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XXVII. Considerations relative to an interesting case in equations

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but still filling up the interstices of the concentric layers, and binding them together, according to the similitude of Grew and Malpighi, as the woof of a web binds together the longitudinal threads of the warp.

[To be continued.]

XXVII. *Considerations relative to an interesting Case in Equations.* *By* W. G. Horner, *Esq.**

PERCEIVING that Professor Moseley has resumed the development of the principle of least pressure, I presume that the discussion, which was introduced by Mr. Earnshaw, respecting the validity of the principle, has terminated. Without entering, therefore, upon the general question, in the fate of which I have no other interest than every lover of science must be supposed to have, I may be allowed to express my disappointment at the unsatisfactory result of that portion of the argument which, if conclusively handled, was likely to have proved the most impressive. I allude to that passage in which the Rev. Professor's reasoning assumed the tangible form of an equation. That this was regarded by both the disputants as a critical point, is abundantly apparent; and, in fact, if the general theory is "such, in its nature, as cannot be submitted to the test of experiment†", it is doubly requisite that the testimony of calculation should be clear. For these reasons, the liberty which I use in recalling attention to that particular point will be the more readily excused by the gentlemen who have already agitated the question.

In page 200 of the Number of this Magazine already referred to, Mr. Moseley, in considering the case of "a pressure equally divided between *three* points of support in the same right line," gives the following as "the two equations of equilibrium:"

$$\text{“}\frac{\alpha}{x} + \frac{\alpha}{x-b} + \frac{\alpha}{x-c} = \mathrm{M}$$

$$\frac{\alpha b}{x-b} + \frac{\alpha c}{x-c} = \mathrm{M}a.\text{”}$$

The resultant of this pair of equations he finds, with the tacit assent of Mr. Earnshaw, to be a mere *quadratic*, which, however, seems to have been regarded by each party as having but one root; and in consequence of this complicated oversight, the dispute settled down into a discussion of the point of legitimacy

* Communicated by the Author.

† See Phil. Mag. and Annals, March 1834, p. 194.

between two roots, neither of which has been proved to be relevant to the argument. What is still more strange, that root which has been most strenuously claimed as auxiliary to one side of the cause, proves to be quite adverse to it. Under all the circumstances, it is not without a sentiment approaching to diffidence that I venture to make these assertions; and especially because the objections I have to offer to the method employed in reducing the proposed equations are so palpable, that it is difficult to dismiss the notion that they have been considered and overruled; but on what sound principle of reasoning I cannot conjecture.

1. If the equations contain but one unknown quantity, one of them suffices for the solution, and the other is either superfluous or contradictory; and so, *à fortiori*, is the third equation, which has been derived from them.

2. If two unknowns are involved in each equation, either two new equations must be formed by elimination, or if only one subsidiary equation is employed, the result must at all events be introduced into the original statements.

3. A third exception applies to the mode of obtaining the subsidiary equation, namely, by taking the quotients of the separate scales. If it were stated that $x^3 = a$, and $x = b$, would it follow that $\therefore x^2 = \frac{a}{b}$, and $x = \pm\sqrt{\frac{a}{b}}$? Assuredly not.

4. This objection is still more valid when the quantities exterminated by division are zero or infinite. That this impediment exists in the present instance will appear on adopting a mode of reduction exempt from the faults above recited; *e. g.*

Dividing by α, the equations become

$$\frac{1}{x} + \frac{1}{x-b} + \frac{1}{x-c} = \frac{M}{\alpha} \quad \ldots \ldots \quad (1.)$$

$$\frac{b}{x-b} + \frac{c}{x-c} = \frac{Ma}{\alpha} \quad \ldots \ldots \quad (2.)$$

Deducting (2.) from a times (1.),

$$\frac{a}{x} + \frac{a-b}{x-b} + \frac{a-c}{x-c} = 0 \quad \ldots \ldots \quad (3.)$$

To prove this to be a complete *cubic* equation, it would be abundantly sufficient to make $x = \frac{1}{z-\gamma}$, and reduce. The result would be a cubic, complete in all its terms. In fact, merely say $x = \frac{1}{z}$, and we have

$$\left(a + \frac{a-b}{1-bz} + \frac{a-c}{1-cz}\right) z = 0 \quad \dots \quad (4.)$$

which, when cleared of fractions, will be an equation of *three* dimensions.

If the parenthetic portion of (4.) is made $= 0$, and reduced, the result will be a quadratic agreeing with Mr. Moseley's. But here is, besides, a third root, $z = 0$, which gives $x = \frac{1}{0}$ and $\frac{M}{\alpha} = 0$ in each equation (1.), (2.). Q. E. D.

Again, this infinite x not only solves both the original equations, it does so to the exclusion of any other infinite x presumed to be deducible from any relation incident to a, b, c. For it reduces (1.) (2.) (3.) to *simple* and *dependent* equations, viz. $\frac{3}{x} = 0$, $\frac{b+c}{x} = 0$, $\frac{3a-b-c}{x} = 0$, which all merge in $\frac{1}{x} = 0$; the condition $3a-b-c = 0$ being quite superfluous and nugatory.

In some views of the equation $\frac{M}{\alpha} = 0$, upon which the infinity of x depends, physical and analytical considerations are inseparable, and results are obtained which confirm what has just been alleged; *e.g.* it distinctly announces either that the *mass* M has *absolutely no* weight, or else that the standard moment of *pressure* α is *infinite*. The latter is of course the alternative to be preferred, as it cannot be Mr. Moseley's design to discuss the relations of weight and pressure in masses absolutely destitute of weight. But if α is infinite, what becomes of its constancy? "Let α represent a constant quantity." Or, granting that, what becomes of the quantities $A = \frac{\alpha}{x}$, $B = \frac{\alpha}{x-b}$, $C = \frac{\alpha}{x-c}$? Can they be severally affirmative and infinite, although their sum is $=$ the finite quantity M? If not, x must have been infinite from the first, and irrespectively of a, b, c.

After all, the only way of arriving at conclusions perfectly satisfactory is by regular elimination; the labour of which, even by the easiest methods, is in this instance considerable. I have, however, undertaken it, not only for the sake of epitomizing the present argument, but in the hope of supplying students with a pretty addition to their collections of examples. Reducing (1.), (2.), and making $y = \frac{a}{M}$, $s = b + c$, $p = b\,c$, the equations to be solved are

$$x^3 - (s+3y)\,x^2 + (p+2sy)\,x - py = 0 \;\ldots\;\ldots\; \text{(A.)}$$

$$x^2 - \left(s - \frac{sy}{a}\right)x + \left(p + \frac{2py}{a}\right) = 0 \;\ldots\;\ldots\; \text{(B.)}$$

Eliminating y by the method of the common measure, we find

$$(3a-s)\,x^2 - 2(as-p)\,x + ap = 0 \;\ldots\;\ldots\; \text{(C.)}$$

as determined by Mr. Moseley. But if we also eliminate x by the method of combinations, we obtain a formula in y which, when arranged, resolves itself into the factors

$$\frac{(4p-s^2)\,py^2}{a^3} = 0 \;\ldots\;\ldots\;\ldots\; \text{(D.)}$$

and

$$3(3a-s)\,y^2 - 2(3a^2 - 2as + p)\,y + (a^2 - as + p)\,a = 0 \quad \text{(E.)}$$

Of these, equation (D.) shows that $y = 0$ yields two solutions, which are readily found to be $x = b$, $x = c$. But when these values are placed in (1.), (2.), the conditions appear to be rather eluded than satisfied, the equations being reduced to a balance of infinity.

Equations (C.) (E.) being compared, in the event of $s = 3\,a$, which leads to infinite values of x and y, which answer to each other, and finite values which likewise correspond, give in the former case $x = 3y$. The same result is obtained directly from either (A.) or (B.), their mutual independence and their dependence upon a, b, c, being simultaneously destroyed by the same hypothesis, viz. that of the infinity of x, necessarily involving that of y.

The corresponding finite values are $y = \frac{p-2a^2}{3a^2-p} \cdot \frac{a}{2}$, and $x = \frac{ap}{2(3a^2-p)}$, which appear, therefore, to furnish the only legitimate solution of the pair of equations in the event which has been insisted upon.

It is for Mr. Moseley to determine how far his principle is affected by these results. My concern has been to solve a curious difficulty connected with equations in a case where the application of some of the ordinary rules has proved illusive. The general equation, of which (3.) is a partial case, deserves attention.