Relative linear velocity of surfaces. . .	3040 cm. per sec.
Distance between cylinders	0·38 cm.
Least relative retardation which could have been detected	0·00005 λ.

Assuming the speed and sensibility attained by Kundt in this case the same as that in case of water—it could not have been greater—the limits of the experiment with solution of sodium hyposulphite are extended by a factor of 100.

LXIII. *On the Theory of the Magnetic Influence on Spectra; and on the Radiation from moving Ions.* *By* J. LARMOR, *D.Sc., F.R.S.**

A THEORETICAL analysis of somewhat general character can be developed in connexion with Zeeman's phenomenon †, which may help to throw light on the nature of the electric vibrations in the molecule. It will be convenient to begin with a simple case.

1. Consider a single ion e, of effective mass M, describing an elliptic orbit under an attraction to a fixed centre proportional to the distance therefrom. The equations of motion will be $(\ddot{x}, \ddot{y}, \ddot{z}) = -a^2(x, y, z)$; and the frequency of oscillation in any direction will be $a/2\pi$. Now suppose that a uniform magnetic field H, in a direction (l, m, n), is introduced : the equations of motion will become

$$\ddot{x} = -a^2x + \kappa(n\dot{y} - m\dot{z})$$
$$\ddot{y} = -a^2y + \kappa(l\dot{z} - n\dot{x})$$
$$\ddot{z} = -a^2z + \kappa(m\dot{x} - l\dot{y})$$

where $\kappa = e\mathrm{H}/\mathrm{M}c^2$, in which c is the velocity of radiation‡. To obtain the frequencies $(p/2\pi)$ of the oscillations thus modified, we make as usual (x, y, z) proportional to e^{ipt}. This gives, after easy reduction, an equation for p,

$$(p^2 - a^2)^3 - \kappa^2p^2(p^2 - a^2) = 0.$$

Thus, corresponding to each original period represented by $p = a$, there are three modified ones represented by $p = a$ and $p^2 \pm \kappa p - a^2 = 0$; when, as in practice, κ is very small the two

* Communicated by the Author.

† Zeeman, Phil. Mag. March and July 1897 ; Michelson, Phil. Mag. May 1897 ; Lodge, Proc. Roy. Soc. Feb. and June, 1897.

‡ *Cf.* Phil. Trans. A, 1895, p. 718. These equations only apply strictly (*infra*, § 10) when the velocity of the ion is small compared with c.

latter will be approximately $p = a + \kappa^2/8a \pm \frac{1}{2}\kappa$, or with sufficient accuracy $p = a \pm \frac{1}{2}\kappa$. Each vibration period will therefore be tripled: and the striking feature is that the modification thus produced is the same whatever be the orientation of the orbit with respect to the magnetic field.

An inquiry into the cause of this feature enables us to generalize the result. Suppose that the original orbit is referred to a system of axes (x, y, z) that are themselves revolving with angular velocity ω round an axis of which the direction is (l, m, n). The component velocities (u, v, w) referred to this moving space are $\dot{x} - y\omega n + z\omega m, \ldots, \ldots$, and the component accelerations are $\dot{u} - v\omega n + w\omega m, \ldots, \ldots$. Thus the component acceleration parallel to x is

$$\ddot{x} - 2\omega(n\dot{y} - m\dot{z}) - \omega^2 x + \omega^2 l(lx + my + nz).$$

If, then, we take ω equal to $\frac{1}{2}\kappa$, and so can neglect ω^2, the equations of the original orbit referred to this revolving space are identical with those of that orbit as modified by the magnetic field. In other words, the oscillation thus modified will be brought back to its original aspect if the observer is attached to a frame which revolves with angular velocity $\frac{1}{2}\kappa$ or $e\mathrm{H}/2\mathrm{M}c^2$ round the axis of the magnetic field. In a circular orbit described one way round this axis the apparent rotation will in fact be retarded, in one described the other way round it will be accelerated, in a linear oscillation along the axis there will be no alteration: hence the three periods found above.

2. Now the argument above given still applies, whatever be the number of revolving ions in the molecule, and however they attract each other or are attracted to fixed centres on the axis, provided κ has the same value for them all. In any such case the actual oscillation in the magnetic field is identical with the unmodified oscillation as seen from a revolving frame; or, more simply, the modification may be represented by imparting an opposite angular velocity $\frac{1}{2}\kappa$ to the vibrating system. Thus the period of a principal oscillation of the system will be affected by the magnetic field in the opposite way to that of its optical image in a plane parallel to the field; and these two oscillations, previously identical as regards period, will be separated on account of their right-handed and left-handed qualities. An oscillation which does not involve rotation round an axis parallel to the field will, however, present the same aspect to the field as its image, and will not be affected at all. This latter type of oscillation, in a compound system, will be a very special one; and when a crowd of vibrators indifferently orientated are

considered, radiation of this kind will usually be practically nonexistent. Thus *each* spectral line of the vibrator will be split up into two with righthanded and lefthanded circular polarizations when seen along the axis, and plane-polarized with phase difference of half a wave-length when seen at right angles to it, and with differences of frequency the same for all lines in the spectrum, as in the special case above. This simple statement applies to all systems in which the electric charge of each mobile ion in the vibrator is proportional to its effective mass, which implies that the charges of the mobile ions are all of the same sign.

3. The characters of the three principal oscillations in § 1 may be determined in the usual manner by substituting in the equations of motion $(x, y, z) = (x_0, y_0, z_0)e^{\iota pt}$ and determining (x_0, y_0, z_0) from the resulting system of linear equations. But algebraic reductions will be avoided by taking the magnetic field to be along the axis of z, so that $(l, m, n) = (0, 0, 1)$, as might in fact have been done from the beginning. The equations of motion are then

$$\ddot{x} = -a^2x + \kappa\dot{y}, \quad \ddot{y} = -a^2y - \kappa\dot{x}, \quad \ddot{z} = -a^2z.$$

They show at once that the unmodified principal vibration is a linear oscillation parallel to the z-axis. As regards the others, writing $(x, y) = (x_0, y_0)e^{\iota pt}$ we have

$$(a^2 - p^2)x = \iota\kappa py, \quad (a^2 - p^2)y = \iota\kappa px;$$

thus $p^2 \pm \kappa p - a^2 = 0$, or very approximately $p = a \pm \frac{1}{2}\kappa$ as before; and separating the real parts of this solution

$$x = \frac{A}{a^2 - p^2}\cos pt, \quad y = \frac{A}{\kappa p}\sin pt,$$

which to our order of approximation represents motion round a circle in the plane of (x, y) righthanded or lefthanded according to the value of p that is taken*. The character of the radiation from such a vibrator is thus precisely independent of the orientation of its orbit with respect to the magnetic field. With a large number of such vibrators, orientated indifferently, every spectral line seen in a direction at right angles to the magnetic field would be split up into three lines, each of the same breadth as the original, the middle one plane-polarized at right angles to the magnetic field, the outer ones in the direction of the field: as the aggregate light must be

* This is no doubt the analysis recently indicated by Prof. FitzGerald in 'Nature,' Sept. 1897.

unpolarized, the intensity of the middle line would be twice that of either of the outer ones. But viewed along the field the middle line would disappear, as the exciting vibration would be end on to the observer and could not therefore send out transverse radiation : the other lines (equally sharp as before) would be circularly polarized, and their directions of polarization would, as Zeeman remarks, determine whether the vibrator involves a positive or a negative electron.

4. A view has been enunciated that it is only one kind of ions, namely the negative ones, that are mobile and free to vibrate in the atom or molecule, the other kind being fixed to the matter and immobile. On such an hypothesis, if the charges of these negative ions are proportional to their effective masses, for example if they are simple electrons without inertia other than that of the electric charge, the intervals (measured in difference of frequency) between magnetic doublets and the outside lines of magnetic triplets in the spectrum should be the same for all lines. Moreover, they should be the same in different spectra. Thus an hypothesis of that kind can be definitely put to the test.

5. When there are ions of different kinds describing orbits in the molecule, these exact results no longer hold : but even then we can assert that the difference of frequency between the lines of a magnetic doublet is of the order $eH/2\pi Mc^2$, and the order of magnitude of e/M can be thence derived. Thus Zeeman concludes from his experiments that the effective mass of a revolving ion, supposed to have the full unitary charge or electron, is about 10^{-3} of the mass of the atom. This is about the same as Professor J. J. Thomson's estimates of the masses of the electric carriers in the cathode rays. If we took these carriers to be simply electrons, as their constancy under various environments tends to indicate, there would thus be about 10^3 electrons in the molecule.

6. In view of the above considerations, the circumstance that in a magnetic field certain lines, viewed transversely, are divided into sharp triplets with perfect plane polarizations, which has been described by Zeeman and assented to by Lodge and by Michelson*, is an important clue to the character of the principal oscillations which emit those lines. In an oscillating molecule undisturbed by a magnetic field there must be three types of vibration which all have the period belonging to that line, namely two types which differ only by involving rotations in opposite directions round the axis of that magnetic field and would naturally have the same

* In Cornu's experiments, *Comptes Rendus*, Oct. 18, the application of the polarizing apparatus seems to have been required to divide the lines.

period, and a third type which does not involve any rotation with respect to that axis. Now that extraneous axis may have any direction with reference to the molecule. Hence a principal oscillation which is thus magnetically tripled must be capable of being excited with reference to any axis in the molecule: otherwise there would be merely hazy broadening or duplication instead of definite triplication.

7. A system of electrons or ions of the same sign, confined to a surface over which they are free to move and constituting an electric charge on it, is an artificial vibrator whose periods illustrate these results. The free periods of such a vibrating system in which the forces acting on the electrons work against the inertia of the moving electrons, would be only theoretically different from the free periods of an actual electric charge on a metallic conductor; although in the latter case the forces acting on the ions work mainly against the ohmic diffusive resistance to their transfer either actual or electrolytic through the crowd of neighbouring molecules, which is far greater than the reaction arising from inertia alone unless it is rapid optical vibrations that are dealt with. In either case the forces acting on the ions are so great compared with the possible kinetic reactions to their motion that their distribution is at each instant practically in equilibrium on the surface, so that there is no electric force along it: and either set of conditions simply reduces to the condition that there shall be no electric or magnetic field in the interior. It follows that the oscillations of an electric charge on any conductor of the form of a surface of revolution are modified by the introduction of a magnetic field along the direction of the axis of the surface, just as if the angular velocity $\frac{1}{2}\kappa$ above given were imparted to the vibrating system. All the free periods except those of zonal oscillations would be duplicated in the manner above explained, the interval measured in difference of frequencies between the components of the doubled vibration being the same for all. In the special case of zonal oscillations on a sphere they would be triplicated, because the period of a zonal vibration along the axis would not be modified: but the middle line would be very weak compared with the flanking ones*.

8. This analysis gives a hint as to one way in which a

* The radiation from a continuously distributed electric charge is, however, known to be so great as to make its oscillations dead-beat; hence these conclusions could only be applied if (i.) there are only a limited number of discrete ions moving on the surface, or (ii.) there are material forces other than electric imagined to act between the ions, whose energy could maintain the vibration for a large number of periods. (See § 10.)

series of double lines in a spectrum, with equidistant frequencies, might be originated. Suppose, as a very rough illustration, that a polar molecule is constituted of a system of positive electrons around one pole and a system of negative ones around the other, the two systems being so far apart as to have practically separate sets of periods for their orbital motions, each of course disturbed by the presence of the other. Each of these systems moves in the magnetic field, more or less constant, arising from the other; and the effect of this disturbing field will, as above, be to duplicate all the periods of that set in the above regular way.

9. It is desirable to precisely formulate the relation between the motions of the electrons and the radiation emitted by them, which has been tacitly employed in the foregoing discussion. The specification of that radiation may be readily assigned by a summation over the different elements of the paths of the oscillating ions. Suppose that an ion e is at A and after a time δt is at B, where $AB = v\delta t$, v being its velocity; the effect of its displacement is the same as that of the creation of an electric doublet AB of moment $ev\delta t$; thus we have only to find the influence propagated from such a doublet, and then integrate along the paths of the ions of the molecule.

Consider now such a doublet at the origin, lying along the axis of z; for it, or indeed for any distribution symmetrical with respect to that axis, the lines of magnetic force will be circles round the axis, and the force will be specified by a single variable, its intensity H. The current, whether in dielectric or in conducting media, will circulate in wedge-shaped sheets with their edges on the axis, and may be specified by a stream function, as in fact will appear below. If we employ cylindrical coordinates ρ, θ, z, and apply the Amperean circuital relation (viz: circulation of magnetic force equals 4π times current) to the faces of the element of volume $\delta\rho.\rho\delta\theta.\delta z$, we obtain for the components P, R of the electric force

$$\frac{dP}{dt} = -c^2 \frac{dH\rho}{\rho dz}, \quad \frac{dR}{dt} = c^2 \frac{dH\rho}{\rho d\rho},$$

so that $H\rho$ plays the part of a stream function; while by the circuital relation of Faraday

$$\frac{dP}{dz} - \frac{dR}{d\rho} = -\frac{dH}{dt}.$$

Thus the characteristic equation for H is

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho H + \frac{d^2H}{dz^2} = c^2\frac{d^2H}{dt^2},$$

which is $$(\nabla^2 - \rho^{-2})H = c^2 d^2H/dt^2,$$

where ∇^2 is Laplace's operator. But a more convenient reduction comes on substituting $H = dY/d\rho$, and then neglecting an irrelevant operator $d/d\rho$ along the equation: this gives

$$\nabla^2 Y = c^2 d^2 Y/dt^2.$$

We can now express the disturbance emitted by an electric doublet situated along the axis of z at the origin, and vibrating so that its moment M is an arbitrary function of the time. As regards places at a finite distance, the doublet may be treated as a linear current-element of strength $d\mathrm{M}/dt$. Close up to such an element in its equatorial plane, the magnetic force H due to it is $-r^{-2}d\mathrm{M}/dt$. The appropriate solution for Y for this simplest case is $Y = r^{-1}f(t-r/c)$, so that

$$H = -\sin\theta\left\{\frac{f(t-r/c)}{r^2} + \frac{f'(t-r/c)}{cr}\right\},$$

giving when θ is $\frac{1}{2}\pi$ and r is very small, $H = -r^{-2}f(t)$: thus $d\mathrm{M}/dt = f(t)$. That is, if the moment of the oscillating doublet is given in the form $d\mathrm{M}/dt = f(t)$, the magnetic force thus originated at the point (r, θ) is

$$H = -\sin\theta\left\{\frac{f(t-r/c)}{r^2} + \frac{f'(t-r/c)}{cr}\right\},$$

or $$\sin\theta\frac{d}{dr}r^{-1}f(t-r/c).$$

The second term is negligible for movements of slow period, as it involves the velocity c of radiation in the denominator. The components of the magnetic field due to a vibrating doublet M at the origin whose direction-cosines are (l, m, n) are then $(mz-ny, nx-lz, ly-mx)r^{-1}d/dr\, r^{-1}f(t-r/c)$, where $d\mathrm{M}/dt = f(t)$; and the components of the magnetic field, and therefore of the radiation emanating from any system of electric oscillators vibrating in any given manner can thence be expressed in a general form by integration. At present we only want the effect of suddenly establishing the doublet $\mathrm{M} = ev\delta t$ at the origin. This comes by integration over the very small time of establishment; there is a thin spherical shell of magnetic force propagated out with velocity c, the total force integrated across the shell being exactly $-\mathrm{M}r^{-2}\sin\theta$ whatever be its radius, for the integral of the second term

in H vanishes because dM/dt is null at the beginning and end of the operation. The aggregate amount of magnetic force thus propagated in the spherical sheet is the same as the steady magnetic force due to a permanent steady current-element of intensity equal to $M/\delta t$, or ev: it is clear, in fact, that this must be so, if we consider a sudden creation of this current-element and remember that its magnetic field establishes itself by spreading out ready formed with the velocity of radiation.

The magnetic force at a point at distance r due to a moving ion thus depends on the state of the ion at a time r/c previously; for near points it is in the plane perpendicular to r, at right angles to the projection v of the velocity of the ion on that plane, and equal to evr^{-2}. For vibrations whose wave-length in free æther is very great compared with the dimensions of the molecular orbit, if we interpret magnetic force as velocity of the æther, the vibration-path of a point attached to the æther, and close to the vibrator, will be in the plane transverse to r, and similar to the projection of the orbit of the electron on that plane when turned round through a right angle *. If the condition of wave-length very large compared with molecular magnitude were not satisfied, phase-differences would sensibly disturb this result, and in effect each spectral line would be accompanied, more or less, by its system of harmonics.

As the vibration of a near point in the æther is thus similar to the projection on the wave-front of the vibration of the electrons in the molecule, it is verified that the free periods of the radiation are those of the system of ions.

10. It might appear also at first sight that every steady orbital motion must rapidly lose its energy by radiation just as vibrations on the conductor in § 7 would do if the ions on it formed a continuous charge. On the other hand it might be argued that what we have really been calculating is the amount added on to the previous motion in the medium by the successive displacements of the electrons; and in the cases of steady motion it is just this amount that is needed to maintain the permanency of the motion in the æther, which of itself has a tendency to be carried away. Thus in the parallel case of the movement of a very long stretched cord when an end of it has a steady circular motion imparted to it, an analysis in the ordinary way leads to a train of circular waves running along the cord; but there

* For a different treatment of similar topics *cf.* H. A. Lorentz, "La Théorie Electromagnetique de Maxwell," §§ 112–119, *Archives Néerlandaises*, 1892; "Versuch...," 1895; quoted by Zeeman, Phil. Mag. March, 1897.

exists a steady motion in which the cord whirls round bodily, and which will be generated when the velocity of the motion imposed on the end is gradually increased from a very small initial value to its final amount.

The difficulty is, however, not thus surmounted; for this steady motion which does not involve radiation is really a state of stationary undulation arising from the superposition of a wave-train travelling outwards on another travelling inwards, and the genesis of the latter one would have to be accounted for. We might assume that these non-radiating vibrations consisted of stationary waves reflected backwards and forwards between two vibrating molecules, or between two ions in the same molecule; but even that would not be satisfactory. As a matter of fact, however, no explanation of this kind is needed. The effective electric inertia of an ion e by itself is $\frac{2}{3}e^2a^{-1}$*, where a is the radius of its nucleus supposed spherical: the rate at which it loses energy by radiation is proportional to e^2, and involves its motion, but does not depend on a at all. The kinetic reaction to change of its velocity which is connected with loss of energy by radiation can thus be made negligible in comparison with the kinetic reaction arising from its inertia. In fact, the energy of the æthereal motion carried along by the moving ion depends on the first term in H, involving r^{-2}, and the radiated energy depends on the second term, involving r^{-1}. But in types of oscillation in which there are crowds of ions moving close together in step, loss of energy by radiation is an important feature in the dynamics of free vibrations.

These considerations can be developed by aid of the analysis of § 9 above. In consequence of the stream-function property of $\mathrm{H}\rho$, the components of d/dt of the electric force, along δr and along $r\delta\theta$, are respectively

$$\frac{c^2}{\rho}\frac{d\mathrm{H}\rho}{rd\theta} \quad \text{and} \quad -\frac{c^2}{\rho}\frac{d\mathrm{H}\rho}{dr},$$

ρ being $r\sin\theta$; thus they are

$$-2c^2\cos\theta\left\{\frac{f(t-r/c)}{r^3}+\frac{f'(t-r/c)}{cr^2}\right\},$$

and

$$-c^2\sin\theta\left\{\frac{f(t-r/c)}{r^3}+\frac{f'(t-r/c)}{cr^2}+\frac{f''(t-r/c)}{c^2r}\right\};$$

and the electric force is obtained by integrating with respect to t.

* Phil. Trans. A, 1894, p. 812. This inertia is no longer quite constant when the velocity of the ion is considerable compared with that of radiation. In that case also the simple computation of the radiation here given would not be exactly applicable; and the problem would have to be treated by continuous differential analysis after the manner of Phil. Trans. A, 1895, p. 718.

At a very great distance the electric force (as well as the magnetic force) is thus perpendicular to r, and is equal to $-r^{-1}\sin\theta f'(t-r/c)$; and the flow of energy is thus by Poynting's principle radial. For the case of an ion e moving with velocity v, $f(t)$ is equal to ev; and in $f(t-r/c)$ the value of the function f belongs to the position of the molecule at a time r/c previous, where r is its distance at that time. The rate of loss of energy by radiation may be computed by Poynting's formula as $(4\pi)^{-1}$ times the product of the above electric and magnetic forces integrated over an infinite sphere: it is thus

$$(4\pi r^2 c)^{-1}\{f'(t-r/c)\}^2\int\sin^2\theta\, d\mathrm{S}, \text{ or } \tfrac{2}{3}e^2c^{-1}\dot{v}^2.$$

In the process of getting up a velocity v of the ion from rest, there is a loss of energy equal to $\frac{2}{3}e^2c^{-1}\int\dot{v}^2\,dt$. In motion with uniform velocity there is no loss; during uniformly accelerated motion the rate of loss is constant.

As the electric and magnetic forces at a great distance are each proportional to the acceleration of the ion and do not involve its velocity, and as we can combine the components of its motion in fixed directions, it follows generally that the rate of loss of energy by radiation is $\frac{2}{3}e^2c^{-1}\times$ (acceleration)^2.

The store of kinetic energy belonging to the ion is $\frac{2}{3}e^2a^{-1}v^2$. Thus the loss of energy by radiation from an undisturbed vibrating molecule would not be sensible compared with its whole intrinsic kinetic energy, when the velocities of the ions are not of the order of magnitude of that of radiation: while for higher velocities the importance of the radiation is, in part at any rate, counteracted by the increase of the inertia coefficient.

11. Finally, it is to be observed that the law of the magnetic vibration excited by a moving ion is stated in § 9 only for the case in which r is small compared with the wave-length. Further away from the ion the law of variation of the magnetic force with distance is $ev/r^2+e\dot{v}/cr$ instead of ev/r^2. Thus at a distance of a large number of wave-lengths, the vibration-curve of the radiation proper is similar to the projection of the hodograph of the orbit of the ion on the wave-front, instead of the projection of the orbit itself.

It would thus appear that when the steady orbital motions in a molecule are so constituted that the vector sum of the accelerations of all its ions or electrons is constantly null, there will be no radiation, or very little, from it, and therefore this steady motion will be permanent. But this is just the condition which holds good so long as the molecule is free from extraneous disturbance.

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