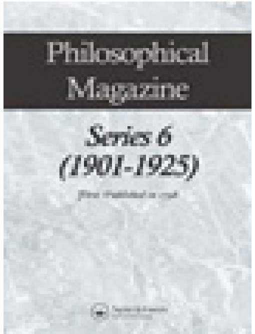

Philosophical Magazine Series 6

ISSN: 1941-5982 (Print) 1941-5990 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm17

XXXIV. On plane waves of sound

J.R. Wilton M.A. B.Sc.

Hence, at a considerable distance from the centre amplitude is $\frac{V^2AK^{\frac{3}{2}}}{r}$, this being where $\frac{1}{Kr}$ is small.

Now returning to equation (25), if plate extends to infinity V must be zero when x is infinite,

$$i.\ e.\quad V = A\{(\gamma - \log 2)\, J_0(p\sqrt{KLx}) + Y_0(p\sqrt{KLx})\}.$$

Hence, when x is large, but not infinite,

$$V \doteq \frac{A\sqrt{\pi}}{\sqrt{\{2\sqrt{(KL)px}\}}} \sin\left(\frac{\pi}{4} - \sqrt{(KL)px}\right). \quad . \quad (31)$$

Hence, comparing (30) and (31), we see that at large distances from the origin on the elementary theory, the amplitude of the oscillation is inversely proportional to the square root of the distance, and when radiation is taken into account, inversely proportional to the distance.

In conclusion I must express my thanks to Prof. Howe, of the City and Guilds (Eng.) College, to whom I am indebted, both for the suggestion of the original problem which led to the deductions in the paper, and also for many helpful criticisms in the course of the work.

XXXIV. *On Plane Waves of Sound.* *By* J. R. Wilton, *M.A., B.Sc., Assistant Lecturer in Mathematics in the University of Sheffield**.

1. IT is well known that the exact equation for plane waves of sound leads to a result which cannot hold beyond a certain time, owing to the fact that the motion becomes discontinuous †. The proof given in the text-books depends on Earnshaw's solution of the equation of motion, which assumes a relation between the velocity and the rate of variation of the displacement with the position. There does not seem to be any reason for this relation, and the fact of discontinuity does not depend upon it. The following paper was undertaken with a view to discovering when discontinuity sets in in any given case, the initial displacement and velocity being arbitrarily assigned. Incidentally it will be shown that the ordinary approximate solution, in which the displacement is regarded as a small quantity whose square may be neglected, does not begin to depart widely from the truth until the

* Communicated by the Author.

† See, for example, Lord Rayleigh's 'Sound,' 2nd edit. vol. ii. p. 36.

motion approaches the stage in which it becomes discontinuous.

2. We take the equation of motion in Earnshaw's form, using the notation of equation (4), paragraph 249, of Lord Rayleigh's 'Sound,' namely,

$$\left(\frac{\partial y}{\partial x}\right)^{\gamma+1}\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $a^2=p_0\gamma/\rho_0$, and γ is the ratio of the specific heats.

The general solution of equation (1) may be put in various forms, of which one is the envelope of the plane (x, t, y being the current coordinates):

$$px+qt-y=p^{\frac{1}{2}}\sum_{n=1}^{\infty}\left\{A_nJ_{\frac{1}{\gamma-1}}\left(\frac{2na}{\gamma-1}p^{-\frac{\gamma-1}{2}}\right)\right.$$
$$\left.+B_nJ_{-\frac{1}{\gamma-1}}\left(\frac{2na}{\gamma-1}p^{-\frac{\gamma-1}{2}}\right)\right\}\cos(nq+c_n),$$

where p and q are considered as variable parameters, and A_n, B_n, c_n are constants.

The form of this solution shows that when $1/(\gamma-1)$ is the half of an odd integer, *e. g.* $\gamma=1{\cdot}4$, the solution will be expressible in finite terms.

We proceed to derive the solution, in the particular case mentioned, in a form which will be more immediately useful. First, by Legendre's transformation (The Principle of Duality), we obtain

$$p^{\gamma+1}\frac{\partial^2 u}{\partial p^2}=a^2\frac{\partial^2 u}{\partial q^2},$$

where

$$p=\frac{\partial y}{\partial x},\quad q=\frac{\partial y}{\partial t},\quad u=px+qt-y.$$

Putting $u=pv$, the equation becomes

$$p^{\gamma+1}\frac{\partial^2 v}{\partial p^2}+2p^{\gamma}\frac{\partial v}{\partial p}=a^2\frac{\partial^2 v}{\partial q^2},$$

and, finally, putting

$$\xi=\frac{2a}{\gamma-1}p^{-\frac{\gamma-1}{2}}+q,\quad \eta=\frac{2a}{\gamma-1}p^{-\frac{\gamma-1}{2}}-q,$$

we have

$$\frac{\partial^2 v}{\partial\xi\partial\eta}=\frac{3-\gamma}{2(\gamma-1)}\frac{\frac{\partial v}{\partial\xi}+\frac{\partial v}{\partial\eta}}{\xi+\eta}. \quad . \quad . \quad . \quad . \quad (2)$$

3. In the particular case when $\gamma=1\cdot 4$, its approximate value, equation (2) becomes

$$\frac{\partial^2 v}{\partial\xi\partial\eta}=2\frac{\left(\frac{\partial v}{\partial\xi}+\frac{\partial v}{\partial\eta}\right)}{\xi+\eta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and in this form it may be solved by Darboux' method, and —though with so much labour as to make it impracticable— the solution corresponding to any initial conditions may be obtained. Each set of characteristics, with the above value of γ, has two integrable combinations, namely,

$$d\xi=0, \qquad d\eta=0,$$

$$d\alpha=2\alpha\, d\eta/(\xi+\eta), \qquad d\delta=2\delta\, d\eta/(\xi+\eta)\;;$$

whence

$$\alpha=(\xi+\eta)^2\phi^{\mathrm{V}}(\xi), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\delta=(\xi+\eta)^2\psi^{\mathrm{V}}(\eta), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where ϕ and ψ are arbitrary functions of their arguments, and α, β, γ, δ are the partial derivatives of the third order. ϕ^{V}, ψ^{V} denote the fifth differential coefficients of their arguments. The general solution of equation (3) is therefore

$$v=(\xi+\eta)^2\phi''(\xi)-6(\xi+\eta)\phi'(\xi)+12\phi(\xi)$$
$$+(\xi+\eta)^2\psi''(\eta)-6(\xi+\eta)\psi'(\eta)+12\psi(\eta). \quad . \quad (6)$$

We have to choose the arbitrary functions ϕ and ψ so as to satisfy given initial conditions. These conditions may be satisfied by making the surface (6) pass through a given curve, and touch a given developable all along that curve. We may take, that is to say, ξ, η, v, $\frac{\partial v}{\partial\xi}$, $\frac{\partial v}{\partial\eta}$, to be, on this curve, given functions of a single variable λ, subject to the condition

$$\frac{dv}{d\lambda}=\frac{\partial v}{\partial\xi}\frac{d\xi}{d\lambda}+\frac{\partial v}{\partial\eta}\frac{d\eta}{d\lambda},$$

say

$$\xi=\xi_0(\lambda), \quad \eta=\eta_0(\lambda), \quad v=v_0(\lambda), \quad \frac{\partial v}{\partial\xi}=p_0(\lambda), \quad \frac{\partial v}{\partial\eta}=q_0(\lambda),$$

where

$$v_0'=p_0\xi_0'+q_0\eta_0'.$$

These equations will give us the values of r, s, t, α, β, γ, δ, on the curve, where r, s, t denote $\frac{\partial^2 v}{\partial \xi^2}$, $\frac{\partial^2 v}{\partial \xi \partial \eta}$, $\frac{\partial^2 v}{\partial \eta^2}$, say r_0, s_0, t_0, α_0, β_0, γ_0, δ_0. For

$$\delta_0 = 2\frac{p_0+q_0}{\xi_0+\eta_0},$$

$$\beta_0 = 2\frac{r_0}{\xi_0+\eta_0}+2\frac{p_0+q_0}{(\xi_0+\eta_0)^2}, \quad \gamma_0 = 2\frac{t_0}{\xi_0+\eta_0}+2\frac{p_0+q_0}{(\xi_0+\eta_0)^2},$$

$$p_0' = r_0\xi_0'+s_0\eta_0', \quad q_0' = s_0\xi_0'+t_0\eta_0',$$

$$r_0' = \alpha_0\xi_0'+\beta_0\eta_0', \quad t_0' = \gamma_0\xi_0'+\delta_0\eta_0'.$$

To determine the forms of ϕ and ψ, we have then, from equations (4) and (5),

$$\phi^{\mathrm{V}}(\xi_0) = \frac{\alpha_0}{(\xi_0+\eta_0)^2}, \quad \psi^{\mathrm{V}}(\eta_0) = \frac{\delta_0}{(\xi_0+\eta_0)^2}. \quad . \quad . \quad (7)$$

ϕ and ψ are thus to be determined by quadratures, and the results when substituted in equation (6) give the solution required.

4. The initial conditions are given in terms of the x, y, and t of equation (1), say,

$$y = f(x), \quad \frac{\partial y}{\partial t} = \mathrm{F}(x), \quad \text{when } t=0\ ^*,$$

These conditions necessitate

$$\frac{\partial y}{\partial x} = f'(x) \text{ when } t=0.$$

When the initial conditions are transformed so as to apply to equation (3), they become

$$\xi_0 = \frac{2a}{\gamma-1}\{f'(\lambda)\}^{-\frac{\gamma-1}{2}}+\mathrm{F}(\lambda) = 5a\{f'(\lambda)\}^{-1/5}+\mathrm{F}(\lambda),$$

$$\eta_0 = \frac{2a}{\gamma-1}\{f'(\lambda)\}^{-\frac{\gamma-1}{2}}-\mathrm{F}(\lambda) = 5a\{f'(\lambda)\}^{-1/5}-\mathrm{F}(\lambda),$$

$$v_0 = \lambda-\frac{f(\lambda)}{f'(\lambda)},$$

$$p_0 = q_0 = -\frac{1}{a}f(\lambda)\{f'(\lambda)\}^{-\frac{3-\gamma}{2}} = -\frac{1}{a}f(\lambda)\{f'(\lambda)\}^{-4/5},$$

where λ is the x of equation (1).

* Earnshaw's solution (which is the same as Poisson's integral) of equation (1) assumes $\mathrm{F}(x)=\frac{2a}{\gamma-1}\{f'(x)\}^{-\frac{\gamma-1}{2}}$, *i. e.*, $\mathrm{F}(x)=5a\{f'(x)\}^{-1/5}$ in this case. There does not seem to be any reason for this relation between the initial displacement and the velocity.

These equations give

$$s_0 \equiv 2(p_0+q_0)/(\xi_0+\eta_0) = -2f(\lambda)/5a^2\{f'(\lambda)\}^{3/5}.$$

The integration of equations (7) cannot, evidently, be completed except when f and F are specified functions ; and even in the simplest case the labour of determining a solution is considerable.

5. It is, however, for our present purpose, unnecessary to go through this labour. Equation (2) reduces to

$$\frac{\partial^2 v}{\partial\xi\,\partial\eta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

when $\gamma=3$; and although this is widely different from its actual value, yet it will appear that, owing to the fact that $\frac{\partial y}{\partial x}$ is very nearly unity, the solution so obtained represents the facts with a fair degree of accuracy, and we shall be led to a second order approximation to the solution of equation (1) which, it so happens, holds for all values of γ.

In this case we may take the conditions to be satisfied in a rather more general form ; *e. g*, we may take

$$\left.\begin{array}{c} y=\lambda+\mathrm{Y}(\lambda)=\int(\mathrm{PX}'+\mathrm{QT}')d\lambda, \quad t=\mathrm{T}(\lambda), \quad x=\mathrm{X}(\lambda), \\ p=\mathrm{P}(\lambda), \qquad q=\mathrm{Q}(\lambda), \end{array}\right\}, \quad (9)$$

where p and q stand for $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$, respectively. From these equations we find, using the notation of the second paragraph, that

$$v = \frac{\mathrm{PX}+\mathrm{QT}-\lambda-\mathrm{Y}}{\mathrm{P}},$$

$$\xi = \frac{a}{\mathrm{P}}+\mathrm{Q}, \qquad \eta = \frac{a}{\mathrm{P}}-\mathrm{Q},$$

$$\frac{\partial v}{\partial \xi} = \frac{\mathrm{T}}{2\mathrm{P}}+\frac{\mathrm{QT}-\lambda-\mathrm{Y}}{2a}, \qquad \frac{\partial v}{\partial \eta} = -\frac{\mathrm{T}}{2\mathrm{P}}+\frac{\mathrm{QT}-\lambda-\mathrm{Y}}{2a}.$$

The solution of equation (8) under these conditions is * given by eliminating θ and τ between the equations

$$v = \int\left\{\frac{\mathrm{T}(\theta)}{2\mathrm{P}(\theta)}-\frac{\mathrm{Q}(\theta)\mathrm{T}(\theta)-\theta-\mathrm{Y}(\theta)}{2a}\right\}\left\{\mathrm{Q}'(\theta)+\frac{a\mathrm{P}'(\theta)}{\mathrm{P}^2(\theta)}\right\}d\theta$$

$$+\int\left\{\frac{\mathrm{T}(\tau)}{2\mathrm{P}(\tau)}+\frac{\mathrm{Q}(\tau)\mathrm{T}(\tau)-\tau-\mathrm{Y}(\tau)}{2a}\right\}\left\{\mathrm{Q}'(\tau)-\frac{a\mathrm{P}'(\tau)}{\mathrm{P}^2(\tau)}\right\}d$$

$$\xi = \frac{a}{\mathrm{P}(\tau} + \mathrm{Q}(\tau), \qquad \eta = \frac{a}{\mathrm{P}(\theta)}-\mathrm{Q}(\theta).$$

* See, for example, Goursat's 'Leçons sur l'Intégration des Equations aux dérivées Partielles du Second Ordre,' tom. i. p. 61 or 121.

Moreover,

$$\frac{\partial v}{\partial \xi}=\frac{\mathrm{T}(\tau)}{2\mathrm{P}(\tau)}+\frac{\mathrm{Q}(\tau)\mathrm{T}(\tau)-\tau-\mathrm{Y}(\tau)}{2a},$$

$$\frac{\partial v}{\partial \eta}=-\frac{\mathrm{T}(\theta)}{2\mathrm{P}(\theta)}+\frac{\mathrm{Q}(\theta)\mathrm{T}(\theta)-\theta-\mathrm{Y}(\theta)}{2a};$$

whence we find the solution of equation (1), when $\gamma=3$, *i. e.*,

$$\left(\frac{\partial y}{\partial x}\right)^4\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

subject to the conditions (9), in the form

$$\left(\frac{a}{\mathrm{P}_2}+\mathrm{Q}_2\right)\left(\frac{a\mathrm{T}_1}{\mathrm{P}_1}-\mathrm{Q}_1\mathrm{T}_1+\theta+\mathrm{Y}_1\right)$$

$$-\left(\frac{a}{\mathrm{P}_1}-\mathrm{Q}_1\right)\left(\frac{a\mathrm{T}_2}{\mathrm{P}_2}+\mathrm{Q}_2\mathrm{T}_2-\tau-\mathrm{Y}_2\right)\Bigg\} \Bigg/ \left\{a\left(\frac{1}{\mathrm{P}_1}+\frac{1}{\mathrm{P}_2}\right)-\mathrm{Q}_1+\mathrm{Q}_2\right\},$$

$$\Bigg\{\int\left(\frac{a\mathrm{T}_1}{\mathrm{P}_1}-\mathrm{Q}_1\mathrm{T}_1+\theta+\mathrm{Y}_1\right)\left(\mathrm{Q}_1'+\frac{a\mathrm{P}_1'}{\mathrm{P}_1^2}\right)d\theta$$

$$+\int\left(\frac{a\mathrm{T}_2}{\mathrm{P}_2}+\mathrm{Q}_2\mathrm{T}_2-\tau-\mathrm{Y}_2\right)\left(\mathrm{Q}_2'-\frac{a\mathrm{P}_2'}{\mathrm{P}_2^2}\right)d\tau$$

$$+\frac{1}{2}\left[a\left(\frac{1}{\mathrm{P}_1}+\frac{1}{\mathrm{P}_2}\right)-\mathrm{Q}_1+\mathrm{Q}_2\right]\left(\frac{a\mathrm{T}_1}{\mathrm{P}_1}-\frac{a\mathrm{T}_2}{\mathrm{P}_2}-\mathrm{Q}_1\mathrm{T}_1-\mathrm{Q}_2\mathrm{T}_2+\theta+\tau+\mathrm{Y}_1+\mathrm{Y}_2\right)\Bigg\}$$

$$\frac{a\mathrm{T}_1}{\mathrm{P}_1}+\frac{a\mathrm{T}_2}{\mathrm{P}_2}-\mathrm{Q}_1\mathrm{T}_1+\mathrm{Q}_2\mathrm{T}_2+\theta-\tau+\mathrm{Y}_1-\mathrm{Y}_2\Bigg)\Bigg/\left\{a\left(\frac{1}{\mathrm{P}_1}+\frac{1}{\mathrm{P}_2}\right)-\mathrm{Q}_1+\mathrm{Q}_2\right\},$$

where the subscript 1 means that the variable is θ, and the subscript 2 that the variable is τ. We also have

$$\frac{\partial y}{\partial x}\equiv p=2a\Big/\left\{a\left(\frac{1}{\mathrm{P}_1}+\frac{1}{\mathrm{P}_2}\right)-\mathrm{Q}_1+\mathrm{Q}_2\right\},$$

$$\frac{\partial y}{\partial t}\equiv q=\frac{1}{2}\left\{a\left(\frac{1}{\mathrm{P}_1}-\frac{1}{\mathrm{P}_2}\right)-\mathrm{Q}_1-\mathrm{Q}_2\right\}.$$

The solution evidently satisfies the conditions (9) when we put $\theta=\tau$ $(=\lambda)$. We obtain the solution corresponding to given initial values of the displacement and velocity by putting $\mathrm{T}=0$. Or we may obtain the solution corresponding to given values of y and $\frac{\partial y}{\partial x}$ when $x=0$, by putting $\mathrm{X}=0$.

The conditions might be given in other forms; for instance, one might wish to find the solution such that the displacement was a given function of x when $t=0$, and a given function of t when $x=0$. Such solutions may be obtained fairly straightforwardly, but their importance is not such as to warrant our delaying to investigate them in this connexion.

Putting $Q=aZ'$, $T=0$, and therefore $P=1+Y'$, we find after a slight reduction that the solution of equation (10), subject to the conditions

$$y = x+Y(x), \quad q = aZ'(x), \quad \text{when } t=0,$$

is given by

$$y = \frac{\left\{\left(\frac{1}{1+Y_1'}-Z_1'\right)(\tau+Y_2)+\left(\frac{1}{1+Y_2'}+Z_2'\right)(\theta+Y_1)\right\}}{\frac{1}{1+Y_1'}+\frac{1}{1+Y_2'}-Z_1'+Z_2'},$$

$$x = \tfrac{1}{2}(\theta+\tau)-\tfrac{1}{2}(Z_1-Z_2)-\tfrac{1}{2}\int(Y_1'Z_1'd\theta-Y_2'Z_2'd\tau)$$
$$-\tfrac{1}{4}(\theta-\tau+Y_1-Y_2)\left(\frac{1}{1+Y_1'}-\frac{1}{1+Y_2'}-Z_1'-Z_2'\right),$$

$$at = (\theta-\tau+Y_1-Y_2)\Big/\left(\frac{1}{1+Y_1'}+\frac{1}{1+Y_2'}-Z_1'+Z_2'\right).$$

This solution cannot always represent the motion, for it becomes discontinuous after a certain time. We shall return to this point in paragraph 7.

6. To compare this with the ordinary approximate solution, namely,

$$y = x+\tfrac{1}{2}Y(x+at)+\tfrac{1}{2}Y(x-at)+\tfrac{1}{2}Z(x+at)-\tfrac{1}{2}Z(x-at),$$

we must expand y in terms of functions of $x+at$ and of $x-at$. To do this we make use of the fact that y is nearly equal to x—i. e., Y is small.

The reduction is long, but it is a good deal simplified by taking it in two stages. We consider first the case where $Z=0$, and retain only terms of the second order in the values of x, y, and t. We find

$$y = \tfrac{1}{2}(\theta+\tau)+\tfrac{1}{4}(\theta-\tau)(Y_1'-Y_2')[1-\tfrac{1}{2}(Y_1'+Y_2')]$$
$$+\tfrac{1}{2}(Y_1+Y_2)+\tfrac{1}{4}(Y_1-Y_2)(Y_1'-Y_2'),$$

$$x+at = \theta+\tfrac{1}{2}(Y_1-Y_2)+\tfrac{1}{2}(Y_1-Y_2)Y_1'$$
$$+\tfrac{1}{2}(\theta-\tau)[Y_1'-\tfrac{1}{4}(Y_1'-Y_2')(3Y_1'+Y_2')],$$

$$x-at = \tau-\tfrac{1}{2}(Y_1-Y_2)-\tfrac{1}{2}(Y_1-Y_2)Y_2'$$
$$-\tfrac{1}{2}(\theta-\tau)[Y_2'+\tfrac{1}{4}(Y_1'-Y_2')(Y_1+3Y_2')].$$

And after reduction we find, as a second order approximation to the solution of equation (10), such that

$$y = x + \mathrm{Y}(x), \quad q = 0, \quad \text{when } t=0,$$

$$\begin{aligned} y = x + \tfrac{1}{2}[\mathrm{Y}(x+at) + \mathrm{Y}(x-at)] \\ -\tfrac{1}{4}[\mathrm{Y}(x+at) - \mathrm{Y}(x-at)][\mathrm{Y}'(x+at) - \mathrm{Y}'(x-at)] \\ -\tfrac{1}{4}at[\mathrm{Y}'^2(x+at) - \mathrm{Y}'^2(x-at)]. \end{aligned}$$

The solution in this form is very easily verified. And in the process of verification we see that the second-order approximation corresponding to the conditions

$$y = x, \quad q = a\mathrm{Z}'(x), \quad \text{when } t=0,$$

is

$$\begin{aligned} y = x + \tfrac{1}{2}[\mathrm{Z}(x+at) - \mathrm{Z}(x-at)] \\ +\tfrac{1}{4}[\mathrm{Z}(x+at) - \mathrm{Z}(x-at)][\mathrm{Z}'(x+at) - \mathrm{Z}'(x-at)] \\ -\tfrac{1}{4}at[\mathrm{Z}'^2(x+at) - \mathrm{Z}'^2(x-at)]. \end{aligned}$$

This solution might, of course, have been obtained in the same way as the other by assuming $\mathrm{Y}=0$.

We note that the last two results are true for the general equation (1), provided that we substitute $\frac{\gamma+1}{16}$ for $\frac{1}{4}$ in the four places in which it occurs. This follows by verification.

We proceed to obtain the second order approximation to that solution of equation (1), which is such that

$$y = x + \mathrm{Y}(x), \quad q = a\mathrm{Z}'(x), \quad \text{when } t=0.$$

We write Y_1 for $\mathrm{Y}(x+at)$, Y_2 for $\mathrm{Y}(x-at)$, &c., and assume

$$\begin{aligned} y = x + \tfrac{1}{2}(\mathrm{Y}_1 + \mathrm{Y}_2) - \frac{\gamma+1}{16}(\mathrm{Y}_1 - \mathrm{Y}_2)(\mathrm{Y}_1' - \mathrm{Y}_2') - \frac{\gamma+1}{16}at(\mathrm{Y}_1'^2 - \mathrm{Y}_2'^2) \\ + \tfrac{1}{2}(\mathrm{Z}_1 - \mathrm{Z}_2) + \frac{\gamma+1}{16}(\mathrm{Z}_1 - \mathrm{Z}_2)(\mathrm{Z}_1' - \mathrm{Z}_2') - \frac{\gamma+1}{16}at(\mathrm{Z}_1'^2 - \mathrm{Z}_2'^2) \\ + \phi(x+at,\ x-at). \end{aligned}$$

The first-order terms in $\frac{\partial y}{\partial x}$ and $\frac{\partial^2 y}{\partial t^2}$ are

$$\frac{\partial y}{\partial x} = \tfrac{1}{2}(\mathrm{Y}_1' + \mathrm{Y}_2') + \tfrac{1}{2}(\mathrm{Z}_1' - \mathrm{Z}_2')$$

$$\frac{\partial^2 y}{\partial t^2} = \tfrac{1}{2}a^2(\mathrm{Y}_1'' + \mathrm{Y}_2'') + \tfrac{1}{2}a^2(\mathrm{Z}_1'' - \mathrm{Z}_2'');$$

wherefore $\frac{\partial y}{\partial x}$ contributes to $\left(\frac{\partial y}{\partial x}\right)^{\gamma+1}\left(\frac{\partial^2 y}{\partial t^2}\right)$ a second order term

$$\frac{\gamma+1}{4}a^2[(Y_1'+Y_2')+(Z_1'-Z_2')][(Y_1''+Y_2'')+(Z_1''-Z_2'')],$$

of which only the parts $\frac{\gamma+1}{4}a^2(Y_1'+Y_2')(Y_1''+Y_2'')$, and $\frac{\gamma+1}{4}a^2(Z_1'-Z_2')(Z_1''-Z_2'')$, are accounted for without invoking the aid of the function ϕ. This function must therefore be such that

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2}=\frac{\gamma+1}{4}[(Y_1'+Y_2')(Z_1''-Z_2'')+(Y_1''+Y_2'')(Z_1'-Z_2')], \quad (11$$

with the conditions that ϕ, $\frac{\partial\phi}{\partial x}$, and $\frac{\partial\phi}{\partial t}$ are all zero when $t=0$.

Put $x+at=\theta$, $x-at=\tau$, so that Y_1 and Z_1 are functions of θ, Y_2 and Z_2 are functions of τ. Equation (11) then becomes

$$\frac{\partial^2\phi}{\partial\theta\,\partial\tau}=\frac{\gamma+1}{16}[(Y_1'+Y_2')(Z_1''-Z_2'')+(Y_1''+Y_2'')((Z_1'-Z_2')],$$

with the conditions

$$\phi=\frac{\partial\phi}{\partial\theta}=\frac{\partial\phi}{\partial\tau}=0, \quad \text{when } \theta=\tau.$$

The solution is

$$\phi=\frac{\gamma+1}{16}[\psi(\theta)-\psi(\tau)-\theta Y_2'Z_2'+\tau Y_1'Z_1'+Y_2Z_1'-Y_1Z_2'+Y_2'Z_1-Y_1'Z_2]$$

where the function ψ is to be determined by the conditions

$$\frac{\partial\phi}{\partial\theta}=\frac{\partial\phi}{\partial\tau}=0, \quad \text{when } \theta=\tau,$$

which reduce to one. We find

$$\psi(\theta)=2\int Y_1'Z_1'\,d\theta-\theta\,Y_1'Z_1'-Y_1Z_1'+Y_1'Z;$$

and, finally, we obtain as the second order approximation to the solution of equation (1)

$$y=x+\tfrac{1}{2}(Y_1+Y_2)+\tfrac{1}{2}(Z_1-Z_2)-\frac{\gamma+1}{16}(Y_1'-Y_2')[Y_1-Y_2+at(Y_1'+Y_2')$$
$$+\frac{\gamma+1}{16}(Z_1'-Z_2')[Z_1-Z_2-at(Z_1'+Z_2')]+\frac{\gamma+1}{16}\left\{2\int_{x-at}^{x+at}Y'(\theta)\,Z'(\theta)\,dt\right.$$
$$\left.-2at(Y_1'Z_1'+Y_2'Z_2')+(Y_1'+Y_2')(Z_1-Z_2)-(Y_1-Y_2)(Z_1'+Z_2')\right\}$$
$$\quad . \quad . \quad . \quad . \quad (1$$

7. We have now to determine when the motion becomes discontinuous. For this purpose we must return to the exact equations at the end of paragraph 5. These equations assume that $\gamma=3$, but it is evident that they give a very fair representation of what actually happens in a plane wave of sound,—assuming, of course, that the equation (1) takes account of all the facts. The form of the second order approximation, for example, in the case when $\gamma=3$ is exactly the same as when γ has its general value.

The motion becomes discontinuous when one of the three partial derivatives of the second order, $\frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial x \partial t}, \frac{\partial^2 y}{\partial t^2}$ becomes infinite. It is evident, however, that they all become infinite together, so that we need consider only one of them.

We have

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial p}{\partial \tau}\frac{\partial t}{\partial \theta} - \frac{\partial p}{\partial \theta}\frac{\partial t}{\partial \tau}\right) \Big/ \left(\frac{\partial x}{\partial \tau}\frac{\partial t}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial t}{\partial \tau}\right),$$

where $p = \frac{\partial y}{\partial x}$; so that $\frac{\partial^2 y}{\partial x^2}$ becomes infinite when

$$\frac{\partial x}{\partial \tau}\frac{\partial t}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial t}{\partial \tau} = 0,$$

which after reduction leads to

$$(1+Y_2')^2[(2+Y_1'+Y_2')-(1+Y_1')(1+Y_2')(Z_1'-Z_2')] \\ =(1+Y_1')(\theta-\tau+Y_1-Y_2)[Y_2''-Z_2''(1+Y_2')^2], \quad (13)$$

or

$$(1+Y_1')^2[(2+Y_1'+Y_2')-(1+Y_1')(1+Y_2')(Z_1'-Z_2')] \\ +(1+Y_2')(\theta-\tau+Y_1-Y_2)[Y_1''+Z_1''(1+Y_1')^2]=0. \quad (14)$$

To find the time corresponding to these equations, we have

$$at = (\theta-\tau+Y_1-Y_2)\Big/\left\{\frac{1}{1+Y_1'}+\frac{1}{1+Y_2'}-Z_1'+Z_2'\right\};$$

i. e.,

$$at = (1+Y_2')^3/[Y_2''-Z_2''(1+Y_2')^2], \quad . \quad . \quad (15)$$

or

$$at = -(1+Y_1')^3/[Y_1''+Z_1''(1+Y_1')^2]. \quad . \quad (16)$$

The time required is the smallest positive value of t

given by equation (15) or (16) when θ and τ are given by equation (13) or (14) and

$$x = \tfrac{1}{2}(\theta+\tau)-\tfrac{1}{2}\int(1+Y_1')Z_1'd\theta+\tfrac{1}{2}\int(1+Y_2')Z_2'd\tau$$
$$-\tfrac{1}{4}(\theta-\tau+Y_1-Y_2)\left\{\frac{1}{1+Y_1'}-\frac{1}{1+Y_2'}-Z_1'-Z_2'\right\}.$$

If we retain only the terms of lowest order,

$$\left.\begin{aligned} at &= \tfrac{1}{2}(\theta-\tau) = 1/(Y_2''-Z_2''), \\ x &= \tfrac{1}{2}(\theta+\tau)+\tfrac{1}{4}(Y_1'-Y_2'+Z_1'+Z_2')(\theta-\tau), \end{aligned}\right\} \quad . \quad . \quad (17)$$

or

$$\left.\begin{aligned} &at = \tfrac{1}{2}(\theta-\tau) = -1/(Y_1''+Z_1''), \\ &\text{with the same value of } x \text{ as in equation (17).} \end{aligned}\right\} \quad . \quad . \quad (18)$$

8. A glance at equation (12) shows that the first order approximation

$$y = x+\tfrac{1}{2}[Y(x+at)+Y(x-at)+Z(x+at)-Z(x-at)]$$

ceases to hold when

$$at(Y_1'^2-Y_2'^2+Z_1'^2-Z_2'^2+2Y_1'Z_1'+2Y_2'Z_2')-2\int_{x-at}^{x+at}Y_1'Z_1'd\theta$$

ceases to be small compared with

$$Y_1+Y_2+Z_1-Z_2,$$

where Y_1 and Z_1 are functions of $x+at$, Y_2, Z_2 of $x-at$.

It is evident that t is in general of the order of magnitude of the least value of the reciprocal of Y, which is the same order as that given by equation (17) or (18).

In the particular case when $Y+Z=0$, which, in the ordinary approximate solution, represents a single progressive wave,

$$y = x+Y(x-at),$$

the second order approximation is

$$y = x+Y(x-at)+\frac{\gamma+1}{8}\left[2atY'^2+\int_{x-at}^{x+at}Y'^2(\theta)d\theta\right],$$

which ceases to hold after a time t whose order of magnitude is given by

$$at = Y_2/Y_2'^2\,;$$

i.e., *at* is of the order of the reciprocal of the amplitude of the wave.

The motion becomes discontinuous after time t_1, where

$$at_1 = 1/2Y_2'' = \tfrac{1}{2}(\theta-\tau),$$

θ and τ being connected by the relation

$$x = \tfrac{1}{2}(\theta+\tau)-\tfrac{1}{2}Y_2'(\theta-\tau) = \tfrac{1}{2}(\theta+\tau)-Y_2'/2Y_2''.$$

To take a simple numerical example, suppose

$$Y(x-at) = A\sin[\pi(x-at)/c].$$

Here

$$at = c^2/\pi^2 A, \qquad at_1 = -c^2/(2\pi^2 A \sin \pi\tau/c),$$

where

$$\theta-\tau = -\frac{c^2}{\pi^2 A}\operatorname{cosec}\frac{\pi\tau}{c},$$

$$\theta+\tau = 2x-\frac{c}{\pi}\cot\frac{\pi\tau}{c};$$

so that

$$\tau = x+\frac{c^2}{2\pi^2 A}\operatorname{cosec}\frac{\pi\tau}{c}-\frac{c}{2\pi}\cot\frac{\pi\tau}{c}.$$

For a low note, just audible, we may take $c=200$ cm., *i.e.* a frequency of 80, and $A=10^{-6}$ cm.*; so that the equation for τ is

$$\tau = x+2\,.\,10^9\operatorname{cosec}\frac{\pi\tau}{200}-\frac{100}{\pi}\cot\frac{\pi\tau}{200},$$

whence, for moderate distances, $\tau=-(2\,.\,10^9+100)$ is the smallest negative root, and

$$at_1 = \tfrac{1}{2}(\theta-\tau) - x-\tau = 2\,.\,10^9,$$

i. e.,

$$t_1 = 10^5 \text{ sec. nearly};$$

while t is of the same order of magnitude. The motion of the air due to a low, barely audible note is therefore such that viscosity and other influences will cause the motion to cease long before discontinuity sets in.

But a high, loud note, on the other hand, gives rise to a motion which instantly becomes discontinuous. Let us take, for instance, $c=2$ cm., *i. e.* a frequency of 8500, and $A=10^{-2}$ cm.† In this case the equation for τ is

$$\tau = x+20\operatorname{cosec}\frac{\pi\tau}{2}-\frac{1}{\pi}\cot\frac{\pi\tau}{2}.$$

* This is well within the range of audibility. See Rayleigh's 'Sound,' vol. ii. § 384, p. 439.

† This is the amplitude of the sound-wave, at a distance of 1 cm. from the source, in the experiment described on pp. 434 & 435 of Lord Rayleigh's 'Sound' (vol. ii.).

Whatever the value of x, the smallest negative root of this equation lies between $\tau=-20$ and $\tau=-22$; so that

$$at_1 = \tfrac{1}{2}(\theta-\tau) = x-\tau;$$

i. e.,
$$t_1 = \frac{x+20}{34{,}000} = 10^{-3} \text{ sec. nearly,}$$

if x is not greater than about 20 cm., and t is of the same order.

It is evident that the value of t_1 cannot greatly vary with x. The reason for the appearance of x in this equation is that $\frac{1}{2}(\theta-\tau)$ was treated as large in forming the approximate values of x and t in equations (17).

The result $t_1=10^{-3}$ lends weight to the hypothesis that a very shrill note, *e. g.* a hiss, is not propagated in the same manner as an ordinary sound-wave.

When conductivity of heat and viscosity are taken into account, it has been shown by G. I. Taylor * that the motion in an ordinary sound-wave does not become discontinuous. The present paper is to be taken merely as an attempt to follow out the motion, before discontinuity ensues, on the assumption that equation (1) takes account of all the facts. With what happens in the neighbourhood of a discontinuity, when viscosity and conductivity are both zero, we are not here concerned, and the problem is one of little physical importance, though from a purely mathematical point of view it possesses a certain interest.

XXXV. *The Ions from Hot Salts.*

By O. W. Richardson, *M.A., D.Sc., F.R.S., Professor of Physics, Princeton University* †.

THIS paper describes the results of experiments which form a continuation of earlier work carried out in this laboratory, partly by the writer ‡ and partly by Dr. C. J. Davisson §. Some of the main points have already been briefly indicated ||. The chief object of the investigation has been the measurement of the specific charge or the electric molecular weight of the emitted ions; but in certain cases other points which seemed to be of interest or importance

* Proceedings of the Royal Society, A. vol. lxxxiv. 1910, pp. 371–7.
† Communicated by the Author.
‡ Phil. Mag. vol. xx. pp. 981, 999 (1910); vol. xxii. p. 669 (1911).
§ Phil. Mag. vol. xxiii. pp. 121, 139 (1912).
|| Phys. Rev. vol. xxxiv. p. 386 (1912).