turnal dews, has been finally washed away by the sea-water. This is rather an unsatisfactory mode of accounting for the absence of the muriate of ammonia, which from what we know of guano, and the existence of oxalate of soda in the saline guano, we may fairly conclude must have been formed, but which in the sample I have examined contains but little more than one equivalent of muriate of ammonia to three of oxalate of soda, so that nearly two equivalents have disappeared.

As to the respective hypotheses of the coprolitic or recent nature of these deposits, I have never held but one opinion, the one I believe generally entertained, that this manure is deposited by sea-fowl inhabiting coasts where no rain falls, and which consequently is never washed away. This view is supported by all ancient and most modern authorities, and has recently received additional confirmation from Mr. Teschemacher of Boston, who has presented specimens of Peruvian guano to the Philosophical Society of that town, containing *feathers*; and after quoting the accounts given by the old Portuguese historian respecting the formation and preservation of this manure, very justly remarks, that the beds of the greatest thickness hitherto observed might, without any extravagant calculation, and at the rate only of two to three inches a year, or less, be deposited in about three thousand years; whilst the theory of its coprolitic origin not merely requires a considerable exercise of the imagination, but is opposed by the direct testimony of eye-witnesses.

P.S. The May Number of the Philosophical Magazine, which has just come into my hands, contains an analysis of African guano by our Foreign Secretary, Mr. E. F. Teschemacher, in which he finds humic acid to exist in a soluble state in the African specimen, but no urate of ammonia. The humus described to exist in the Peruvian specimens analysed by me was extracted by dilute potash from the residue insoluble in water; and if it be true humus, and it possesses all the characters assigned to that substance, the South American varies from the African guano, amongst other differences, in containing uncombined humus.

XIII. *On certain Results relating to Quaternions.* *By* ARTHUR CAYLEY, *Esq., B.A., Fellow of Trinity College, Cambridge.*

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

IN his last paper on Quaternions, Sir William R. Hamilton has alluded to a paper of mine on the Analytical Geometry of (n) dimensions, in the Cambridge Mathematical

Journal, as one that might refer to the same subject. It may perhaps be as well to notice that the investigations there contained have no reference whatever to Sir William Hamilton's very beautiful theory; a more correct title for them would have been, a Generalization of the Analysis which occurs in ordinary Analytical Geometry.

I will, with your permission, take this opportunity of communicating one or two results relating to quaternions; the first of them does appear to me rather a curious one.

Observing that

$$\left.\begin{aligned}&(\mathrm{A}+\mathrm{B}i+\mathrm{C}j+\mathrm{D}k)^{-1}\\&\quad=(\mathrm{A}-\mathrm{B}i-\mathrm{C}j-\mathrm{D}k)\div(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2+\mathrm{D}^2)\end{aligned}\right\}, \quad . \quad (1.)$$

it is easy to form the equation

$$\left.\begin{aligned}&(\mathrm{A}+\mathrm{B}i+\mathrm{C}j+\mathrm{D}k)^{-1}(\alpha+\beta i+\gamma j+\delta k)(\mathrm{A}+\mathrm{B}i+\mathrm{C}j+\mathrm{D}k)\\&=\frac{1}{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2+\mathrm{D}^2}\\&\left\{\begin{aligned}&\alpha.(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2+\mathrm{D}^2)\\&+i[\beta(\mathrm{A}^2+\mathrm{B}^2-\mathrm{C}^2-\mathrm{D}^2)+2\gamma.\overline{\mathrm{BC}+\mathrm{AD}}+2\delta.\overline{\mathrm{BD}-\mathrm{AC}}]\\&+j[2\beta(\mathrm{BC}-\mathrm{AD})+\gamma(\mathrm{A}^2-\mathrm{B}^2+\mathrm{C}^2-\mathrm{D}^2)+2\delta.\overline{\mathrm{CD}+\mathrm{AB}}]\\&+k[2\beta(\mathrm{BD}+\mathrm{AC})+2\gamma(\mathrm{CD}-\mathrm{AB})+\delta.\overline{\mathrm{A}^2-\mathrm{B}^2-\mathrm{C}^2+\mathrm{D}^2}]\end{aligned}\right.\end{aligned}\right\}, \quad (2.)$$

which I have given with these letters for the sake of reference; it will be convenient to change the notation and write

$$\left.\begin{aligned}&(1+\lambda i+\mu j+\nu k)^{-1}.(ix+jy+kz)(1+\lambda i+\mu j+\nu k)\\&=\frac{1}{1+\lambda^2+\mu^2+\nu^2}\\&\left\{\begin{aligned}&i[x(1+\lambda^2-\mu^2-\nu^2)+2y(\lambda\mu+\nu)+2z(\lambda\nu-\mu)]+\\&j[2x(\lambda\mu-\nu)+y(1-\lambda^2+\mu^2-\nu^2)+2z(\mu\nu+\lambda)]+\\&k[2x(\lambda\nu+\mu)+2y.(\mu\nu-\lambda)+z(1-\lambda^2-\mu^2+\nu^2)]\end{aligned}\right.\end{aligned}\right\} \quad (3.)$$

$$\left.\begin{aligned}&i(\alpha x+\alpha' y+\alpha'' z)\\&+j.(\beta x+\beta' y+\beta'' z)\\&+k(\gamma x+\gamma' y+\gamma'' z)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

suppose. The peculiarity of this formula is, that the coefficients α, β ... are precisely such that a system of formulæ

$$\left.\begin{aligned}x_{\prime}&=\alpha x+\alpha' y+\alpha'' z\\y_{\prime}&=\beta' x+\beta' y+\beta'' z\\z_{\prime}&=\gamma x+\gamma' y+\gamma'' z\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

denote the transformation from one set of rectangular axes to another set, also rectangular. Nor is this all, the quantities λ, μ, ν may be geometrically interpreted. Suppose the axes Ax, Ay, Az could be made to coincide with the axes A$x_{\prime}$, A$y_{\prime}$, A$z_{\prime}$ by means of a revolution through an angle θ round an axis A P inclined to A x, A y, A z, at angles f, g, h, then

$$\lambda = \tan\tfrac{1}{2}\theta\cos f,\quad \mu = \tan\tfrac{1}{2}\theta\cos g,\quad \nu = \tan\tfrac{1}{2}\theta\cos h.$$

In fact the formulæ are precisely those given for such a transformation by M. Olinde Rodrigues Liouville, t. v., "Des lois géometriques qui régissent les déplacemens d'un système solide" (or Camb. Math. Journal, t. iii. p. 224). It would be an interesting question to account, *à priori*, for the appearance of these coefficients here.

The ordinary definition of a determinant naturally leads to that of a quaternion determinant. We have, for instance,

$$\begin{vmatrix} \varpi, \phi \\ \varpi', \phi' \end{vmatrix} = \varpi\phi' - \varpi'\phi, \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

$$\begin{vmatrix} \varpi, & \phi, & \chi \\ \varpi', & \phi', & \chi' \\ \varpi'', & \phi'', & \chi'' \end{vmatrix} = \varpi(\phi'\chi'' - \phi''\chi') + \varpi'(\phi''\chi - \phi\chi'') + \varpi''(\phi\chi' - \phi'\chi),$$

&c., the same as for common determinants, only here the order of the factors on each term of the second side of the equation is essential, and not, as in the other case, arbitrary. Thus, for instance,

$$\begin{vmatrix} \varpi, \varpi' \\ \varpi, \varpi' \end{vmatrix} = \varpi\varpi' - \varpi\varpi' = 0, \quad . \quad . \quad . \quad . \quad (7.)$$

but

$$\begin{vmatrix} \varpi, \varpi \\ \varpi', \varpi' \end{vmatrix} = \varpi\varpi' - \varpi'\varpi = 0. \quad . \quad . \quad . \quad . \quad (8.)$$

Or a quaternion determinant does not vanish when two vertical rows become identical. One is immediately led to inquire what the value of such determinants is. Suppose

$$\varpi = x + iy + jz + kw,\quad \varpi' = x' + iy' + jz' + kw', \text{ \&c.},$$

is it easy to prove

$$\left.\begin{vmatrix} \varpi & \varpi \\ \varpi' & \varpi' \end{vmatrix} = -2 \begin{vmatrix} i, & j, & k \\ x, & y, & z \\ x', & y', & z' \end{vmatrix}\right\}, \quad . \quad . \quad . \quad . \quad (9.)$$

$$\left.\begin{vmatrix} \varpi, & \varpi, & \varpi \\ \varpi', & \varpi', & \varpi' \\ \varpi'', & \varpi'', & \varpi'' \end{vmatrix} = -2 \begin{vmatrix} 3, & i, & j, & k \\ x, & y, & z, & w \\ x', & y', & z', & w' \\ x'', & y'', & z'', & w'' \end{vmatrix}\right\}, \quad . \quad (10.)$$

$$\left.\begin{vmatrix} \varpi & \varpi & \varpi & \varpi \\ \varpi' & \varpi' & \varpi' & \varpi' \\ \varpi'' & \varpi'' & \varpi'' & \varpi'' \\ \varpi''' & \varpi''' & \varpi''' & \varpi''' \end{vmatrix} = 0\right\}. \quad . \quad . \quad . \quad . \quad (11.)$$

Or a quaternion determinant vanishes when four or more of its vertical rows become identical.

Again, it is immediately seen that

$$\left.\begin{vmatrix} \varpi, \phi \\ \varpi', \phi' \end{vmatrix} + \begin{vmatrix} \phi, \varpi \\ \phi', \varpi' \end{vmatrix} = \begin{vmatrix} \varpi, \varpi \\ \phi', \phi' \end{vmatrix} - \begin{vmatrix} \varpi', \varpi' \\ \phi, \phi \end{vmatrix}\right\}, \quad (12.)$$

&c. for determinants of any order, whence the theorem, if any four (or more) adjacent vertical columns of a quaternion determinant be transposed in every possible manner, the sum of all these determinants vanishes, which is a much less simple property than the one which exists for the horizontal rows, viz. the same that in ordinary determinants exists for the horizontal or vertical rows indifferently. It is important to remark that the equations

$$\begin{vmatrix} \varpi, & \phi \\ \varpi', & \phi' \end{vmatrix} = 0 \text{ or } \begin{vmatrix} \varpi, & \varpi' \\ \phi, & \phi' \end{vmatrix} = 0, \text{ \&c.} \quad . \quad . \quad . \quad (13.)$$

i. e. $\varpi \phi' - \varpi' \phi = 0$, or $\varpi \phi' - \phi \varpi' = 0$, &c.

are none of them the result of the elimination of Π, Φ, from the two equations

$$\left.\begin{array}{l} \varpi \Pi + \phi \Phi = 0, \\ \varpi' \Pi + \phi' \Phi = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

On the contrary, the result of this elimination is the very different equation

$$\varpi^{-1} . \phi - \varpi'^{-1} . \phi' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

equivalent of course to four independent equations, one of which may evidently be replaced by

$$\text{M} \varpi . \text{M} \phi' - \text{M} \varpi' . \text{M} \phi = 0, \quad . \quad . \quad . \quad (16.)$$

if $\text{M} \varpi$, &c. denotes the modulus of ϖ, &c. An equation analogous to this last will undoubtedly hold for any number of equations, but it is difficult to say what is the equation analogous to the one immediately preceding this, in the case of a greater number of equations, or rather, it is difficult to give the result in a symmetrical form independent of extraneous factors.

I may just, in conclusion, mention what appears to me a possible application of Sir William Hamilton's interesting discovery. In the same way that the circular functions depend on infinite products, such as

$$x \Pi \left(1 + \frac{x}{m \pi}\right), \text{ \&c.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

[m any integer from ∞ to $-\infty$, omitting $m=0$]

and the inverse elliptic functions on the doubly infinite products

$$x \Pi \left(1 + \frac{x}{m \omega + n \varpi i}\right), \text{ \&c.} \quad . \quad . \quad . \quad . \quad (18.)$$

[m and n integers from ∞ to $-\infty$, omitting $m=0$, $n=0$], may not the inverse ultra-elliptic functions of the next order of complexity depend on the quadruply infinite products

$$x \Pi \left(1 + \frac{x}{m\varpi + n\varpi i + o\phi j + p\psi k}\right)? \quad . \quad . \quad (19.)$$

[m, n, o, p integers from ∞ to $-\infty$, omitting $m=0$, $n=0$, $o=0$, $p=0$].

It seems as if some supposition of this kind would remove a difficulty started by Jacobi (Crelle, t. ix.) with respect to the multiple periodicity of these functions. Of course this must remain a mere suggestion until the theory of quaternions is very much more developed than it is at present; in particular the theory of quaternion exponentials would have to be developed, for even in a product, such as (18.), there is a certain singular exponential factor running through the theory, as appears from some formulæ in Jacobi's *Fund. Nova* (relative to his functions Θ, H), the occurrence of which may be accounted for, *à priori*, as I have succeeded in doing in a paper to be published shortly in the Cambridge Mathematical Journal.

I remain, Gentlemen,
Your obedient Servant,

Cambridge, December 4, 1844. A. CAYLEY.

XIV. *Correction of an Error in the author's* "Strictures on Professor Dove's Essay on the Law of Storms." *By* ROBERT HARE, *M.D., Professor of Chemistry in the University of Pennsylvania.*

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

I BEG leave to correct an error, committed in my Strictures on Dove's Law of Storms, Phil. Mag. S. 3. vol. xxv. p. 100, in assuming that the contents of the zones in a circular area between equidistant concentric parallel lines are to each other as the squares of their mean distances from the common centre; instead of assuming them to be simply as those distances. Of course the velocity of the air in the zone nearest the upward columnar current, in a tornado or hurricane, will not be to the velocity of any greater zone inversely as the *squares* of the mean distances from the axis of the column, but simply in the inverse ratio of those distances.

Hence, supposing the centripetal velocity at a mile from the centre, or say five thousand feet to be one hundred miles per hour, at twenty miles, it would be five miles per hour, or merely a breeze. By this amendment of the calculation my argument is strengthened, so far as it was an object of it to prove that in an extensive hurricane the central area protected