The Roots of the Neumann and Bessel Functions

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XX. *The Roots of the Neumann and Bessel Functions.* By

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THE first 40 values of x for which the Bessel function of the zeroth order, $J_0(x)$ vanishes, have been calculated by Willson and Peirce,* whilst a general expression for the roots of $J_p(x)=(0)$ has been given by MacMahon.† The second solution of the Bessel differential equation—Neumann's solution—is often referred to as a Neumann function, and is usually denoted by $Y_p(x)$. Only the first four roots of $Y_0(x)=0$ and the first three roots of $Y_1(x)=0$ appear to have been given. These were found by graphic interpolation from Smith's tables ‡ by Kalähne §:—

Roots of $Y_0(x)=0$.	Roots of $Y_1(x)=0$.
0·826(0)	2·11(8)
3·88(5)	5·35(5)
7·01(3)	8·52(1)
10·14(9)	—

The general expression for the calculation of the roots of $Y_p(x)$ and the roots of $\mu J_p(x)+Y_p(x)$, where μ is any real number, positive or negative, can be found by adopting the method employed in determining the roots of $J_p(x)$.

Roots of $\mu J_p(x)+Y_p(x)=0$.

The equation may be written in the form

$$\mu J_p(x)+(\log 2-\gamma)J_p(x)+\frac{\pi}{2}N_p(x)=0. \qquad . \quad . \quad . \quad (1)$$

The function ¶ $N_p(x)$ is the same as $-Y_p(x)$ [Schafheitlin] $K_p(x)$ [Graf. u. Gubler] and the Neumann function of Nielsen, $Y_p(x)$.

* "Bulletin" of the American Mathematical Society, Vol. III., 1896-7.
† "Annals of Mathematics," Vol. IX., 1895.
‡ B. A. Smith, "Messenger of Mathematics," 26, 1897.
§ Kalähne, "Zeitschrift für Mathematik u. Physik," 54, 1907.
¶ Jahnke u. Emde. Funktionentafeln, 1909.

Writing a for $\mu+\log 2-\gamma$, and substituting the semi-convergent series for the functions, equation (1) becomes

$$\left[aP_p+\frac{\pi}{2}Q_p\right]\cos\left(x-\frac{2p+1}{4}\pi\right)$$
$$+\left[\frac{\pi}{2}P_p-aQ_p\right]\sin\left(x-\frac{2p+1}{4}\pi\right)=0. \quad . \quad (2)$$

Case (i.), when a is equal to or less than $\frac{\pi}{2}$.

Divide by $\frac{\pi}{2}$, write b for $\frac{2a}{\pi}$, and put

$$bP_p+Q_p=R\sin\theta_p,$$
$$P_p-bQ_p=R\cos\theta_p.$$

Then $$\tan\theta_p=(bP_p+Q_p)/(P_p-bQ_p). \quad . \quad (3)$$

and (2) becomes $\sin\left(x-\frac{2p+1}{4}\pi+\theta_p\right)=0.$

$$x=\frac{2p+1}{4}\pi+\theta_p=n\pi. \quad (n=0,\ 1,\ 2\ .\ .\ )$$

$$x=(4n+2p+1)\frac{\pi}{4}-\theta_p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The expression for θ_p in powers of x is found by first calculating the value of $\tan\theta_p$ from (3) and applying Gregory's series to determine the angle from its tangent.

Writing m for $4p^2$ and y for $\frac{1}{8x}$, we get

$$\tan\theta_p=b+(1+b^2)(m-1)y+b(1+b^2)(m-1)^2\,.\,y^2+\ .\ \&c.,$$

and $$\theta_p=\text{arc}\tan b+(m-1)y+\frac{4}{3}(m-1)(m-25)y^3+\ldots \quad . \quad (5)$$

The coefficients of the even powers of y and all terms containing b in the coefficients of the odd powers of y vanish. Equation (4) takes the form

$$x=(4n+2p+1)\frac{\pi}{4}-\text{arc}\tan b-\frac{m-1}{8x}-\frac{4}{3}\frac{(m-1)(m-25)}{(8x)^3}\cdots$$

or, writing β for $(4n+2p+1)\frac{\pi}{4}-\text{arc}\tan b$, and applying Lagrange's theorem,

$$x=\beta-\frac{m-1}{8\beta}-\frac{4(m-1)(7m-31)}{3(8\beta)^3}$$
$$-\frac{32(m-1)(83m^2-982m+3779)}{15(8\beta)^5}, \quad . \quad (6)$$

where $$m=4p^2,\ b=\frac{2a}{\pi}=\frac{2(\mu+\log 2-\gamma)}{\pi}.$$

Case (*ii.*) When a is equal to or greater than $\frac{\pi}{2}$, it is readily shown, as above, that x can be expressed in the same form as (6), if

$$b=\frac{\pi}{2a}=\frac{\pi}{2(\mu+\log 2-\gamma)} \text{ and } \beta=(4n+2p+3)\frac{\pi}{4}+\text{arc tan}\, b.$$

These results can be tested when p is half an odd integer.

Roots of $Y_p(x)=0$.

The expression for the roots of this equation is

$$x=\beta-\frac{m-1}{8\beta}-\frac{4(m-1)(7m-31)}{3(8\beta)^3}\ \cdot\ \cdot$$

where $$\beta=(4n+2p+1)\frac{\pi}{4}-\text{arc tan}\, b$$

$$=(4n+2p+1)\frac{\pi}{4}-0\cdot073671.$$

The following tables of the roots of $Y_0(x)$, $Y_1(x)$ and $Y_2(x)$ have been calculated from this formula, except in the case of some of the earlier roots. These were found by calculating the values of the Neumann functions to seven places of decimals in the neighbourhood of the roots from the expressions

$$Y_0(x)=J_0(x)\,.\,\log_e x+\left(\frac{x}{2}\right)^2-\frac{3}{8}\left(\frac{x}{2}\right)^4+\frac{11}{216}\left(\frac{x}{2}\right)^5 \ldots \ \&c.,$$

and then graphically representing these results.

TABLE *of First Forty Roots of* $Y_0(x)=0$.

No. of root.	Root.	No. of root.	Root.
1	0·82601	21	63·54555
2	3·88456	22	66·68705
3	7·01256	23	69·82856
4	10·14876	24	72·97007
5	13·28748	25	76·11159
6	16·42729	26	79·25312
7	19·56766	27	82·39465
8	22·70837	28	85·53619
9	25·84930	29	88·67773
10	28·99037	30	91·81928
11	32·13154	31	94·96082
12	35·27278	32	98·10237
13	38·41409	33	101·24393
14	41·55544	34	104·38548
15	44·69682	35	107·52704
16	47·83823	36	110·66860
17	50·97966	37	113·81016
18	54·12111	38	116·95172
19	57·26258	39	120·09329
20	60·40406	40	123·23486

TABLE *of Roots of* $Y_1(x)=0$.

No. of root.	Root.	No. of root.	Root.
1	2·11827	21	65·10862
2	5·35509	22	68·25048
3	8·52196	23	71·39231
4	11·67528	24	74·53412
5	14·82365	25	77·67592
6	17·96965	26	80·81770
7	21·11434	27	83·95947
8	24·25823	28	87·10123
9	27·40159	29	90·24296
10	30·54459	30	93·38470
11	33·68732	31	96·52642
12	36·82986	32	99·66813
13	39·97226	33	102·80984
14	43·11453	34	105·95154
15	46·25671	35	109·09324
16	49·39882	36	112·23493
17	52·54087	37	115·37661
18	55·68287	38	118·51829
19	58·82482	39	121·65996
20	61·96673	40	124·80163

TABLE *of Roots of* $Y_2(x)=0$.

No. of root.	Root.	No. of root.	Root.
1	3·29690	21	66·65705
2	6·71703	22	69·79990
3	9·94840	23	72·94265
4	13·13551	24	76·08531
5	16·30478	25	79·22788
6	19·46500	26	82·37037
7	22·62002	27	85·51280
8	25·77173	28	88·65517
9	28·92124	29	91·79749
10	32·06920	30	94·93976
11	35·21601	31	98·08199
12	38·36197	32	101·22417
13	41·50726	33	104·36632
14	44·65204	34	107·50844
15	47·79639	35	110·65053
16	50·94040	36	113·79259
17	54·08414	37	116·93462
18	57·22763	38	120·07664
19	60·37093	39	123·21863
20	63·51406	40	126·36060

Roots of $N_p(x)$ [Nielsen's $Y_p(x)$, &c.].

By making $\mu=\gamma-\log 2$, $a=0$ and $b=0$, equation $\mu J_p(x)+Y_p(x)=0$ becomes $N_p(x)=0$.

The expression for the roots of this function is found from (9) and agrees with that given by Prof. MacMahon, viz.:

$$x=\beta-\frac{m-1}{8\beta}-\frac{4(m-1)(7m-31)}{3(8\beta)^3}-\dots,$$

where $$\beta=(4n+2p+1)\frac{\pi}{4}=(4n+2p-1)\frac{\pi}{4}+\frac{\pi}{2}.$$

Roots of $N_0(x)$ and $N_1(x)$.

The first two roots of $N_0(x)$ and the first root of $N_1(x)$ were found by interpolation from values of these functions in the neighbourhood of the roots. These values were calculated from the expressions

$$N_0(x)=\frac{2}{\pi}\left[\left(\frac{x}{2}\right)^2-(1+\tfrac{1}{2})\left(\frac{x}{2}\right)^4\Big/(2!)^2+(1+\tfrac{1}{2}+\tfrac{1}{3})\left(\frac{x}{2}\right)^6\Big/(3!)^2-\dots\right.$$
$$\left.-(\log 2-\gamma-\log_e x)\,J_0(x)\right].$$

$$N_1(x)=\frac{2}{\pi}\left[-\frac{x}{2}+\frac{3}{4}\left(\frac{x}{2}\right)^3-\frac{11}{72}\left(\frac{x}{2}\right)^5+\dots\right.$$
$$\left.-(\log 2-\gamma-\log_e x)\,J_1(x)-\frac{J_0(x)}{x}\right].$$

No. of root.	$N_0(x)$.	$N_1(x)$.
1	0·89358	2·19685
2	3·95769	5·42968
3	7·08605	8·59601
4	10·22234	11·74915
5	13·36110	14·89744
6	16·50092	18·04340
7	19·64131	21·18807
8	22·78203	24·33194
9	25·92296	27·47529
10	29·06403	30·61829

Roots of $\mu J_p(x)+Y_p(x)=0$.

The first two or three roots of the equation, when $\mu=1$, 2, 3 . . . and $p=0$ and $p=1$, have been found graphically by Kalähne from tables of the Bessel and Neumann functions.

Roots of $J_0(x)+Y_0(x)=0$. 0·356(5), 3·34(4)

Roots of $J_1(x)+Y_1(x)=0$. 1·52(2), 4·80(2).

TABLE *of First Ten Roots of* $J_0(x)+Y_0(x)=0$.

No. of root.	Root.	No. of root.	Root.
1	0·35617	6	15·88353
2	3·34532	7	19·02383
3	6·46999	8	22·16449
4	9·60543	9	25·30539
5	12·74386	10	28·44644

TABLE *of First Ten Roots of* $J_1(x)+Y_1(x)=0$.

No. of root.	Root.	No. of root.	Root.
1	1·52101	6	17·42498
2	4·80339	7	20·56985
3	7·97498	8	23·71385
4	11·12971	9	26·85729
5	14·27867	10	30·00035

Roots of $J_0(x)-Y_0(x)=0$, &c.

No. of root.	J_0-Y_0.	J_0-2Y_0.	$10J_0+Y_0$.	$10J_0-Y_0$.
1	1·37616	1·11707	2·55629	2·25096
2	4·46696	4·19582	5·67354	5·36311
3	7·59752	7·32529	8·80753	8·49639
4	10·73440	10·46187	11·94545	11·63407
5	13·87338	13·60073	15·08488	14·77345
6	17·01333	16·74062	18·22505	17·91352
7	20·15378	19·88103	21·36564	21·05408
8	23·29454	23·02177	24·50649	24·19490
9	26·43549	26·16271	27·64750	27·33590
10	29·57658	29·30380	30·78864	30·47702