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II. On quaternions; or on a new system of imaginaries in algebra

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tithonization is not a transient effect which at once passes away, but is, on the contrary, a persistent change.

I tithonized the chlorine and hydrogen contained in the instrument, and kept it in the dark for ten hours. On exposure to the lamp rays it moved after a few seconds, showing, therefore, that the change which had been impressed on the chlorine was not lost. In the former case 600 seconds had elapsed before any movement was visible.

When, however, we remember that the invisible images on Daguerreotype plates, and even photographic impressions on surfaces of resin, and probably all other similar changes are slowly effaced, it would be premature to conclude that tithonized chlorine does not revert to its original condition. I have sometimes thought that there were in several of my experiments indications that this was taking place, but would not be understood to assert it positively. Whether it be so or not, one thing is certain, that the taking on of this condition and the loss of it is a very different affair from any transient exaltation of action due to a temporary elevation of temperature, or the contrary effect produced by cooling.

April 26, 1844.

II. *On Quaternions; or on a new System of Imaginaries in Algebra**. *By* Sir William Rowan Hamilton, *LL.D., P.R.I.A., F.R.A.S., Hon. M. R. Soc. Ed. and Dub., Hon. or Corr. M. of the Royal or Imperial Academies of St. Petersburgh, Berlin, Turin, and Paris, Member of the American Academy of Arts and Sciences, and of other Scientific Societies at Home and Abroad, Andrews' Prof. of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

1. Let an expression of the form

$$Q = w + ix + jy + kz$$

be called a *quaternion*, when w, x, y, z, which we shall call the four *constituents* of the quaternion Q, denote any real quantities, positive or negative or null, but i, j, k are symbols of three imaginary quantities, which we shall call *imaginary units*, and shall suppose to be unconnected by any linear relation with each other; in such a manner that if there be another expression of the same form,

$$Q' = w' + ix' + jy' + kz',$$

the supposition of an equality between these two quaternions,

$$Q = Q',$$

* A communication, substantially the same with that here published, was made by the present writer to the Royal Irish Academy, at the first meeting of that body after the last summer recess, in November 1843.

shall be understood to involve four separate equations between their respective constituents, namely, the four following,

$$w = w', \quad x = x', \quad y = y', \quad z = z'.$$

It will then be natural to define that the *addition* or *subtraction* of quaternions is effected by the formula

$$Q \pm Q' = w \pm w' + i(x \pm x') + j(y \pm y') + k(z \pm z');$$

or, in words, by the rule, that *the sums or differences of the constituents of any two quaternions, are the constituents of the sum or difference of those two quaternions themselves.* It will also be natural to define that the product $Q\,Q'$, of the multiplication of Q as a multiplier into Q' as a multiplicand, is capable of being thus expressed:

$$\begin{aligned} Q\,Q' = {} & ww' + iwx' + jwy' + kwz' \\ & + ixw' + i^2xx' + ijxy' + ikxz' \\ & + jyw' + jiyx' + j^2yy' + jkyz' \\ & + kzw' + kizx' + kjzy' + k^2zz'; \end{aligned}$$

but before we can reduce this product to an expression of the quaternion form, such as

$$Q\,Q' = Q'' = w'' + ix'' + jy'' + kz'',$$

it is necessary to fix on quaternion-expressions (or on real values) for the nine squares or products,

$$i^2, \ ij, \ ik, \ ji, \ j^2, \ jk, \ ki, \ kj, \ k^2.$$

2. Considerations, which it might occupy too much space to give an account of on the present occasion, have led the writer to adopt the following system of values or expressions for these nine squares or products:

$$i^2 = j^2 = k^2 = -1; \quad \ldots \ldots \ldots \quad \text{(A.)}$$

$$ij = k, \ jk = i, \ ki = j; \quad \ldots \ldots \ldots \quad \text{(B.)}$$

$$ji = -k, \ kj = -i, \ ik = -j; \quad \ldots \ldots \quad \text{(C.)}$$

though it must, at first sight, seem strange and almost unallowable, to define that the product of two imaginary factors in one order differs (in sign) from the product of the same factors in the opposite order ($ji = -ij$). It will, however, it is hoped, be allowed, that in entering on the discussion of a new system of imaginaries, it may be found necessary or convenient to surrender *some* of the expectations suggested by the previous study of products of real quantities, or even of expressions of the form $x + iy$, in which $i^2 = -1$. And whether the choice of the system of definitional equations, (A.), (B.), (C.), has been a judicious, or at least a happy one, will probably be judged by the event, that is, by trying whether those equations conduct to results of sufficient consistency and elegance.

3. With the assumed relations (A.), (B.), (C.), we have the four following expressions for the four constituents of the product of two quaternions, as functions of the constituents of the multiplier and multiplicand:

$$\left.\begin{aligned} w'' &= w w' - x x' - y y' - z z', \\ x'' &= w x' + x w' + y z' - z y', \\ y'' &= w y' + y w' + z x' - x z', \\ z'' &= w z' + z w' + x y' - y x'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D.)}$$

These equations give

$$w''^2 + x''^2 + y''^2 + z''^2 = (w^2 + x^2 + y^2 + z^2)\,(w'^2 + x'^2 + y'^2 + z'^2);$$

and therefore

$$\mu'' = \mu\,\mu', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(E.)}$$

if we introduce a system of expressions for the constituents, of the forms

$$\left.\begin{aligned} w &= \mu \cos\theta, \\ x &= \mu \sin\theta \cos\phi, \\ y &= \mu \sin\theta \sin\phi \cos\psi, \\ z &= \mu \sin\theta \sin\phi \sin\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(F.)}$$

and suppose each μ to be positive. Calling, therefore, μ the *modulus* of the quaternion Q, we have this theorem: that *the modulus of the product* Q$''$ *of any two quaternions* Q *and* Q$'$, *is equal to the product of their moduli.*

4. The equations (D.) give also

$$w' w'' + x' x'' + y' y'' + z' z'' = w\,(w'^2 + x'^2 + y'^2 + z'^2),$$
$$w w'' + x x'' + y y'' + z z'' = w'\,(w^2 + x^2 + y^2 + z^2);$$

combining, therefore, these results with the first of those equations (D.), and with the trigonometrical expressions (F.), and the relation (E.) between the moduli, we obtain the three following relations between the angular co-ordinates $\theta\ \phi\ \psi$, $\theta'\ \phi'\ \psi'$, $\theta''\ \phi''\ \psi''$ of the two factors and the product:

$$\left.\begin{aligned} \cos\theta'' &= \cos\theta\cos\theta' - \sin\theta\sin\theta'\,(\cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\psi - \psi')), \\ \cos\theta\ &= \cos\theta'\cos\theta'' + \sin\theta'\sin\theta''(\cos\phi'\cos\phi'' + \sin\phi'\sin\phi''\cos(\psi' - \psi'')), \\ \cos\theta'\ &= \cos\theta''\cos\theta + \sin\theta''\sin\theta\,(\cos\phi''\cos\phi + \sin\phi''\sin\phi\cos(\psi'' - \psi)). \end{aligned}\right\} \text{(G.)}$$

These equations (G.) admit of a simple geometrical construction. Let $x\,y\,z$ be considered as the three rectangular co-ordinates of a point in space, of which the radius vector is $= \mu\sin\theta$, the longitude $= \psi$, and the co-latitude $= \phi$; and let these three latter quantities be called also the *radius vector*, the *longitude* and the *co-latitude of the quaternion* Q; while θ shall be called the *amplitude* of that quaternion. Let R be the point where the radius vector, prolonged if necessary, intersects the spheric surface described about the origin of co-ordinates with a radius = unity, so that ϕ is the co-latitude and

ψ is the longitude of R; and let this point R be called the *representative point* of the quaternion Q. Let R′ and R″ be, in like manner, the representative points of Q′ and Q″; then the equations (G.) express that *in the spherical triangle* R R′ R″, *formed by the representative points of the two factors and the product* (in any multiplication of two quaternions), *the angles are respectively equal to the amplitudes of those two factors, and the supplement of the amplitude of the product* (to two right angles); in such a manner that we have the three equations:

$$\mathrm{R} = \theta, \quad \mathrm{R}' = \theta', \quad \mathrm{R}'' = \pi - \theta''. \quad . \quad . \quad . \quad . \quad \text{(H.)}$$

5. The system of the four very simple and easily remembered equations (E.) and (H.), may be considered as equivalent to the system of the four more complex equations (D.), and as containing within themselves a sufficient expression of the rules of multiplication of quaternions; with this exception, that they leave undetermined the *hemisphere* to which the point R″ belongs, or the *side* of the arc R R′ on which that *product-point* R″ falls, after the *factor-points* R and R′, and the amplitudes θ and θ' have been assigned. In fact, the equations (E.) and (H.) have been obtained, not immediately from the equations (D.), but from certain combinations of the last-mentioned equations, which combinations would have been unchanged, if the signs of the three functions,

$$y z' - z y', \quad z x' - x z', \quad x y' - y x',$$

had all been changed together. This latter change would correspond to an alteration in the assumed conditions (B.) and (C.), such as would have consisted in assuming $i j = -k$, $j i = +k$, &c., that is, in taking the cyclical order $k j i$ (instead of $i j k$), as that in which the product of any two imaginary units (considered as multiplier and multiplicand) is equal to the imaginary unit following, taken positively. With this remark, it is not difficult to perceive that *the product-point* R″ *is always to be taken to the right, or always to the left of the multiplicand-point* R′, *with respect to the multiplier-point* R, according as the semiaxis of $+z$ is to the right or left of the semiaxis of $+y$, with respect to the semiaxis of $+x$; or, in other words, according as the positive direction of rotation in longitude is to the right or to the left. This *rule of rotation*, combined with the *law of the moduli* and with the *theorem of the spherical triangle*, completes the transformed system of conditions, connecting the product of any two quaternions with the factors, and with their order.

[To be continued.]