and finally

$$P'(x) = a_1 + 2a_2x + \cdots + (n-1)a_{n-1}x^{n-2} + \epsilon_1(x)x^{n-2},$$

which is the asymptotic development sought. Also it results easily from the fact that $P'(x)$ approaches a_1 as x goes to zero that the derivative at the origin exists and is equal to a_1.

It is interesting to notice that a single differentiation has lost us two terms of the development of $P(x)$.* However, if $P(x)$ has an infinite asymptotic development, it is clear that $P'(x)$ will also have an infinite development, and in fact that the development of $P(x)$ can be differentiated formally any number of times.

The above proof applies, of course, to other domains than sectors; for instance, we might use the horn angle obtained by an inversion relative to the origin, and a reflection across the real axis of the infinite strip between any two parallel lines.

Lastly, it is important to notice that we have also established above the differentiability in the complex domain of asymptotic developments in descending powers of the variable.

COLUMBIA UNIVERSITY.

DARBOUX'S CONTRIBUTION TO GEOMETRY.

BY PROFESSOR L. P. EISENHART.

(Read before the joint session of the American Mathematical Society and the Mathematical Association of America at Cleveland, Ohio, September 6, 1917.)

GASTON DARBOUX was born in 1842 at Nîmes, a place of interest to mathematicians because here from 1819 to 1831 Gergonne edited his *Annales,* and incidentally exerted a great influence on the development of geometry. At the age of eighteen Darboux went to Paris, in whose intellectual life he had a prominent part for fifty-seven years. As a student, first at the Ecole Polytechnique and then at the Ecole Normale, his unusual mathematical ability made him conspicuous. His

* The loss of the term $na_n x^{n-1}$ is only apparent. This can be shown by taking the radius of the circle of integration above equal to $k\,|\,y\,|$, where k is some number independent of y. I prefer the proof above because of its applicability to more general domains than sectors.

rare powers of expression soon won for him a reputation as a lecturer and expositor, with the consequence that he received desirable teaching posts from the beginning. In 1880 he succeeded Chasles in the chair of Géométrie supérieure at the Sorbonne. Four years later he became a Membre de l'Institut, and in 1889 assumed the duties of Doyen de la Faculté des Sciences. At the death of Bertrand in 1900 Darboux was elected Secrétaire Perpétuel de l'Académie des Sciences, which post he administered with distinction until his death. We cannot go further into detail concerning the many honors and responsibilities which came to him in his very full life. For we are concerned with the work he did in the field of geometry, to a review of which we now turn.

While a student at the Ecole Normale in 1864, Darboux published his first papers, two notes in the *Nouvelles Annales*.* It is important that we consider these papers in some detail, as they contain germs of his subsequent work. The first dealt with the plane sections of the tore. It was shown that the curve is of the fourth order with the circular points at infinity for double points; that there are sixteen foci, in the sense of Plücker, of which four are real and lie on a circle; that there is a homogeneous linear relation between the distances of any point of the curve from three of these foci; and that an inversion with respect to any of these foci as pole transforms the curve into an oval of Descartes.

In his second note Darboux considered the curves of intersection of a sphere and a quadric. Through such a curve pass four quadric cones; the tangent planes of the cone meet the sphere in circles doubly tangent to the curve; the four of these planes tangent also to the sphere determine four null circles doubly tangent to the curve and lying on a circle; the distances of any three of them from any point of the curve are in homogeneous linear relation. The curve has, therefore, sixteen foci, lying in sets of four on four circles, such that any two circles meet orthogonally. To these curves and their transforms by inversion Darboux has given the name cyclics; they are spherical or plane and are of the fourth order. In particular, the plane sections of the tore and the ovals of Descartes are cyclics, as are also the cissoid, lemniscate, circular cubic, and other well-known curves.

In 1872 Darboux published his first treatise, entitled Sur une

* Ser. 2, vol. 3 (1864), pp. 156, 199.

Classe remarquable de Courbes et de Surfaces algébriques et sur la Théorie des Imaginaires. It consists of five parts, one of which is devoted to a discussion of cyclics.

When Darboux was at the Ecole Normale, he became familiar with the writings of Lamé, Dupin, and Bonnet on triply orthogonal systems of surfaces, the field with which his name will ever be associated. At that time the best example of an orthogonal system of surfaces was afforded by the confocal quadrics. Years before Kummer had sought families of plane curves defined by an equation $f(x, y, a) = 0$, where a is the parameter of the family, such that through each point of the plane pass two curves of the family and that they meet orthogonally. He found that such families have the property that all the curves are confocal. Darboux, seeking to generalize this result, sought the triply orthogonal systems of surfaces, each of which can be defined by a single equation involving a parameter λ. In particular he sought the generalization of the orthogonal family of ovals of Descartes having three common foci. As a result he found the system defined by the single equation

$$(1)\qquad (x^2 + y^2 + z^2)^2 + \frac{\alpha\lambda - 4h}{\alpha - \lambda} x^2 + \frac{\beta\lambda - 4h}{\beta - \lambda} y^2 + \frac{\gamma\lambda - 4h}{\gamma - \lambda} z^2 - h = 0,$$

where α, β, γ and h are constants, and λ is the parameter of the system. Each surface of the system so defined, subsequently called cyclides by Darboux, has the following properties: it is of the fourth order and has the imaginary circle at infinity for a double curve; it is cut by any sphere in a cyclic; in five different ways it is the envelope of a two-parameter family of spheres with their centers on a fixed quadric and orthogonal to a fixed sphere, each of the doubly tangent spheres meeting the surface in two circles. These results were presented to the French Academy of Sciences on August 1, 1864.* On the same day Moutard announced to the Academy the discovery of the same system. He had just been studying surfaces transformable into themselves by inversion, and had found that the surfaces of the fourth order with the circle at

* *Comptes Rendus*, vol. 59 (1864), p. 240. The details of this paper were published in *Annales de l'Ecole Normale*, vol. 2 (1865), pp. 55–69.

infinity for double curve are transformable in five different ways. While seeking the lines of curvature of these surfaces he found the triple system of cyclides.

After finding this orthogonal system, Darboux determined the linear element of space in terms of parameters referring to such a system of surfaces, and found, as in the case of confocal quadrics, that the curves of intersection form an isothermic system of curves on each surface. The celebrated theorem of Dupin, that in a triply orthogonal system any two surfaces of different families cut in a line of curvature for both, was made complete by the following addition due to Darboux:

When two families of orthogonal surfaces cut along the lines of curvature of these surfaces, there is a third family orthogonal to the first two families.

With the aid of this theorem Darboux was enabled to determine the condition to be satisfied in order that the family of surfaces defined by an equation of the form $\varphi(x, y, z) = \alpha$ can be part of a triple system; he found that φ must be any solution of a partial differential equation of the third order in two independent variables, but he did not calculate the equation, because of its complicated form. These results were published by Darboux in 1866 in his classic memoir "Sur les surfaces orthogonales"* subsequently presented as his thesis for the doctorate. This memoir contained also the determination of the orthogonal systems for which the lines of curvature are plane, and the erroneous theorem that the triple system of cyclides (1), which includes the system of confocal quadrics as a special case, affords the only example of a triple system of isothermic surfaces, a mistake corrected by Darboux in a later paper.

In 1872 Cayley attacked the problem of orthogonal systems, and gave the differential equation of triple systems the form now associated with his name, and valuable because of its adaptability to the determination of particular orthogonal systems. Darboux was quick to see the value of Cayley's work. On the one hand he made a study of the analytical processes underlying Cayley's equation and extended them to systems in n variables. On the other hand he made use of this equation for the determination of orthogonal systems for which the surfaces in one family are quadrics; those for which

* *Annales de l'Ecole Normale*, vol. 3 (1866), pp. 97–141.

the surfaces in one family have a plane of symmetry; and orthogonal systems containing a given surface and involving in their equation four arbitrary functions of a single variable. These results were published in the first and second parts of his second great memoir on orthogonal systems.* In the third part he generalized the equations of Lamé to space of n dimensions and studied certain special problems in this connection. The last fifty pages of this memoir are devoted to the problem which he thought he had solved in his thesis, namely the determination of orthogonal systems of isothermic surfaces. He found other systems than those of cyclides, but we cannot go further into detail. As it is, we have devoted a great deal of space to these two memoirs, which, however, seems justified in view of their importance in the development of the theory of orthogonal systems, and of their place among Darboux's memoirs. Twenty years later he published his Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes, which includes much of the foregoing theory.† In the second memoir there appears also a reference to triply conjugate systems of surfaces, studied subsequently by Guichard and Tzitzeica and discussed at length by Darboux in the second edition of the above treatise issued in 1910.‡ In this volume and in subsequent papers Darboux made further contributions to the theory of orthogonal systems.

From the time when Gauss introduced the idea of curvilinear coordinates of a surface and discovered the absolute invariant of the linear element which measures the curvature of the surface, geometers have been interested in the problem, as yet unsolved, of finding all surfaces with a given linear element, that is, the problem of applicable surfaces. In 1872 Darboux§ showed that the rectangular point coordinates of a surface with a given linear element are solutions of a partial differential equation of the second order of the Ampère type; also that when one solution of this equation is known two others can be found by quadratures, and that these three solutions are the coordinates of a surface with the given linear element.

* *Annales de l'Ecole Normale*, ser. 2, vol. 7 (1878), pp. 97–151, 227–261, 275–349.

† A full analysis of this treatise, by E. O. Lovett, was published in this BULLETIN, vol. 5 (1899), pp. 185–202.

‡ A review by Wilczynski of this edition is published in the BULLETIN, vol. 20 (1914), pp. 247–253.

§ Sur une Classe remarquable, etc., pp. 14, 181.

Moutard and Ribaucour, who were at work on the problem of applicable surfaces in the late sixties, showed that when two applicable surfaces are known one can find at once two other surfaces S and S_1 whose rectangular coordinates x, y, z and x_1, y_1, z_1 satisfy the condition

$$(2) \qquad dxdx_1 + dydy_1 + dzdz_1 = 0.$$

In 1873 Darboux announced to the Société mathématique de France* that the problem of the infinitesimal deformation of a surface S is equivalent to the determination of the surfaces S_1 whose coordinates satisfy equation (2). In his well-known treatise Leçons sur la Théorie générale des Surfaces† he showed that when the equation (2) for a surface S has been integrated, the infinitesimal deformation of S, to terms of any order, is a problem of quadratures alone. Weingarten‡ reduced the solution of equation (2) to the integration of a linear partial differential equation of the second order, whose equation of characteristics defines the asymptotic lines of the given surface. When the latter are parametric, the equation is of the form

$$(3) \qquad \partial^2\theta/\partial u\partial v = \mu(u, v)\theta.$$

A surface S_0 corresponding to a given surface in such a way that the tangent planes to the two surfaces at corresponding points are parallel and asymptotic lines on each surface correspond to a conjugate system on the other is said to be associate to it. This idea is due to Bianchi,§ who proved that the determination of the surfaces associate to a given surface is equivalent to the integration of equation (2). Having noted the reciprocal character of the relations between S and S_1 and between S and S_0, Darboux found nine other surfaces arising from each solution of equation (2), the whole group of twelve surfaces forming a closed system.

It would be a great oversight were we not to refer to the charming chapters in the fourth volume of the Leçons in which Darboux treats the rolling of one applicable surface upon another, and the geometrical configurations thus generated by points, lines, etc., invariably fixed with respect to the rolling surface. Ribaucour also had a part in the develop-

* This paper was not published.

† Vol. 4, p. 5. Hereafter this treatise will be referred to as the Leçons.

‡ *Crelle*, vol. 100 (1886), pp. 296–310.

§ Lezioni di Geometria differenziale, Pisa, 1894, p. 279.

ment of this field and recently Bianchi. As an example of the beauty of some of the theorems in this theory we cite the following: Let S and S_1 be two applicable surfaces; as S rolls over S_1, a null sphere with center invariably fixed with respect to S cuts the common tangent plane to the two surfaces in a circle; these circles form a cyclic system.

The idea of the spherical representation of a surface is due to Gauss. The lines of curvature of a surface are represented on the sphere by an orthogonal system of curves. Darboux early interested himself in the solution of the problem of finding the surfaces whose lines of curvature are represented on the sphere by a given orthogonal system, which he called the "problem of spherical representation." By making use of the simple expressions for the rectangular coordinates of a point of the sphere in terms of parameters referring to the imaginary generators, he showed that the problem is reducible to the solution of a partial differential equation of the form (3). He wrote a number of papers dealing with special forms of this equation, and with the geometrical significance of this work and of the researches of Moutard on the equation (3).

We recall that Moutard showed that a cyclide of the fourth order is, in five different ways, the envelope of a two parameter family of spheres orthogonal to a fixed sphere S, the centers of the spheres being on a quadric; that the five quadrics thus associated with the cyclide are confocal; and that any two of the five fixed spheres S cut one another orthogonally. When he announced his discovery of the orthogonal system of cyclides, Moutard remarked that for all the cyclides of the system these five spheres and five quadrics are the same. Making use of this result Darboux showed* that the equation of the system can be written in the form

$$\sum_{i=1}^{5} \frac{\left(\frac{S_i}{R_i}\right)^2}{\lambda - a_i} = 0, \tag{4}$$

where S_i is the power of the point with respect to the sphere S_i, R_i is its radius, a_i is a constant and λ the parameter of the system. This equation being necessarily of the third order the coefficient of λ^4 is identically zero. Hence we have the

* Sur une Classe remarquable, etc., p. 134. The fourth and fifth parts of this treatise are devoted to an exposition of the geometry of cyclides.

identity

$$\sum_{i=1}^{5}\left(\frac{S_i}{R_i}\right)^2 = 0 \tag{5}$$

for any point in space. Darboux saw in these quantities S_i/R_i, subject to the above condition, a new type of coordinate which could be used to good purpose in certain problems. Thus were discovered *pentaspherical coordinates*. An important theorem involving them is the following:

If five particular solutions $x_1, \cdots, x_5$ of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v} + C\theta$$

satisfy the relation $\Sigma x_i^2 = 0$, the quantities x_i are the pentaspherical coordinates of a surface upon which the parametric curves are the lines of curvature. When, in particular, the above equation is reducible to the form (3), the surface is isothermic. Important consequences of this result have been obtained by Darboux and Guichard.

From the equations of Gauss it follows that the nonhomogeneous coordinates x, y, z of a surface, referred to a conjugate system of curves $u = \text{const.}$, $v = \text{const.}$, satisfy an equation of the Laplace form

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{6}$$

called the point equation of the system. However, Darboux called attention to the fact that each function $\varphi(x, y, z)$, in general, determines a conjugate system, found by the solution of an ordinary differential equation of the first order and second degree, in the sense that the corresponding equation (6) admits the solution φ as well as x, y, and z. In particular, the lines of curvature of any surface are characterized by the property that they form the conjugate system for which $\varphi \equiv x^2 + y^2 + z^2$. Guichard has extended this conception and considered the remarkable class of conjugate systems n, O, defined by the property that the point equation of such a system admits $n - 1$ solutions $t_1, \cdots, t_{n-1}$ and that $x^2 + y^2 + z^2 - t_1^2 - \cdots - t_{n-1}^2$ also is a solution.

Darboux has been such a conspicuous worker in differential geometry that one may make the mistake of thinking that his geometrical work has been confined to this field. To offset this impression we recall that the greater part of his treatment of cyclics and cyclides is not differential. We mention also his studies of Poncelet polygons, and their generalization to polygons inscribed and circumscribed to ellipsoids, and some of his researches on the wave surface. It is an interesting fact that his first treatise, previously referred to, and his last, Principes de Géométrie analytique, just published,* deal almost entirely with finite geometry.

In 1868 Beltrami published† his striking interpretation of the geometry of Lobachevsky, Bolyai and Gauss in terms of geodesics on a pseudospherical surface. Three years later Klein‡ pointed out the significance for non-euclidean geometry of the concept of an absolute and of the associated metric which Cayley had developed in his Sixth Memoir on Quantics, twelve years before. These new ideas made a decided impression on Darboux. He made use of the Cayley measurement in the theory of cyclides as presented in his first treatise, and in a note appended to this volume he studied geodesics and lines of curvature in Cayleyan geometry. These ideas are developed more fully in the last chapter of the third volume of his Leçons, and are applied there to the determination of surfaces upon which there is a conjugate system of curves all of whose tangents are tangent to a quadric also. One of the five parts of the Principes de Géométrie analytique is devoted to a very clear exposition of Cayleyan geometry, displacements in this geometry and its trigonometry. But for one reason or another Darboux confined his researches to euclidean space, and he had no part in the development of the theory of non-euclidean differential geometry.

Darboux was a strong advocate of the use of imaginary elements in the study of geometry. He believed that their use was as necessary in geometry as in analysis. He had been impressed by the success with which they had been employed in the solution of the problem of minimal surfaces. From the very beginning he made use in his papers of the isotropic line, the null sphere (the isotropic cone) and the general isotropic developable. In his first memoir on orthogonal systems of

* This treatise is in fact an enlarged edition of the first. The writer will shortly publish a detailed review of it.

† *Giornale di Mathematiche*, vol. 6 (1868), pp. 284–312.

‡ "Ueber die sogenannte Nicht-Euklidische Geometrie," *Math. Annalen*, vol. 4 (1871), pp. 573–625.

surfaces he showed that the envelope of the surfaces of such a system, when defined by a single equation, is an isotropic developable. We have given examples of the use of these elements in the theorem cited for rolling surfaces, and likewise in the solution of the problem of spherical representation. Another striking example is afforded by the following theorem, which gives a general method of obtaining surfaces with plane lines of curvature in one system: If a developable D rolls over a developable D_1 in such a way that all the points of a generator are in contact at any instant, an isotropic developable invariably fixed with respect to D will be cut by successive planes of contact along lines of curvature of the surface which is their locus.

In a discussion of so prolific a writer as Darboux one can merely mention the results of some of his investigations and cannot give an idea of the details. However, it would be a mistake in his case not to refer to the character of the methods used. Darboux gives to Combescure the credit of being the first to apply the considerations of kinematics to the study of the theory of surfaces with the consequent use of moving coordinate axes. But to Darboux we are indebted for a realization of the power of this method, and for its systematic development and exposition. This exposition is to be found in the first two volumes of his Leçons, with applications later to the discussion of particular types of surfaces and of orthogonal systems. The method has been used extensively by his students and followers, and it is sure to be used by any worker in the field of differential geometry who is familiar with its processes.

Darboux's ability was based on a rare combination of geometrical fancy and analytical power. He did not sympathize with those who use only geometrical reasoning in attacking geometrical problems, nor with those who feel that there is a certain virtue in adhering strictly to analytical processes. His geometrical proofs of the theorems dealing with rolling surfaces referred to above are as pure as they are simple and beautiful. No less brilliant are his reductions of various geometrical problems to a common analytical basis, and their solution and development from a common point of view. At times in the solution of a problem we find a combination of the two methods. This is seen in various parts of his treatises, and in particular in the memoir in which he solved the problem of finding the two-parameter families of spheres for which the correspondence between the two sheets of the envelope is conformal.* By geometrical considerations it is shown that the lines of curvature on the two sheets are in correspondence, and then by analysis that they are isothermic surfaces. Here is established the transformations of isothermic surfaces, which Bianchi has called the transformations D_m, and the associated deformations of quadrics. However, we believe that Darboux's interest lay rather in analytical geometry. This was natural in one who possessed such remarkable powers as an analyst. We have not the time to discuss, nor indeed to enumerate, the contributions to analysis which Darboux made while in pursuit of the solution of geometrical problems. Surely we find realized in his works his belief that "the alliance between geometry and analysis is useful and productive; perhaps this alliance is a condition for the success of both."

In the foregoing sections we have tried to give an idea of the scope of Darboux's contribution to geometry. His writings possess not only content but singular finish and refinement of style. In the presentation of results the form of exposition was carefully studied. Darboux's varied powers combined with his personality in making him a great teacher, so that he always had about him a group of able students. In common with Monge he was not content with discoveries, but felt that it was equally important to make disciples. Like this distinguished predecessor he developed a large group of geometers, including Guichard, Koenigs, Cosserat, Demoulin, Tzitzeica, and Demartres. Their brilliant researches are the best tribute to his teaching. His spirit will continue to live in these men, and in the many who will turn to his works for direction and inspiration. Hence of Darboux also the prophecy may be true which Lagrange made of Monge, when he said "With his geometry this devil of a man will make himself immortal."

PRINCETON UNIVERSITY.

* "Sur les surfaces isothermiques," *Annales de l'Ecole Normale*, ser. 3, vol. 16 (1899), pp. 491–508.