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LVII. *On the Dilatancy of Media composed of Rigid Particles in Contact. With Experimental Illustrations. By* Professor OSBORNE REYNOLDS, *LL.D., F.R.S.**

[Plate X.]

IDEAL rigid particles have been used in almost all attempts to build fundamental dynamical hypotheses of matter: these particles have generally been supposed smooth.

Actual media composed of approximately rigid particles exist in the shape of sand, shingle, grain, and piles of shot; all which media are influenced by friction between the particles.

The dynamical properties of media composed of ideal smooth particles *in a high state of agitation* have formed the subject of very long and successful investigations, resulting in the dynamical theory of fluids. Also the limiting conditions of equilibrium of such media as sand have been made the subject of theoretical treatment by the aid of certain assumptions.

These investigations, however, by no means constitute a complete theory of granular masses; nor does it appear that any attempts have been made to investigate the dynamical properties of a medium consisting of smooth hard particles, held in contact by forces transmitted through the medium. It has sometimes been assumed that such a medium would possess the properties of a liquid, although in the molecular hypothesis of liquids now accepted the particles are assumed to be in a high state of motion, holding each other apart by

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collisions; such motion being rendered necessary to account for the property of diffusion.

Without attempting anything like a complete dynamical theory, which will require a large development of mathematics, I would point out the existence of a singular fundamental property of such granular media which is not possessed by known fluids or solids. On perceiving something which resembles nothing within the limits of one's knowledge, a name is a matter of great difficulty. I have called this unique property of granular masses "dilatancy," because the property consists in a definite change of bulk, consequent on a definite change of shape or distortional strain, any disturbance whatever causing a change of volume and generally dilation.

In the case of fluids, volume and shape are perfectly independent; and although in practice it is often difficult to alter the shape of an elastic body without altering its volume, yet the properties of dilation and distortion are essentially distinct, and are so considered in the theory of elasticity. In fact there are very few solid bodies which are to any extent dilatable at all.

With granular media, the grains being sensibly hard, the case is, according to the results I have obtained, entirely different. So long as the grains are held in mutual equilibrium by stresses transmitted through the mass, every change of relative position of the grains is attended by a consequent change of volume; and if in any way the volume be fixed, then all change of shape is prevented.

In speaking of a granular medium, it is assumed to be in such a condition that the position of any internal particle becomes fixed when the positions of the surrounding particles are fixed.

This condition is very generally fulfilled, but not always where there is friction; without friction it would be always fulfilled.

From this assumption it at once follows that no grain in the interior can change its position in the mass by passing between the contiguous grains without disturbing these; hence, whatever alterations the medium may undergo, the same particle will always be in the same neighbourhood.

If, then, the medium is subject to an internal strain, the shapes of the internal groups of molecules will all be altered, the shape of each elementary group being determined by the shape of the surrounding particles. This will be rendered most intelligible by considering instances; that of equal spheres is the most general, and presents least difficulty.

A group of such spheres being arranged in such a manner that, if the external spheres are fixed, the internal ones cannot move, any distortion of the boundaries will cause an alteration of the mean density, depending on the distortion and the arrangement of the spheres. For example :—

If arranged as a pile of shot (Plate X. fig. 2), which is an arrangement of tetrahedra and octahedra, the density of the media is $\frac{1}{\sqrt{2}}\frac{\pi}{3}$, taking the density of the sphere as unity.

If arranged in a cubical formation, as in fig. 1, the density is $\frac{\pi}{6}$, or $\sqrt{2}$ times less than in the former case.

These arrangements are both controlled by the bounding spheres ; and in either case the distortion necessitates a change of volume.

Either of these forms can be changed into the other by changing the shape of the bounding surface.

In both these cases the structure of the group is crystalline, but that is on account of the plane boundaries.

Practically, when the boundaries are not plane, or when the grains are of various sizes or shapes, such media consist of more or less crystalline groups having their axes in different directions, so that their mean condition is amorphous.

The dilation consequent on any distortion for a crystalline group may be definitely expressed. When the mean condition is amorphous, it becomes difficult to ascertain definitely what the relations between distortion and dilation are. But if, when at maximum density, the mean condition is not only amorphous but isotropic, a natural assumption seems to be that any small contraction from the condition of maximum density in one direction means an equal extension in two others at right angles.

As such a contraction in one direction continues, the condition of the medium ceases to be isotropic, and the relation changes until dilation ceases. Then a minimum density is reached ; after this, further contraction in the same direction causes a contraction of volume, which continues until a maximum density is reached. Such a relation between the contraction in one direction and the consequent dilation would be expressed by

$$e-1=e_1\sqrt{\sin^2\frac{a}{e_1}}\,;$$

e being the coefficient of dilation, a that of contraction, and e_1 the maximum dilation ; the $+$ve root only to be taken.

The amorphous condition of minimum volume is a very stable condition ; but there would be a direct relation between

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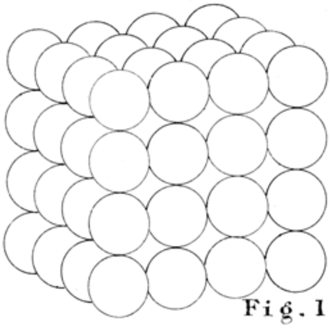

Fig. 1.

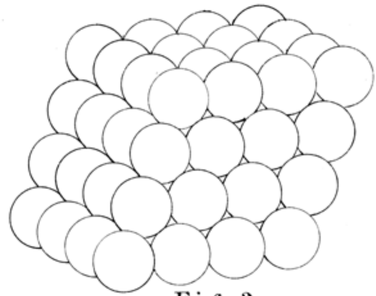

Fig. 2.

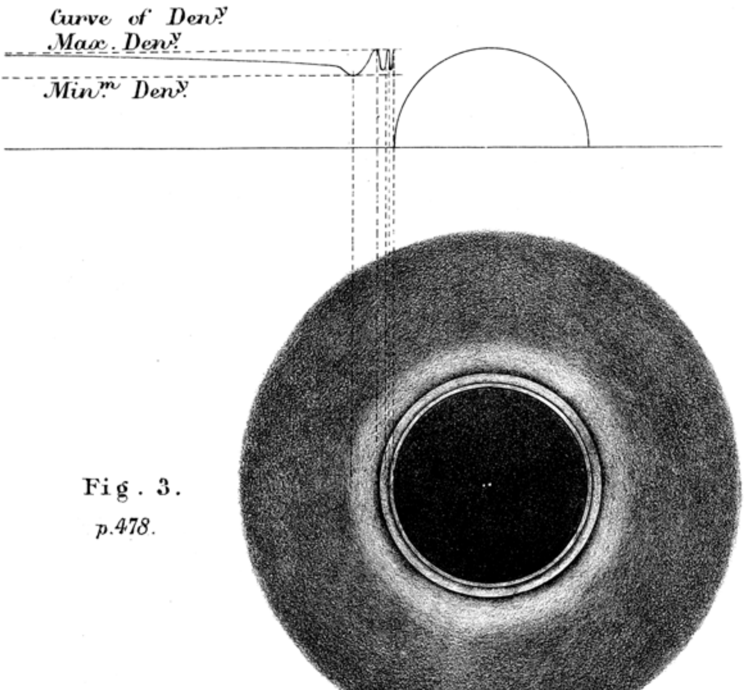

Fig. 3.

p.478.

Fig. 4.

Fig. 5.

Mintern Bros. lith.

the strains and stresses in any other condition if the particles were frictionless and rigid.

If the particles were rigid the medium would be absolutely without resilience, and hence the only energy of which it would be susceptible would be kinetic energy; so that, supposing the motion slow, the work done upon any group in distorting it would be zero. Thus, supposing a contraction in one direction and expansion at right angles, then if p_x be the stress in the direction of contraction, and p_y, p_z the stress at right angles, a being the contraction, b and c expansions,

$$p_x a + p_y b + p_z c = 0;$$

or, supposing $b=c$, $p_y=p_z$,

$$p_x a + p_y (a+c) = 0.$$

With friction the relation will be different; the friction always opposes strain, *i. e.* tends to give stability.

It is a very difficult question to say exactly what part friction plays; for although we may perhaps still assume without error,

$$\frac{p_y}{p_x} = \frac{1 - \sin\phi}{1 + \sin\phi},$$

where ϕ is the angle of repose, we cannot assume that $\tan\phi$ has any relation to the actual friction between the molecules.

The extreme value of ϕ is a matter of arrangement; as in the case of shot, which would pile equally well although without friction.

Supposing the grains rigid, the relations between distortion and dilation are independent of friction; that is to say, the same distortion of any bounding surfaces must mean the same internal distortion whatever the friction may be.

The only possible effect of friction would be to render the grains stable under circumstances under which they would not otherwise be stable; and hence we might with friction be able to bring about an alteration of the boundaries other than the alteration possible without friction; and thus we might possibly obtain a dilation due to friction. How far this is the case can be best ascertained by experiment.

In the case of a granular medium, friction may always be relaxed by relieving the mass of stress, and any stability due to this cause would be shown by shaking the mass when in a condition of no stress.

But before applying this test, it is necessary to make perfectly sure that during the shaking the boundary spheres do not change position.

Another test of the effect of friction is by comparing the relative dilation and distortion with different degrees of friction. If the dilation were in any sense a consequence of friction, it would be greater when the coefficient of friction between the spheres was greater. Where the granular mass is bounded by solid surfaces, the friction of the grains against these surfaces will considerably modify the results.

The problem presented by frictionless balls is much simpler than that presented in the case of friction. In the former case the theoretical problem may be attacked with some hope of success. With friction the property is most easily studied by experiment.

As a matter of fact, if we take means to measure the volume of a mass of solid grains more or less approximately spheres, the property of dilatancy is evident enough, and its effects are very striking, affording an explanation of many well-known phenomena.

If we have in a canvas bag any hard grains or balls, so long as the bag is not nearly full it will change its shape as it is moved about; but when the sack is approximately full a small change of shape causes it to become perfectly hard. There is perhaps nothing surprising in this, even apart from familiarity; because an inextensible sack has a rigid shape when extended to the full, any deformation diminishing its capacity, so that contents which did not fill the sack at its greatest extension fill it when deformed. On careful consideration, however, many curious questions present themselves.

If, instead of a canvas bag, we have an extremely flexible bag of india-rubber, this envelope, when filled with heavy spheres (No. 6 shot), imposes no sensible restraint on their distortion; standing on the table it takes nearly the form of a heap of shot. This is apparently accounted for by the fact that the capacity of the bag does not diminish as it is deformed. In this condition it really shows us less of the qualities of its granular contents than the canvas bag. But as it is impervious to fluid, it will enable me to measure exactly the volume of its contents.

Filling up the interstices between the shot with water so that the bag is quite full of water and shot, no bubble of air in it, and carefully closing the mouth, I now find that the bag has become absolutely rigid in whatever form it happened to be when closed.

It is clear that the envelope now imposes no distortional constraint on the shot within it, nor does the water. What, then, converts the heap of loose shot into an absolutely rigid body? Clearly the limit which is imposed on the volume by the pressure of the atmosphere.

So long as the arrangement of the shot is such that there is enough water to fill the interstices the shot are free, but any arrangement which requires more room is absolutely prevented by the pressure of the atmosphere.

If there is an excess of water in the bag when the shot are in their maximum density, the bag will change its shape quite freely for a limited extent, but then becomes instantly rigid, supporting 56 lb. without further change. By connecting the bag with a graduated vessel of water so that the quantity which flows in and out can be measured, the bag again becomes susceptible of any amount of distortion.

Getting the bag into a spherical form and its contents at maximum density, and then squeezing it between two planes, the moment the squeezing begins the water begins to flow in, and flows in at a diminishing rate until it ceases to draw more water.

The material in the bag is in a condition of minimum density under the circumstances. This does not mean that all the parts are in a condition of minimum density because the distortion is not the same in all the parts; but some parts have passed through the condition of maximum while others have not reached it, so that on further distortion the dilations of the latter balance the contractions of the former. If we continue to squeeze, water begins to flow out until about half as much has run out as came in; then again it begins to flow in. We cannot by squeezing get it back into a condition of uniform maximum density, because the strain is not homogeneous. This is just what would occur if the shot were frictionless; so that it is not surprising to find that, using oil instead of water, or, better (on account of the india-rubber), a strong solution of soap and water, which greatly diminishes the friction, the results are not altered.

On measuring the quantities of water, we find that the greatest quantity drawn in is about 10 per cent. of the volume of the bag; this is about one third of the difference between the volumes of the shot at minimum and maximum density.

$$\frac{1}{\sqrt{2}} : 1, \text{ or 30 per cent. of the latter.}$$

On easing the bag it might be supposed that the shot would return to their initial condition. But that does not follow: the elasticity of form of the bag is so slight compared with its elasticity of volume, that restitution will only take place as long as it is accompanied with contraction of volume.

So long as the point of maximum volume has not been reached, approximate restitution follows quite as nearly as could be expected, considering that friction opposes restitution. But when the squeezing has been carried past the point of maximum volume, then restitution requires expansion; and this the elasticity of shape is not equal to accomplish, so that the bag retains its flattened condition. This experiment has been varied in a great variety of ways.

The very finest quartz sand, or glass balls $\frac{3}{4}$ inch in diameter, all give the same results. Sand is, on the whole, the most convenient material, and its extreme fineness reduces any effect of the squeezing of the india-rubber between the interstices of the balls at the boundaries; which effect is very apparent with the balloon bags, and shot as large as No. 6.

A well-marked phenomenon receives its explanation at once from the existence of dilatancy in sand. When the falling tide leaves the sand firm, as the foot falls on it the sand whitens, or appears momentarily to dry round the foot. When this happens the sand is full of water, the surface of which is kept up to that of the sand by capillary attraction; the pressure of the foot causing dilation of the sand, more water is required, which has to be obtained either by depressing the level of the surface against the capillary attraction, or by drawing water through the interstices of the surrounding sand. This latter requires time to accomplish, so that for the moment the capillary forces are overcome; the surface of the water is lowered below that of the sand, leaving the latter white or dryer until a sufficient supply has been obtained from below, when the surface rises and wets the sand again. On raising the foot it is generally seen that the sand under the foot and around becomes momentarily wet; this is because, on the distorting forces being removed, the sand again contracts, and the excess of water finds momentary relief at the surface.

Leaving out of account the effect of friction between the balls and the envelope, the results obtained with actual balls, as regards the relation between distortion and dilation, appear to be the same as would follow if the balls were smooth.

The friction at the boundaries is not important as long as the strain over the boundaries is homogeneous, and particularly if the balls indent themselves into the boundaries, as they do in the case of india-rubber. But with a plane surface the balls at the boundaries are in another condition from the balls within. The layer of balls at the surface can only vary its density from $2/\sqrt{3}$ to 1. This means that the layer of balls at a surface can slide between that surface and the adjacent layer, causing much less dilation than would be caused by the sliding of an internal layer within the mass. Hence where two parts of the mass are connected by such a surface, certain conditions of strain of the boundaries may be accommodated by a continuous stream of balls adjacent to the surface. This fact made itself evident in two very different experiments.

In order to examine the formation which the shot went through, an ordinary glass funnel was filled with shot and oil, and held vertical while more shot were forced up the spout of the funnel. It was expected that the shot in the funnel would rise as a body, expanding laterally so as to keep the funnel full. This seems to have been the effect at the commencement of the experiment; but after a small quantity had passed up it appeared, looking at the side of the funnel, that the shot were rising much too fast, for which, on looking into the top of the funnel, the reason became apparent. A sheet of shot adjacent to the funnel were rising steadily all round, leaving the interior shot at the same level with only a slight disturbance.

In another experiment one india-rubber bag was filled with sand and water; at the centre of this ball was another much smaller ball, communicating through the sides of the outer envelope by means of a glass pipe with an hydraulic pump. It was expected that, on expanding the interior ball by water, the sand in the outer ball would dilate, expanding the outer ball and drawing more water into the intervening sand. This it did, but not to the extent expected. It was then observed that the outer envelope, instead of expanding, generally bulged in the immediate neighbourhood of the point where the glass tube passed through it; showing that this tube acted as a conductor for the sand from the immediate neighbourhood of the interior ball to the outer envelope, just as the glass sides of the funnel had acted for the shot.

As regards any results which may be expected to follow from the recognition of this property of dilatancy,—

In a practical point of view, it will place the theory of earth-pressures on a true foundation. But inasmuch as the present theory is founded on the angle of repose, which is certainly not altered by the recognition of dilatancy, its effect will be mainly to show the real reason for the angle of repose.

The greatest results are likely to follow in philosophy, and it was with a view to these results that the investigation was undertaken.

The recognition of this property of dilatancy places a hitherto unrecognized mechanical contrivance at the command of those who would explain the fundamental arrangement of the universe, and one which, so far as I have been able to look into it, seems to promise great things, besides possessing the inherent advantage of extreme simplicity.

Hitherto no medium has ever been suggested which would cause a statical force of attraction between two bodies at a distance. Such attraction would be caused by granular media in virtue of this dilatancy and stress. More than this, when two bodies in a granular medium under stress are near together, the effect of dilatancy is to cause forces between the bodies in very striking accordance with those necessary to explain coherence of matter.

Suppose an outer envelope of sufficiently large extent, at first not absolutely rigid, filled with granular media, at its maximum density. Suppose one of the grains of the media commences to grow into a larger sphere; as it grows, the surrounding medium will be pushed outwards radially from the centre of the expanding sphere. Considering spherical envelopes following the grains of the medium, these will expand as the grains move outwards. This fixes the distortion of the medium, which must be contraction along the radii, and expansion along all tangents.

The consequent amount of dilation depends on the relation of distortion and dilation, and on the arrangement of the grains in the medium. At first the entire medium will undergo dilation, which will diminish as the distance from the centre increases. As the expansion goes on, the medium immediately adjacent to the sphere will first arrive at a condition of minimum density; and for further expansion this will be returning to a maximum density, while that a little further away will have reached a minimum. The effect of continued growth will therefore be to institute concentric undulations of density from maximum to minimum density, which will move outwards; so that after considerable growth the sphere will be surrounded with a series of envelopes of alternately maximum and minimum density, the medium at a great distance being at maximum density. At a definite distance from the centre of the sphere not more than

$$1.4\mathrm{R},$$

where R is the radius of the sphere, the density will be a minimum, and between this and the sphere there may be a number of alternations depending on the relative diameters of the grains and the spheres.

The distance between these alternations will diminish rapidly as the sphere is approached. The distance of the next maximum is 1.2 R, the next minimum is given by 1.09 R, and the next maximum 1.06 R.

The general condition of the medium around a sphere which has expanded in the medium is shown in Plate X. fig. 3, which has been arrived at on the supposition that the sphere is large compared with the grains.

From a radius about 1.4 R outwards the density gradually increases, reaching a maximum density at infinity; and at all distances greater than 1.8 R the law is expressed by

$$\frac{de}{dr}=\frac{1}{r^n},$$

where n has some value greater than 3 depending on the structure of the medium.

Within the distance 1.4 R the variation is periodic, with a rapidly diminishing period. In this condition, supposing the medium of unlimited extent and the sphere smooth, the sphere may move without causing further expansion, merely changing the position of the distortion in the medium; for the grains, slipping over the sphere, would come back to their original positions. It thus appears that smooth bodies would move without resistance if the relation between the size of the grains and bodies is such that the energy due to the relative motion of the grains in immediate proximity may be neglected. The kinetic energy of the motion of the medium would be proportional to the volume of the ball multiplied by the density of the medium and the square of the velocity.

But the momentum might be infinite supposing the medium infinite in extent, in which case a single sphere would be held rigidly fixed.

If we suppose two balls to expand instead of one, and suppose the distortion of the medium for one ball to be the same as if the other were not there, the result will be a compound distortion. Since, however, the dilation does not bear a linear relation to the distortion, the dilation resulting from the compound distortion will not be the sum of the dilations for the separate distortions unless we neglect the squares and products of the distortions as small.

Supposing the bodies so far apart that one or other of the separate distortions caused at any point is small, then, retaining squares and products, it appears that the resultant dilation at any point will be less than the sum of the separate dilations by quantities which are proportional to the products of the separate distortions.

The integrals of these terms through the space bounded by spheres of radii R and L are expressed by finite terms, and terms inversely proportional to L, which latter vanish if L is infinite. Thus, while the total separate dilations are infinite, the compound dilations differ from the sum of the separate by finite terms, and these are functions of the product of the volumes and the reciprocal of the distance.

Assuming stress in the medium, the difference in the value of these finite terms for two relative positions of the bodies multiplied by the stresses, represents an amount of work which must be done by the bodies on the medium in moving from one position to another.

To get rid of the difficulty of infinite extent of medium, if for the moment we assume the envelope sufficiently large and imposing a normal pressure upon the medium, then, since the work done will be proportional to the dilation, the force between the bodies will be proportional to the rate at which this dilation varies with the distance between them.

The force between the bodies would depend on the character of the elasticity as well as on the dilation.

It is not necessary to assume the outer envelope elastic; this may be absolutely rigid and one or both the balls elastic.

In such case the two balls are connected by a definite kinematic relation. As they approach they must expand, doing work which is spent in producing energy of motion; as they recede, the kinetic energy is spent in the work of compressing the balls.

As already stated, the momentum of the infinite medium for a single ball in finite motion may be infinite, and proportional to the product of the volume of the ball by the velocity; but with two balls moving in opposite directions, with velocities inversely as the masses, the momentum of the system is zero. Therefore such motion may be the only motion possible in a medium of infinite extent.

When the distance between the balls is of the same order as their dimensions, the law of attraction changes with the law of the compound dilations and becomes periodic, corresponding to the undulations of density surrounding the balls. Thus, before actual contact were reached, the balls would suffer alternate repulsion and attraction, with positions of equilibrium more or less stable between, as shown in figs. 4 and 5 (Pl. X.).

We have thus a possible explanation of the cohesion and chemical combination of molecules, which I think is far more in accordance with actual experience than anything hitherto suggested.

It was the observation of these envelopes of maximum and minimum density which led me to look more fully into the property of dilatancy.

The assumed elasticity of the surrounding envelope, or of the balls, has only been introduced to make the argument clear.

The medium itself may be supposed to possess kinetic elasticity arising from internal distortional motion, such as would arise from the transmission of waves in which the motion of the medium is in the plane of their fronts.

The fitness of a dilatant medium to transmit such waves is only less striking than its property of causing attraction, because in the first respect it is not unique.

But as far as I can see such transmission is not possible in a medium composed of uniform grains. If, however, we have comparatively large grains uniformly interspersed, then such transmission becomes possible. If, notwithstanding the large grains, the medium is at maximum density, the large grains will not be free to move without causing further dilation; and it seems that the medium would transmit distortional vibrations in which the distortions of the two sets of grains are opposite.

Such waves, although the motion would be essentially in the plane of the wave, would cause dilation, just as waves in a chain cause contraction in the reach of the chain. They would in fact impart elasticity to the medium, exactly as, in the case of a slack chain having its ends fixed but otherwise not subject to forces, any lateral motion imparted to the chain will cause tension proportional to the energy of disturbance divided by the slackness or free length of chain.

Distortional waves therefore, travelling through dilatant material which does not quite occupy the space in which it is confined when at maximum density, would render the medium uniformly elastic to distortion, but not in the same degree to compression or extension. The tension caused by such waves would depend on the gross energy of motion of the waves divided by the total dilation from maximum density consequent on the wave-motion. All such waves, whatever might be their length, would therefore move with the same velocity.

If, when rendered elastic by such waves, the medium were thrown into a state of distortion by some external cause, this would diminish the possible dilation caused by the waves. Thus work would have to be done on the medium in producing the external distortion which would be spent in increasing the energy of the waves. For instance, the separation of two bodies in such a medium, which, as already shown, would increase the statical distortion, would increase the energy of the waves and *vice versâ*.

As far as the integrations have been carried for this condition of elasticity, it appears, with a certain arrangement of large and small grains, that the forces between the bodies would be proportional to the product of the volumes divided by the square of the distance ; *i. e.* that the state of stress of the medium may be the same as Maxwell has shown must exist in the æther to account for gravity. We have thus an instance of a medium transmitting waves similar to heat-waves and causing force between bodies similar to the forces of gravitation and cohesion, in such a manner as to constitute a conservative system. More than this, by the separation of the two sets of grains, there would result phenomena similar to those resulting from the separation of the two electricities. The observed conducting power of a continuous surface for the grains of a medium closely resembles the conduction of electricity. And such a composite medium would be susceptible of a state in which the arrangement of the two sets of grains were thrown into opposite distortions, which state, so far as it has yet been examined, appears to coincide with the state of a medium necessary to explain electrodynamic and magnetic phenomena according to Maxwell's theory.

In this short sketch of the results which it appears to me may follow from the recognition of the property of dilatancy, I have not attempted to follow the exact reasoning even so far as I have carried it.

In the preliminary acceptance of a theory the mind must be guided rather by a general view of its adaptability than by its definite accordance with some out of many observed facts. And as it seems, after a preliminary investigation, that in space filled with discrete particles, endowed with rigidity, smoothness, and inertia, the property of dilatancy would cause amongst other bodies not only one property but all the fundamental properties of matter, I have, in pointing out the existence of dilatancy, ventured to call attention to this dilatant or kinematic theory of æther without waiting for the completion of the definite integrations, which must take long, although it is by these that the fitness of the hypotheses must be eventually tested.

LVIII. *On the Refraction of Fluorine.*

By GEORGE GLADSTONE, *F.C.S.**

IN his paper on the Refraction-Equivalents of the Elements, published in the Phil. Trans. of 1869, Dr. Gladstone estimated the equivalent of fluorine at 1·45, from the results

* Communicated by the Author, having been read at the Meeting of the British Association at Aberdeen, September 1885.