The Vibrations of Circular Plates and their Relation to Bessel Functions

1910 Proc. Phys. Soc. London 23 225

(http://iopscience.iop.org/1478-7814/23/1/322)

XXI. *The Vibrations of Circular Plates and their Relation to Bessel Functions.* By JOHN R. AIREY, *M.A., B.Sc., late Scholar of St. John's College, Cambridge.*

FIRST RECEIVED FEBRUARY 15, 1911. RECEIVED IN FINAL FORM MARCH 7, 1911.

THE vibrations of circular plates were first investigated by Poisson * in a celebrated memoir read before the French Academy of Sciences in 1829. Three cases were considered: (*a*) when the circumference was fixed; (*b*) when the plate was "supported"; (*c*) when the plate was free. The ratios of the radii of the nodal circles to the radius of the plate were calculated when the vibrating plate had no nodal diameter and one or two nodal circles. Kirchhoff † extended Poisson's results for the free plate by calculating six ratios of the radii when the plate vibrated with one, two or three nodal diameters, whilst Schulze ‡ found eight more values of the ratios for a plate with fixed circumference.

The calculation of these ratios required the determination of the roots of equations involving Bessel functions with real and imaginary arguments. These appear to have been found by a "trial and error" method or by interpolation from tables of these functions.

The object of the present Paper is to give a general method of solving these equations—viz., equation (4A) for a circular plate with fixed circumference (Table I.), and equation (9A) for a free circular plate (Tables II. and III.). From the roots so calculated, the radii of the nodal circles and the times of vibration in any given mode are readily found.

(A) *Vibrations of a Circular Plate with Fixed Circumference.*

The displacement of a point on the plate from its position of equilibrium is given by

$$w = \mathrm{A}\cos(p\theta)\,\{\mathrm{J}_p(\kappa r)+\lambda \mathrm{J}_p(i\kappa r)\}\cos(qt-\epsilon). \quad . \quad . \quad (1)$$

The boundary conditions in this case are $w=0$ and $\frac{dw}{dr}=0$ when $r=a$, or, if the radius of the plate is equal to unity, when $r=1$.

* "Mémoires de l'Académie royale des Sciences de l'Institut de France," tome VIII., 1829.

† Kirchhoff, "Pogg. Annalen," 1850. Strehlke, "Pogg. Annalen," 1855.

‡ Schulze, "Ann. der Physik," XXIV., 1907.

Hence (1) gives

$$J_p(\kappa a)+\lambda J_p(i\kappa a)=0, \quad \ldots \quad (2)$$

$$J_p'(\kappa a)+\lambda J_p'(i\kappa a)=0. \quad \ldots \quad (3)$$

Eliminating λ and writing x for κa, we get

$$\frac{J_p'(x)}{J_p(x)}=\frac{J_p'(ix)}{J_p(ix)} \quad \ldots \quad (4)$$

or

$$\frac{J_{p+1}(x)}{J_p(x)}+\frac{I_{p+1}(x)}{I_p(x)}=0. \quad \ldots \quad (4\text{A})$$

This becomes, after the substitution of the semi-convergent series for $J_p(x)$, &c.,

$$\tan\left(x-\frac{2p+1}{4}\pi\right)=\frac{I_{p+1}\cdot P_p+I_p\cdot Q_{p+1}}{I_{p+1}\cdot Q_p-I_p\cdot P_{p+1}}. \quad \ldots \quad (5)$$

$$\text{or}\quad \tan\left(x-\frac{p\pi}{2}-n\pi\right)=\frac{I_p(P_{p+1}-Q_{p+1})-I_{p+1}(P_p+Q_p)}{I_p(P_{p+1}+Q_{p+1})+I_{p+1}(P_p-Q_p)} \quad (5\text{A})$$

$$=-ay-8ay^2-\frac{a}{3}(m^2+2m+93)y^3-8a^3y^4$$

$$-\frac{2a}{15}(m^4+6m^3+744m^2-10726m+56055)y^5\ldots,$$

where $a=4p^2-1$, $m=4p^2$ and $y=\frac{1}{8x}$.

Then, by Gregory's series,

$$x=n\pi+\frac{p\pi}{2}-a\{y+8y^2+\tfrac{4}{3}(m+23)y^3$$

$$+\tfrac{1}{15}(96m^2-18624m+110688)y^5\ldots\}$$

$$=\beta-\frac{a}{8x}-\frac{8a}{(8x)^2}-\frac{4a(m+23)}{3(8x)^3}$$

$$-\frac{a}{15}\cdot\frac{(96m^2-18624m+110688)}{(8x)^5}\ldots. \quad (6)$$

By Lagrange's theorem, if

$$x=\beta+\frac{p}{x}+\frac{q}{x^2}+\frac{r}{x^3}+\frac{s}{x^4}+\frac{t}{x^5}\ldots,$$

$$x=\beta+\frac{p}{\beta}+\frac{q}{\beta^2}+\frac{r-p^2}{\beta^3}+\frac{s-3pq}{\beta^4}+\frac{t-2q^2-4pr+2p^3}{\beta^5}\ldots.$$

Hence

$$x=\beta-(m-1)\left[\frac{1}{8\beta}+\frac{8}{(8\beta)^2}+\frac{4(7m+17)}{3(8\beta)^3}+\frac{192(m-1)}{(8\beta)^4}+\frac{32(83m^2+218m+2579)}{15(8\beta)^5}\ldots\right], \quad (7)$$

where

$$\beta=\left(\frac{2n+p}{2}\right)\pi.$$

Two roots of equation (4A) were given by Poisson when $p=0$, viz., 3·196 and 6·292, whilst Lord Rayleigh gave the values 3·20 and 6·3. For other values of p, Schulze found the following roots: When $p=1$, 4·612, 7·80, 10·95; and when $p=2$, 5·904 and 9·40.

The roots of equation (4) have been calculated from the expression given in (7), when $p=0$, 1, 2 and 3, some of the earlier roots by interpolation.

TABLE I.

Roots of $\frac{J_{p+1}(x)}{J_p(x)}+\frac{I_{p+1}(x)}{I_p(x)}=0.$

No. of root.	$p=0$.	$p=1$.	$p=2$.	$p=3$.
1	3·1955	4·611	5·906	7·144
2	6·3064	7·799	9·197	10·536
3	9·4395	10·958	12·402	13·795
4	12·5771	14·109	15·579	17·005
5	15·7164	17·256	18·745	20·192
6	18·8565	20·401	21·901	23·366
7	21·9971	23·545	25·055	26·532
8	25·1379	26·689	28·205	29·693
9	28·2790	29·832	31·354	32·849
10	31·4200	32·975	34·502	36·003

By substituting one of these values in (2), the value of λ corresponding to this root can be found—*e.g.*, when the plate is vibrating with one nodal diameter and three nodal circles, $\kappa a=x=10·958$, and

$$\lambda=-\frac{J_1(10·958)}{J_1(10·958i)}$$

or

$$i\lambda=0·0000255\ldots.$$

Equation (2) becomes

$$J_1(\kappa r)+0·0000255I_1(\kappa r)=0.$$

The roots of this equation are 3·8312, 7·0024 and 10·958.

Therefore, the ratios of the radii of the nodal circles to that of the plate are

$$\frac{3\cdot8312}{10\cdot958}=0\cdot3496 \text{ and } \frac{7\cdot0024}{10\cdot958}=0\cdot6390,$$

Schulze ("Ann. der Physik.," 1907) gives the values

0·350 and 0·640.

The expression for the frequency of vibration of the plate is

$$\mathrm{N}=\frac{ax^2}{\sqrt{1-\mu^2}}.$$

where a is constant for the same plate (Lord Rayleigh, "Theory of Sound," Vol. I., § 217), μ is Poisson's ratio, and x is a root of equation (4). Since x is independent of μ, N is only affected by a change in the assumed value of μ through the factor $\frac{1}{\sqrt{1-\mu^2}}$.

(B) *Vibrations of a Free Circular Plate.*

The boundary conditions require

$$-\lambda=\frac{p^2(\mu-1)\{x\mathrm{J}_p'(x)-\mathrm{J}_p(x)\}-x^3\mathrm{J}_p'(x)}{p^2(\mu-1)\{ix\mathrm{J}_p'(ix)-\mathrm{J}_p(ix)\}+ix^3\mathrm{J}_p'(ix)} \quad . \quad . \quad (8)$$

and

$$-\lambda=\frac{(\mu-1)\{x\mathrm{J}_p'(x)-p^2\mathrm{J}_p(x)\}-x^2\mathrm{J}_p(x)}{(\mu-1)\{ix\mathrm{J}_p'(x)-p^2\mathrm{J}_p(ix)\}+x^2\mathrm{J}_p(ix)}, \quad . \quad . \quad (9)$$

where x is written for κa.

Eliminating λ, the expressions on the right of (8) and (9) are equal. (9A)

(i.) When $p=0$, *i.e.*, when there is no nodal diameter, equation (9A) becomes

$$2(1-\mu)+ix\frac{\mathrm{J}_0(ix)}{\mathrm{J}_0'(ix)}+x\frac{\mathrm{J}_0(x)}{\mathrm{J}_0'(x)}=0. \quad . \quad . \quad . \quad (10)$$

or, using Poisson's value for $\mu(\mu=\frac{1}{4})$

$$\frac{\mathrm{J}_0(x)}{\mathrm{J}_1(x)}=\frac{3}{2x}-\frac{\mathrm{I}_0(x)}{\mathrm{I}_1(x)}. \quad . \quad . \quad . \quad . \quad . \quad (10\text{A})$$

Substituting the semi-convergent series for $\mathrm{J}_0(x)$, $\mathrm{J}_1(x)$, &c., we get

$$\frac{\mathrm{P}_0\cdot\cos\left(x-\frac{\pi}{4}\right)-\mathrm{Q}_0\sin\left(x-\frac{\pi}{4}\right)}{\mathrm{P}_1\cdot\sin\left(x-\frac{\pi}{4}\right)+\mathrm{Q}_1\cos\left(x-\frac{\pi}{4}\right)}=\frac{3}{2x}-\frac{\mathrm{I}_0(x)}{\mathrm{I}_1(x)}=a \quad . \quad (11)$$

or
$$\tan\left(x-\frac{\pi}{4}\right)=\frac{P_0-aQ_1}{Q_0+aP_1}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Putting $\quad y=\frac{1}{8x},\ a=12y-\frac{I_0(x)}{I_1(x)}$

$$=-1+8y-24y-192y^3-2016y^4-27648y\ldots$$

This value substituted in (12) gives, after simplification,

$$\tan\left(x-\frac{\pi}{4}\right)=-1-10y-10y^2+320y^3+3650y^4+59840y^5\ldots$$

$$\therefore \quad \tan(x-n\pi)=-5y+20y^2+85y^3+500y^4+21070y^5\ldots.$$

Expressing the angle in terms of the tangent, we find

$$x-n\pi=-5y+20y^2+\frac{380}{3}y^3+20320y^5\ldots$$

or
$$x=n\pi-\frac{5}{8x}+\frac{20}{(8x)^2}+\frac{380}{3(8x)^3}+\frac{20320}{(8x)^5}\ldots.$$

Then, as before, by Lagrange's theorem,

$$x=\beta-\frac{5}{8\beta}+\frac{20}{(8\beta)^2}-\frac{220}{3(8\beta)^3}+\frac{2400}{(8\beta)^4}+\frac{54560}{3(8\beta)^5}\ldots$$

(ii.) When $p=1$—*i.e.*, when the vibrating plate has one nodal diameter, equation (9A) gives

$$\frac{(6+4x^2)J_1-(3x+4x^3)J_0}{(6-4x^2)J_1-3xJ_0}=\frac{(6-4x^2)I_1-(3x-4x^3)I_0}{(6+4x^2)I_1-3xI_0}. \quad (13)$$

This reduces to

$$\tan\left(\frac{\pi}{2}-x+n\pi\right)=9y-84y^2+639y^3-6804y^4+168714y^5\ldots.$$

Hence $\quad x=(2n+1)\frac{\pi}{2}-\frac{9}{8x}+\frac{84}{(8x)^2}-\frac{396}{(8x)^3}-\frac{65261}{(8x)^5}\ldots$

Writing β for $(2n+1)\frac{\pi}{2}$, Lagrange's theorem then gives

$$x=\beta-\frac{9}{8\beta}+\frac{84}{(8\beta)^2}-\frac{1044}{(8\beta)^3}+\frac{18144}{(8\beta)^4}-\frac{385877}{(8\beta)^5}\ldots.$$

(iii.) The general expression for the roots of equation (9A), as far as the term containing $1/(8\beta)^4$ can be obtained from a result of Kirchhoff's, viz.,

$$\tan\left(x-n\pi-\frac{p\pi}{2}\right)=\frac{B/(8x)+C/(8x)^2-D/(8x)^3\ldots}{A+B/(8x)+D/(8x)^3\ldots}\ldots,$$

where $A=\gamma=\frac{4}{3}$, using Poisson's value of $\mu=\frac{1}{4}$,

$$B=\gamma\,(1-4p^2)-8,$$

$$C=\gamma\,(1-4p^2)(9-4p^2)+48(1+4p^2),$$

$$D=-\frac{\gamma}{3}\{(1-4p^2)(9-4p^2)(13-4p^2)\}+8(9+136p^2+80p^4)$$

$$\tan\left(x-n\pi-\frac{p\pi}{2}\right)=-(4p^2+5)\,y+4(16p^2+5)y^2$$

$$-\tfrac{1}{3}\,(64p^6+304p^4+1804p^2-255)y^3\ldots.$$

Writing β for $(2n+p)\frac{\pi}{2}$ and m for $4p^2$, it is easily shown that

$$x=\beta-\frac{m+5}{8x}+\frac{4(4m+5)}{(8x)^2}-\frac{4(m-1)\,(m+95)}{(8x)^3}+\frac{0}{(8x)^4}\cdots$$

$$\therefore\ x=\beta-\frac{m+5}{8\beta}+\frac{4(4m+5)}{(8\beta)^2}-\frac{4(7m^2+154m+55)}{3(8\beta)^3}$$

$$+\frac{96(m+5)\,(4m+5)}{(8\beta)^4}\ldots.\qquad(14)$$

This series is not convergent enough to give the earlier roots of equation (9A). These can be obtained without difficulty from tables of Bessel functions.

Poisson (Mém. Acad., 1829) found the first two roots of equation (9A), when $p=0$, viz., 2·9815 and 6·1936.

Kirchhoff calculated some of the roots of the general equation (9A) by expressing it in the form

$$0=1-\frac{x^4}{A_1}+\frac{x^8}{A_2}-\frac{x^{12}}{A_3}+\ldots$$

and finding the roots by "trial." Only the first two roots, $(\lambda l)^4$, in each case ($p=0$, 1, 2, 3) could be calculated from the table of values of A_1, A_2, &c., given in the Paper. Kirchhoff's roots are readily expressed in the same form as those calculated from (14). Several of these values have been verified and are included in the following table :—

TABLE II.
Roots of Equation (9A).
When $p=0, 1, 2, 3$. $\mu=\frac{1}{4}$ (Poisson's value).

n.	$p=0$.	$p=1$.	$p=2$.	$p=3$.
0	...	...	2·348	3·571
1	2·982	4·518	5·940	7·291
2	6·192	7·729	9·186	10·600
3	9·362	10·903	12·381	13·821
4	12·519	14·024	15·556	17·015
5	15·669	17·218	18·721	20·203
6	18·817	20·368	21·880	23·363
7	21·963	23·516	25·035	26·526
8	25·108	26·663	28·187	29·685
9	28·253	29·809	31·337	32·841

The values of $\lambda(p=0, \mu=\frac{1}{4})$ can be found from (8)

$$\lambda=\frac{J_0'(x)}{iJ_0'(ix)}=-\frac{J_1(x)}{I_1(x)},$$

and x has one of the values in column 2 of Table II. When $x=\kappa a=9\cdot362$, for example, $\lambda=-0\cdot0001299\ldots$. The radii of the nodal circles are readily found from the roots of the equation

$$J_0(\kappa r)-0\cdot0001299-J_0(i\kappa r)=0. \quad . \quad . \quad . \quad (15)$$

The four roots of (15) are 2·40406, 5·5369, 8·3662 and 9·3620. Hence the radius of the plate being taken as unity,

$$r_1=0\cdot25679 \qquad 0\cdot25679,$$
$$r_2=0\cdot59143 \qquad 0\cdot59147,$$
$$r_3=0\cdot89365 \qquad 0\cdot89381.$$

For comparison, Kirchhoff's calculated results * are given in the second column.

The roots of equation (9A) vary with the assumed value of μ, but, with the exception of some of the earlier roots, the variations are comparatively small. The value of the roots for any given value of μ can be obtained without difficulty. For example, if $\mu=\frac{1}{3}$ (Wertheim's value),

$$x=\beta-\frac{3m+13}{3(8\beta)}+\frac{40(3m+5)}{9(8\beta)^2}\cdots$$

* Strehlke, "Pogg. Ann.," 1855. Lord Rayleigh, "Theory of Sound," Vol. I.

If this expression be divided by the value of x in (14), the new series of roots can be found with considerable accuracy by multiplying the roots in Table II. by this quotient, viz. :—

$$1+\frac{2}{24\beta^2}-\frac{4\beta(6m-5)-6(m+5)}{(24\beta^2)^2}\cdots$$

TABLE III.
Roots of Equation (9A).
$\mu=\frac{1}{3}$ (Wertheim's value).

n.	$p=0$.	$p=1$.	$p=2$.	$p=3$.
0	...	...	2·292	3·496
1	3·013	4·530	5·936	7·274
2	6·206	7·737	9·188	10·595
3	9·371	10·910	12·386	13·820
4	12·526	14·029	15·559	17·015

The first two roots in each column are those given by Kirchhoff (Crelle, 1850, § 85).

The calculated values of the radii of the nodal circles vary very little for different values of μ. Taking the case where the variation is greatest, viz., when $p=3$ and $n=0$, the change in the value of the radius when μ is changed from $\frac{1}{4}$ to $\frac{1}{3}$ is less than 1 in 500. (Lord Rayleigh, " Theory of Sound," Vol. I., p. 363.)

The change in the calculated value of the frequency of vibration of a " free " plate for a given change in μ is easily found from a consideration of the expression for the frequency, viz. :—

$$N=\frac{ax^2}{\sqrt{1-\mu^2}},$$

a being constant for the same plate and x one of the roots of equation (9A). When the value of μ is changed x also changes and both contribute to the variation in the value of N. The factor $1/\sqrt{1-\mu^2}$ introduces a change of about 2·7 per cent. in the frequency when the value of μ is changed from $\frac{1}{4}$ to $\frac{1}{3}$. Since N varies as the square of the root (x), the variation due to this is easily found from Tables II. and III. For example, when $p=3$, a change in μ from $\frac{1}{4}$ to $\frac{1}{3}$ diminishes the first root by about 2 per cent., and, therefore, the calculated value of the frequency is about 4 per cent. less. In the second mode of vibration, the decrease in the frequency is less than 1 in 200, in the third mode less than 1 in 1,000, and so on.