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LX. On the potential of an ellipsoid at an external point

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ence is collected on C_2 when C_1 is charged up to a certain potential-level, and that which in the charging of C_2 up to the same potential is through influence collected on C_1, are equal the one to the other.

Beside these two conclusions, here cited as examples, referable to two very simple special cases, from equation (I.) many other similar inferences can of course be drawn.

LX. *On the Potential of an Ellipsoid at an External Point.* *By* Colonel A. R. Clarke, *C.B., F.R.S.**

IN connexion with the subject of the attraction of the earth upon external objects, there is a series which is usually brought forward by writers on astronomy, viz. that which results from expanding into a series in descending powers of r the expression for V,

$$V=\iiint\frac{dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{1}{2}}},$$

r being the distance $=(f^2+g^2+h^2)^{\frac{1}{2}}$ of the attracted external particle from the origin, x, y, z the coordinates of any element of the mass, the density being unity throughout. Put $x^2+y^2+z^2=\rho^2$, $\alpha x+\beta y+\gamma z=\sigma$, where α, β, γ are the direction-cosines of r; then the radical in the expression for V may be expanded in the form

$$\frac{1}{r}+\frac{Q_1}{r^2}+\frac{Q_2}{r^3}+\frac{Q_3}{r^4}+\ldots,$$

where Q_i is a homogeneous function of x, y, z of the degree i. When the body is an ellipsoid referred to its principal axes, the terms in which i is odd will disappear in integrating over the volume of the ellipsoid; so that in this case, M being the mass of the ellipsoid,

$$V=\frac{M}{r}+\frac{1}{r^3}\int Q_2 dm+\frac{1}{r^5}\int Q_4 dm+\frac{1}{r^7}\int Q_6 dm+\ldots, \quad (1)$$

which is the series alluded to. Lagrange's investigation of the terms (as far as just written down) of this series is referred to in Todhunter's 'History of the Theory of Attraction and the Figure of the Earth,' vol. ii. p. 161. See also Thomson and Tait, 'Natural Philosophy,' pp. 401, 402; Pontécoulant, *Théorie Analytique du Système du Monde*, ii. p. 233, where the above expression for the potential is given as one suitable

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only when r, as compared with the dimensions of the attracting body, is very large—as, for instance, in the investigation of the disturbance of the moon's motion produced by the non-sphericity of the earth, and of the reaction of the same disturbing force on the earth, causing lunar nutation and precession. But the fact is, that the first two terms of the series are sufficient for any external point, however near, provided the square of the earth's ellipticity be neglected; for the ratio of the successive terms is not of the order $\frac{a^2}{r^2}$, but $\frac{a^2-b^2}{r^2}$, a, b being the semidiameters of the earth.

If P_n be Legendre's coefficient of the order n, then we have, μ being the cosine of the angle between r and ρ,

$$(r^2-2r\rho\mu+\rho^2)^{-\frac{1}{2}}=\frac{P_0}{r}+\frac{P_1\rho}{r^2}+\frac{P_2\rho^2}{r^3}+\frac{P_3\rho^3}{r^4}+\dots;$$

consequently

$$\int Q_i dm=\int P_i\rho^i dm.$$

Now we require only the even coefficients; and their values (Todhunter's 'Functions of Laplace, Lamé, and Bessel,' p. 4) are:—

$$P_2=\frac{3}{2}\mu^2-\frac{1}{2},$$

$$P_4=\frac{5.7}{2.4}\mu^4-\frac{3.5}{2.4}2\mu^2+\frac{1.3}{2.4},$$

$$P_6=\frac{7.9.11}{2.4.6}\mu^6-\frac{5.7.9}{2.4.6}3\mu^4+\frac{3.5.7}{2.4.6}3\mu^2-\frac{1.3.5}{2.4.6};$$

whence the values of Q, since $\rho\mu=\sigma$, are

$$Q_2=\frac{3}{2}\sigma^2-\frac{1}{2}\rho^2,$$

$$Q_4=\frac{5.7}{2.4}\sigma^4-\frac{3.5}{2.4}2\sigma^2\rho^2+\frac{1.3}{2.4}\rho^4,$$

$$Q_6=\frac{7.9.11}{2.4.6}\sigma^6-\frac{5.7.9}{2.4.6}3\sigma^4\rho^2+\frac{3.5.7}{2.4.6}3\sigma^2\rho^4-\frac{1.3.5}{2.4.6}\rho^6.$$

For the integration of Q_2 we have

$$\int x^2dm=\frac{M}{5}a^2:\int y^2dm=\frac{M}{5}b^2:\int z^2dm=\frac{M}{5}c^2;$$

and there is no difficulty in arriving at the equation

$$\int Q_2\,dm=-\frac{M}{15}\{P_2'(e_2^2-e_3^2)+P_2''(e_3^2-e_1^2)+P_2'''(e_1^2-e_2^2)\},\ (2)$$

where $e_1^2=b^2-c^2$; $e_2^2=c^2-a^2$; $e_3^2=a^2-b^2$; and P_2', P_2'', P_2''' are what P_2 becomes when α, β, γ are respectively written for μ.

We may write the next term thus,

$$Q_4=Ax^4+By^4+Cz^4+A'x^2y^2+B'y^2z^2+C'z^2x^2+W,$$

where W includes all terms in which the exponents are odd, which disappear in integration. But Q_i must satisfy the differential equation

$$\frac{d^2Q_i}{dx^2}+\frac{d^2Q_i}{dy^2}+\frac{d^2Q_i}{dz^2}=0. \quad . \quad . \quad . \quad . \quad (3)$$

Applying this to Q_4, and remembering that the equation is identically true, we get these equations:—

$$6A+A'+C'=0,$$
$$6B+A'+B'=0,$$
$$6C+B'+C'=0,$$

whence

$$A'+3(\ \ A+B-C)=0,$$
$$B'+3(-A+B+C)=0,$$
$$C'+3(\ \ A-B+C)=0.$$

On integrating the expression for Q_4, after substituting these values of A′, B′, C′, it is to be observed that

$$\int x^4dm=\frac{1\,.\,3}{5\,.\,7}Ma^4;\quad \int x^2y^2dm=\frac{1}{5\,.\,7}Ma^2b^2,$$

with corresponding values for the other integrals. Also A, B, C are equal respectively to P_4', P_4'', P_4''', where the accents have the meaning already explained. Thus we get

$$\int Q_4\,dm=-\frac{3M}{5\,.\,7}\{P_4'e_3^2e_2^2+P_4''e_1^2e_3^2+P_4'''e_2^2e_1^2\}.\ .\quad (4)$$

For the next term put $Q_6=I_1\sigma^6+I_2\sigma^4\rho^2+I_3\sigma^2\rho^4+I_4\rho^6$; and then, expanding, we get

$$Q_6=Ax^6+By^6+Cz^6+A_1x^2y^4+A_2x^4y^2+B_1y^2z^4+B_2y^4z^2$$
$$+C_1z^2x^4+C_2z^4x^2+Ex^2y^2z^2$$

plus terms involving odd powers of x, y, z, which disappear in integration. On substituting Q_6 in the differential equation (3), which it has to satisfy, the following relations are found amongst the coefficients with which we are concerned:—

$$15A+A_2+C_1=0,\qquad 6A_1+6A_2+E=0,$$
$$15B+A_1+B_2=0,\qquad 6B_1+6B_2+E=0,$$
$$15C+C_2+B_1=0,\qquad 6C_1+6C_2+E=0.$$

We cannot from these six equations express A_1, A_2, B_1, B_2, C_1, C_2 in terms of the other four coefficients, because on comparing the sums of the two sets of equations we get $E=30(A+B+C)$; therefore we must seek some simple relation amongst the coefficients that may be used as another equation. The coefficients are

$$A=I_1\alpha^6+I_2\alpha^4+I_3\alpha^2+I_4,$$
$$A_1=15I_1\alpha^2\beta^4+I_2(6\alpha^2\beta^2+\beta^4)+I_3(\alpha^2+2\beta^2)+3I_4,$$
$$A_2=15I_1\alpha^4\beta^2+I_2(6\alpha^2\beta^2+\alpha^4)+I_3(2\alpha^2+\beta^2)+3I_4,$$

from which the others may be written down. The simplest relation that suggests itself arises from adding the three differences $A_1-A_2+B_1-B_2+C_1-C_2$; this is equal to

$$15I_1(\beta^2-\gamma^2)(\gamma^2-\alpha^2)(\alpha^2-\beta^2)=15P. \quad . \quad . \quad (5)$$

Also $A=P_6'$: $B=P_6''$: $C=P_6'''$, the accents having the meaning already explained. Thus we get

$$\tfrac{2}{5}A_1=\ \ P_0+\ P_6'-3P_6''-\ P_6''',$$
$$\tfrac{2}{5}A_2=-P_0-3P_6'+\ P_6''-\ P_6''',$$
$$\tfrac{2}{5}B_1=\ \ P_0-\ P_6'+\ P_6''-3P_6''',$$
$$\tfrac{2}{5}B_2=-P_0-\ P_6'-3P_6''+\ P_6''',$$
$$\tfrac{2}{5}C_1=\ \ P_0-3P_6'-\ P_6''+\ P_6''',$$
$$\tfrac{2}{5}C_2=-P_0+\ P_6'-\ P_6''-3P_6'''.$$

These have to be substituted in Q_6, with the following values for the integrals:—

$$\int x^6dm=\frac{1.3.5}{5.7.9}Ma^6 : \int x^2y^4dm=\frac{1.3}{5.7.9}Ma^2b^4 :$$
$$\int x^2y^2z^2dm=\frac{M}{5.7.9}a^2b^2c^2.$$

The result is

$$\int Q_6dm=\frac{1.3}{7.9}\frac{M}{2}\{P_6'e_2^2e_3^2(e_2^2-e_3^2)+P_6''e_3^2e_1^2(e_3^2-e_1^2)$$
$$+P_6'''e_1^2e_2^2(e_1^2-e_2^2)+Pe_1^2e_2^2e_3^2\}. \quad (6)$$

The sequence of the terms (2), (4), (6), which complete the series (1) as far as written down, is very remarkable, and suggests the idea that possibly an expression might be obtained for the general term. When the ellipsoid is one of revolution, so that $c=b$, and $e^2=c^2-a^2$,

$$V=\frac{M}{r}\left\{1-\frac{3P_2'}{3.5}\cdot\frac{e^2}{r^2}+\frac{3P_4'}{5.7}\cdot\frac{e^4}{r^4}-\frac{3P_6'}{7.9}\cdot\frac{e^6}{r^6}+\ldots\right\}.$$