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# XVI. *On the Instability of a Cylinder of Viscous Liquid under Capillary Force.* By Lord Rayleigh, *Sec. R.S.**

THE main outline of the theory of the instability of a long cylinder of liquid is due to Plateau, who showed that if the equilibrium surface $r=a$ be slightly deformed so as to become

$$r=a+\alpha\cos kz, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $z$ is measured parallel to the axis, the deformation is stable or unstable according as $ka$ is greater or less than unity; that is, according as the wave-length $\lambda$ of the varicosity is less or greater than $2\pi a$, the circumference of the cylinder. The solution of the merely statical question is, however, insufficient for the application to the important problem of the disintegration of a jet of liquid. A deformation of any wave-length exceeding $2\pi a$ increases exponentially with the time $(e^{qt})$; and what we require to know is the relation between $q$ and $\lambda$. A value of $\lambda$, if any, for which $q$ is a maximum, determines the mode of maximum instability; and this in general tends more and more to be the actual mode of disintegration as the initial disturbances are reduced without limit.

As resisting the capillary force, Plateau seems to have had in view only viscosity. "Par suite des viscosités intérieure et superficielle, le rapport entre la longueur normale des divisions et le diamètre du cylindre surpasse toujours la limite de la stabilité. Il se réduirait sans doute à cette limite

* Communicated by the Author.

même si le liquide était exempt de toute viscosité"*. In the case of jets composed of liquids such as water, the influence of viscosity might be expected to be small, but it would be erroneous to conclude that the limit $ka=1$ would then correspond to the mode of maximum instability. *Inertia* would necessarily play a leading part, and from a complete investigation† it appears that the mode of maximum instability corresponds with $\lambda=4\cdot51\times2a$, exceeding very considerably the circumference of the cylinder.

The importance attached by Plateau to viscosity suggested an investigation in which this property should be included; and the results would at any rate find application to threads of materials like glass and sealing-wax, in which viscosity would predominate over inertia. Having in my mind some old experiments upon the behaviour of fine threads of treacle deposited upon paper, which slowly resolve themselves into drops having a very similar appearance to those obtained from a jet of water, I rather expected to find that under the influence of viscosity alone the mode of resolution would be nearly the same as under the influence of inertia alone. This anticipation proved to be wide of the mark, the result showing that under viscosity alone the value of $\lambda$ for maximum instability would be very great. And a little consideration shows that the retarding forces exerted by the paper support may well be of quite a different character from those due to mere fluid viscosity. In the latter case the gathering together from considerable distances is but little resisted, the motion not differing greatly from that of a solid body, whereas such a mode of resolution would be greatly impeded by the contact with paper. In order better to represent such contact forces, I have considered the problem in the form which it assumes when the resistances are proportional to the absolute velocities of the parts. This admits of easy solution, and the result illustrates the behaviour of the thread of treacle in contact with paper, and shows that there is a marked difference between this case and that of a thread whose disintegration is resisted by true fluid viscosity.

The introduction of resistances proportional to absolute velocities does not interfere with the irrotational character of the motion of otherwise frictionless fluid‡. The radial and axial velocities $u$, $w$ may thus, as usual, be regarded as derived from a velocity-potential according to the equation

* *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires*, 1873, vol. ii. p. 231.

† Proc. Math. Soc. November 1878. See also below.

‡ 'Theory of Sound,' vol. ii. § 239 (1878).

$$u = d\phi/dr, \quad w = d\phi/dz. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If the resistance is $\mu'$ times the velocity, the general equation of pressure, viz.

$$p/\rho = \mathrm{R} - d\phi/dt - \tfrac{1}{2}\mathrm{U}^2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

becomes for the present purpose, where $\mathrm{U}^2$ may be neglected,

$$p = -\mu'\phi - \rho\, d\phi/dt. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The quantities defining the motion are as functions of $z$ proportional to $e^{ikz}$, and as functions of $t$ proportional to $e^{int}$, where $k$ is real, but $n$ may be complex. The general equation for the velocity-potential of an incompressible fluid, viz. $\nabla^2\phi = 0$, thus becomes

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - k^2\phi = 0,$$

of which the solution, subject to the conditions to be imposed when $r=0$, is

$$\phi = \mathrm{A}\,\mathrm{J}_0(ikr),$$

or rather

$$\phi = \mathrm{A}\, e^{i(nt+kz)}\, \mathrm{J}_0(ikr). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

At the same time $p$ is given by

$$p = -(\mu' + in\rho)\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

We have now to consider the boundary condition, applicable when $r=a$. The displacement $\xi$ at the surface is connected with $\phi$ by the equation

$$\xi = \int u\, dt = \int \frac{d\phi}{dr}\, dt = \frac{1}{in}\frac{d\phi}{dr}. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The variable part of the pressure is due to the tension T, which is supposed to be constant, as is practically the case in the absence of surface-contamination. The curvature in the plane of the axis is $-d^2\xi/dz^2$, or $k^2\xi$. The curvature in the perpendicular direction is $(a+\xi)^{-1}$, or $1/a - \xi/a^2$. Thus

$$p = \frac{\mathrm{T}\xi(k^2a^2 - 1)}{a^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and the boundary condition is

$$\frac{\mathrm{T}(k^2a^2-1)}{ina^2}\frac{d\phi}{dr} = -(\mu' + in\rho)\phi;$$

or by (5),

$$\frac{\mathrm{T}}{\rho a^3}\frac{(k^2a^2-1)\,ika\,.\,\mathrm{J}_0'(ika)}{\mathrm{J}_0(ika)} + in(in + \mu'/\rho) = 0, \quad . \qquad (9)$$

a quadratic equation by which $n$ is determined.

If $\mu'=0$,

$$(in)^2=\frac{T}{\rho a^3}\frac{(1-k^2a^2)\,ika\,J_0'}{J_0}, \quad . \quad . \quad . \quad . \quad (10)$$

as found in the former paper. In this expression $ika\,J_0'/J_0$ is a real positive quantity for all (real) values of $ka$; so that the displacement is exponentially unstable if $ka<1$, and periodic if $ka>1$, as was to be expected. In the former case the values of $in$ are numerically greatest when $ka=\pi/4{\cdot}5$.

In the other extreme case where inertia may be neglected in comparison with viscosity, we have

$$in=\frac{T}{\rho a^3}\frac{(1-k^2a^2)\,J_0'}{\mu'/\rho\,.\,J_0}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so that the instability is greatest when $ka$ has the same value as in the first case.

The general form of the quadratic is

$$(in)^2+in\,.\,\mu'/\rho+H(k^2a^2-1)=0, \quad . \quad . \quad . \quad (12)$$

where H is positive.

If $ka<1$, both values of $in$ are real, one being positive and the other negative. The displacement is accordingly unstable, and the greatest instability occurs with the above-defined value of $ka$. If, on the other hand, $ka>1$, the values of $in$ may be either real or imaginary. In the former case both values are negative, and in the latter the real parts are negative, so that the deformations are stable.

The investigation applicable to a real viscous liquid of viscosity $\mu$, or $\rho\nu$, is much more complicated than the foregoing, mainly in consequence of the non-existence of a velocity-potential. But inasmuch as the motion is still supposed to be symmetrical about the axis, the equation of continuity gives

$$u=\frac{1}{r}\frac{d\psi}{dz}, \quad w=-\frac{1}{r}\frac{d\psi}{dr}, \quad . \quad . \quad . \quad . \quad (13)$$

where $\psi$ is Stokes's current function. For small motions $\psi$ satisfies the equation*

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{d^2}{dz^2}-\frac{1}{\nu}\frac{d}{dt}\right)\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{d^2}{dz^2}\right)\psi=0. \quad . \quad (14)$$

In the present question $\psi$ as a function of $z$ and $t$ is proportional to $e^{i(nt+kz)}$, and it may be separated into two parts, $\psi_1$

* Camb. Trans. 1850. See also Basset's 'Hydrodynamics,' vol. ii. p. 259.

and $\psi_2$, of which $\psi_1$ satisfies

$$\frac{d^2\psi_1}{dr^2}-\frac{1}{r}\frac{d\psi_1}{dr}-k^2\psi_1=r\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi_1}{dr}\right)-k^2\psi_1=0, \quad . \quad (15)$$

and $\psi_2$ satisfies

$$\frac{d^2\psi_2}{dr^2}-\frac{1}{r}\frac{d\psi_2}{dr}-k'^2\psi_2=r\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi_2}{dr}\right)-k'^2\psi_2=0, \quad . \quad (16)$$

where

$$k'^2=k^2+in/\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

At the surface we have to consider the normal stress P, and the tangential stresses. Of the latter one vanishes in virtue of the symmetry, and the other is to be made to vanish in conformity with the condition that there is to be no impressed tangential force*. Thus

$$\frac{du}{dz}+\frac{dw}{dr}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

or in terms of $\psi$ by (13)

$$\frac{d^2\psi}{dr^2}-\frac{1}{r}\frac{d\psi}{dr}+k^2\psi=0. \quad . \quad . \quad . \quad . \quad (19)$$

Introducing $\psi_1$, $\psi_2$, and having regard to (15), (16), we may express this condition in the form

$$2k^2\psi_1+(k'^2+k^2)\psi_2=0, \quad . \quad . \quad . \quad . \quad (20)$$

which is to be satisfied when $r=a$.

Again, for the normal stress,

$$\begin{aligned} \mathrm{P} &= -p+2\mu\, du/dr \\ &= \rho\left(\frac{u}{k}w-\frac{\nu}{ik}\nabla^2 w\right)+2\mu\frac{du}{dr} \\ &= \mu\left\{\frac{1}{ik}\nabla^2\left(\frac{1}{r}\frac{d\psi}{dr}\right)+2ik\frac{d}{dr}\left(\frac{\psi}{r}\right)\right\}-\frac{n\rho}{kr}\frac{d\psi}{dr}. \quad . \quad (21) \end{aligned}$$

Herein

$$\nabla^2\left(\frac{1}{r}\frac{d\psi}{dr}\right)=\frac{1}{r}\frac{d}{dr}\,r\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{dr}\right)-\frac{k^2}{r}\frac{d\psi}{dr}.$$

For $\psi_1$,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\frac{1}{r}\frac{d\psi_1}{dr}\right)-\frac{k^2}{r}\frac{d\psi_1}{dr}=0,$$

* It is here assumed that there is no "superficial viscosity."

for $\psi_2$,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\frac{1}{r}\frac{d\psi_2}{dr}\right)-\frac{k^2}{r}\frac{d\psi_2}{dr}=\frac{k'^2-k^2}{r}\frac{d\psi_2}{dr};$$

so that

$$\mathrm{P}=\mu\left\{\frac{k'^2-k^2}{ikr}\frac{d\psi_2}{dr}+2ik\frac{d}{dr}\frac{\psi_1+\psi_2}{r}\right\}-\frac{np}{kr}\frac{d(\psi_1+\psi_2)}{dr}. \quad .\,(22)$$

The variable part of the capillary pressure is, as we have already seen,

$$\frac{\mathrm{T}\xi(k^2a^2-1)}{a^2},$$

in which

$$\xi=\int u\,dt=\frac{k\psi}{na},$$

Thus, the condition to be satisfied when $r=a$ is

$$\frac{\mathrm{T}(1-k^2a^2)}{a^2}\frac{k\psi}{na}=\mu\left\{\frac{k'^2-k^2}{ika}\frac{d\psi_2}{dr}+2ik\frac{d}{dr}\frac{\psi}{r}\right\}-\frac{np}{ka}\frac{d\psi}{dr}. \quad (23)$$

The forms of $\psi_1, \psi_2$ are to be determined by the equations (15), (16), and by the conditions to be satisfied when $r=0$. It will be observed that $\psi_1$ satisfies the condition appropriate to the stream function when there is a velocity-potential. This would be of the form

$$\phi=e^{ikz}\,\mathrm{J}_0(ikr), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

so that

$$\psi_1=\int(ru)\,dz=\frac{r}{ik}\frac{d\phi}{dr}=re^{ikz}\,\mathrm{J}_0'(ikr).$$

Thus

$$\psi_1=\mathrm{A}\,r\,\mathrm{J}_0'(ikr) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

is the most general form admissible, as may be verified by differentiation. In this $\mathrm{J}_0(ikr)$ satisfies the equation

$$\mathrm{J}_0''(ikr)+\frac{1}{ikr}\mathrm{J}_0'(ikr)+\mathrm{J}_0(ikr)=0. \quad . \quad . \quad . \quad (26)$$

Since (16) differ from (15) only by the substitution of $k'$ for $k$, the general form for $\psi_2$ is

$$\psi_2=\mathrm{B}\,r\,\mathrm{J}_0'(ik'r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

By use of these values the first boundary condition (20) becomes

$$2k^2\mathrm{A}\,\mathrm{J}_0'(ika)+(k'^2+k^2)\mathrm{B}\,\mathrm{J}_0'(ik'a)=0. \quad . \quad . \quad (28)$$

We have next to introduce the same values into the second boundary condition (23). In this

$$\frac{d\psi_1}{dr}=ikr\mathrm{A}\left[\mathrm{J}_0''(ikr)+\frac{1}{ikr}\mathrm{J}_0'(ikr)\right]=-\mathrm{A}\,ika\,\mathrm{J}_0(ika)$$

by (26). In like manner,

$$\frac{d\psi_2}{dr} = -\mathrm{B}\, ik'a\, \mathrm{J}_0(ik'a).$$

Thus

$$\frac{\mathrm{T}(1-k^2a^2)}{\rho a^3}\frac{ka}{n}[\mathrm{A}\,\mathrm{J}_0'(ika)+\mathrm{B}\,\mathrm{J}_0'(ik'a)]$$

$$= -\nu\Big[\mathrm{B}\frac{k'(k'^2-k^2)}{k}\mathrm{J}_0(ik'a) + 2k^2\mathrm{A}\,\mathrm{J}_0''(ika) + 2kk'\mathrm{B}\,\mathrm{J}_0''(ik'a)\Big]$$

$$+ \frac{n}{ka}[\mathrm{A}\, ika\, \mathrm{J}_0(ika) + \mathrm{B}\, ik'a\,\mathrm{J}_0(ik'a)]. \quad . \quad . \quad . \quad (29)$$

Between (28) and (29) we now eliminate the ratio A/B, and thus obtain as the equation by which [in conjunction with (17)] the value of $n$ is to be determined

$$\frac{\mathrm{T}(1-k^2a^2)}{\rho a^3}\frac{ka}{n}\frac{k'^2-k^2}{k'^2+k^2}\mathrm{J}_0'(ika)$$

$$= -2k^2\nu\Big\{\mathrm{J}_0''(ika) - \frac{2kk'}{k'^2+k^2}\frac{\mathrm{J}_0'(ika)}{\mathrm{J}_0'(ik'a)}\mathrm{J}_0''(ik'a)$$

$$- \frac{k'(k'^2-k^2)}{k(k'^2+k^2)}\frac{\mathrm{J}_0'(ika)}{\mathrm{J}_0'(ik'a)}\mathrm{J}_0(ik'a)\Big\}$$

$$+ \frac{n}{ka}\Big\{ika\mathrm{J}_0(ika) - \frac{2k^2}{k'^2+k^2}\frac{\mathrm{J}_0'(ika)}{\mathrm{J}_0'(ik'a)}ik'a\,\mathrm{J}_0(ik'a)\Big\}. \quad (30)$$

We shall now apply this result to the particular case where the viscosity is very great in comparison with the inertia. The third part of (30) may then be omitted, and we have to seek the limiting form of the remainder when $k'$ is nearly equal to $k$, as we see must happen by (17). In the first part,

$$\frac{k'^2-k^2}{k'^2+k^2} = \frac{\delta k}{k}.$$

In the second,

$$\mathrm{J}_0''(ika) - \frac{2kk'}{k'^2+k^2}\frac{\mathrm{J}_0'(ika)}{\mathrm{J}_0'(ik'a)}\mathrm{J}_0''(ik'a) = \frac{ika\{\mathrm{J}_0''^2-\mathrm{J}_0'\mathrm{J}_0'''\}\delta k}{k\mathrm{J}_0'},$$

and

$$-\frac{k'(k'^2-k^2)}{k(k'^2+k^2)}\frac{\mathrm{J}_0'(ika)}{\mathrm{J}_0'(ik'a)}\mathrm{J}_0(ik'a) = -\frac{\mathrm{J}_0\,\delta k}{k}.$$

Thus the limiting form is

$$\frac{\mathrm{T}(1-k^2a^2)}{\mu a\,.\,n} = -\frac{2ka\,.\,ika}{\mathrm{J}_0'^2}\Big\{\mathrm{J}_0''^2 - \mathrm{J}_0'\mathrm{J}_0''' - \frac{\mathrm{J}_0\mathrm{J}_0'}{ika}\Big\},$$

in which, however, we may effect further simplifications by means of the properties of $J_0$. We find by use of (26)

$$J_0''^2 - J_0'J_0''' - \frac{J_0J_0'}{ika} = J_0^2 + J_0'^2\left(1 + \frac{1}{k^2a^2}\right),$$

so that, finally,

$$in = -\frac{T(1-k^2a^2)}{2\mu a \,.\, k^2a^2\{J_0^2/J_0'^2 + 1 + 1/k^2a^2\}}. \quad . \quad . \quad (31)$$

In (31) the argument of $J_0$, $J_0'$ is $ika$, or $z$ as we will call it for brevity. And by a known property $J_0' = -J_1$. Now

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \,.\, 4^2} - \ldots,$$

$$J_1(z) = \frac{z}{2}\left\{1 - \frac{z^2}{2 \,.\, 4} + \frac{z^4}{2 \,.\, 4^2 \,.\, 6} - \ldots\right\};$$

so that if $x = ka$

$$J_0(ix) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \,.\, 4^2} - \ldots,$$

$$J_1(ix) = \frac{ix}{2}\left\{1 + \frac{x^2}{2 \,.\, 4} + \frac{x^4}{2 \,.\, 4^2 \,.\, 6} + \ldots\right\}.$$

These functions have been tabulated by Prof. A. Lodge * under the notation $I_0(x)$, $I_1(x)$, where

$$I_0(x) = J_0(ix) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \,.\, 4^2} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$I_1(x) = -iJ_1(ix) = \frac{x}{2}\left\{1 + \frac{x^2}{2 \,.\, 4} + \frac{x^4}{2 \,.\, 4^2 \,.\, 6} + \ldots\right\}. \quad . \quad (33)$$

In this notation

$$x^2\{J_0^2(ix)/J_1^2(ix) + 1 + 1/x^2\} = x^2 + 1 - x^2I_0^2(x)/I_1^2(x), \quad . \quad (34)$$

and we have to consider the march of (34) as a function of $x$.

When $x$ is very small,

$$I_0(x) = 1 - \tfrac{1}{4}x^2, \quad I_1(x) = \tfrac{1}{2}x + \tfrac{1}{16}x^3,$$

so that

$$(34) = -3 + \text{terms in } x^4;$$

and then from (31)

$$in = \frac{T}{6\mu a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

* Brit. Ass. Report, 1889, p. 28.

We shall see that this corresponds to the maximum instability, and it occurs when the wave-length of the varicosity is very large in comparison with the diameter of the cylinder. The following table gives the values of (34) for specified values of $x$ :—

| $x$. | (34). | $x$. | (34). |
|---|---|---|---|
| 0·0 | −3·0000 | 1·0 | −3·0188 |
| 0·2 | −3·0000 | 2·0 | −3·2160 |
| 0·4 | −3·0004 | 4·0 | −4·458 |
| 0·6 | −3·0023 | 6·0 | −6·247 |

It will be seen that the numerical value of (34) is least when $x=0$, which is also the value of $x$ for which the numerator of (31) is greatest. On both accounts, therefore, $in$ is greatest when $x$ or $ka=0$. But over the whole range of the instability from $ka=0$ to $ka=1$, (34) differs but little from $-3$, so that we may take as approximately applicable

$$in=\frac{T(1-k^2a^2)}{6\mu a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The result of the investigation is to show that when viscosity is paramount long threads do not tend to divide themselves into drops at mutual distances comparable with the diameter of the cylinder, but rather to give way by attenuation at few and distant places. This is, I think, in agreement with the observed behaviour of highly viscous threads of glass or treacle when supported only at the terminals. A separation into numerous drops, or a varicosity pointing to such a resolution, may thus be taken as evidence that the fluidity has been sufficient to bring inertia into play.

The application of (31) to the case of stability ($ka>1$) is of less interest, but it may be worth while to refer to the extreme case where the wave-length of the varicosity is *very small* in comparison with the diameter. We then fall upon the particular case of a plane surface disturbed by waves of length $\lambda$. The result, applicable when the viscosity is so great that inertia may be left out of account, is the limit of (31) when $a$, or $x$, is infinite, while $k$ remains constant, or

$$in=\frac{Tk}{2\mu}\div \text{Lim}\, x\{J_0^2(ix)/J_1^2(ix)+1\}.$$

By means of the expressions appropriate when the argument is large, the limit in question may be proved to be $-1$; so that

$$in=-\frac{Tk}{2\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

If gravity be supposed operative in aid of the restoration of equilibrium, we should have to include in the boundary condition relative to pressure a term $g\rho\xi$ in addition to $Tk^2\xi$; so that the more general result is obtainable by adding $g\rho/k^2$ to T. Thus

$$in = -\frac{k}{2\mu}\left(T + \frac{g\rho}{k^2}\right), \quad . \quad . \quad . \quad . \quad (38)$$

giving the rate of subsidence of waves upon the surface of a highly viscous material. It could of course be more readily obtained directly.

When gravity operates alone,

$$in = -\frac{g\rho}{2\mu k} = -\frac{g}{2\nu k}, \ldots \quad . \quad . \quad . \quad (39)$$

which agrees with a conclusion of Prof. Darwin*. A like result may be obtained from equations given by Mr. Basset†.

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## XVII. *Rotating Elastic Solid Cylinders of Elliptic Section. By* C. Chree, *M.A., Fellow of King's College, Cambridge*‡.

### Part II.—*The Long Elliptic Cylinder.*

§ 35. By a *long* cylinder is here meant one whose length $2l$ bears to its greatest diameter $2a$ a ratio such as is required for the legitimate application of Saint-Venant's solution for beams. What this ratio may be depends on the degree of accuracy aimed at, but the best authorities seem satisfied with values of $l/a$ which are not markedly less than 10. The cylinder is supposed to be rotating uniformly, and to be free from all but "centrifugal" forces. In the paper in the Quarterly Journal, already referred to, I obtained a solution for a rotating elliptic cylinder, but its length was supposed to be maintained constant by the application of suitable forces over the ends. This is a totally different case from the present, in which the cylinder is supposed free from all surface forces and capable of altering alike in length and diameter. The present solution is thus completely new, except for the case of a circular section which I have already treated elsewhere §, and for the limiting value 0 of $\eta$ when the alteration

* Phil. Trans. 1879, p. 10. In equation (12) write $i/a = k$, and make $i = \infty$.

† Hydrodynamics, vol. ii. § 520, equations (21), (27). See also Tait, Edinb. Proc. 1890, p. 110.

‡ Communicated by the Author.

§ Cambridge Philosophical Society's Proceedings, vol. vii. part vi.