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XVII. *On a General Theorem of the Stability of the Motion of a Viscous Fluid.* By D. J. Korteweg, *Professor of Mathematics at the University of Amsterdam**.

THE interesting experimental investigation of Prof. O. Reynolds† on sinuous fluid-motion and the origin of eddies induces me to communicate a very general theorem on the stability of fluid-motion, which I published some months ago in the Transactions of the Royal Academy of Amsterdam. Conceiving that the origin of eddies was to be explained by the existence of unstable solutions of the equations of motion, I endeavoured to find such solutions by means of the well-known equations for slow motion of viscous fluid, till I found out that *in any simply connected region, when the velocities along the boundary are given, there exists, as far as the squares and products of the velocities may be neglected, only* one *solution of the equations for the steady motion of an incompressible viscous fluid, and that this solution is always stable.*

The first part of this theorem is due to Helmholtz‡. He shows it to be a very simple consequence of another theorem, stating that "*if the motion be steady, the currents in a viscous fluid are so distributed that the loss of energy due to viscosity is a minimum, on the supposition that the velocities along the boundaries of the fluid are given.*"

Though my demonstration of this theorem is somewhat less general than that of Helmholtz, I will write it down here, as it leads to a very simple and symmetrical expression for the difference between the real dissipation of energy by internal friction with any motion and the minimum dissipation consistent with the same velocities at the boundary, which expression may be useful in other cases.

Theorem I.—*Let* M_0 *represent a mode of motion of an incompressible fluid answering to the equations*

$$\left.\begin{aligned} \mu\nabla^2 u_0 &= \frac{\delta(V\rho+p_0)}{\delta x}, \\ \mu\nabla^2 v_0 &= \frac{\delta(V\rho+p_0)}{\delta y}, \\ \mu\nabla^2 w_0 &= \frac{\delta(V\rho+p_0)}{\delta z}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

M *another mode of motion of incompressible fluid, consistent*

* Communicated by the Author.

† Proceedings of the Royal Society, vol. xxxv. 1883, p. 84.

‡ *Verh. des naturh.-med. Vereins zu Heidelberg*, Bd. v. S. 1–7; Collected Works, i. p. 223.

with the same velocities along the boundary, then

$$A = A_0 + 4\mu \int (\Omega')^2 . \, dx\, dy\, dz; \quad . \quad . \quad . \quad . \quad (2)$$

where A_0 *is the dissipation of energy in unit of time at the mode of motion* M_0, A *at the mode of motion* M, Ω' *the angular velocity corresponding at any point of the region to the mode of motion indicated by* $M - M_0$.

Proof. "Let u', v', w' represent the component velocities, A' the dissipation of energy by friction at the mode of motion $M - M_0$, then at every point of the region occupied by the fluid we have by definition,

$$\left.\begin{aligned} u' &= u - u_0, \\ v' &= v - v_0, \\ w' &= w - w_0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

" Along the boundary,

$$u' = 0, \quad v' = 0, \quad w' = 0. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

" Substituting the values of u, v, w from (3) in the well-known expression for the dissipation-function,

$$\left.\begin{aligned} A = \mu \int \Big[& 2\left(\frac{\delta u}{\delta x}\right)^2 + 2\left(\frac{\delta v}{\delta y}\right)^2 + 2\left(\frac{\delta w}{dz}\right)^2 + \left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right)^2 \\ & + \left(\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)^2 \Big] dx\, dy\, dz \\ = 2\mu \int \Big[& \left\{ \left(\frac{\delta u}{dx}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2 \right\} + \left\{ \left(\frac{\delta v}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta y}\right)^2 + \left(\frac{\delta v}{\delta z}\right)^2 \right\} \\ & + \left\{ \left(\frac{\delta w}{\delta x}\right)^2 + \left(\frac{\delta w}{\delta y}\right)^2 + \left(\frac{\delta w}{\delta z}\right)^2 \right\} \Big] dx\, dy\, dz \\ - \mu \int \Big[& \left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right)^2 + \left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)^2 \Big] dx\, dy\, dz, \end{aligned}\right\} \quad (5)$$

we find

$$\left.\begin{aligned} A = A_0 + A' + 4\mu \int \Big[& \left(\frac{\delta u_0}{\delta x} \cdot \frac{\delta u'}{\delta x} + \frac{\delta u_0}{\delta y} \cdot \frac{\delta u'}{\delta y} + \frac{\delta u_0}{\delta z} \cdot \frac{\delta u'}{\delta z}\right) \\ & + \left(\frac{\delta v_0}{\delta x} \cdot \frac{\delta v'}{\delta x} + \frac{\delta v_0}{\delta y} \cdot \frac{\delta v'}{\delta y} + \frac{\delta v_0}{\delta z} \cdot \frac{\delta v'}{\delta z}\right) \\ & + \left(\frac{\delta w_0}{\delta x} \cdot \frac{dw'}{\delta x} + \frac{\delta w_0}{\delta y} \cdot \frac{\delta w'}{\delta y} + \frac{\delta w_0}{\delta z} \cdot \frac{\delta w'}{\delta z}\right) \Big] dx\, dy\, dz \\ - 2\mu \int \Big[& \left(\frac{\delta w_0}{\delta y} - \frac{\delta v_0}{\delta z}\right)\left(\frac{\delta w'}{\delta y} - \frac{\delta v'}{\delta z}\right) + \left(\frac{\delta u_0}{\delta z} - \frac{\delta w_0}{\delta x}\right)\left(\frac{\delta u'}{\delta z} - \frac{\delta u'}{\delta x}\right) \\ & + \left(\frac{\delta v_0}{\delta x} - \frac{\delta u_0}{\delta y}\right)\left(\frac{\delta v'}{\delta x} - \frac{\delta u'}{\delta y}\right) \Big] dx\, dy\, dz. \end{aligned}\right\} \quad (6)$$

"To the third term of the right-hand side of this equation we apply the theorem of Green. With respect to the boundary conditions (4), it then takes the form

$$-4\mu\int(u'\nabla^2u_0+v'\nabla^2v_0+w'\nabla^2w_0)\,dx\,dy\,dz.$$

"In virtue of (1), this may be written

$$-4\int\left(u'\frac{\delta(V\rho+p_0)}{\delta x}+v'\frac{\delta(V\rho+p_0)}{\delta y}+w'\frac{\delta(V\rho+p_0)}{\delta z}\right)dx\,dy\,dz,$$

or, by partial integration,

$$4\int(V\rho+p_0)\left(\frac{\delta u'}{\delta x}+\frac{\delta v'}{\delta y}+\frac{\delta w'}{\delta z}\right)dx\,dy\,dz,$$

which expression is identically zero, having regard to the equation of continuity.

"In the last term of the equation (6) the multiplications must be effected. By partial integration we then can give it the form

$$\begin{aligned}-2\mu\int\Big[&u'\left(\frac{\delta^2w_0}{\delta x\,\delta z}+\frac{\delta^2v_0}{\delta x\,\delta y}-\frac{\delta^2u_0}{\delta z^2}-\frac{\delta^2u_0}{\delta y^2}\right)\\&+v'\left(\frac{\delta^2u_0}{\delta y\,\delta x}+\frac{\delta^2w_0}{\delta y\,\delta x}-\frac{\delta^2v_0}{\delta x^2}-\frac{\delta^2v_0}{\delta z^2}\right)\\&+w'\left(\frac{\delta^2v_0}{\delta z\,\delta y}+\frac{\delta^2u_0}{\delta z\,\delta x}-\frac{\delta^2w_0}{\delta y^2}-\frac{\delta^2w_0}{\delta x^2}\right)\Big]\,dx\,dy\,dz.\end{aligned}$$

"By means of the equation of continuity this may be reduced to

$$2\mu\int(u'\nabla^2u_0+v'\nabla^2v_0+w'\nabla^2w_0)\,dx\,dy\,dz,$$

which expression has already been seen to vanish. Therefore

$$A=A_0+A'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

"Now, if we effect quite similar transformations with the terms of the expression for A′,

$$\begin{aligned}A'=2\mu\int\Big[&\left\{\left(\frac{\delta u'}{\delta x}\right)^2+\left(\frac{\delta u'}{\delta y}\right)^2+\left(\frac{\delta u'}{\delta z}\right)^2\right\}\\&+\left\{\left(\frac{\delta v'}{dx}\right)^2+\left(\frac{\delta v'}{\delta y}\right)^2+\left(\frac{\delta v'}{\delta z}\right)^2\right\}\\&+\left\{\left(\frac{\delta w'}{\delta x}\right)^2+\left(\frac{\delta w'}{\delta y}\right)^2+\left(\frac{\delta w'}{\delta z}\right)^2\right\}\Big]\,dx\,dy\,dz\\-\mu\int\Big[&\left(\frac{\delta w'}{\delta y}-\frac{\delta v'}{\delta z}\right)^2+\left(\frac{\delta u'}{\delta z}-\frac{\delta w'}{\delta x}\right)^2+\left(\frac{\delta v'}{\delta x}-\frac{\delta u'}{\delta y}\right)^2\Big]\,dv\,dy\,dz,\end{aligned}$$

the first may be reduced to

$$-2\mu\int(u'\nabla^2u'+v'\nabla^2v'+w'\nabla^2w')\,dx\,dy\,dz,$$

the second to

$$+\ \mu\int(u'\nabla^2u'+v'\nabla^2v'+w'\nabla^2w')\,dx\,dy\,dz;$$

but then it is obvious that A′ may be put at choice under one of two forms:—

$$A'=-\mu\int(u'\nabla^2u'+v'\nabla v'+w'\nabla^2w')\,dx\,dy\,dz, \quad . \quad (8)$$

or

$$A'=\mu\int\left[\left(\frac{\delta w'}{\delta y}-\frac{\delta v'}{\delta z}\right)^2+\left(\frac{\delta u'}{\delta z}-\frac{\delta w'}{\delta x}\right)\right.$$

$$\left.+\left(\frac{\delta v'}{\delta x}-\frac{\delta u'}{\delta y}\right)^2\right]dx\,dy\,dz=4\mu\int(\Omega')^2\,dx\,dy\,dz.\text{''} \quad (9)$$

Theorem II.—*The mode of motion M_0 is such that the dissipation of energy by internal friction is the least possible consistent with the same velocities along the boundary. It is unique in every simply connected region when these velocities are given.*

Proof. "The first part of this theorem follows immediately from the equation (2), the second term of the right-hand side of this equation being necessarily positive or zero.

"Now let M'_0 be a second mode of motion, consistent with the given velocities along the boundary and satisfying the equations (1). Then, as in both motions the dissipation of energy must be the least possible, A'_0 is equal to A_0. This can be so only when, at every point,

$$\Omega'=0;$$

but then $M'_0 - M_0$ should represent an irrotational motion with zero-velocities all over a closed boundary; and such a motion is known to be impossible."

Theorem III.—*When in a given region occupied by viscous incompressible fluid, there exists at a certain moment a mode of motion M, which does not satisfy the equations (1), then, the velocities along the boundary being maintained constant, the change which must occur in the mode of motion will be such (neglecting squares and products of velocities) that the dissipation of energy by external friction is constantly decreasing till it reaches the value A_0 and the mode of motion becomes identical with M_0.*

Proof. "The change occurring at any moment in the mode of motion is determined by the equations

$$\left.\begin{aligned}\rho\frac{\delta u}{\delta t}-\mu\nabla^2 u+\frac{\delta(\mathrm{V}\rho+p)}{\delta x}&=0,\\ \rho\frac{\delta v}{\delta t}-\mu\nabla^2 v+\frac{\delta(\mathrm{V}\rho+p)}{\delta y}&=0,\\ \rho\frac{\delta w}{\delta t}-\mu\nabla^2 w+\frac{\delta(\mathrm{V}\rho+p)}{\delta z}&=0.\end{aligned}\right\}\quad . \quad . \quad . \quad (10)$$

"Along the boundary,

$$\frac{\delta u}{\delta t}=\frac{\delta v}{\delta t}=\frac{\delta w}{\delta t}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

" By means of these relations we have to prove that $\frac{\delta \mathrm{A}}{\delta t}$ is constantly *negative.*

" Now, in virtue of (5),

$$\left.\begin{aligned}\frac{\delta \mathrm{A}}{\delta t}=4\mu\int\Bigg[&\left(\frac{\delta u}{\delta x}\cdot\frac{\delta^2 u}{\delta x\,\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta^2 u}{\delta y\,\delta t}+\frac{\delta u}{\delta z}\cdot\frac{\delta^2 u}{\delta z\,\delta t}\right)\\ &+\left(\frac{\delta v}{\delta x}\cdot\frac{\delta^2 v}{\delta x\,\delta t}+\frac{\delta v}{\delta y}\cdot\frac{\delta^2 v}{\delta y\,\delta t}+\frac{\delta v}{\delta z}\cdot\frac{\delta^2 v}{\delta z\,\delta t}\right)\\ &+\left(\frac{\delta w}{\delta x}\cdot\frac{\delta^2 w}{\delta x\,\delta t}+\frac{\delta w}{\delta y}\cdot\frac{\delta^2 w}{\delta y\,\delta t}+\frac{\delta w}{\delta z}\cdot\frac{\delta^2 w}{\delta z\,\delta t}\right)\Bigg]\,dx\,dy\,dz\\ -2\mu\int\Bigg[&\left(\frac{\delta w}{\delta y}-\frac{\delta v}{\delta z}\right)\left(\frac{\delta^2 w}{\delta y\,\delta t}-\frac{\delta^2 v}{\delta z\,\delta t}\right)+\left(\frac{\delta u}{\delta z}-\frac{\delta w}{\delta x}\right)\left(\frac{\delta^2 u}{\delta z\,\delta t}-\frac{\delta^2 w}{\delta x\,\delta t}\right)\\ &+\left(\frac{\delta v}{\delta x}-\frac{\delta u}{\delta y}\right)\left(\frac{\delta^2 v}{\delta x\,\delta t}-\frac{\delta^2 u}{\delta y\,\delta t}\right)\Bigg]\,dx\,dy\,dz.\end{aligned}\right\}\ (12)$$

" Applying the theorem of Green to the first term of the right-hand side of this equation, we get

$$-4\mu\int\left(\frac{\delta u}{\delta t}\cdot\nabla^2 u+\frac{\delta v}{\delta t}\nabla^2 v+\frac{\delta w}{\delta t}\nabla^2 w\right)dx\,dy\,dz.$$

" With respect to (10), this may be written

$$-4\rho\int\left[\left(\frac{\delta u}{\delta t}\right)^2+\left(\frac{\delta v}{\delta t}\right)^2+\left(\frac{\delta w}{\delta t}\right)^2\right]dx\,dy\,dz$$

$$-4\int\left(\frac{\delta u}{\delta t}\cdot\frac{\delta(\mathrm{V}\rho+p)}{\delta x}+\frac{\delta v}{\delta t}\cdot\frac{\delta(\mathrm{V}\rho+p)}{\delta y}+\frac{\delta w}{\delta t}\cdot\frac{\delta(\mathrm{V}\rho+p)}{\delta z}\right)dx\,dy\,dz,$$

the second term of which expression vanishes after partial integration by virtue of the equation of continuity.

" As for the second term of the right-hand side of (12), effecting the multiplications and applying partial integration, it takes the form

$$-2\mu\int\left(\frac{\delta u}{\delta t}\left(\frac{\delta^2 w}{\delta x\,\delta z}+\frac{\delta^2 v}{\delta x\,\delta y}-\frac{\delta^2 u}{\delta z^2}-\frac{\delta^2 u}{\delta y^2}\right)\right.$$
$$+\frac{\delta v}{\delta t}\left(\frac{\delta^2 u}{\delta y\,\delta x}+\frac{\delta^2 w}{\delta y\,\delta z}-\frac{\delta^2 v}{\delta x^2}-\frac{\delta^2 v}{\delta z^2}\right)$$
$$\left.+\frac{\delta w}{\delta t}\left(\frac{\delta^2 v}{\delta z\,\delta y}+\frac{\delta^2 u}{\delta z\,\delta x}-\frac{\delta^2 w}{\delta y^2}-\frac{\delta^2 w}{\delta x^2}\right)\right)dx\,dy\,dz$$
$$=2\mu\int\left(\frac{\delta u}{\delta t}\nabla^2 u+\frac{\delta v}{\delta t}\nabla^2 v+\frac{\delta^2 w}{\delta t}\nabla^2 w\right)dx\,dy\,dz$$
$$=2\rho\int\left(\frac{\delta u}{\delta t}\right)^2+\left(\frac{\delta v}{\delta t}\right)^2+\left(\frac{\delta w}{\delta t}\right)^2\Big]\,dx\,dy\,dz.$$

"Combining the values of both terms, we have

$$\frac{\delta \mathrm{A}}{\delta t}=-2\rho\int\left[\left(\frac{\delta u}{\delta t}\right)^2+\left(\frac{\delta v}{\delta t}\right)^2+\left(\frac{dw}{dt}\right)^2\right]dx\,dy\,dz. \quad . \quad . \quad (13)$$

"This expression remains negative, and therefore the dissipation of energy is decreasing, till everywhere in the fluid

$$\frac{\delta u}{\delta t}=\frac{\delta v}{\delta t}=\frac{\delta w}{\delta t}=0;$$

but then the motion has become steady, and is necessarily identical with the motion represented by M_0."

THEOREM IV.—*The mode of motion represented by* M_0 *is always stable as far as squares and products of velocities may be neglected.*

Proof. "Let the mode of motion M_0 be disturbed by any cause whatever. Then the dissipation of energy by internal friction is necessarily increased (Theorem II.); but as soon as the cause of disturbance ceases it must decrease again (Theorem III.) till it reaches the value A_0, and then the mode of motion M_0 is restored."

From this theorem I draw the following conclusions:—

1st. That the existence of unstable modes of fluid-motion and the origin of eddies cannot be explained without taking into account squares and products of velocities; for that the equations (1) for steady motion with low velocities cannot lead directly to eddying fluid-motion, whatever the velocities along the boundary be, is a consequence of the well-known relations

$$\nabla^2\xi=0,\quad \nabla^2\eta=0,\quad \nabla^2\zeta=0.$$

According to these relations, ξ, η, and ζ can have no maximum or minimum values in the interior of the fluid ; but then the angular velocity Ω cannot have there a maximum value, which, taking the axis of x parallel to the direction of the rotation-axis, would correspond obviously with a maximum value of ξ.

2nd. That though the idea of the possible existence of unstable solutions of the equations of motion (alluded to, as far as I know, for the first time by Prof. Stokes*) was very just and fertile in itself, yet the case of motion which suggested it to him is not one of unstable motion, at least not so unless the squares and products of velocities be taken into account. It is perfectly true that, when a cylinder of infinite length moves with uniform velocity through an incompressible viscous fluid, the state of steady motion never can be reached, and an ever increasing quantity of fluid will be carried on by the cylinder. Yet as the dissipation of energy will be ever decreasing, and even, as may be proved, tending to zero, as the motion proceeds, such a change in the state of motion as Prof. Stokes alludes to, and by which the dissipation of energy could only be augmented, cannot occur.

When, on the contrary, the squares and higher powers of the velocities are taken into account, I have my reasons for supposing that, even in the case of a sphere moving with uniform velocity, no state of steady motion can be reached, and the motion must finally become unstable.

Amsterdam, June 4, 1883.

XVIII. *On the Critical Point of Liquefiable Gases.* *By* William Ramsay.

To the Editors of the Philosophical Magazine and Journal.

Gentlemen,

In the July number of this Magazine there is a translation of a memoir by M. J. Jamin, presented by him to the French Academy. As he has taken no notice of the views expressed by me in the 'Proceedings of the Royal Society' for 1880, April 22nd and December 16th, I think it right to point out that the substance of this memoir has been anticipated ; and in support of this statement let me quote the following passages. M. Jamin states (1) that the critical point of a liquid is not to be regarded, as hitherto, as the temperature at

* Trans. of the Camb. Phil. Soc. vol. ix. 1849, "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," p. 56.