates, then postulates 6 and 7 are independent of each other. This is proved by the following systems, each of which satisfies all the postulates 1–7 except the one for which it is numbered.

(6) $K =$ a class composed of the following areas: all the squares which lie within a given square (with sides parallel to the sides of the given square); the given square itself, and the "null" square; a fixed circle, lying wholly within the given square; and all areas formed by the addition of two or more of these areas;— with ⧀ interpreted as "includes or coincides with."

Postulates 1–5 clearly hold. (In 4, $\wedge$ = the whole square; in 5, $\vee$ = the null-square). Postulate 6 fails when a = the circle and b = a square which overlaps the circle; for there is no *largest* area (belonging to the class) which lies within both a and b. Postulate 7 holds, the area p being the combined area of a and b.

(7) K = the same class as used above in the proof of the independence of 6; with $\otimes$ interpreted as "within or coincident with."

Postulates 1–5 clearly hold. (In 4, $\wedge$ = the null-square; in 5, $\vee$ = the whole square). Postulate 6 holds, the area s being the combined area of a and b. Postulate 7 fails when a = the circle and b = a square which overlaps the circle; for there is no *largest* area (belonging to the class) which lies within both a and b.

A problem connected with postulate 9.

Other forms which may be used in place of postulate 9 are the following [assuming such of the postulates 1–8 as may be necessary, and defining $a \oplus b$ and $a \odot b (= ab)$ as above]:

9_1. If $bc = \wedge$, then $b \otimes \bar{c}$.

9_2.* If $bc = \wedge$, then $a(b \oplus c) \otimes ab \oplus ac$ [whence $a(b \oplus c) = ab \oplus ac$, by 2 and 21$a$].

9_3.† $a \otimes ab \oplus a\bar{b}$ [whence $a = ab \oplus a\bar{b}$, by 2 and 21a].

The form 9_1 can be deduced from 9_2 or 9_3 as follows: if $bc = \wedge$, then $b\bar{c} = bc \oplus b\bar{c} = b(c \oplus \bar{c}) = b \vee = b$; whence, $b \otimes \bar{c}$.

The form 9 can be deduced from 9_1 as follows: if $a \otimes \bar{b}$ is false, there must be some element $x \neq \wedge$, such that $x \otimes a$ and $x \otimes b$; for, if there were no element except $\wedge$ which is $\otimes a$ and $\otimes b$, then by 12b, $ab = \wedge$, whence, by 9_1, $a \otimes \bar{b}$, which contradicts the hypothesis.

The form 9_1 is clearly simpler than 9_2 or even than 9_3; but all these forms are in so far unsatisfactory as they lack the symmetry which corresponds to the principle of duality between $\oplus$ and $\odot$.

I therefore propose the following problem: *if postulate* 9 *is replaced by* 9_4, *namely:*

> 9_4. If the elements $\wedge$ and $\vee$ in postulates 4 and 5 exist and are unique, and if postulates 8 is true, then the element $\bar{a}$ in 8 is uniquely determined by the element a;

can 9 *then be deduced from* 9_4, *or must some other postulate be added?*

In this connection, 19 is clearly of special importance.

* This is SCHRÖDER'S "Prinzip $\text{III}_\times$" (*loc. cit.*, p. 293), from which he showed that the distributive law, 23, can be deduced.

† This is SCHRÖDER'S "Prinzip $\text{III}^0_\times$," a weaker form of his "Prinzip $\text{III}_\times$," and not, as far as he could see without the knowledge of a proof like that of PEIRCE'S in the present paper, sufficient for his purpose.

§ 3. The Third Set of Postulates.

In § 3 we take as the fundamental concepts a class, K, with a rule of combination, $\oplus$; and as the fundamental propositions, the following nine postulates:

A. $a \oplus a = a$ *whenever* a *and* $a \oplus a$ *belong to the class.*

B. $a \oplus b = b \oplus a$ *whenever* a, b, $a \oplus b$, *and* $b \oplus a$ *belong to the class.*

C. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ *whenever* a, b, c, $a \oplus b$, $b \oplus c$, $(a \oplus b) \oplus c$, *and* $a \oplus (b \oplus c)$ *belong to the class.*

D. There is an element $\wedge$ *such that* $a \oplus \wedge = a$ *for every element* a.

E. There is an element $\vee$ *such that* $\vee \oplus a = \vee$ *for every element* a.

F. If a *and* b *belong to the class, then* $a \oplus b$ *belongs to the class.*

G. If the elements $\wedge$ *and* $\vee$ *in postulates D and E exist and are unique, then for every element* a *there is an element* $\bar{a}$ *such that*

$$1°)\ \textit{if}\ x \oplus a = a\ \textit{and}\ x \oplus \bar{a} = \bar{a},\ \textit{then}\ x = \wedge;$$

$$\textit{and} \qquad 2°)\ a \oplus \bar{a} = \vee.$$

H. If postulates A, D, E, and G hold, and if $a \oplus \bar{b} \neq \bar{b}$ *then there is an element* $x \neq \wedge$ *such that* $a \oplus x = a$ *and* $b \oplus x = b$.

J. There are at least two elements, x *and* y, *such that* $x \neq y$.

Consistency of the postulates of the third set.

The consistency of the postulates is shown by the existence of the following system $(K, \oplus)$, in which all the postulates are satisfied:

$K =$ the class of regions in the plane (including the null region and the whole plane); $a \oplus b =$ the "logical sum" of the regions a and b, that is, the smallest region which includes them both.

Another such system is the class composed of two elements, 0 and 1, with $\oplus$ defined by the table

$\oplus$	0	1
0	0	1
1	1	1

For other such systems, see the appendix.

Deductions from the postulates of the third set.

All the postulates of § 2 are very easily deduced from the postulates of § 3 when $\otimes$ is defined as follows:

DEFINITION. We shall write $a \circledless b$ (or $b \circledgtr a$) when and only when $a \oplus b = b$.

The proof of 6, for example, is as follows: By F, B, C and A,

$$(a \oplus b) \oplus a = a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b,$$

and similarly $(a \oplus b) \oplus b = a \oplus b$; whence, $a \oplus b \circledless a$ and $a \oplus b \circledless b$. Further, if $y \circledless a$ and $y \circledless b$, then $a \oplus y = y$ and $b \oplus y = y$, whence $(a \oplus y) \oplus (b \oplus y) = y \oplus y$, or $(a \oplus b) \oplus y = y$, or $y \circledless (a \oplus b)$.

Postulate 7 follows as in 25b, and we may define $a \odot b$ thus:

DEFINITION. $a \odot b = \overline{\bar{a} \oplus \bar{b}}$.

The equivalence between the algebra of § 3 and the algebras of § 1 and § 2 is thus readily established.

Independence of the postulates of the third set.

The independence of the nine postulates of the third set is shown by the following systems $(K, \oplus)$, each of which satisfies all the other postulates, but not the one for which it is lettered.

(A) The class of positive integers, and 0 and ∞; with $a \oplus b = a + b$.

Here postulates B–F clearly hold. In G take $\bar{a} = \infty$ when $a \neq \infty$, and $\bar{a}$ = any other element when $a = \infty$. Postulate H is satisfied vacuously.

(B) A class of two or more elements, with $a \oplus b = a$.

Here B clearly fails. In D, E, and F, any elements will answer as $\wedge$ and $\vee$. Postulate G is satisfied vacuously, since $\wedge$ and $\vee$ are not uniquely determined. In H, x = any element.

(C) A class of six elements, $0, 1, 2, \cdots, 5$, with $\oplus$ defined as in the accompanying table.

$\oplus$	4	2	0	1	3	5
4	4	4	4	1	3	1
2	4	2	2	1	1	2
0	4	2	0	1	3	5
1	1	1	1	1	1	1
3	3	1	3	1	3	5
5	1	2	5	1	5	5

Here C fails, since $(2 \oplus 4) \oplus 3 = 4 \oplus 3 = 3$, while $2 \oplus (4 \oplus 3) = 2 \oplus 3 = 1$. In D and E, take $\wedge = 0$ and $\vee = 1$. Postulate G holds: $\bar{0} = 1$, $\bar{2} = 3$, $\bar{4} = 5$, and conversely. Postulate H also holds.

(D) The class of all the finite sets of integers which include the integer 1; with $a \oplus b$ defined as the set of integers common to the sets a and b.

Here D fails, since no set includes all the rest. In E, take $\vee = 1$. Postulates G and H are satisfied vacuously.

(E) The class of all the finite sets of integers which include the integer 0; with $a \oplus b$ defined as the set composed of the sets a and b together.

Here E fails, since no set includes all the rest. In D, take $\wedge = 0$. Postulates G and H are satisfied vacuously.

(F) The class of fourteen elements used in proving the independence of 6 and 7 in § 2; with $\oplus$ defined as follows: $u_A \oplus u_B = u_S$, where the subscript S includes all the digits in the subscript A and also all those in the subscript B.

Here F fails when $a = u_{01}$, $b = u_{04}$, and also when $a = u_{02}$, $b = u_{03}$.

(G) A class of three elements, $0, 1, 2$, with $\oplus$ defined as in the accompanying table.

$\oplus$	0	2	1
0	0	2	1
2	2	2	1
1	1	1	1

Here G fails when $a = 2$. In D and E, take $\wedge = 0$ and $\vee = 1$. Postulate H is satisfied vacuously.

(H) A class of five elements, $0, 1, 2, 3, 4$ with $\oplus$ defined as in the accompanying table.

$\oplus$	0	2	3	4	1
0	0	2	3	4	1
2	2	2	1	4	1
3	3	1	3	1	1
4	4	4	1	4	1
1	1	1	1	1	1

Here A and B clearly hold. Postulate C holds, when a or b or c is 0 or 1; when $a = c$; and in the other cases by trial. Postulate G holds: $\bar{0} = 1$, $\bar{2} = 3$, $\bar{3} = 2$ or 4, $\bar{4} = 3$, $\bar{1} = 0$; but notice that $\bar{a}$ is not uniquely determined by a in the case $a = 3$. Postulate H fails in the case $a = 4$, $b = 3$, $\bar{b} = 2$. (Compare proof of independence of postulate 9 in § 2, and the unsolved problem proposed at the end of that section.)

(J) A class composed of a single element a, with $a \oplus a = a$.

Appendix.

Any system (K, $\oplus$, $\odot$, $\otimes$) which obeys the laws of the algebra of logic may be called a *logical field* (a term which I venture to suggest as analogous to

"Galois field"). In this appendix * we consider all possible *finite* logical fields, that is, all possible finite classes, K, in which the rules of combination, $\oplus$ and $\odot$, and the dyadic relation, $\otimes$, can be so defined as to satisfy the postulates of § 1, § 2, or § 3.

1. *The number of elements in every finite logical field must be* 2^m, where $m = 1, 2, 3, \cdots$.

For, if $a \otimes b$, and $a \neq b$, we can always find an element $x \neq \wedge$, namely $x = \bar{a}b$, such that $a \oplus x = b$ and $ax = \wedge$. Hence, in any finite logical field we can find a set of m elements different from $\wedge$, say $u_1, u_2, \cdots, u_m$, such that

$$u_1 + u_2 + u_3 + \cdots + u_m = \vee$$

while

$$u_i u_j = \wedge \qquad (i \neq j).$$

These m elements may be called *irreducible.*

Every element except $\wedge$ is then the sum of k of these irreducible elements $(1 \leqq k \leqq m)$, whence, by a familiar theorem in combinations,† the total number of elements is 2^m.

2. *Any class the number of whose elements is a power of* 2, *say* 2^m, *can be made into a logical field by properly defining* $\oplus$, $\odot$ *and* $\otimes$; *and this in essentially only one way.*

The process of constructing the requisite "addition-," "multiplication-," and "relation-tables" is the following:

Select one element to serve as $\wedge$. Select m other elements to serve as the "irreducible" elements of the system, and denote them by $u_1, u_2, u_3, \cdots, u_m$. Select ${}_mC_2$ other elements to serve as the elements which are the sums of *two* of the irreducible elements, and denote them by u_{12}, u_{13}, u_{23}, etc., so that we shall have $u_{12} = u_1 \oplus u_2$, etc. Select ${}_mC_3$ other elements to serve as the elements which are the sums of *three* of the irreducible elements, and denote them by $u_{123}, u_{124}, u_{234}$, etc.; so that we shall have $u_{123} = u_1 \oplus u_2 \oplus u_3$, etc. And so on. Finally, $u_{123\cdots m} = \vee$.

The construction of the tables is then obvious. Thus, $u_A \otimes u_B$ when and only when the digits in the subscript A are all contained among the digits in the subscript B; $u_A \oplus u_B = u_S$ where the subscript S contains all the digits that occur in A and also all that occur in B; $u_A \odot u_B = u_P$, where the subscript P contains only those digits which are common to the subscripts A and B.

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* Cf. P. PORETSKII, *Théorie des égalités logiques à trois termes a, b et c;* Bibliothèque du congrès international de philosophie, Paris 1900, vol. 3 (1901), pp. 201–233; and SCHRÖDER, loc. cit., vol. 1, p. 658. The notation and method of proof here used were suggested to me by Professor E. H. MOORE.

† This theorem: $1 + {}_mC_1 + {}_mC_2 + {}_mC_3 + \cdots + {}_mC_m = 2^m$, is, oddly enough, not explicitly mentioned in CHRYSTAL's *Algebra;* see vol. 2 (2d edition). chap. XXIII, § 13.