When $p=3$ there are 32 such groups. They occur as follows:—

Degree of groups	4	6	8	12	24
Number ,, ...	1	3	3	10	15

When $p=7$ there are 22 such groups. They occur as follows:—

Degree of groups	8	14	28	56
Number ,, ...	1	1	7	13

When $p-1$ is divisible by 8 there are 27 such groups. They occur as follows:—

Degree of groups	p	$2p$	$4p$	$8p$
Number ,, ...	1	2	9	15

When $p-1$ is divisible by 4 but not by 8 there are 23 such groups. They occur as follows —

Degree of groups	$2p$	$4p$	$8p$
Number ,, ...	1	8	14

When $p-1$ is not divisible by 4 and p does not have one of the three values 2, 3, 7, there are 18 such groups. They occur as follows:—

Degree of groups	$4p$	$8p$
Number ,, ...	6	12

The three groups whose degrees are 4, 8, and p respectively are primitive. All the others are nonprimitive. When $p=2$ there are five commutative groups, but when $p>2$ there are only three such groups. When $p=2$ there are three non-commutative groups that are not simply isomorphic to any non-regular transitive group. When $p-1$ is divisible by 4 there are five such groups. When this condition is not satisfied and $p>2$ there are four such groups.

Paris, December 1896.

XVIII. *On the Passage of Electric Waves through Tubes, or the Vibrations of Dielectric Cylinders.* By Lord Rayleigh, *F.R.S.**

General Analytical Investigation.

THE problem here proposed bears affinity to that of the vibrations of a cylindrical solid treated by Pochhammer† and others, but when the bounding conductor is

* Communicated by the Author.

† Crelle, vol. xxxi. 1876.

regarded as perfect it is so much simpler in its conditions as to justify a separate treatment. Some particular cases of it have already been considered by Prof. J. J. Thomson*. The cylinder is supposed to be infinitely long and of arbitrary section; and the vibrations to be investigated are assumed to be periodic with regard both to the time (t) and to the coordinate (z) measured parallel to the axis of the cylinder, *i. e.*, to be proportional to $e^{i(mz+pt)}$.

By Maxwell's Theory, the components of electromotive intensity in the dielectric (P, Q, R) and those of magnetic induction (a, b, c) all satisfy equations such as

$$\frac{d^2R}{dx^2}+\frac{d^2R}{dy^2}+\frac{d^2R}{dz^2}=\frac{1}{V^2}\frac{d^2R}{dt^2}, \quad . \quad . \quad . \quad . \quad (1)$$

V being the velocity of light; or since by supposition

$$\frac{d^2R}{dz^2}=-m^2R, \qquad \frac{d^2R}{dt^2}=-p^2R,$$

$$\frac{d^2R}{dx^2}+\frac{d^2R}{dy^2}+k^2R=0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$k^2=p^2/V^2-m^2. \quad . \quad . \quad . \quad . \quad . \quad (3)\dagger$$

The relations between P, Q, R and a, b, c are expressed as usual by

$$\frac{da}{dt}=\frac{dQ}{dz}-\frac{dR}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and two similar equations; while

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The conditions to be satisfied at the boundary are that the components of electromotive intensity parallel to the surface shall vanish. Accordingly

$$R=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$P\frac{dx}{ds}+Q\frac{dy}{ds}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* 'Recent Researches in Electricity and Magnetism,' 1893, § 300.

† The k^2 of Prof. J. J. Thomson (*loc. cit.* § 262) is the negative of that here chosen for convenience.

dx/ds, dy/ds being the cosines of the angles which the tangent (ds) at any point of the section makes with the axes of x and y.

Equations (2) and (7) are met with in various two-dimensional problems of mathematical physics. They are the equations which determine the free transverse vibrations of a stretched membrane whose fixed boundary coincides with that of the section of the cylinder. The quantity k^2 is limited to certain definite values, k_1^2, k_2^2, . . ., and to each of these corresponds a certain normal function. In this way the possible forms of R are determined. A value of R which is zero throughout is also possible.

With respect to P and Q we may write

$$P = \frac{d\phi}{dx} + \frac{d\psi}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$Q = \frac{d\phi}{dy} - \frac{d\psi}{dx}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where ϕ and ψ are certain functions, of which the former is given by

$$\nabla^2\phi = \frac{dP}{dx} + \frac{dQ}{dy} = -\frac{dR}{dz} = -im\mathrm{R}. \quad . \quad . \quad (11)$$

There are thus two distinct classes of solutions; the first dependent upon ϕ, in which R has a finite value, while $\psi = 0$; the second dependent upon ψ, in which R and ϕ vanish.

For a vibration of the first class we have

$$P = d\phi/dx, \qquad Q = d\phi/dy, \quad . \quad . \quad . \quad . \quad (12)$$

and

$$(\nabla^2 + k^2)\phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Accordingly by (11)

$$\phi = \frac{im}{k^2}\mathrm{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$P = \frac{im}{k^2}\frac{dR}{dx}, \qquad Q = \frac{im}{k^2}\frac{dR}{dy}, \quad . \quad . \quad . \quad (15)$$

by which P and Q are expressed in terms of R supposed already known.

The boundary condition (7) is satisfied by the value ascribed to R, and the same value suffices also to secure the fulfilment of (8), inasmuch as

$$P\frac{dx}{ds} + Q\frac{dy}{ds} = \frac{im}{k^2}\frac{dR}{ds} = 0.$$

The functions P, Q, R being now known, we may express a, b, c. From (4)

$$\frac{da}{dt}=ipa=im\mathrm{Q}-\frac{d\mathrm{R}}{dy}=-\frac{m^2+k^2}{k^2}\frac{d\mathrm{R}}{dy};$$

so that

$$a=-\frac{m^2+k^2}{ipk^2}\frac{d\mathrm{R}}{dy},\qquad b=\frac{m^2+k^2}{ipk^2}\frac{d\mathrm{R}}{dx},\qquad c=0.\,. \quad (16)$$

In vibrations of the second class R=0 throughout, so that (2) and (7) are satisfied, while k^2 is still at disposal. In this case

$$\mathrm{P}=d\psi/dy,\qquad \mathrm{Q}=-d\psi/dx,\quad .\ .\ .\ (17)$$

and

$$(\nabla^2+k^2)\psi=0.\quad .\ .\ .\ .\ .\ .\ .\ (18)$$

By the third of equations (4)

$$\frac{dc}{dt}=ipc=\frac{d\mathrm{P}}{dy}-\frac{d\mathrm{Q}}{dx}=\nabla^2\psi=-k^2\psi;$$

so that $\psi=-ipc/k^2$, and

$$\mathrm{P}=-\frac{ip}{k^2}\frac{dc}{dy},\quad \mathrm{Q}=\frac{ip}{k^2}\frac{dc}{dx},\quad \mathrm{R}=0.\quad .\ .\ .\ (19)$$

Also by (4)

$$a=\frac{im}{k^2}\frac{dc}{dx},\quad b=\frac{im}{k^2}\frac{dc}{dy}.\quad .\ .\ .\ .\ (20)$$

Thus all the functions are expressed by means of c, which itself satisfies

$$(\nabla^2+k^2)c=0.\quad .\ .\ .\ .\ .\ .\ .\ (21)$$

We have still to consider the second boundary condition (8). This takes the form

$$\frac{dc}{dy}\frac{dx}{ds}-\frac{dc}{dx}\frac{dy}{ds}=0,$$

requiring that dc/dn, the variation of c along the normal to the boundary at any point, shall vanish. By (21) and the boundary condition

$$dc/dn=0,\quad .\ .\ .\ .\ .\ .\ .\ .\ (22)$$

the form of c is determined, as well as the admissible values of k^2. The problem as regards c is thus the same as for the two-dimensional vibrations of gas within a cylinder which is bounded by rigid walls coincident with the conductor, or for the vibrations of a liquid under gravity in a vessel of the same form *.

* Phil. Mag. vol. i. p. 272 (1876).

All the values of k determined by (2) and (7), or by (21) and (22), are real, but the reality of k still leaves it open whether m in (3) shall be real or imaginary. If we are dealing with free stationary vibrations m is given and real, from which it follows that p is also real. But if it be p that is given, m^2 may be either positive or negative. In the former case the motion is really periodic with respect to z; but in the latter z enters in the forms $e^{m'z}$, $e^{-m'z}$, and the motion becomes infinite when $z=+\infty$, or when $z=-\infty$, or in both cases. If the smallest of the possible values of k^2 exceeds p^2/V^2, m is necessarily imaginary, that is to say no periodic waves of the frequency in question can be propagated along the cylinder.

Rectangular Section.

The simplest case to which these formulæ can be applied is when the section of the cylinder is rectangular, bounded, we may suppose, by the lines $x=0$, $x=\alpha$, $y=0$, $y=\beta$.

As for the vibrations of stretched membranes,* the appropriate value of R applicable to solutions of the first class is

$$\mathrm{R}=e^{i(mz+pt)}\sin(\mu\pi x/\alpha)\sin(\nu\pi y/\beta)\ ; \quad . \quad . \quad . \quad (23)$$

from which the remaining functions are deduced so easily by (15), (16) that it is hardly necessary to write down the expressions. In (23) μ and ν are integers, and by (13)

$$k^2=\pi^2\left(\frac{\mu^2}{\alpha^2}+\frac{\nu^2}{\beta^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

whence

$$m^2=p^2/\mathrm{V}^2-\pi^2\left(\frac{\mu^2}{\alpha^2}+\frac{\nu^2}{\beta^2}\right). \quad . \quad . \quad . \quad (25)$$

The lowest frequency which allows of the propagation of periodic waves along the cylinder is given by

$$\frac{p^2}{\mathrm{V}^2}=\frac{\pi^2}{{}^2}+\frac{\pi^2}{\beta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If the actual frequency of a vibration having its origin at any part of the cylinder be much less than the above, the resulting disturbance is practically limited to a neighbouring finite length of the cylinder.

For vibrations of the second class we have

$$c=e^{i(mz+pt)}\cos(\mu\pi x/\alpha)\cos(\nu\pi y/\beta), \quad . \quad . \quad . \quad (27)$$

the remaining functions being at once deducible by means of (19), (20). The satisfaction of (22) requires that here again

* 'Theory of Sound,' § 195.

μ, ν be integers, and (21) gives

$$k^2=\pi^2\left(\frac{\mu^2}{\alpha^2}+\frac{\nu^2}{\beta^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

identical with (24).

If $\alpha>\beta$, the smallest value of k corresponds to $\mu=1$, $\nu=0$. When $\nu=0$, we have $k=\mu\pi/\alpha$, and if the factor $e^{i(mz+pt)}$ be omitted,

$$a=-\frac{im}{k}\sin kx, \quad b=0, \quad c=\cos kx, \quad . \quad . \quad (29)$$

$$\mathrm{P}=0, \quad \mathrm{Q}=-\frac{ip}{k}\sin kx, \quad \mathrm{R}=0\,; \quad . \quad . \quad . \quad (30)$$

a solution independent of the value of β. There is no solution derivable from $\mu=0$, $\nu=0$, $k=0$*.

Circular Section.

For the vibrations of the first class we have as the solution of (2) by means of Bessel's functions,

$$\mathrm{R}=\mathrm{J}_n(kr)\cos n\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

n being an integer, and the factor $e^{i(mz+pt)}$ being dropped for the sake of brevity. In (31) an arbitrary multiplier and an arbitrary addition to θ are of course admissible. The value of k is limited to be one of those for which

$$\mathrm{J}_n(kr')=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

at the boundary where $r=r'$.

The expressions for P, Q, a, b, c in (15), (16) involve only $d\mathrm{R}/dx$, $d\mathrm{R}/dy$. For these we have

$$\frac{d\mathrm{R}}{dx}=\frac{d\mathrm{R}}{dr}\cos\theta-\frac{d\mathrm{R}}{r\,d\theta}\sin\theta=k\,\mathrm{J}_n'(kr)\cos n\theta\cos\theta$$

$$+\frac{n}{r}\mathrm{J}_n(kr)\sin n\theta\sin\theta$$

$$=\tfrac{1}{2}k\cos(n-1)\theta\left\{\mathrm{J}_n'+\frac{\mathrm{J}_n}{kr}\right\}+\tfrac{1}{2}k\cos(n+1)\theta\left\{\mathrm{J}_n'-\frac{\mathrm{J}_n}{kr}\right\}$$

$$=\tfrac{1}{2}k\cos(n-1)\theta\,\mathrm{J}_{n-1}(kr)-\tfrac{1}{2}k\cos(n+1)\theta\,\mathrm{J}_{n+1}(kr), \quad . \quad (33)$$

according to known properties of these functions; and in

* For (18) would then become $\nabla^2\psi=0$; and this, with the boundary condition $d\psi/dn=0$, would require that P and Q, as well as R, vanish throughout.

like manner

$$\frac{d\mathrm{R}}{dy}=\frac{d\mathrm{R}}{dr}\sin\theta+\frac{d\mathrm{R}}{r\,d\theta}\cos\theta=-\tfrac{1}{2}k\sin(n-1)\theta\,\mathrm{J}_{n-1}(kr)$$
$$-\tfrac{1}{2}k\sin(n+1)\theta\,\mathrm{J}_{n+1}(kr). \quad . \quad (34)$$

These forms show directly that $d\mathrm{R}/dx$, $d\mathrm{R}/dy$ satisfy the fundamental equation (2). They apply when n is equal to unity or any greater integer. When $n=0$, we have

$$\mathrm{R}=\mathrm{J}_0(kr), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\frac{d\mathrm{R}}{dx}=-k\,\mathrm{J}_1(kr)\cos\theta, \quad \frac{d\mathrm{R}}{dy}=-k\,\mathrm{J}_1(kr)\sin\theta. \quad . \quad (36)$$

The expressions for the electromotive intensity are somewhat simpler when the resolution is circumferential and radial:

$$\text{circumf. component } =\mathrm{Q}\cos\theta-\mathrm{P}\sin\theta=\frac{im}{k^2}\frac{d\mathrm{R}}{r\,d\theta}$$
$$=-\frac{imn}{k^2r}\mathrm{J}_n(kr)\sin n\theta, \quad . \quad . \quad (37)$$

$$\text{radial component } = \mathrm{P}\cos\theta+\mathrm{Q}\sin\theta=\frac{im}{k^2}\frac{d\mathrm{R}}{dr}$$
$$=\frac{im}{k}\mathrm{J}_n'(kr)\cos n\theta. \quad . \quad . \quad . \quad (38)$$

If $n=0$, the circumferential component vanishes.

Also for the magnetization

$$\text{circ. comp. of mag. } =b\cos\theta-a\sin\theta=\frac{m^2+k^2}{ipk^2}\frac{d\mathrm{R}}{dr}$$
$$=\frac{m^2+k^2}{ipk}\mathrm{J}_n'(kr)\cos n\theta, \quad . \quad . \quad (39)$$

$$\text{rad. comp. of mag. } =a\cos\theta+b\sin\theta=-\frac{m^2+k^2}{ipk^2}\frac{d\mathrm{R}}{r\,d\theta}$$
$$=\frac{n(m^2+k^2)}{ipk^2r}\mathrm{J}_n(kr)\sin n\theta. \quad . \quad . \quad (40)$$

The smallest value of k for vibrations of this class belongs to the series $n=0$, and is such that $kr=2{\cdot}404$, r being the radius of the cylinder.

For the vibrations of the second class $\mathrm{R}=0$, and by (21),

$$c=\mathrm{J}_n(kr)\cos n\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

k being subject to the boundary condition

$$\mathrm{J}_n'(kr')=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

As in (33), (34),

$$\frac{dc}{dx}=\frac{dc}{dr}\cos\theta-\frac{dc}{rd\theta}\sin\theta=\tfrac{1}{2}k\cos(n-1)\theta\,J_{n-1}(kr)$$
$$-\tfrac{1}{2}k\cos(n+1)\theta\,J_{n+1}(kr), \quad . \quad . \quad (43)$$

$$\frac{dc}{dy}=\frac{dc}{dr}\sin\theta+\frac{dc}{rd\theta}\cos\theta=-\tfrac{1}{2}k\sin(n-1)\theta\,J_{n-1}(kr)$$
$$-\tfrac{1}{2}k\sin(n+1)\theta\,J_{n+1}(kr), \quad . \quad . \quad (44)$$

so that by (19), (20) all the functions are readily expressed.

When $n=0$, we have

$$\frac{dc}{dx}=-kJ_1(kr)\cos\theta, \quad \frac{dc}{dy}=-kJ_1(kr)\sin\theta. \quad . \quad . \quad (45)$$

For the circumferential and radial components of magnetization we get

$$\text{circ. comp. of mag.} = b\cos\theta-a\sin\theta=\frac{im}{k^2}\frac{dc}{rd\theta}$$
$$=-\frac{imn}{k^2r}J_n(kr)\sin n\theta, \quad . \quad . \quad (46)$$

$$\text{rad. comp. of mag.} = a\cos\theta+b\sin\theta=\frac{im}{k^2}\frac{dc}{dr}$$
$$=\frac{im}{k}J_n'(kr)\cos n\theta, \quad . \quad . \quad (47)$$

corresponding to (37), (38) for vibrations of the first class.

In like manner equations analogous to (39), (40) now give the components of electromotive intensity. Thus

$$\text{circ. comp.} = Q\cos\theta-P\sin\theta=\frac{ip}{k^2}\frac{dc}{dr}=\frac{ip}{k}J_n'(kr)\cos n\theta, \quad (48)$$

$$\text{rad. comp.} = P\cos\theta+Q\sin\theta=-\frac{ip}{k^2}\frac{dc}{rd\theta}=\frac{ipn}{k^2r}J_n(kr)\sin n\theta. \quad . \quad . \quad . \quad . \quad (49)$$

The smallest value of k admissible for vibrations of the second class is of the series belonging to $n=1$, and is such that $kr'=1{\cdot}841$, a smaller value than is admissible for any vibration of the first class. Accordingly no real wave of any kind can be propagated along the cylinder for which p/V is less than $1{\cdot}841/r'$, where r' denotes the radius. The transition case is the two-dimensional vibration for which

$$c=e^{ipt}J_1(1{\cdot}841\,r/r')\cos\theta, \quad . \quad . \quad . \quad . \quad (50)$$

$$p=1{\cdot}841\,V/r'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$