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XLIV. On the vibrations of a loaded spiral spring

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mercial" copper and that which is sold by chemical supply houses under the label "chemically pure."

From the tables it will be seen that the plates are practically clear except for the impurity lines, which are very weak, many of them not showing on a silver print. In any case a table of the impurity lines and "ghosts" might accompany each map. A few years hence, when the spectra of the metals are more completely measured, such a table will be easily made.

North-Western University,
Evanston, Illinois, U.S.A.
July, 1894.

XLIV. *On the Vibrations of a Loaded Spiral Spring.* By
L. R. WILBERFORCE, M.A., *Demonstrator in Physics at the
Cavendish Laboratory, Cambridge*.*.

IT has been pointed out by Profs. Ayrton and Perry † that, by comparing the axial elongation and the twisting produced in a spiral spring of finite angle by the action of an axial force, we can deduce the ratio of the torsional and flexural rigidities of the wire or strip of which the spring is made, and hence obtain the ratio of the rigidity to the Young's modulus of its material.

This method is very interesting and instructive; but as it is not easy to produce springs of convenient and yet sufficiently uniform angles, nor to determine accurately a small axial elongation, it seemed to me that it might be worth while to modify it by attaching a mass to the spring and observing the periods of the vibrations which it executes when displaced. In this case it will be found convenient to use a spring of an angle so small that its square may be neglected.

Apart from their use in comparing moduli of elasticity, the vibrations of such a system present some rather interesting features, of which a detailed consideration may not be out of place.

If we have a spiral spring made of a length l of wire, and wound on a cylinder of radius r , so that the distance between the ends of the spring is x , and if ϕ is the angle between the planes through the axis of the spiral and the two ends of the wire, the force and couple required to produce a deformation from the equilibrium state (x_0, ϕ_0) to the state

* Communicated by the Author.

† Proc. Roy. Soc. vol. xxxvi. p. 311.

(x, ϕ) are respectively given by*

$$F_{x, \phi} = -\frac{B}{l^3} \left\{ \sqrt{l^2 - x^2} \phi - \sqrt{l^2 - x_0^2} \phi_0 \right\} \frac{x\phi}{\sqrt{l^2 - x^2}} + \frac{A}{l^3} (x\phi - x_0\phi_0) \phi,$$

$$C_{x, \phi} = \frac{B}{l^3} \left\{ \sqrt{l^2 - x^2} \phi - \sqrt{l^2 - x_0^2} \phi_0 \right\} \sqrt{l^2 - x^2} + \frac{A}{l^3} (x\phi - x_0\phi_0) x,$$

where A is the torsional and B the flexural rigidity of the wire.

If the spring is hung up in a vertical position with its upper end rigidly fixed, and a mass M attached to its lower end so as to be symmetrical about its axis, then, neglecting the mass of the spring, we have

$$F_{x, \phi} = Mg, \quad C_{x, \phi} = 0.$$

Now let the spring be displaced to the configuration $(x + \delta x, \phi + \delta \phi)$ and released, then the equations of motion are

$$M \frac{d^2}{dt^2} \delta x = Mg - F_{x+\delta x, \phi+\delta \phi},$$

$$M k^2 \frac{d^2}{dt^2} \delta \phi = -C_{x+\delta x, \phi+\delta \phi},$$

where Mk^2 is the moment of inertia of M about the axis of the spring; that is,

$$-M \frac{d^2}{dt^2} \delta x = \delta x \frac{d}{dx} F_{x, \phi} + \delta \phi \frac{d}{d\phi} F_{x, \phi},$$

$$-Mk^2 \frac{d^2}{dt^2} \delta \phi = \delta x \frac{d}{dx} C_{x, \phi} + \delta \phi \frac{d}{d\phi} C_{x, \phi};$$

or

$$\begin{aligned} -M \frac{d^2}{dt^2} \delta x = & \delta x \left[-\frac{B}{l^3} \phi^2 + \frac{B}{l^3} \phi \phi_0 \sqrt{l^2 - x_0^2} \left\{ \frac{1}{\sqrt{l^2 - x^2}} + \frac{x^2}{(l^2 - x^2)^{\frac{3}{2}}} \right\} + \frac{A}{l^3} \phi^2 \right] \\ & + \delta \phi \left[-\frac{B}{l^3} \left\{ 2\sqrt{l^2 - x^2} \phi - \sqrt{l^2 - x_0^2} \phi_0 \right\} \frac{x}{\sqrt{l^2 - x^2}} + \frac{A}{l^3} (2x\phi - x_0\phi_0) \right] \\ -Mk^2 \frac{d^2}{dt^2} \delta \phi = & \delta x \left[\frac{B}{l^3} \left\{ -2x\phi + x \frac{\sqrt{l^2 - x_0^2}}{\sqrt{l^2 - x^2}} \phi_0 \right\} + \frac{A}{l^3} (2x\phi - x_0\phi_0) \right] \\ & + \delta \phi \left[\frac{B}{l^3} (l^2 - x^2) + \frac{A}{l^3} x^2 \right]. \end{aligned}$$

If the angles of the unstretched and of the stretched spring are both so small that we can neglect $\frac{x^2}{l^2}$, it follows from the

* Thomson and Tait's 'Natural Philosophy,' vol. i. part ii. p. 141.

equation $C_{x,\phi}=0$ that we can neglect $\frac{\phi-\phi_0}{\phi}$, and then the above equations reduce to

$$\begin{aligned} -M \frac{d^2}{dt^2} \delta x &= \delta x \cdot \frac{A\phi^2}{l^3} + \delta\phi \left[\frac{(2A-B)x\phi}{l^3} - \frac{Ax_0\phi}{l^3} \right], \\ -Mk^2 \frac{d^2}{dt^2} \delta\phi &= \delta x \left[\frac{(2A-B)x\phi}{l^3} - \frac{Ax_0\phi}{l^3} \right] + \delta\phi \cdot \frac{B}{l}; \end{aligned}$$

which may be written

$$\left. \begin{aligned} -\frac{d^2}{dt^2} \delta x &= a\delta x + b\delta k\phi, \\ -\frac{d^2}{dt^2} \delta k\phi &= b\delta x + c\delta k\phi, \end{aligned} \right\}$$

where a and c are necessarily positive, and b is small.

The most general form of solution is

$$\delta x = A_1 \sin pt + A_2 \cos pt + B_1 \sin qt + B_2 \cos qt,$$

$$\delta k\phi = \frac{p^2-a}{b}(A_1 \sin pt + A_2 \cos pt) + \frac{q^2-a}{b}(B_1 \sin qt + B_2 \cos qt),$$

where p^2 and q^2 are the roots of the quadratic $(x-a)(x-c)=b^2$. We will suppose p^2 to be the greater. The solution may be put into the form

$$\frac{p^2-a}{b} \delta x - \delta k\phi = L_1 \sin (qt + \epsilon_1),$$

$$\frac{q^2-a}{b} \delta x - \delta k\phi = L_2 \sin (pt + \epsilon_2).$$

Thus the motion consists of two normal harmonic vibrations of periods $\frac{2\pi}{p}$ and $\frac{2\pi}{q}$; in the former $\frac{p^2-a}{b} \delta x = \delta k\phi$ throughout the motion, and in the latter $\frac{q^2-a}{b} \delta x = \delta k\phi$.

It is easily seen that p^2-a is positive and q^2-a negative; therefore we conclude that, if b is a positive quantity, the shorter period of vibration corresponds to a screwing motion similar to the screw of the spring, and the longer to a screwing motion opposite to the screw of the spring, while if b is negative the reverse is the case.

It is also clear that if $(c-a)$ is large compared with b and is positive $\left. \begin{array}{l} \text{is positive} \\ \text{negative} \end{array} \right\}$, $\frac{p^2-a}{b}$ is $\left. \begin{array}{l} \text{large} \\ \text{small} \end{array} \right\}$, and $\frac{q^2-a}{b}$ is $\left. \begin{array}{l} \text{small} \\ \text{large} \end{array} \right\}$; thus in the former $\left. \begin{array}{l} \text{former} \\ \text{latter} \end{array} \right\}$ case the vibrations of shorter $\left. \begin{array}{l} \text{shorter} \\ \text{longer} \end{array} \right\}$ period cor-

respond to $\delta x = 0$, and those of $\left. \begin{array}{l} \text{longer} \\ \text{shorter} \end{array} \right\}$ period to $\delta \phi = 0$, approximately.

If, however, $\frac{c \sim a}{b}$ is finite and equal to 2λ , we have, when the system is vibrating in one of its normal modes, either

$$(\lambda + \sqrt{\lambda^2 + 1})\delta x = \delta k\phi,$$

or

$$(\lambda - \sqrt{\lambda^2 + 1})\delta x = \delta k\phi$$

throughout the motion, the periods corresponding to these modes being nearly equal. In this case, if the system receives a displacement not represented by either of these equations, the subsequent motion will be compounded of two vibrations, one of which slowly gains upon the other, and will thus exhibit phenomena of intermittence.

For example, if the displacement ($\delta x = X$, $\delta \phi = 0$) be given, this may be resolved into

$$\delta_1 x = X \frac{\sqrt{\lambda^2 + 1} - \lambda}{2\sqrt{\lambda^2 + 1}}, \quad \delta_1 k\phi = X \frac{1}{2\sqrt{\lambda^2 + 1}},$$

and

$$\delta_2 x = X \frac{\sqrt{\lambda^2 + 1} + \lambda}{2\sqrt{\lambda^2 + 1}}, \quad \delta_2 k\phi = -X \frac{1}{2\sqrt{\lambda^2 + 1}};$$

and therefore, when the vibrations of one normal mode have gained half a period on those of the other, the half-amplitude of the x -vibration will have decreased from X to $X \frac{\lambda}{\sqrt{\lambda^2 + 1}}$

and a $k\phi$ -vibration of half-amplitude $X \frac{1}{\sqrt{\lambda^2 + 1}}$ will have

appeared, while when another half-period is gained the initial conditions will be restored. Thus, while at first the system moves simply with an x -vibration, this gradually diminishes to a minimum value, and at the same time a ϕ -vibration is gradually set up and grows to a maximum; the latter vibration then decreases and finally vanishes, while the former increases until it reaches its initial value, and then the phenomena recur.

It is easy to see that a similar intermittence will be exhibited if the system is started with a ϕ -vibration only.

The above results may readily be verified experimentally by employing as the mass M a body of adjustable moment of inertia. The most interesting case is that in which k is adjusted so that λ is rendered very small, when the energy

is seen to be transferred with almost perfect completeness from x -vibrations to ϕ -vibrations and back again.

The condition for the vanishing of λ is of course

$$\frac{A\phi^2}{l^3} = \frac{B}{lk^2};$$

or, since to our order of approximation $l = r\phi$,

$$k^2 = \frac{B}{A} r^2.$$

It is also of interest to produce, by means of a series of suitably timed small impulses, the normal modes of vibration of such a system and to demonstrate the permanence of each.

If, however, the object in view is the determination of elastic constants, it is convenient to arrange that $(c-a)$ shall be large compared with b . In this case, as we have seen, pure x -vibrations and pure ϕ -vibrations are practically the two normal modes, and the periodic times of the former and the latter are given by

$$t_1 = \frac{2\pi}{\sqrt{a}} = 2\pi \sqrt{\frac{Mlr^2}{A}},$$

$$t_2 = \frac{2\pi}{\sqrt{c}} = 2\pi \sqrt{\frac{Mlk^2}{B}}.$$

If the mass m of the spring itself cannot be neglected, we can allow for it, if small compared with M , by taking $M + \frac{1}{3}m$ as the vibrating mass, and $Mk^2 + \frac{1}{3}mr^2$ as its moment of inertia*.

Let us consider the case of a spring made of circular wire of radius ρ . If we may assume the material to be homogeneous and isotropic, an assumption which is undoubtedly a weak point of all methods of determining the elastic constants of a material by experiments on wires, we have

$$A = \frac{\pi}{2} n\rho^4, \quad B = \frac{\pi}{4} E\rho^4,$$

where E and n are respectively the Young's modulus and the rigidity of the material. From the above we obtain

$$\frac{E}{n} = \frac{2B}{A} = 2 \times \frac{Mk^2 + \frac{1}{3}mr^2}{Mr^2 + \frac{1}{3}mr^2} \cdot \frac{t_1^2}{t_2^2},$$

an equation involving only quantities easy of measurement, and hence Poisson's ratio, which is equal to $\frac{1}{2}\left(\frac{E}{n} - 2\right)$, is determined.

* Lord Rayleigh's 'Theory of Sound,' § 156.

In addition, if the values of l and ρ^4 are obtained, E and n can be separately calculated.

Some observations were taken upon different lengths of one of Salter's steel springs ($r=1.494$ cm.), using as the vibrating body one whose moment of inertia could be varied by known amounts from an arbitrary value K by moving two equal masses in and out along a bar.

The following is a specimen of the numbers obtained, in C.G.S. measure :—

Exp. 1.— $\phi=300\pi$, $l=1408$, $x=78$, $m=130.5$, $M=267$.

Moment of Inertia.		t_1 .	t_2 .
$K + \frac{1}{3}mr^2 + 586$. . .	1.473	1.888
„ + 1700	. . .	1.477	2.505
„ + 3400	. . .	1.474	3.228
„ + 5680	. . .	1.475	3.986

Whence

$$\frac{B}{4\pi^2 l} = 413, 410, 416; \text{ mean } 413.$$

$$\frac{A}{4\pi^2 l} = 319, 318, 318, 318; \text{ mean } 318.$$

And as a verification we can deduce

$$K + \frac{1}{3}mr^2 = 886, 892, 903, 884; \text{ mean } 891,$$

which gives for K the value 794.

The results of the experiments are exhibited in the following table :—

ϕ .	l .	M .	m .	A .	B .	K .
300π	1408	267	130.5	1.77×10^7	2.30×10^7	794
200π	939	267	87	1.78×10^7	2.29×10^7	785
100π	469	267	43.5	1.77×10^7	2.28×10^7	791
300π	1408	533	130.5	1.79×10^7	2.28×10^7	

The last experiment of the above series was made by attaching an additional mass to the vibrating body, so that x was increased to 125, and of course K was changed.

Thus we see that the method furnishes consistent results and we deduce for this specimen of steel,

$$\frac{E}{n} = \frac{2B}{A} = 2.57.$$

Also, since for our wire ρ is about .0617 centim., we obtain as approximate values,

$$E = 2.00 \times 10^{12},$$

$$n = 7.79 \times 10^{11}.$$

An experiment with a spring of hard-drawn copper wire gave

$$\frac{E}{n} = 2.76,$$

with the approximate values

$$E = 1.13 \times 10^{12},$$

$$n = 4.10 \times 10^{11}.$$

XLV. On the Velocities of the Ions and the Relative Ionization-Power of Solvents. By W. C. DAMPIER WHETHAM, M.A., Fellow of Trinity College, Cambridge*.

FROM a knowledge of the electrical conductivity and migration-constant of a solution, Prof. F. Kohlrausch has shown us how to calculate the velocity with which its ions must travel in order that, in accordance with Faraday's law, a given current should be carried (*Wied. Ann.* xxvi.).

Prof. O. Lodge experimentally determined the velocity of the hydrogen ion as it travelled through a jelly solution of sodium chloride and so formed hydrochloric acid, the presence of which was indicated by the decolorization of phenolphthalein. When the ion was driven by a potential gradient of one volt per centimetre the speed came out 0.0029 centimetre per second, a number agreeing in a most remarkable manner with Kohlrausch's theoretical value 0.0030 for a decinormal solution (*B.A. Report*, 1886).

The author of this paper has observed the specific ionic velocity of other ions, such as copper and the bichromic-acid group (Cr_2O_7), by tracing the motion of the junction of two salt-solutions (one of which is of different colour from the other) under the influence of an electric current (*Trans. Roy. Soc.* 1893 A.). The results agree with Kohlrausch's numbers even in the case of alcoholic solutions, the conductivities of which are much less than those of the corresponding aqueous solutions.

Certain substances, *e. g.* ammonia and acetic acid, have been regarded as exceptions to the application of the theory. From a knowledge of the conductivity and migration-constants of acids such as nitric and hydrochloric, we can

* Communicated by the Author.